ANALYSIS OF THE HODGE LAPLACIAN ON THE HEISENBERG GROUP

DETLEF MÜLLER, MARCO M. PELOSO, AND FULVIO RICCI

Abstract. We consider the Hodge Laplacian Δ on the Heisenberg group H_n, endowed with a left-invariant and $U(n)$-invariant Riemannian metric. For $0 \le k \le 2n+1$, let Δ_k denote the Hodge Laplacian restricted to k-forms.

Our first main result shows that $L^2\Lambda^k(H_n)$ decomposes into finitely many mutually orthogonal subspaces $\mathcal{V}_\nu$ with the properties:

- $\operatorname{dom}\Delta_k$ splits along the $\mathcal{V}_\nu$'s as $\sum_\nu(\operatorname{dom}\Delta_k \cap \mathcal{V}_\nu)$;
- $\Delta_k : (\operatorname{dom}\Delta_k \cap \mathcal{V}_\nu) \longrightarrow \mathcal{V}_\nu$ for every ν;
- for each ν, there is a Hilbert space $\mathcal{H}_\nu$ of L^2-sections of a $U(n)$-homogeneous vector bundle over H_n such that the restriction of Δ_k to $\mathcal{V}_\nu$ is unitarily equivalent to an explicit scalar operator.

Next, we consider $L^p\Lambda^k$, $1<p<\infty$, and prove that the same kind of decomposition holds true. More precisely we show that:

- the Riesz transforms $d\Delta_k^{-\frac12}$ are L^p-bounded;
- the orthogonal projection onto $\mathcal{V}_\nu$ extends from $(L^2\cap L^p)\Lambda^k$ to a bounded operator from $L^p\Lambda^k$ to the the L^p-closure $\mathcal{V}_\nu^p$ of $\mathcal{V}_\nu \cap L^p\Lambda^k$.

We then use this decomposition to prove a Mihlin–Hörmander multiplier theorem for each Δ_k. We show that the operator $m(\Delta_k)$ is bounded on $L^p\Lambda^k(H_n)$ for all $p\in(1,\infty)$ and all $k=0,\dots,2n+1$, provided m satisfies a Mihlin–Hörmander condition of order $\rho>(2n+1)/2$. We also prove that this restriction on ρ is optimal and extend this result to the Dirac operator.

Contents

2000 *Mathematical Subject Classification.* 43A80, 42B15.

Key words and phrases. Hodge Laplacian, Heisenberg group, spectral multiplier.

This work has been supported by the IHP network HARP "Harmonic Analysis and Related problems" of the European Union and by the project PRIN 2007 "Analisi Armonica". Part of this work was done during a visit at the Centro De Giorgi, Pisa. We thank this institution for the generous hospitality and the support provided.

Introduction

The theory of the Hodge Laplacian Δ on a complete Riemannian manifold M shows deep connections between geometry, topology and analysis on M. While this theory is well developed in the case of functions, i.e., for the Laplace–Beltrami operator, much less is known for forms of higher degree on a non-compact manifold. In particular, one basic question that one would like to answer is whether the Riesz transform $d\Delta^{-\frac{1}{2}}$ is L^p-bounded in the range $1 < p < \infty$. According to [St], this property is relevant for establishing the Hodge decomposition in L^p for differential forms, cf. [ACDH, Li, Loh] and references therein.

In a similar way, functional calculus on self-adjoint, left-invariant Laplacians and sublaplacians L on Lie groups and more general manifolds has been widely studied, cf. [A, An, AnLoh, Ch, ChM, ClS, CoKSi, He, HeZ, Hu, Mar, LuM, LuMS, MauMe, MS, Si, Ta]. A key question concerns the possibility that, for a given L, a Mihlin–Hörmander condition of finite order on the multiplier $m(\lambda)$ implies that the operator $m(L)$ is bounded on L^p for $1 < p < \infty$. A second fundamental question is the L^p-boundedness, in the same range of p, of the Riesz trasforms $XL^{-\frac{1}{2}}$ for appropriate left-invariant vector fields X [CD, CMZ, GSj, Loh2, LohMu].

Also in these situations, not much is known for operators which act on sections of some homogeneous linear bundle over a given group. The most notable case is that of sublaplacians associated to the $\bar\partial_b$-complex on homogeneous CR-manifold [CoKSi, FS]

In this paper we consider the Hodge Laplacian Δ on the Heisenberg group H_n, endowed with a left-invariant and $U(n)$-invariant Riemannian metric, and give answers to the above questions.

The rich structure of the Heisenberg group makes it a natural model to explore such questions in detail. First of all, it has a natural CR-structure, with a well-understood Kohn Laplacian [FS], and nice interactions with the Riemannian structure [MPR1].

For operators on H_n which act on scalar-valued functions and are left- and $U(n)$-invariant, the methods of Fourier analysis are quite handy to study spectral resolution, and sharp multiplier theorems for differential operators of this kind are known [MRS1, MRS2]. This class of operators is based on two commuting differential operators, namely the *sublaplacian* L and the *central derivative* T, in the following sense:

– the left- and $U(n)$-invariant differential operators on H_n are the polynomials in L and T;
– the left- and $U(n)$-invariant self-adjoint operators on $L^2(H_n)$ containing the Schwartz space in their domain are the operators $m(L, i^{-1}T)$, with m a real spectral multiplier.

The same methods also allow to study operators acting on differential forms, like the Kohn Laplacian, which have the property of acting componentwise with respect to a canonical basis of left-invariant forms, cf. (1.4).

On the other hand, the Hodge Laplacian restricted to k-forms, which we denote by Δ_k and whose explicit expression is given in (1.22) below, is far from acting componentwise.

Nevertheless, we are able to reduce the spectral analysis of Δ_k to that of a finite family of explicit *scalar* operators. We call scalar an operator on some space of differential forms which can be expressed as $D \otimes I$, i.e., which acts separately on each scalar component of a given form by the same operator D.

We do so by introducing a decomposition of $L^2\Lambda^k(H_n)$ into finitely many mutually orthogonal subspaces $\mathcal{V}_\nu$ with the following properties:

(i) $\mathrm{dom}\,\Delta_k$ splits along the $\mathcal{V}_\nu$'s as $\sum_\nu(\mathrm{dom}\,\Delta_k \cap \mathcal{V}_\nu)$;
(ii) $\Delta_k : (\mathrm{dom}\,\Delta_k \cap \mathcal{V}_\nu) \longrightarrow \mathcal{V}_\nu$ for every ν;
(iii) for each ν, there is a Hilbert space $\mathcal{H}_\nu$ of L^2-sections of a $U(n)$-homogeneous vector bundle over H_n such that the restriction of Δ_k to $\mathcal{V}_\nu$ is unitarily equivalent to a scalar operator $m_\nu(L, i^{-1}T)$ acting componentwise on $\mathcal{H}_\nu$;
(iv) there exist unitary operators $U_\nu : \mathcal{H}_\nu \longrightarrow \mathcal{V}_\nu$ intertwining $m_\nu(L, i^{-1}T)$ and Δ_k which are either bounded multiplier operators $u_\nu(L, i^{-1}T)$, or compositions of such operators with the Riesz transforms

$$R = d\Delta^{-\frac{1}{2}} , \qquad \mathcal{R} = \partial\Box^{-\frac{1}{2}} , \qquad \bar{\mathcal{R}} = \bar{\partial}\overline{\Box}^{-\frac{1}{2}} .$$

This is done in the first part of the paper (Sections 3-8), and we refer to this part as to the "L^2-theory". The main results in this context are Theorems 8.1 and 8.6, resp., where we obtain the decomposition of $L^2\Lambda^k$ into the Δ_k-invariant subspaces $\mathcal{V}_\nu$, respectively for $0 \le k \le n$ and $n+1 \le k \le 2n+1$.

This decomposition is fundamental for all the second part of the paper, which we are going to describe next. A quick description of the logic and the basic ideas in the construction of the $\mathcal{V}_\nu$ is postponed to the last part of this introduction.

In Sections 9-12, we develop the "L^p-theory"[1]. We prove that, for $1 < p < \infty$, the same kind of decomposition also takes place in $L^p\Lambda^k$. Precisely:

(a) the intertwining operators U_ν in (iv) have L^p-bounded extensions;
(b) consequently, the orthogonal projections $U_\nu^* U_\nu$ from $L^2\Lambda^k$ to $\mathcal{V}_\nu$ extend to bounded operators from $L^p\Lambda^k$ to the the L^p-closure $\mathcal{V}_\nu^p$ of $\mathcal{V}_\nu \cap L^p\Lambda^k$;
(c) the Riesz transforms $R_k = d\Delta_k^{-\frac{1}{2}}$ are L^p-bounded;
(d) the L^p-strong Hodge decomposition holds true for $k = 0, \dots, 2n+1$, and more precisely $L^p\Lambda^k$ is direct sums of the subspaces V_ν^p's.

The (much simpler) case of 1-forms was already considered in [MPR1]. We expect that our results can be applied to the study on conformal invariants on quotients of H_n, along the lines of [MPR2], cf. [Lot, Lü].

In our last main result, as consequence of the L^p-theory we develop, we prove a Mihlin–Hörmander multiplier theorem for Δ_k, for all $k = 0, \dots, 2n+1$. We show that, if $m : \mathbb{R} \to \mathbb{C}$ is a bounded, continuous function satisfying a Mihlin–Hörmander condition of order $\rho > (2n+1)/2$, then, for $1 < p < \infty$, the operator $m(\Delta_k)$ is bounded on $L^p(H_n)\Lambda^k$, with norm bounded by the appropriate norm of m (cf. Theorem 11.1).

We briefly comment on some interesting aspects of the proof and on some consequences and applications. It is always assumed that $1 < p < \infty$.

- Our inductive strategy requires that two statements be proved simultaneously at each step: property (a) above for the given k and L^p-boundedness of the Riesz transform $R_k = d\Delta_k^{-1/2}$. Precisely, the validity of (a) for a given k implies L^p-boundedness of R_k, and this, in turn, is required to prove (a) for $k+1$.
- In order to handle the complicated expressions of the intertwining operators U_ν, we identify certain symbol classes, denoted by $\Psi_\tau^{\rho,\sigma}$, which satisfy simple composition properties, contain all the scalar components of the U_ν, and, when bounded, give L^p-bounded operators (cf. Subsection 9.2).
- Taking as the initial definition of "exact L^p-form" a form ω which is the L^p-limit of a sequence of exact test forms (cf. Proposition 4.5 for $p = 2$), we prove in Subsection 11.2 that this condition is equivalent to saying that ω is in L^p and a differential in the sense of distributions. Incidentally, this allows to prove that the reduced L^p-cohomology of H_n is trivial for every k.
- The Mihlin–Hörmander theorem for spectral multipliers of Δ_k, proved in [MPR1] for $k = 1$, extends to every k.
- Our analysis of Δ easily yields analogous results for the Dirac operator $d + d^*$. Studying the Hodge laplacian first has the advantage of isolating one order of forms at a time. Corollary 11.5 is a multiplier theorem for the Dirac operator completely analogous to Theorem 11.1.

Outline of the decomposition of $L^2\Lambda^k$.

[1]There is an unfortunate notational conflict, due to the fact that the letter p is the commonly used symbol for both Lebesgue spaces and bi-degrees of forms. In this introduction and in the titles of sections we keep the notation L^p, while in the body of the paper we will denote by L^r the generic Lebesgue space.

We go back now to the construction of the subspaces $\mathcal{V}_\nu$ of $L^2\Lambda^k(H_n)$.

First of all, by Hodge duality, we may restrict ourselves to form of degree $k \le n$. We start from the primary decomposition into exact and d^*-closed forms:

$$L^2\Lambda^k(H_n) = (L^2\Lambda^k)_{d\text{-ex}} \oplus (L^2\Lambda^k)_{d^*\text{-cl}} \ ,$$

where each summand is Δ_k-invariant.

Since the Riesz transform $R_{k-1} = d\Delta_{k-1}^{-\frac{1}{2}}$ commutes with Δ and transforms $(L^2\Lambda^{k-1})_{d^*\text{-cl}}$ onto $(L^2\Lambda^k)_{d\text{-ex}}$ unitarily, any Δ-invariant subspace $\mathcal{V}_\nu$ of $(L^2\Lambda^{k-1})_{d^*\text{-cl}}$ has a twin Δ-invariant subspace $R_{k-1}\mathcal{V}_\nu$ inside $(L^2\Lambda^k)_{d\text{-ex}}$.

The analysis is so reduced to the space of d^*-closed forms. Associated with the CR-structure of H_n, there is a natural notion of *horizontal* (p,q)*-form* as a section of the bundle

$$\Lambda^{p,q} = \Lambda^{p,q} T^*_{\mathbb{C}} H_n \ ,$$

and of *horizontal* k*-form* as a section of $\Lambda^k_H = \sum^{\oplus}_{p+q=k} \Lambda^{p,q}$.

Every differential form ω decomposes uniquely as

$$\omega = \omega_1 + \theta \wedge \omega_2 \ ,$$

where ω_1, ω_2 are horizontal and θ is the contact form. Moreover, a d^*-closed form ω is uniquely determined by its horizontal component ω_1.

From now on it is very convenient to introduce a special "test space" $\mathcal{S}_0$, contained in the Schwartz space, together with its corresponding spaces of forms, $\mathcal{S}_0\Lambda^k$, $\mathcal{S}_0\Lambda^{p,q}$ etc., which are cores for Δ_k and the other self-adjoint operators that will appear.

For forms in the core, we have enough flexibility to perform all the required operations in a rather formal way, leaving the extensions to L^2-closures for the very end. For instance, we can say that to every horizontal form ω_1 in the core we can associate a "vertical component" $\theta \wedge \omega_2$, also in the core, to form a d^*-closed form $\omega_1 + \theta \wedge \omega_2$ in the core.

Setting $\Phi(\omega_1) = \omega_1 + \theta \wedge \omega_2$, we can replace Δ_k by the conjugated (but no longer differential) operator $D_k = \Phi^{-1} \circ \Delta_k \circ \Phi$, which acts now on the space of horizontal k-forms in the core and globally defined.

Here comes into play another invariance property of Δ_k, which is easily read as a property of D_k and involves the horizontal symplectic form $d\theta$. The following identity holds (cf. Lemma 5.11) for a horizontal form ω of degree k:

$$D_k(d\theta \wedge \omega) = d\theta \wedge (D_{k-2} + n - k + 1)\omega \ . \tag{0.1}$$

This brings in the Lefschetz decomposition of the space of horizontal forms, as adapted in [MPR1] from the classical context of Kähler manifolds [W]. Denoting by $e(d\theta)$ the operator of exterior multiplication by $d\theta$ and by $i(d\theta)$ its adjoint, it is then natural to think of the core $\mathcal{S}_0\Lambda^{p,q}$ in the space of horizontal (p,q)-forms as the direct sum

$$\mathcal{S}_0\Lambda^{p,q} = \sum_{j=0}^{\min\{p,q\}} e(d\theta)^j \ker i(d\theta) \ .$$

Here each summand is D_k-invariant, and the conjugation formula (0.1) allows us to focus our attention on $\ker i(d\theta)$.

Nevertheless, $\ker i(d\theta)$ still is too big a space to allow a reduction of D_k to scalar operators. It is however easy to identify, for each pair (p,q) with $p+q=k$, a proper D_k-invariant subspace of $\ker i(d\theta)\cap\mathcal{S}_0\Lambda^{p,q}$, namely

$$W_0^{p,q}=\{\omega\in\mathcal{S}_0\Lambda^{p,q}:\partial^*\omega=\bar\partial^*\omega=0\}\ .$$

It turns out that D_k acts as a scalar operator on $W_0^{p,q}$, so that the L^2-closure $\mathcal{V}_0^{p,q}$ of $\Phi(W_0^{p,q})$ will be one of the spaces $\mathcal{V}_\nu$ we are looking for[2]

Next, we take to the orthogonal complement of

$$W_0^k=\sum_{p+q=k}^{\oplus}W_0^{p,q}$$

in $\mathcal{S}_0\Lambda_H^k$. We have

$$(W_0^k)^\perp=\left\{\partial\xi+\bar\partial\eta:\xi,\eta\in\mathcal{S}_0\Lambda_H^{k-1}\right\}\ ,$$

and we can telescopically expand this splitting to obtain that

$$\begin{aligned}\mathcal{S}_0\Lambda_H^k&=W_0^k\oplus\left\{\partial\xi+\bar\partial\eta:\xi,\eta\in\mathcal{S}_0\Lambda_H^{k-1}\right\}\\&=W_0^k\oplus\left\{\partial\xi+\bar\partial\eta:\xi,\eta\in W_0^{k-1}\right\}\oplus\left\{\bar\partial\partial\xi+\partial\bar\partial\eta:\xi,\eta\in\mathcal{S}_0\Lambda_H^{k-2}\right\}\\&=W_0^k\oplus\left\{\partial\xi+\bar\partial\eta:\xi,\eta\in W_0^{k-1}\right\}\oplus\left\{\bar\partial\partial\xi+\partial\bar\partial\eta:\xi,\eta\in W_0^{k-2}\right\}\oplus\cdots\\&=\sum_{p+q=k}^{\oplus}W_0^{p,q}\oplus\sum_{p+q=k-1}^{\oplus}\left\{\partial\xi+\bar\partial\eta:\xi,\eta\in W_0^{p,q}\right\}\\&\qquad\qquad\oplus\sum_{p+q=k-2}^{\oplus}\left\{\bar\partial\partial\xi+\partial\bar\partial\eta:\xi,\eta\in W_0^{p,q}\right\}\oplus\cdots\end{aligned}$$

The subspaces

$$\begin{aligned}&W_1^{p,q}=\left\{\partial\xi+\bar\partial\eta:\xi,\eta\in W_0^{p,q}\right\}\qquad(p+q=k-1)\\&W_2^{p,q}=\left\{\bar\partial\partial\xi+\partial\bar\partial\eta:\xi,\eta\in W_0^{p,q}\right\}\qquad(p+q=k-2)\\&\text{etc.}\end{aligned}$$

generated in this way are D_k-invariant and mutually orthogonal.

Matters are simplified by the fact that, for $j\geq 1$,

$$W_{j+2}^{p,q}=e(d\theta)W_j^{p,q}\ .$$

So only $W_0^{p,q}$, $W_1^{p,q}$ and part of $W_2^{p,q}$ are contained in $\ker i(d\theta)$. Setting

$$W_{1,\ell}^{p,q}=e(d\theta)^\ell W_1^{p,q}\ ,\qquad W_{2,\ell}^{p,q}=e(d\theta)^\ell W_2^{p,q}\ ,$$

we obtain that

$$\mathcal{S}_0\Lambda_H^k\ =\ \sum_{p+q=k}^{\oplus}W_0^{p,q}\oplus\sum_{p+q+2\ell=k-1}^{\oplus}W_{1,\ell}^{p,q}\oplus\sum_{p+q+2\ell=k-2}^{\oplus}W_{2,\ell}^{p,q}\ .$$

On each $W_0^{p,q}$ and $W_2^{p,q}$, D_k acts as a scalar operator, as required.

[2] $W_0^{p,q}$ is nontrivial, unless $p+q=n$ and $0<p<n$, cf. Proposition 5.3.

The situation is not so simple with $W_1^{p,q}$, because the best one can obtain is a representation of D_k as a 2×2 matrix of scalar operators, after parametrizing the elements of $W_1^{p,q}$ with pairs (ξ,η) of forms in $W_0^{p,q}$:

$$\begin{pmatrix}\xi'\\ \eta'\end{pmatrix}=\begin{pmatrix}m_{11}(L,i^{-1}T) & m_{12}(L,i^{-1}T)\\ m_{21}(L,i^{-1}T) & m_{22}(L,i^{-1}T)\end{pmatrix}\begin{pmatrix}\xi\\ \eta\end{pmatrix}\ .$$

A formal computation can be used on the core to produce "eigenvalues" $\lambda_\pm(L,i^{-1}T)$ and the splitting of $W_1^{p,q}$ as the sum of the two "eigenspaces" $W_1^{p,q,\pm}$.

The final decomposition is in formula (10.2).

1. Differential forms and the Hodge Laplacian on H_n

The Heisenberg group H_n is $\mathbb{C}^n\times\mathbb{R}$ with product

$$(z,t)(z',t')=\Big(z+z',t+t'-\frac{1}{2}\Im\mathrm{m}\,\langle z,z'\rangle\Big)\ . \tag{1.2}$$

On its Lie algebra, also identified with $\mathbb{C}^n\times\mathbb{R}$, we introduce the standard Euclidean inner product, and we consider the left-invariant Riemannian metric on H_n induced by it. The complex vector fields

$$Z_j=\sqrt{2}\Big(\partial_{z_j}-\frac{i}{4}\bar z_j\partial_t\Big)\ ,\qquad \bar Z_j=\sqrt{2}\Big(\partial_{\bar z_j}+\frac{i}{4}z_j\partial_t\Big)\ ,\qquad T=\partial_t$$

(with $1\le j\le n$) form an orthonormal basis of the complexified tangent space at each point, and the only nontrivial commutators involving the basis elements are

$$[Z_j,\bar Z_j]=iT\ . \tag{1.3}$$

The dual basis of complex 1-forms is

$$\zeta_j=\frac{1}{\sqrt{2}}dz_j\ ,\qquad \bar\zeta_j=\frac{1}{\sqrt{2}}d\bar z_j\ ,\qquad \theta=dt+\frac{i}{4}\sum_{j=1}^n(\bar z_j dz_j-z_j d\bar z_j)\ . \tag{1.4}$$

The differential of a function f is therefore

$$df=\sum_{j=1}^n\big(Z_jf\zeta_j+\bar Z_jf\bar\zeta_j\big)+Tf\theta\ .$$

This formula extends to forms, once we observe that $d\zeta_j=d\bar\zeta_j=0$ and the differential of the contact form θ is the symplectic form on $\mathbb{C}^n$,

$$d\theta=-i\sum_{j=1}^n\zeta_j\wedge\bar\zeta_j\ . \tag{1.5}$$

A differential form ω is *horizontal* if $\theta\lrcorner\omega=0$, i.e. if

$$\omega=\sum_{I,I'}f_{I,I'}\zeta^I\wedge\bar\zeta^{I'}\ . \tag{1.6}$$

Every form ω decomposes as

$$\omega=\omega_1+\theta\wedge\omega_2\ , \tag{1.7}$$

with ω_1, ω_2 horizontal.

A differential operator D acting on scalar-valued functions is extended to forms by letting D act separately on each scalar component (1.6) of each horizontal component (1.7). Such operators will be called *scalar operators*.

The partial differentials ∂, $\bar\partial$, d_H (resp. *holomorphic, antiholomorphic, horizontal* differential) of a form ω are defined as

$$\partial\omega = \sum_{j=1}^n \zeta_j \wedge Z_j\omega \ , \qquad \bar\partial\omega = \sum_{j=1}^n \bar\zeta_j \wedge \bar Z_j\omega \ , \qquad d_H\omega = \partial\omega + \bar\partial\omega \ . \tag{1.8}$$

As in [MPR1], Prop. 2.2, for $\omega = f\zeta^I \wedge \bar\zeta^{I'}$,

$$\partial\omega = \sum_{\ell,J} \varepsilon^J_{\ell,I}(Z_\ell f)\zeta^J \wedge \bar\zeta^{I'} \ , \qquad \bar\partial\omega = (-1)^{|I|}\sum_{\ell,J'} \varepsilon^{J'}_{\ell,I'}(\bar Z_\ell f)\zeta^I \wedge \bar\zeta^{J'} \ , \tag{1.9}$$

where $\varepsilon^J_{\ell,I} = 0$ unless $\ell \notin I$ and $\{\ell\} \cup I = J$, in which case

$$\varepsilon^J_{\ell,I} = \prod_{i\in I} \mathrm{sgn}\,(i-\ell) \ .$$

Obviously, they act separately on each horizontal component (1.7) of ω, and the same is true for their adjoints $\partial^*, \bar\partial^*, d_H^*$, where

$$\partial^*\omega = -\sum_{\ell,J} \varepsilon^I_{\ell,J}(\bar Z_\ell f)\zeta^J \wedge \bar\zeta^{I'} \ , \qquad \bar\partial^*\omega = (-1)^{|I|+1}\sum_{\ell,J'} \varepsilon^{I'}_{\ell,J'}(Z_\ell f)\zeta^I \wedge \bar\zeta^{J'} \ . \tag{1.10}$$

Moreover, $\partial^2 = \bar\partial^2 = {\partial^*}^2 = {\bar\partial^*}{}^2 = 0$.

Two operators that will play a fundamental role in this paper are

$$e(d\theta)\omega = d\theta \wedge \omega \qquad \text{and} \quad i(d\theta)\omega = e(d\theta)^*\omega = d\theta \lrcorner \omega \ . \tag{1.11}$$

Together with ∂ and $\bar\partial$, they satisfy the following identities:

$$\begin{aligned} \partial\bar\partial + \bar\partial\partial &= d_H^2 = -Te(d\theta) \ , \\ \partial^*\bar\partial^* + \bar\partial^*\partial^* &= {d_H^*}^2 = Ti(d\theta) \ , \end{aligned} \tag{1.12}$$

and

$$\partial\bar\partial^* = -\bar\partial^*\partial \qquad \text{and} \quad \partial^*\bar\partial = -\bar\partial\partial^* \ . \tag{1.13}$$

Other formulas involving $\partial, \bar\partial, e(d\theta)$ and their adjoints are

$$\begin{aligned} \big[i(d\theta), \partial\big] &= -i\bar\partial^* \ , \qquad & \big[i(d\theta), \bar\partial\big] &= i\partial^* \ , \\ \big[\partial^*, e(d\theta)\big] &= i\bar\partial \ , \qquad & \big[\bar\partial^*, e(d\theta)\big] &= -i\partial \ , \end{aligned} \tag{1.14}$$

$$\big[i(d\theta), \partial^*\big] = \big[i(d\theta), \bar\partial^*\big] = 0 = \big[e(d\theta), \partial\big] = \big[e(d\theta), \bar\partial\big] \tag{1.15}$$

and

$$\big[i(d\theta), e(d\theta)\big] = (n-k)I \ . \tag{1.16}$$

For these formulas and the following in this section, we refer to [MPR1].[3]

[3]Perhaps we should add a few more formulas, while moving them into a section, later in the paper.

We define the *holomorphic, antiholomorphic and horizontal Laplacians* as

$$\begin{aligned} \Box &= \partial\partial^* + \partial^*\partial \ , \\ \overline{\Box} &= \bar\partial\bar\partial^* + \bar\partial^*\bar\partial \ , \\ \Delta_H &= d_H d_H^* + d_H^* d_H = \Box + \overline{\Box} \ . \end{aligned} \tag{1.17}$$

Each of these Laplacians acts componentwise. Calling (p,q)-form a horizontal form of type

$$\omega = \sum_{|I|=p\,,\,|I'|=q} f_{I,I'}\zeta^I \wedge \bar\zeta^{I'} \ ,$$

and introducing the *sublaplacian*

$$L = -\sum_{j=1}^n (Z_j\bar Z_j + \bar Z_j Z_j) \ , \tag{1.18}$$

the operators $\Box, \overline{\Box}, \Delta_H$ coincide on (p,q)-forms with the following scalar operators:

$$\begin{aligned} \Box &= \frac12 L + i\Big(\frac n2 - p\Big)T \ , \\ \overline{\Box} &= \frac12 L - i\Big(\frac n2 - q\Big)T \ , \\ \Delta_H &= L + i(q-p)T \ . \end{aligned} \tag{1.19}$$

To be more explicit, we shall occasionally denote the "box"' operators by $\Box_p$ and $\overline{\Box}_q$. Some commutation relations that we will use are (see [MPR1])

$$\begin{aligned} \Box\bar\partial &= \bar\partial(\Box - iT)\,, \qquad & \Box\bar\partial^* &= \bar\partial^*(\Box + iT)\,, \\ \overline{\Box}\partial &= \partial(\overline{\Box} + iT)\,, \qquad & \overline{\Box}\partial^* &= \partial^*(\overline{\Box} - iT)\,. \end{aligned} \tag{1.20}$$

The full differential d of a form $\omega = \omega_1 + \theta\wedge\omega_2$ and its adjoint d^* are represented, in terms of the pair (ω_1,ω_2), by the matrices

$$\begin{pmatrix} d_H & e(d\theta) \\ T & -d_H \end{pmatrix} \ , \qquad d^* = \begin{pmatrix} d_H^* & -T \\ i(d\theta) & -d_H^* \end{pmatrix} \ , \tag{1.21}$$

and the *Hodge Laplacian* $\Delta = dd^* + d^*d$ by the matrix

$$\begin{aligned} \Delta &= \begin{pmatrix} \Delta_H - T^2 + e(d\theta)i(d\theta) & \big[d_H^*, e(d\theta)\big] \\ \\ \big[i(d\theta), d_H\big] & \Delta_H - T^2 + i(d\theta)e(d\theta) \end{pmatrix} \\ &= \begin{pmatrix} \Delta_H - T^2 + e(d\theta)i(d\theta) & i\bar\partial - i\partial \\ \\ i\partial^* - i\bar\partial^* & \Delta_H - T^2 + i(d\theta)e(d\theta) \end{pmatrix} \ . \end{aligned} \tag{1.22}$$

When Δ acts on k-forms, it will be denoted by Δ_k. In particular,

$$\Delta_0 = L - T^2 \ .$$

We denote by Λ^k the k-th exterior product of the dual $\mathfrak{h}_n^*$ of the Lie algebra of H_n (identified with the linear span of $\zeta_1,\dots,\zeta_n,\bar\zeta_1,\dots,\bar\zeta_n,\theta$), by Λ^k_H the k-th exterior product of the horizontal distribution (i.e. the linear span of the $\zeta_j,\bar\zeta_j$), and by $\Lambda^{p,q}$ the space of elements of bidegree (p,q) in Λ^k_H. Symbols like $L^p\Lambda^k$, $\mathcal{S}\Lambda^{p,q}$ etc., denote the space of L^p-sections, $\mathcal{S}$-sections etc., of the corresponding bundle over H_n. Clearly, $L^p\Lambda^k \cong L^p\otimes\Lambda^k$ etc..

2. Bargmann representations and sections of homogeneous bundles

The L^2-Fourier analysis on the Heisenberg group involves the family of infinite dimensional irreducible unitary representations $\{\pi_\lambda\}_{\lambda\neq 0}$ such that $\pi_\lambda(0,t)=e^{i\lambda t}I$. These representations are most conveniently realized for our purposes in a modified version of the Bargmann form [F].

Let $\mathcal{F}=\mathcal{F}(\mathbb{C}^n)$ be the space of entire functions F on $\mathbb{C}^n$ such that
$$\|F\|^2_{\mathcal{F}}=\int_{\mathbb{C}^n}|F(w)|^2e^{-\frac12|w|^2}\,dw<\infty\ .$$

The family of Bargmann representations π_λ on $\mathcal{F}$ is defined, for $\lambda\neq 0$, as follows:

(i) for $\lambda=1$,
$$\big(\pi_1(z,t)F\big)(w)=e^{it}e^{-\frac12\langle w,z\rangle-\frac14|z|^2}F(w+z)\ . \tag{2.1}$$

(ii) For $\lambda>0$,
$$\pi_\lambda(z,t)=\pi_1(\lambda^{\frac12}z,\lambda t)\ ; \tag{2.2}$$

(iii) for $\lambda<0$,
$$\pi_\lambda(z,t)=\pi_{-\lambda}(\bar z,-t)\ . \tag{2.3}$$

The unitary group $U(n)$ acts on H_n through the automorphisms
$$(z,t)\longmapsto(z,t)^g=(gz,t)\ ,\qquad \big(g\in U(n)\big)\ ,$$
and on $L^2(H_n)$ through the representation
$$\big(\alpha(g)f\big)(z,t)=f\big((z,t)^{g^{-1}}\big)\ .$$

We also consider the pair of contragradient representations $U,\bar U$ of $U(n)$ on $\mathcal{F}$, given by
$$U_gF=F\circ g^{-1}\ ,\qquad \bar U_g=U_{\bar g}\ . \tag{2.4}$$

Then
$$\begin{aligned}\pi_\lambda(gz,t)&=U_g\,\pi_\lambda(z,t)\,U_{g^{-1}}\ ,\quad \text{for } \lambda>0\ ,\\ \pi_\lambda(gz,t)&=\bar U_g\,\pi_\lambda(z,t)\,\bar U_{g^{-1}}\ ,\quad \text{for } \lambda<0\ .\end{aligned} \tag{2.5}$$

The representation U in (2.4) splits into irreducibles according to the decomposition of $\mathcal{F}$
$$\mathcal{F}=\sum_{j\geq 0}\mathcal{P}_j\ , \tag{2.6}$$
where $\mathcal{P}_j$ denotes the space of homogeneous polynomials of degree j.

We denote by P_j the orthogonal projection of $\mathcal{F}$ on $\mathcal{P}_j$, and by $\mathcal{F}^\infty$ the space of functions $F \in \mathcal{F}$ such that

$$\|P_j F\|_{\mathcal{F}} = o(j^{-N}) \ , \qquad \forall\, N \in \mathbb{N} \ . \tag{2.7}$$

Then $\mathcal{F}^\infty$ is the space of C^∞-vectors for all representations π_λ.

The differential of π_λ is given by[4] $\pi_\lambda(T) = i\lambda$ and

$$\pi_\lambda(Z_\ell) = \begin{cases} \sqrt{2\lambda}\,\partial_{w_\ell} & \text{if } \lambda > 0 \\ -\sqrt{\frac{|\lambda|}{2}}\, w_\ell & \text{if } \lambda < 0 \ ; \end{cases} \qquad \pi_\lambda(\bar{Z}_\ell) = \begin{cases} -\sqrt{\frac{\lambda}{2}}\, w_\ell & \text{if } \lambda > 0 \\ \sqrt{2|\lambda|}\,\partial_{w_\ell} & \text{if } \lambda < 0 \ . \end{cases} \tag{2.8}$$

We adopt the following definition of $\pi_\lambda(f)$:

$$\pi_\lambda(f) = \int_{H_n} f(x)\pi_\lambda(x)^{-1}\, dx \in Ł(\mathcal{F}, \mathcal{F}) \ . \tag{2.9}$$

Notice that $\pi_\lambda(f * g) = \pi_\lambda(g)\pi_\lambda(f)$, but this disadvantage is compensated by a simpler formalism when dealing with forms or more general vector-valued functions.

The Plancherel formula for $f \in L^2$ is

$$\|f\|_2^2 = c_n \int_{-\infty}^{+\infty} \|\pi_\lambda(f)\|_{HS}^2\, |\lambda|^n\, d\lambda = c_n \int_{-\infty}^{+\infty} \sum_{j,j'} \|P_j \pi_\lambda(f) P_{j'}\|_{HS}^2\, |\lambda|^n\, d\lambda \ .$$

Let V be a finite dimensional Hilbert space. Defining $\pi_\lambda(f)$ for V-valued functions f by (2.9), we have

$$\pi_\lambda(f) \in Ł(\mathcal{F}, \mathcal{F}) \otimes V \cong Ł(\mathcal{F}, \mathcal{F} \otimes V) \ .$$

Suppose now that V is the representation space of a unitary representation ρ of $U(n)$, and consider the two representations $U \otimes \rho$, $\bar{U} \otimes \rho$ of $U(n)$ on $\mathcal{F} \otimes V$. Denote by $\Sigma^+ = \Sigma^{\rho,+}$ (resp. $\Sigma^- = \Sigma^{\rho,-}$) the set of irreducible representations $\sigma \in \widehat{U(n)}$ contained in $U \otimes \rho$ (resp. in $\bar{U} \otimes \rho$), and let

$$\mathcal{F} \otimes V = \bigoplus_{\sigma \in \Sigma^\pm} \mathcal{E}_\sigma^\pm \tag{2.10}$$

be the corresponding orthogonal decompositions into $U(n)$-types. When $V = \mathbb{C}$, the decomposition (2.10) reduces to (2.6). To indicate the dependence on ρ, we shall sometime also write $\mathcal{E}_\sigma^\pm = \mathcal{E}_\sigma^{\rho,\pm}$.

Lemma 2.1. *Each $\mathcal{E}_\sigma^\pm$ is finite dimensional and decomposes into $U(n)$-invariant subspaces*

$$\mathcal{E}_\sigma^\pm = \bigoplus_j \mathcal{E}_\sigma^\pm \cap \big(\mathcal{P}_j \otimes V\big) \ .$$

In particular $\mathcal{E}_\sigma^\pm \subset \mathcal{F} \otimes V$. More precisely, $\mathcal{E}_\sigma^\pm \subset \mathcal{F}^\infty \otimes V$, where $\mathcal{F}^\infty$ is defined in (2.7).

[4] Even though $d\pi_\lambda(D)$ is the more standard notation, we prefer to reduce the number of d's around.

Proof. We only discuss the case of $U\otimes\rho$. For every j, $\mathcal{P}_j\otimes V$ is an invariant subspace. Therefore, for $\sigma\in\widehat{U(n)}$, $\mathcal{E}^+_\sigma=\sum_j\mathcal{E}^+_\sigma\cap(\mathcal{P}_j\otimes V)$. Let χ_j, χ_ρ, χ_σ be the characters of $U_{|\mathcal{P}_j}, \rho, \sigma$ respectively. The multiplicity of σ in $\mathcal{P}_j\otimes V$ is then given by

$$\int_{U(n)}\chi_j(g)\chi_\rho(g)\overline{\chi_\sigma(g)}\,dg=\langle\chi_j,\overline{\chi_\rho}\chi_\sigma\rangle\ ,$$

which is the multiplicity of $U_{|\mathcal{P}_j}$ in $\bar\rho\otimes\sigma$ (with $\bar\rho$ denoting the contragredient of ρ). Since this representation is finite dimensional, the multiplicity can be positive only for a finite number of j. It follows that $\mathcal{E}^+_\sigma$ has finite dimension.

Since $\mathcal{E}^\pm_\sigma$ consists of V-valued polynomials, it is obviously contained in $G\otimes V$. Q.E.D.

The decomposition of $\mathcal{F}\otimes V$ given above leads to the following form of the Plancherel formula for L^2V, with $P^\pm_\sigma$ denoting the orthogonal projection of $\mathcal{F}\otimes V$ onto $\mathcal{E}^\pm_\sigma$:

$$\begin{aligned}\|f\|_2^2&=c_n\int_{-\infty}^{+\infty}\sum_{j\in\mathbb{N}\,,\,\sigma\in\Sigma^{\operatorname{sgn}\lambda}}\|P^{\operatorname{sgn}\lambda}_\sigma\pi_\lambda(f)P_j\|^2_{HS}\,|\lambda|^n\,d\lambda\\ &=c_n\int_{-\infty}^{+\infty}\sum_{\sigma\in\Sigma^{\operatorname{sgn}\lambda}}\|P^{\operatorname{sgn}\lambda}_\sigma\pi_\lambda(f)\|^2_{HS}\,|\lambda|^n\,d\lambda\ .\end{aligned}\tag{2.11}$$

Let ρ' be another unitary representation of $U(n)$ on a finite dimensional Hilbert space V'. The convolution

$$f*K(x)=\int_{H_n}K(y^{-1}x)f(y)\,dy$$

of integrable functions f with values in V and K with values in $Ł(V,V')$ produces a function taking values in V'. In the representations π_λ, $\lambda\neq0$,

$$\pi_\lambda(K)\in Ł(\mathcal{F},\mathcal{F})\otimes Ł(V,V')\cong Ł(\mathcal{F}\otimes V,\mathcal{F}\otimes V')\ ,$$

and

$$\pi_\lambda(f*K)=\pi_\lambda(K)\pi_\lambda(f)\in Ł(\mathcal{F},\mathcal{F}\otimes V')\ .$$

Let $\tilde\rho$ (resp. $\tilde\rho'$) be the representation $\alpha\otimes\rho$ on L^2V (resp. $\alpha\otimes\rho'$ on L^2V') of $U(n)$ and suppose that convolution by K is an equivariant operator, i.e.

$$\tilde\rho'(g)(f*K)=\big(\tilde\rho(g)f\big)*K\tag{2.12}$$

for $g\in U(n)$ and $f\in\mathcal{S}V$. Since for $f\in\mathcal{S}V$ and $\xi\in\mathcal{F}$, with $\lambda>0$,

$$\begin{aligned}\pi_\lambda\Big(\tilde\rho'(g)(f*K)\Big)\xi&=\iint\rho'(g)K(y^{-1}x)f(y)U_g\pi_\lambda(x^{-1})U_{g^{-1}}\xi\,dydx,\\ \pi_\lambda\Big((\tilde\rho(g)f)*K\Big)\xi&=\iint K(y^{-1}x)\rho(g)f(y^{g^{-1}})\pi_\lambda(x^{-1})\xi\,dydx,\end{aligned}$$

by letting f tend weakly to $\delta_0\otimes v$, with $v\in V$, we see that (2.12) implies

$$\int\rho'(g)K(x)v\,U_g\pi_\lambda(x^{-1})U_{g^{-1}}\xi\,dx=\int K(x)\rho(g)v\,\pi_\lambda(x^{-1})\xi\,dx.$$

Replacing ξ by $U_g\xi$, we obtain

$$U_g \otimes \rho'(g)\Big(\pi_\lambda(K)(\xi \otimes v)\Big) = \pi_\lambda(K)(U_g\,\xi \otimes \rho(g)v)$$

for every $\xi \in \mathcal{F}, v \in V$. A similar formula holds for $\lambda < 0$ with $\bar{U}$ in place of U. Thus (2.12) implies the following identities, for K defining an equivariant convolution operator:

$$\begin{aligned}(U \otimes \rho')(g)\pi_\lambda(K) &= \pi_\lambda(K)(U \otimes \rho)(g)\ , \qquad \lambda > 0\\ (\bar{U} \otimes \rho')(g)\pi_\lambda(K) &= \pi_\lambda(K)(\bar{U} \otimes \rho)(g)\ , \qquad \lambda < 0\end{aligned} \tag{2.13}$$

for $g \in U(n)$, i.e. $\pi_\lambda(K)$ intertwines $U \otimes \rho$ and $U \otimes \rho'$, or $\bar{U} \otimes \rho$ and $\bar{U} \otimes \rho'$ depending on the sign of λ. The following is an immediate consequence.

Lemma 2.2. *Assume that convolution by $K \in L^1 \otimes Ł(V, V')$ is an equivariant operator. Then, setting $\Sigma^{\rho,\rho',\mathrm{sgn}\,\lambda} = \Sigma^{\rho,\mathrm{sgn}\,\lambda} \cap \Sigma^{\rho',\mathrm{sgn}\,\lambda}$,*

$$\pi_\lambda(K) = \bigoplus_{\sigma \in \Sigma^{\rho,\rho',\mathrm{sgn}\,\lambda}} \pi_{\lambda,\sigma}(K)\ ,$$

with $\pi_{\lambda,\sigma}(K) : \mathcal{E}_\sigma^{\rho,\mathrm{sgn}\,\lambda} \to \mathcal{E}_\sigma^{\rho',\mathrm{sgn}\,\lambda}$.

By a variant of Schwartz's Kernel Theorem, the convolution operators D with kernels $K \in \mathcal{S}'(H_n) \otimes Ł(V, V')$ are characterized as the continuous operators from $\mathcal{S}(H_n) \otimes V$ to $\mathcal{S}'(H_n) \otimes V'$ that commute with left translations on H_n. Lemma 2.2 applies to operators of this kind, provided that the Fourier transform $\pi_\lambda(K)$ is well defined for $\lambda \neq 0$. This is surely the case if K has compact support, and in particular for a left-invariant differential operator $Df = f * (D\delta_0)$. We then have

$$\pi_\lambda(Df) = \pi_\lambda(D\delta_0)\pi_\lambda(f) = \pi_\lambda(D)\pi_\lambda(f)\ .$$

We apply these remarks to the differentials and Laplacians introduced in Section 1.

With ρ_k denoting the representation of $U(n)$ on Λ^k induced from its action on H_n by automorphisms, and, as before let $\tilde{\rho}_k = \alpha \otimes \rho_k$ be the tensor product acting on $L^2\Lambda^k$. Then d, d^*, Δ_k are equivariant operators. The same applies to $\partial, \bar{\partial}, d_H$ etc. on the appropriate L^2-subbundles.

Notice that $\Box, \overline{\Box}$ and Δ_H have the special property of acting scalarly on (p, q)-forms, by (1.19). Since the sublaplacian L has the property that $\pi_\lambda(L)$ acts as a scalar multiple of the identity (namely, as $|\lambda|(2m+n)I$) on $\mathcal{P}_m \subset \mathcal{F}$, the same is true for the image of $\Box, \overline{\Box}, \Delta_H$ under π_λ.

3. Cores, domains and self-adjoint extensions

For $0 < \delta < R$ and $N \in \mathbb{N}$, denote by $\mathcal{S}_{\delta,R,N}(H_n)$ the space of functions f satisfying the following properties:

(i) $f \in \mathcal{S}(H_n)$;
(ii) $\pi_\lambda(f) = 0$ for $|\lambda| \leq \delta$ and $|\lambda| \geq R$;
(iii) for $\delta < |\lambda| < R$, $P_j\pi_\lambda(f) = 0$ for $j > N$.

We set $\mathcal{S}_0 = \bigcup_{\delta,R,N} \mathcal{S}_{\delta,R,N}$.

Lemma 3.1. *$\mathcal{S}_0$ is invariant under left translations, and dense in L^2.*

Proof. The first statement follows from the identity $\pi_\lambda(L_{(z,t)}f) = \pi_\lambda(f)\pi_\lambda(z,t)^{-1}$, where $L_{(z,t)}f$ is the left translate of $f \in \mathcal{S}_0$ by $(z,t)^{-1}$. Take now $f \in \mathcal{S}$. For $\delta, R > 0$, fix a C^∞-function $u_{\delta,R}(\lambda)$ on $\mathbb{R}$, with values in $[0,1]$, supported where $\delta \leq |\lambda| \leq R$ and equal to 1 where $2\delta \leq |\lambda| \leq R/2$.

Given $\varepsilon > 0$, by Plancherel's formula it is possible to find $\delta, R > 0$ and $N \in \mathbb{N}$ such that the L^2- function g such that $\pi_\lambda(g) = u_{\delta,R}(\lambda) \sum_{j \leq N} P_j \pi_\lambda(f)$ approximates f in L^2 by less than ε.

We claim that g is in $\mathcal{S}$, hence in $\mathcal{S}_0$, and this will conclude the proof, by the density of $\mathcal{S}$ in L^2.

By definition, $g = f * h$, where h is the function with $\pi_\lambda(h) = u_{\delta,R}(\lambda) \sum_{j \leq N} P_j$. By explicit computation of the matrix entries of the representations of H_n [Th], the Fourier transform $h(z, \hat{\lambda})$ of h in the t-variable equals

$$h(z, \hat{\lambda}) = u_{\delta,R}(\lambda) \sum_{j \leq N} \psi_j(|\lambda|^{\frac{1}{2}} z) \ ,$$

where the ψ_j are Schwartz functions on $\mathbb{C}^n$. Hence $h \in \mathcal{S}(H_n)$ and so is g. Q.E.D.

We regard $\mathcal{S}_0$ as the inductive limit of the spaces $\mathcal{S}_{\delta,R,N}$, each with the topology induced from $\mathcal{S}$.

Obviously, $\mathcal{S}_0 V = \mathcal{S}_0 \otimes V$ is contained in $\mathcal{S}V$ and dense in $L^2 V$. Assume, as in Section 2, that V is a finite dimensional Hilbert space on which $U(n)$ acts unitarily by the representation ρ. Taking into account the action of $U(n)$, one can then introduce a different chain of subspaces filling up $\mathcal{S}_0 V$. Given $0 < \delta < R$, $N \in \mathbb{N}$ and finite subsets $X^\pm$ of $\Sigma^\pm$, define $\mathcal{S}_{\delta,R,X^\pm} V$ as the space of functions f such that

(i') $f \in \mathcal{S}(H_n) \otimes V$;
(ii') $\pi_\lambda(f) = 0$ for $|\lambda| \leq \delta$ and $|\lambda| \geq R$;
(iii') for $\delta < |\lambda| < R$, $P_\sigma^{\operatorname{sgn}\lambda} \pi_\lambda(f) = 0$ for $\sigma \notin X^{\operatorname{sgn}\lambda}$;

It follows from Lemma 2.1 that finite unions of the $\mathcal{S}_{\delta,R,X^\pm} V$ exhaust finite unions of the $\mathcal{S}_{\delta,R,N} V$ and viceversa.

In order to develop the L^2-analysis of differentials and Laplacians, we establish some general facts about densely defined operators from $L^2 V$ to $L^2 V'$, wih (V, ρ), (V', ρ') finite-dimensional representation spaces of $U(n)$. Precisely, we consider operators whose initial domain is $\mathcal{S}_0 V$ and which map $\mathcal{S}_0 V$ into $\mathcal{S}_0 V'$, continuously with respect to the Schwartz topologies. Most of the operators to be considered in this paper will belong to this class.

Lemma 3.2.

(i) *Let (V, ρ), (V', ρ') be finite-dimensional representation spaces of $U(n)$, and let*

$$B : \mathcal{S}_0 V \longrightarrow \mathcal{S}_0 V' \ ,$$

be a left-invariant linear operator, $U(n)$-equivariant and continuous with respect to the $\mathcal{S}_0$-topologies. Then there exists a family of linear operators $B_{\lambda,\sigma} : \mathcal{E}_\sigma^{\rho,\operatorname{sgn}\lambda} \longrightarrow \mathcal{E}_\sigma^{\rho',\operatorname{sgn}\lambda}$, depending smoothly on $\lambda \neq 0$, such that

$$\pi_\lambda(Bf) = \bigoplus_{\sigma \in \Sigma^{\rho,\rho',\operatorname{sgn}\lambda}} B_{\lambda,\sigma} P_\sigma^{\operatorname{sgn}\lambda} \pi_\lambda(f) \ , \tag{3.1}$$

(ii) *Conversely, given any family of linear operators* $B_{\lambda,\sigma} : \mathcal{E}_\sigma^{\rho,\operatorname{sgn}\lambda} \longrightarrow \mathcal{E}_\sigma^{\rho',\operatorname{sgn}\lambda}$*, depending smoothly on* $\lambda \neq 0$*, there is a unique left-invariant operator* $B : \mathcal{S}_0 V \longrightarrow \mathcal{S}_0 V'$*,* $U(n)$*-equivariant and continuous with respect to the* $\mathcal{S}_0$*-topologies, such that* (3.1) *holds for every* $f \in \mathcal{S}_0 V$*. We set*

$$\pi_{\lambda,\sigma}(B) = B_{\lambda,\sigma} \ , \qquad B_\lambda = \sum_{\sigma\in\Sigma^{\rho,\rho',\operatorname{sgn}\lambda}} B_{\lambda,\sigma} P_\sigma^{\operatorname{sgn}\lambda} \ , \qquad \pi_\lambda(B) = B_\lambda \ .$$

(iii) *The closure of* B *as an operator from* L^2V *to* L^2V' *has domain* $\operatorname{dom}(B)$ *consisting of those* $f \in L^2V$ *such that*

$$\int_{-\infty}^{+\infty} \sum_\sigma \|B_{\lambda,\sigma} P_\sigma^{\operatorname{sgn}\lambda}\pi_\lambda(f)\|_{HS}^2 \, |\lambda|^n \, d\lambda < \infty \ . \tag{3.2}$$

(iv) *If* $(V,\rho) = (V',\rho')$ *and* B *is symmetric (equivalently,* $B_{\lambda,\sigma}$ *is symmetric for every* λ*,* σ*), then* B *is essentially self-adjoint.*

(v) *If* B *is symmetric and* m *is a Borel function on the real line such that* $m(B_{\lambda,\sigma})$ *is well defined for every* λ*,* σ*, the domain of* $m(B)$ *is the space of those* $f \in L^2V$ *such that*

$$\int_{-\infty}^{+\infty} \sum_\sigma \big\| m(B_{\lambda,\sigma}) P_\sigma^{\operatorname{sgn}\lambda}\pi_\lambda(f)\big\|_{HS}^2 \, |\lambda|^n \, d\lambda < \infty \ .$$

Moreover, the space $\mathcal{S}_0 V \cap \operatorname{dom}\big(m(B)\big)$ *is a core for* $m(B)$ *and the identity*

$$\langle m(B)f, g\rangle = c_n \int_{-\infty}^{+\infty} \sum_\sigma \operatorname{tr}\big(m(B_{\lambda,\sigma}) P_\sigma^{\operatorname{sgn}\lambda}\pi_\lambda(f)\pi_\lambda(g)^*\big) \, |\lambda|^n \, d\lambda$$

holds for $f \in \operatorname{dom}\big(m(B)\big)$ *and* $g \in L^2V$*.*

Proof. To prove (i), let $\{\Phi_\ell\}_{\ell\in\mathbb{N}}$ be an enumeration of the orthonormal basis of monomials in $\mathcal{F}$. Define linear operators $E_{\ell,\ell'} : \mathcal{F} \to \mathcal{F}$ by setting $E_{\ell,\ell'}F = \langle F, \Phi_{\ell'}\rangle_{\mathcal{F}}\Phi_\ell$. Let also $\{e_i\}$ and $\{e'_j\}$ be (finite) bases of V and V' respectively.

Given two compact intervals $[a,b]$ and $[a',b']$ such that $[a,b] \subset [a',b']^0$, with $a' > 0$ (for intervals contained in $\mathbb{R}^-$ the proof is similar) and $\ell \in \mathbb{N}$, there exists $g_\ell \in \mathcal{S}_0$ such that $\pi_\lambda(g_\ell) = E_{\ell,\ell}$ for $\lambda \in [a,b]$ and $\pi_\lambda(g_\ell) = 0$ for $\lambda \notin [a',b']$ (cf. the proof of Lemma 3.1 and [Th]).

Then $B(g_\ell \otimes e_i) \in \mathcal{S}_0 V'$ and

$$\pi_\lambda\big(B(g_\ell\otimes e_i)\big) = \sum_j \Big(\sum_{h,k} c_{h,k,j}^{\ell,i}(\lambda) E_{h,k}\Big) \otimes e'_j \ , \tag{3.3}$$

where the sum in h ranges over a fixed finite set of indices independent of λ. The coefficients $c_{\ell,\ell',j}^{\ell,i}$ are smooth in λ and, since B is left-invariant, supported in $[a',b']$.

Take now $f = \sum_i f_i \otimes e_i \in \mathcal{S}_0 V$, with $\pi_\lambda(f) = 0$ for $\lambda \notin [a,b]$. Then

$$f = \sum_{i,\ell} (f_i * g_\ell * g_\ell) \otimes e_i \ ,$$

where the sum is finite. Hence, by the continuity assumption on B,

$$Bf = \sum_{i,\ell} (f_i * g_\ell) * B(g_\ell \otimes e_i) \ .$$

Introducing the notation

$$\hat{f}_i(\lambda,\ell,\ell') = \langle \pi_\lambda(f_i)\Phi_{\ell'},\Phi_\ell\rangle_{\mathcal{F}} \ ,$$

we have

$$\pi_\lambda(f_i * g_\ell) = E_{\ell,\ell}\,\pi_\lambda(f_i) = \sum_{\ell'} \hat{f}_i(\lambda,\ell,\ell')E_{\ell,\ell'} \ .$$

The composition $\pi_\lambda\big(B(g_\ell\otimes e_i)\big)\pi_\lambda(f_i * g_\ell)$ is well defined, because the second factor has the one-dimensional range $\mathbb{C}\Phi_\ell$. Therefore the index k in (3.3) can only assume the value ℓ, and

$$\begin{aligned}\pi_\lambda(Bf) &= \sum_j \Big(\sum_{i,\ell}\sum_h c^{\ell,i}_{h,\ell,j}(\lambda)E_{h,\ell}\sum_{\ell'}\hat{f}_i(\lambda,\ell,\ell')E_{\ell,\ell'}\Big)\otimes e'_j\\ &= \sum_j \Big(\sum_i\sum_{h,\ell'}\Big(\sum_\ell c^{\ell,i}_{h,\ell,j}(\lambda)\hat{f}_i(\lambda,\ell,\ell')\Big)E_{h,\ell'}\Big)\otimes e'_j \ .\end{aligned}$$

The infinite matrix $C_{i,j}(\lambda) = \big(c^{\ell,i}_{h,\ell,j}(\lambda)\big)_{h,\ell}$ has only a finite number of nonzero entries, hence it defines a linear operator $B^\lambda_{i,j}$ from the linear span of the Φ_ℓ (i.e. the space of polynomials inside $\mathcal{F}$) into itself, by setting

$$B^\lambda_{i,j}\Phi_\ell = \sum_h c^{\ell,i}_{h,\ell,j}(\lambda)\Phi_h.$$

Notice that $B^\lambda_{i,j}E_{\ell,\ell'} = \sum_h c^{\ell,i}_{h,\ell,j}(\lambda)E_{h,\ell'}$. Then, setting $E'_{j,i} = \langle\cdot,e_i\rangle_V e'_j \in \mathrm{L}(V,V')$, one easily verifies that

$$B_\lambda = \sum_{i,j} B^\lambda_{i,j}\otimes E'_{j,i} \ , \tag{3.4}$$

maps V-valued polynomials into V'-valued polynomials and

$$\pi_\lambda(Bf) = B_\lambda\pi_\lambda(f) \ , \tag{3.5}$$

for every $f\in\mathcal{S}_0V$ with $\pi_\lambda(f)=0$ for $\lambda\notin[a,b]$.

It is now easy to prove that for $\lambda\in[a,b]$, B_λ is uniquely defined by the identity (3.5), which shows that it does not depend on the choice of the functions g_ℓ, and that if we repeat the same argument starting with a larger interval $[a^\#,b^\#]\supset[a,b]$ contained in $\mathbb{R}^+$, the new operators $B^\#_\lambda$ coincide with B_λ for $\lambda\in[a,b]$. Covering the positive half-line by compact intervals of this type and repeating the same argument on the negative half-line, we find a unique map $\lambda\longmapsto B_\lambda$ defined for $\lambda\neq 0$ and for which (3.5) holds for every $f\in\mathcal{S}_0V$.

Since B is $U(n)$-equivariant, a repetition of the proof of Lemma 2.2 shows that B_λ maps $\mathcal{E}^{\rho,\operatorname{sgn}\lambda}_\sigma$ into $\mathcal{E}^{\rho',\operatorname{sgn}\lambda}_\sigma$ for every $\sigma\in\Sigma^{\rho,\rho',\operatorname{sgn}\lambda}$. It is obvious from the smoothness of the coefficients $c^{\ell,i}_{h,\ell,j}$ that the restricted operators $B_{\lambda,\sigma}$ depend smoothly on λ.

The proof of (ii) is quite obvious.

To prove (iii), denote by $\tilde{B}$ be the operator on $\operatorname{dom}(B)$, defined in (3.2), such that $\pi_\lambda(\tilde{B}f) = B_\lambda\pi_\lambda(f)$. It is easy to verify that $\tilde{B}$ is closed and that $\mathcal{S}_0V$ is dense in $\operatorname{dom}(B)$ in the graph norm of $\tilde{B}$. Since $\tilde{B}$ coincides with B on $\mathcal{S}_0V\subset\mathcal{S}V$, $\tilde{B}$ is the closure of B.

To prove (iv), assume that B is symmetric. Then each operator $B_{\lambda,\sigma}$ is self-adjoint, hence so is $\pi_\lambda(B)$. If B' is the adjoint of B, taking g in the domain of B' and $f \in \mathcal{S}_0 V$, we have

$$\begin{aligned}\langle B'g, f\rangle &= \langle g, Bf\rangle \\ &= c_n \int_{-\infty}^{+\infty} \sum_\sigma \operatorname{tr}\big(\pi_\lambda(f)^*\pi_\lambda(B)P_\sigma^{\operatorname{sgn}\lambda}\pi_\lambda(g)\big)\,|\lambda|^n\,d\lambda \\ &= c_n \int_{-\infty}^{+\infty} \sum_\sigma \operatorname{tr}\big(\pi_\lambda(f)^* B_{\lambda,\sigma}P_\sigma^{\operatorname{sgn}\lambda}\pi_\lambda(g)\big)\,|\lambda|^n\,d\lambda\ .\end{aligned}$$

By the arbitrariness of $\pi_\lambda(f)$ subject to conditions (i')-(iii'), we conclude that

$$\int_{-\infty}^{+\infty} \sum_\sigma \|B_{\lambda,\sigma}P_\sigma^{\operatorname{sgn}\lambda}\pi_\lambda(g)\|_{HS}^2\,|\lambda|^n\,d\lambda < \infty\ ,$$

i.e. $g \in \operatorname{dom}(B)$, and that $\pi_\lambda(B'g) = \pi_\lambda(B)\pi_\lambda(g)$, i.e. $B'g = \tilde{B}g$.

Finally, (v) is proved in a similar way. Q.E.D.

Consistently with the identity $\pi_\lambda\big(m(B)\big) = m\big(\pi_\lambda(B)\big)$, we write $\pi_{\lambda,\sigma}\big(m(B)\big)$ for $m\big(\pi_{\lambda,\sigma}(B)\big)$.

Remark 3.3. As a typical instance of operations that will be done in the sequel, consider an expression like $d\Delta_k^{-1}$. As soon as we find out that the finite-dimensional operators $\pi_{\lambda,\sigma}(\Delta_k)$ are invertible and depend smoothly on λ (see the next Section), an operator Ψ satisfying the identity $\Psi\Delta_k = d$ is automatically defined on $\mathcal{S}_0\Lambda^k$ with values in $\mathcal{S}_0\Lambda^{k+1}$ by imposing that

$$\pi_\lambda(\Psi\omega) = \sum_\sigma \pi_{\lambda,\sigma}(d)\pi_{\lambda,\sigma}(\Delta_k)^{-1}P_\sigma^\lambda\pi_\lambda(\omega)\ .$$

Its closure $\bar{\Psi}$ is defined on the space consists of the $\omega \in L^2\Lambda^k$ such that

$$\int_{-\infty}^{+\infty} \sum_\sigma \|\pi_{\lambda,\sigma}(d)\pi_{\lambda,\sigma}(\Delta_k)^{-1}P_\sigma^\lambda\pi_\lambda(\omega)\|_{HS}^2\,|\lambda|^n\,d\lambda < \infty\ .$$

Notice that formal identities, like

(i) $d\Delta_k^{-1} = (d\Delta_k^{-\frac12})\Delta_k^{-\frac12}$;

(ii) $\pi_\lambda(d\Delta_k^{-1}) = \pi_\lambda(d)\pi_\lambda(\Delta_k^{-1})$;

(iii) $\pi_{\lambda,\sigma}(d\Delta_k^{-1}) = \pi_{\lambda,\sigma}(d)\pi_{\lambda,\sigma}(\Delta_k^{-1})$;

are fully justified on $\mathcal{S}_0\Lambda^k$.

In many instances we will make use of homogeneity properties of operators B as those considered in Lemma 3.2. As before, we assume that $(V,\rho), (V',\rho')$ are finite dimensional representations of $U(n)$.

We assume that the multiplicative group $\mathbb{R}_+$ acts on V by means of the linear representation $\gamma : \mathbb{R}_+ \to Ł(V)$ and on V' by means of the linear representation $\gamma' : \mathbb{R}_+ \to Ł(V')$ such that the operators $\gamma(r)$ and $\gamma'(r)$ are self-adjoint, and in such a way that each of these actions commutes with the corresponding action of $U(n)$ (on V given by ρ and on V' given by ρ').

We also denote by δ_r the dilating automorphism of H_n defined by

$$\delta_r(z,t) := (r^{1/2}z, rt), \qquad r>0\ ,$$

and let $\mathbb{R}_+$ act on functions on H_n by the representation

$$\beta(r)f := f\circ\delta_{r^{-1}}\ .$$

Then B is said to be *homogeneous of degree* a if

$$B\circ(\beta\otimes\gamma)(r) = r^{-a}\,(\beta\otimes\gamma')(r)\circ B \qquad \text{on } \mathcal{S}_0V, \text{ for every } r>0\ . \tag{3.6}$$

We shall repeatedly use the following lemma, which applies in particular to operators such as $\partial, \bar\partial, d_H$ etc.

Lemma 3.4. *Let $(V,\rho),(V',\rho')$ be finite dimensional unitary representations of $U(n)$ and let β,γ,γ' be as above.*

If B is a $U(n)$- equivariant, left-invariant operator as in Lemma 3.2 (i), homogeneous in the sense of (3.6) *for some $a\in\mathbb{R}$, then*

$$\overline{\operatorname{ran} B}\cap\mathcal{S}_0V' = B(\mathcal{S}_0V).$$

Proof. We just have to verify that

$$\overline{\operatorname{ran} B}\cap\mathcal{S}_0V' \subseteq B(\mathcal{S}_0V)\ ,$$

the other implication being contained in the assumptions.

The homogeneity of B implies that

$$\pi_\lambda(B) = |\lambda|^a\,(I\otimes\gamma'(|\lambda|^{-1}))\,\pi_{\operatorname{sgn}\lambda}(B)\,(I\otimes\gamma(|\lambda|))\ . \tag{3.7}$$

Assume in fact that $f\in\mathcal{S}_0V$. For $\lambda\neq 0$, $\pi_\lambda(f)\in Ł(\mathcal{F},\mathcal{F}\otimes V)$ and, by (2.2) and (2.3),

$$(I\otimes\gamma(|\lambda|))\,\pi_\lambda(f) = (I\otimes\gamma(|\lambda|))\,\pi_{\operatorname{sgn}\lambda}\big(\beta(\lambda)f\big) = \pi_{\operatorname{sgn}\lambda}\big((\beta\otimes\gamma)(|\lambda|)f\big).$$

Similarly,

$$(I\otimes\gamma'(|\lambda|))\,\pi_\lambda(Bf) = \pi_{\operatorname{sgn}\lambda}((\beta\otimes\gamma')(|\lambda|)(Bf)),$$

and thus, by the homogeneity (3.6) of B,

$$(I\otimes\gamma'(|\lambda|))\,\pi_\lambda(Bf) = |\lambda|^a\,\pi_{\operatorname{sgn}\lambda}(B\,((\beta\otimes\gamma)(|\lambda|)f\,)\,) = |\lambda|^a\,\pi_{\operatorname{sgn}\lambda}(B)(I\otimes\gamma(|\lambda|))\,\pi_\lambda(f).$$

This yields (3.7).

Since γ and γ' commute with ρ and ρ', $I\otimes\gamma$ respects the decomposition

$$\mathcal{F}\otimes V = \bigoplus_{\sigma\in\Sigma^{\rho,\pm}}\mathcal{E}_\sigma^\pm\ ,$$

in (2.10), we have

$$B_{\lambda,\sigma} = |\lambda|^a\,\big(I\otimes\gamma'(|\lambda|^{-1})\big)\,B_{\operatorname{sgn}\lambda,\sigma}\,\big(I\otimes\gamma(|\lambda|)\big)\ , \tag{3.8}$$

where $B_{\lambda,\sigma} = \pi_{\lambda,\sigma}(B) : \mathcal{E}_\sigma^{\rho,\operatorname{sgn}\lambda} \longrightarrow \mathcal{E}_\sigma^{\rho',\operatorname{sgn}\lambda}$ is the operator defined in (3.1).

We restrict now our attention to $\lambda=\pm1$ and write, for simplicity, $B_\sigma^\pm$ instead of $B_{\pm1,\sigma}$.

Since domain and codomain are finite dimensional, we have an inverse $Q_\sigma^\pm : \operatorname{ran} B_\sigma^\pm \longrightarrow (\ker B_\sigma^\pm)^\perp$ of $B_\sigma^\pm|_{(\ker B_\sigma^\pm)^\perp}$. Denote by $\tilde{Q}_\sigma^\pm$ the extension of $Q_\sigma^\pm$ to $\mathcal{E}_\sigma^{\rho,\pm}$ equal to 0 on $(\operatorname{ran} B_\sigma^\pm)^\perp$. If $f \in \mathcal{S}_0 V'$, define the function Jf by requiring that

$$P_\sigma^{\operatorname{sgn}\lambda}\pi_\lambda(Jf) = |\lambda|^{-a}(I\otimes\gamma(|\lambda|^{-1}))\tilde{Q}_\sigma^\pm(I\otimes\gamma'(|\lambda|))P_\sigma^{\operatorname{sgn}\lambda}\pi_\lambda(f).$$

If $f \in \overline{\operatorname{ran} B} \cap \mathcal{S}_0 V'$, then the range of $P_\sigma^{\operatorname{sgn}\lambda}\pi_\lambda(f)$ is contained in the range of $\pi_{\lambda,\sigma}(B)$.

Choose $0 < \delta < R$ such that (i) in the definition of $\mathcal{S}_0 V$ holds for f. Since for $\delta \le |\lambda| \le R$ the functions $\gamma'(|\lambda|)$ and $\gamma(|\lambda|^{-1})$ are smooth in λ, it is easy to see that $Jf \in \mathcal{S}_0 V$. Moreover, applying (3.8) to $g := Jf$, we see that $\pi_\lambda(B(Jf)) = \pi_\lambda(f)$, hence $f = B(Jf) \in B(\mathcal{S}_0 V)$. Q.E.D.

Proposition 3.5. *Let $(V,\rho),(V',\rho')$, β,γ,γ' be as above. Then the following hold.*

(i) *If B is a $U(n)$- equivariant, left-invariant linear operator, homogeneous in the sense of* (3.6)*, and bounded from L^2V to L^2V', then B satisfies the assumptions of Lemma 3.2 (i).*

(ii) *Assume that $H \subset L^2 V$ is a closed subspace, which is invariant under left-translation by elements of H_n, under the action of $U(n)$ and invariant under the dilations $(\beta\otimes\gamma)(r)$, $r>0$. Then*

$$\mathcal{S}_0 V = (\mathcal{S}_0 V \cap H) \oplus (\mathcal{S}_0 V \cap H^\perp)\ ,$$

where $\mathcal{S}_0 V \cap H$ is dense in H and $\mathcal{S}_0 V\cap H^\perp$ is dense in $H^\perp$.

Proof. As in the proof of Lemma 3.2, let $\{\Phi_\ell\}_{\ell\in\mathbb{N}}$ be an enumeration of the orthonormal basis of monomials in $\mathcal{F}$ and set $E_{\ell,\ell'}F = \langle F, \Phi_{\ell'}\rangle_{\mathcal{F}}\Phi_\ell$. Let also $\{e_i\}$ and $\{e'_j\}$ be (finite) bases of V and V' respectively.

We fix an interval $I = [a,b]$ with $0<a<b$ and, for every $\ell \in \mathbb{N}$, a function $g_\ell \in \mathcal{S}_0$ such that $\pi_\lambda(g_\ell) = E_{\ell,\ell}$ for $\lambda \in I$. Then $B(g_\ell \otimes e_i) \in L^2 V'$ and

$$\pi_\lambda\big(B(g_\ell\otimes e_i)\big) = \sum_j \Big(\sum_{h,k} c_{h,k,j}^{\ell,i}(\lambda)E_{h,k}\Big)\otimes e'_j\ , \tag{3.9}$$

with $c_{h,k,j}^{\ell,i} \in L^2(I)$ for every choice of the indices. Then almost every point $\lambda \in I$ is a Lebesgue point for all $c_{h,k,j}^{\ell,i}$ and for $\sum_{h,k}|c_{h,k,j}^{\ell,i}(\lambda)|^2$.

For every $f = \sum_i f_i \otimes e_i \in \mathcal{S}_0 V$ and for a.e. $\lambda \in I$,

$$\pi_\lambda(f) = \sum_{i,\ell}\pi_\lambda\big((f_i * g_\ell) * (g_\ell\otimes e_i)\big)\ ,$$

where the sum is finite (say over $\ell \le N$). The invariance of B under translations by elements $(0,t)$ of the center of H_n implies that B preserves the λ-support of the group Fourier transform. Therefore, we also have

$$\pi_\lambda(Bf) = \sum_{i,\ell}\pi_\lambda\big(B((f_i*g_\ell)*(g_\ell\otimes e_i))\big)\ ,$$

for a.e. $\lambda \in I$. On the other hand, $B\big((f_i * g_\ell) * (g_\ell \otimes e_i)\big) = (f_i * g_\ell) * B(g_\ell \otimes e_i)$. Hence, for a.e. $\lambda \in I$ (say, $\lambda \in \Lambda$),

$$\pi_\lambda(Bf) = \sum_{i,\ell} \pi_\lambda\big(B(g_\ell \otimes e_i)\big)\pi_\lambda(f_i * g_\ell)\ .$$

The same computations in the proof of Lemma 3.2 produce an infinite matrix $C_{i,j}(\lambda) = \big(c^{\ell,i}_{h,\ell,j}(\lambda)\big)_{h,\ell}$ with at most N nonzero entries on each row, defined for $\lambda \in \Lambda$. Defining B_λ by (3.4), we have that

$$\pi_\lambda(Bf) = B_\lambda \pi_\lambda(f)\ , \tag{3.10}$$

for $\lambda \in \Lambda$.

Now, the homogeneity of B easily implies that, for $\lambda, \lambda' \in \Lambda$,

$$B_\lambda = (\lambda/\lambda')^a \left(I \otimes \gamma'(\lambda'/\lambda)\right) B_{\lambda'} \left(I \otimes \gamma(\lambda/\lambda')\right)\ .$$

This identity allows to extend B_λ as a smooth function of λ to every $\lambda > 0$.

Obviously, the same construction can be made for $\lambda < 0$. Then, for every $f \in \mathcal{S}_0V$, the identity (3.10) holds for every $\lambda \neq 0$, which shows that $B(\mathcal{S}_0V) \subset \mathcal{S}_0V'$. Then Lemma 2.2 and the following remarks imply that we are in the hypotheses of Lemma 3.2 (ii).

In order to prove (ii), let us denote by P the orthogonal projection from L^2V onto H. Since H is invariant under left-translations, $U(n)$-invariant and dilation invariant, P is a left-invariant operator which is $U(n)$-equivariant and homogeneous of degree 0. Moreover, by the Schwartz kernel theorem, it is given by the convolution $Pf = f * K$ with a tempered distribution kernel K taking values in $\mathrm{Ł}(V,V)$. We may therefore apply (i) to $B := P$ and conclude by means of Lemma 3.4 that $P(\mathcal{S}_0V) \subset \mathcal{S}_0V$, and similarly, $(I-P)(\mathcal{S}_0V) \subset \mathcal{S}_0V$. Q.E.D.

4. First properties of Δ_k; exact and closed forms

The domain dom$\,(\Delta_0)$, defined according to Lemma 3.2, is the "left-invariant Sobolev space" H^2 consisting of those $f \in L^2$ such that $Xf, XYf \in L^2$ for every $X, Y \in \mathfrak{h}_n$. This follows from the L^2 boundedness of the operators $X(1+\Delta_0)^{-\frac12}$, $XY(1+\Delta_0)^{-1}$ [MPR1]. We also recall that the operators $XY\Delta_0^{-1}$, $X\Delta_0^{-\frac12}$ are bounded on L^2 for every $X, Y \in \mathfrak{h}_n$.

For $k \geq 1$, we have the analogous description of dom$\,(\Delta_k)$.

Lemma 4.1. *For every k,* dom$\,(\Delta_k) = H^2\Lambda^k$.

Proof. It is evident from (1.22) that $H^2\Lambda^k \subset$ dom$\,(\Delta_k)$.

Since $\Delta_0 = L - T^2$, identifying Δ_k with the matrix (1.22) we have

$$\begin{aligned}\Delta_k &= \begin{pmatrix}\Delta_0 & 0\\ 0 & \Delta_0\end{pmatrix} + \begin{pmatrix}\Delta_H - L + e(d\theta)i(d\theta) & i\bar\partial - i\partial \\ i\partial^* - i\bar\partial^* & \Delta_H - L + i(d\theta)e(d\theta)\end{pmatrix}\\ &= \Delta_0 + P\ .\end{aligned}$$

where P is symmetric on $H^2\Lambda^k$. By (1.19), each entry in P involves at most first-order derivatives in the left-invariant vector fields. Therefore, for $\omega \in H^2\Lambda^k$,

$$\begin{aligned}\|P\omega\|_2 &\le C\big(\|\omega\|_2 + \|\Delta_0^{\frac12}\omega\|_2\big)\\ &\le C\big(\|\omega\|_2 + \|\Delta_0\omega\|_2^{\frac12}\|\omega\|_2^{\frac12}\big)\\ &\le C(1+\varepsilon^{-1})\|\omega\|_2 + C\varepsilon\|\Delta_0\omega\|_2\ ,\end{aligned}$$

for every $\varepsilon > 0$. By the Kato-Rellich theorem [Ka], $\Delta_0 + P$ is self-adjoint on $\mathrm{dom}\,(\Delta_0) = H^2\Lambda^k$. Q.E.D.

The following statement is an immediate consequence.

Proposition 4.2. Δ_k *is injective on its domain.*

Proof. Let $\omega = \omega_1 + \theta \wedge \omega_2 \in \mathrm{dom}\,(\Delta_k)$, with ω_1, ω_2 horizontal. Then

$$\begin{aligned}\langle \Delta_k\omega,\omega\rangle &= \langle \Delta_H\omega_1,\omega_1\rangle + \|T\omega_1\|_2^2 + \|i(d\theta)\omega_1\|_2^2 + \big\langle \big[d_H^*, e(d\theta)\big]\omega_2,\omega_1\big\rangle\\ &\qquad + \big\langle \big[i(d\theta), d_H\big]\omega_1,\omega_2\big\rangle + \langle \Delta_H\omega_2,\omega_2\rangle + \|T\omega_2\|_2^2 + \|e(d\theta)\omega_2\|_2^2\ .\end{aligned}$$

Notice that

$$\langle \Delta_H\omega_1,\omega_1\rangle = \|d_H\omega_1\|_2^2 + \|d_H^*\omega_1\|_2^2\ ,$$

and the same holds for ω_2. Moreover,

$$\begin{aligned}&\big\langle \big[d_H^*, e(d\theta)\big]\omega_2,\omega_1\big\rangle + \big\langle \big[i(d\theta), d_H\big]\omega_1,\omega_2\big\rangle\\ &\qquad = 2\Re\mathrm{e}\,\big\langle \big[i(d\theta), d_H\big]\omega_1,\omega_2\big\rangle\\ &\qquad = 2\Re\mathrm{e}\,\big\langle d_H\omega_1, e(d\theta)\omega_2\big\rangle - 2\Re\mathrm{e}\,\big\langle i(d\theta)\omega_1, d_H^*\omega_2\big\rangle\\ &\qquad \ge -\|d_H\omega_1\|_2^2 - \|e(d\theta)\omega_2\|_2^2 - \|i(d\theta)\omega_1\|_2^2 - \|d_H^*\omega_2\|_2^2\ .\end{aligned}$$

It follows that

$$\langle \Delta_k\omega,\omega\rangle \ge \|d_H^*\omega_1\|_2^2 + \|d_H\omega_2\|_2^2 + \|T\omega_1\|_2^2 + \|T\omega_2\|_2^2\ . \tag{4.1}$$

Therefore, if $\Delta_k\omega = 0$, then $T\omega = 0$. Since $\pi_\lambda(T) = i\lambda\, I$, this implies that $\pi_\lambda(\omega) = 0$ for almost every λ, and finally that $\omega = 0$. Q.E.D.

Corollary 4.3. *For every $\lambda > 0$ and $\sigma \in \Sigma^\pm$, $d\pi_{\pm\lambda,\sigma}(\Delta_k)$ is invertible and for every pair of elements $u, v \in \mathcal{E}_\sigma^\pm$, $\langle \pi_{\pm\lambda,\sigma}(\Delta_k)u, v\rangle$ is a polynomial in λ. For every $\alpha > 0$, $\Delta_k^{-\alpha}$ maps $\mathcal{S}_0\Lambda^k$ into itself.*

Proof. By (4.1), $\|\Delta_k^{\frac12}\omega\|_2 \ge \|T\omega\|_2$ for every $\omega \in \mathcal{S}_0\Lambda^k$. This implies that

$$\|\pi_{\lambda,\sigma}(\Delta_k)^{\frac12}\xi\| \ge |\lambda|\|\xi\|, \quad \xi \in \mathcal{E}_\sigma^{\mathrm{sgn}\,\lambda},$$

for every λ, σ with $\lambda \ne 0$. The rest is obvious. Q.E.D.

We call *Riesz transforms* the operators

$$R_k = d\Delta_k^{-\frac{1}{2}} : \mathcal{S}_0\Lambda^k \longrightarrow \mathcal{S}_0\Lambda^{k+1} \ ,$$

and their adjoints

$$R_k^* = \Delta_k^{-\frac{1}{2}} d^* : \mathcal{S}_0\Lambda^{k+1} \longrightarrow \mathcal{S}_0\Lambda^k \ .$$

Lemma 4.4. *The following identities hold (with the convention that $R_{-1} = R_{2n+1} = 0$):*

$$R_k = \Delta_{k+1}^{-\frac{1}{2}} d \ , \quad R_k^* = d^* \Delta_{k+1}^{-\frac{1}{2}} \ , \tag{4.2}$$

$$R_{k+1} R_k = R_k^* R_{k+1}^* = 0 \ , \tag{4.3}$$

$$R_k^* R_k + R_{k-1} R_{k-1}^* = I \ . \tag{4.4}$$

In particular, $R_k R_{k-1} = 0$.

Moreover, if $1 \leq k \leq 2n$, $R_{k-1}R_{k-1}^*$, $R_k^* R_k$ are orthogonal projections on complementary orthogonal subspaces of $L^2\Lambda^k$ and R_k and R_k^* are partial isometries.

Proof. From the identity $d\Delta_k = \Delta_{k+1} d$ on test functions we derive that

$$\pi_{\lambda,\sigma}(d)\pi_{\lambda,\sigma}(\Delta_k) = \pi_{\lambda,\sigma}(\Delta_{k+1})\pi_{\lambda,\sigma}(d)$$

for all λ, σ. Hence

$$\pi_{\lambda,\sigma}(d)\pi_{\lambda,\sigma}(\Delta_k)^{-\frac{1}{2}} = \pi_{\lambda,\sigma}(\Delta_{k+1})^{-\frac{1}{2}}\pi_{\lambda,\sigma}(d)$$

by finite-dimensional linear algebra. In turn, this gives the first identity of (4.2) on $\mathcal{S}_0\Lambda^k$. The second identity is proved in the same way.

Then (4.3) follows from (4.2) and the identity $d^2 = 0$.

On $\mathcal{S}_0\Lambda^k$, by applying again $\pi_{\lambda,\sigma}$ to each term,

$$\begin{aligned} R_k^* R_k + R_{k-1} R_{k-1}^* &= \Delta_k^{-\frac{1}{2}} d^* d \Delta_k^{-\frac{1}{2}} + \Delta_k^{-\frac{1}{2}} d d^* \Delta_k^{-\frac{1}{2}} \\ &= \Delta_k^{-\frac{1}{2}} \Delta_k \Delta_k^{-\frac{1}{2}} \\ &= I \ , \end{aligned}$$

which gives (4.4).

Since the two summands on the left-hand side of (4.4) are positive operators, they are L^2-contractions. Since their sum is the identity and their product is zero by (4.3), they are idempotent. This proves that they are orthogonal projections. It follows that R_k and R_{k-1}^* are partial isometries. Q.E.D.

The following statement says in particular that the cohomology groups of the De Rham complex are trivial.

Proposition 4.5. *Let $1 \leq k \leq 2n$. The following subspaces of $L^2\Lambda^k$ are the same:*

(i) *the range of $R_{k-1}R_{k-1}^*$;*
(ii) *the range of R_{k-1};*
(iii) $\ker R_k$*;*
(iv) $\ker d$*;*

(v) $\overline{d(\mathcal{S}_0\Lambda^{k-1})}$;

(vi) $\overline{d(\mathcal{D}\Lambda^{k-1})}$;

(vii) $\{\omega \in L^2\Lambda^k : \omega = du$ *in the sense of distributions for some* $u \in \mathcal{D}'\Lambda^{k-1}\}$.

We call this space $(L^2\Lambda^k)_{d\text{-ex}}$ *or* $(L^2\Lambda^k)_{d\text{-cl}}$. *Similarly, the following spaces*

(i') *the range of* $R_k^* R_k$;

(ii') *the range of* R_k^*;

(iii') $\ker R_{k-1}^*$;

(iv') $\ker d^*$;

(v') $\overline{d^*(\mathcal{S}_0\Lambda^{k+1})}$;

(vi') $\overline{d^*(\mathcal{D}\Lambda^{k+1})}$;

(vii') $\{\omega \in L^2\Lambda^k : \omega = d^*v$ *in the sense of distributions for some* $v \in \mathcal{D}'\Lambda^{k+1}\}$

are the same; we call them $(L^2\Lambda^k)_{d^*\text{-ex}}$ *or* $(L^2\Lambda^k)_{d^*\text{-cl}}$.

Proof. Since R_{k-1} is a partial isometry, its range is closed, and

$$\operatorname{ran} R_{k-1} = (\ker R_{k-1}^*)^\perp = (\ker R_{k-1}R_{k-1}^*)^\perp = \operatorname{ran} R_{k-1}R_{k-1}^* .$$

This proves the identity of the spaces in (i) and (ii). In the same way one proves the same for (i') and (ii'). From Lemma 4.4 we then obtain the orthogonal decomposition

$$L^2\Lambda^k = \operatorname{ran} R_{k-1} \oplus \operatorname{ran} R_k^* .$$

But $\operatorname{ran} R_k^* = (\ker R_k)^\perp$, so that $\operatorname{ran} R_{k-1} = \ker R_k$, i.e. (ii)=(iii).

By Plancherel's formula and Lemma 3.2, $\omega \in \ker d$ if and only if $\pi_{\lambda,\sigma}(d)\pi_{\lambda,\sigma}(\omega) = 0$ for a.e. λ and every σ. By Corollary 4.3 and (4.2), this is equivalent to saying that $\pi_{\lambda,\sigma}(R_k)\pi_{\lambda,\sigma}(\omega) = 0$ for a.e. λ and every σ, i.e. that $R_k\omega = 0$. So (iii)=(iv). By Corollary 4.3, $d(\mathcal{S}_0\Lambda^{k-1}) = R_{k-1}(\mathcal{S}_0\Lambda^{k-1})$ and this implies that (v)=(ii).

We thus have shown that the spaces (i) - (v) are the same, and the equality of the spaces (i')-(v') are proved in the same way.

In order to prove that the spaces (i) - (v) agree also with the space (vi), we first observe that $\overline{d(\mathcal{D}\Lambda^{k-1})} \subset \ker d$, since $d^2 = 0$ on $\mathcal{D}\Lambda^{k-1}$. We thus have $\overline{d(\mathcal{D}\Lambda^{k-1})} \subset R_{k-1}(L^2\Lambda^{k-1})$. To prove that these spaces are indeed the same, it will suffice to prove that $\sigma \perp R_{k-1}(L^2\Lambda^{k-1})$ whenever $\sigma \in L^2\Lambda^k$ satisfies $\sigma \perp d(\mathcal{D}\Lambda^{k-1})$. But, the latter condition means that $d^*\sigma = 0$ in the sense of distributions. So, by Lemma 3.2, $\sigma \in \operatorname{dom} d^*$, and since (iv')=(ii'), we see that $\sigma = R_k^*\xi$ for some $\xi \in L^2\Lambda^{k+1}$. This implies that for every $R_{k-1}\mu \in R_{k-1}(L^2\Lambda^{k-1})$

$$\langle \sigma, R_{k-1}\mu \rangle = \langle R_k^*\xi, R_{k-1}\mu \rangle = \langle \xi, R_k R_{k-1}\mu \rangle = 0.$$

We have thus seen that the spaces (i) - (vi) all agree, and in a similar way one proves that the spaces (i') - (vi') are all the same.

The proof that these spaces also do agree with the space (vii) respectively (vii') will require deeper L^p-methods, and will therefore be postponed to Section 11 (see Corollary 11.3).

Q.E.D.

Working out the same program for $\partial, \bar\partial$, their adjoints and the box-operators one encounters some differences. One simplification comes from the fact that $\Box$ and $\overline{\Box}$ act as scalar operators on horizontal forms of a given bi-degree.

On the other hand, a complication comes from the fact they have a non-trivial null space in L^2 for certain values of p or q. It is well known since [FS] that $L+i\alpha T$ is injective on L^2 if and only if $\alpha \neq \pm(n+2j)$, $j\in\mathbb{N}$, and that it is hypoelliptic under the same restriction. It follows from (1.19) that $\Box$ (resp. $\overline{\Box}$) is injective, and hypoelliptic, on (p,q)-forms provided that $p\neq 0,n$ (resp. $q\neq 0,n$).

For $p=0$, $\Box_0=\partial^*\partial$ and $\ker\Box=\ker\partial$, while, for $p=n$, $\Box=\partial\partial^*$ and $\ker\Box=\ker\partial^*$. Similarly, $\ker\overline{\Box}=\ker\bar\partial$ for $q=0$, and $\ker\overline{\Box}=\ker\bar\partial^*$ for $q=n$.

For these values of p (resp. q), we shall denote by $\Box'$ (resp. $\overline{\Box}'$) the unprimed operator with domain and range restricted to the orthogonal complement of the corresponding null space. Notice that the core $\mathcal{S}_0\Lambda^{p,q}$ splits according to the decompositions $\ker\partial\oplus(\ker\partial)^\perp$, $\ker\bar\partial\oplus(\ker\bar\partial)^\perp$. The negative powers $\Box'^{-\alpha}$(resp. $\overline{\Box}'^{-\alpha}$) are then well defined on $\mathcal{S}_0\Lambda^{p,q}\cap(\ker\partial)^\perp$ (resp. $\mathcal{S}_0\Lambda^{p,q}\cap(\ker\bar\partial)^\perp$).

By (1.19)

$$\Box_0=\overline{\Box}_n=\frac12(L+inT)\ ,\qquad \Box_n=\overline{\Box}_0=\frac12(L-inT)\ .$$

We denote by $\mathcal{C}$ (resp. $\bar{\mathcal{C}}$) the orthogonal projection from scalar L^2 onto $\ker(L+inT)$ (resp. $\ker(L-inT)$). The same symbols will be used to denote the extension to forms by componentwise application.

Thus, $\mathcal{C}$ is the orthogonal projection onto $\ker\partial$ when acting on $(0,q)$-forms as well as onto $\ker\partial^*$ when acting on (n,q)-forms, and $\bar{\mathcal{C}}$ is the orthogonal projection onto $\ker\bar\partial$ when acting on $(p,0)$-forms as well as onto $\ker\bar\partial^*$ when acting on (p,n)-forms.

Regard ∂ as a closed operator from $L^2\Lambda^{p,q}$ to $L^2\Lambda^{p+1,q}$. The *holomorphic Riesz transforms* are defined on $\mathcal{S}_0\Lambda^{p,q}$ (with values in $\mathcal{S}_0\Lambda^{p+1,q}$) by

$$\mathcal{R}_p=\begin{cases}\partial\Box_p^{-\frac12}=\Box_{p+1}^{-\frac12}\partial & \text{for } 1\le p\le n-2\ ,\\ \partial{\Box'_0}^{-\frac12}(I-\mathcal{C})=\Box_1^{-\frac12}\partial & \text{for } p=0\ ,\\ \partial\Box_{n-1}^{-\frac12}={\Box'_n}^{-\frac12}\partial & \text{for } p=n-1\ .\end{cases}\tag{4.5}$$

We observe that, is all cases,

$$\mathcal{R}_p\Box_p^{\frac12}=\Box_{p+1}^{\frac12}\mathcal{R}_p=\partial\ .\tag{4.6}$$

The adjoint operators $\mathcal{R}_p^*$ from $\mathcal{S}_0\Lambda^{p+1,q}$ to $\mathcal{S}_0\Lambda^{p,q}$ are

$$\mathcal{R}_p^*=\begin{cases}\Box_p^{-\frac12}\partial^*=\partial^*\Box_{p+1}^{-\frac12} & \text{for } 1\le p\le n-2\ ,\\ {\Box'_0}^{-\frac12}\partial^*=\partial^*\Box_1^{-\frac12} & \text{for } p=0\ ,\\ \Box_{n-1}^{-\frac12}\partial^*=\partial^*{\Box'_n}^{-\frac12}(I-\bar{\mathcal{C}}) & \text{for } p=n-1\ .\end{cases}\tag{4.7}$$

The analogues of (4.3) and (4.4) are

$$
\begin{aligned}
&\mathcal{R}_{p+1}\mathcal{R}_p = \mathcal{R}_p^*\mathcal{R}_{p+1}^* = 0 \ , \\
&\mathcal{R}_p^*\mathcal{R}_p + \mathcal{R}_{p-1}\mathcal{R}_{p-1}^* = I \ , \qquad (1 \le p \le n-1) \\
&\mathcal{R}_0^*\mathcal{R}_0 = I - \mathcal{C} \ , \quad \mathcal{R}_{n-1}\mathcal{R}_{n-1}^* = I - \bar{\mathcal{C}} \ .
\end{aligned} \tag{4.8}
$$

Proposition 4.5 has the following analogue.

Proposition 4.6. *For $0 \le p \le n-1$, the following subspaces of $L^2\Lambda^{p,q}$ are the same:*

(i) $\ker \mathcal{R}_p$*;*
(ii) $\ker \partial$*;*

We call this space $(L^2\Lambda^{p,q})_{\partial\text{-cl}}$.

For $1 \le p \le n$, the following subspaces of $L^2\Lambda^{p,q}$ are the same:

(iii) the range of $\mathcal{R}_{p-1}\mathcal{R}_{p-1}^*$;
(iv) $\overline{\text{the range of } \mathcal{R}_{p-1}}$;
(v) $\overline{\partial(\mathcal{S}_0\Lambda^{p-1,q})}$.

We call this subspace $(L^2\Lambda^{p,q})_{\partial\text{-ex}}$.

For $1 \le p \le n-1$, $(L^2\Lambda^{p,q})_{\partial\text{-cl}} = (L^2\Lambda^{p,q})_{\partial\text{-ex}}$.

Similarly, for $1 \le p \le n$, the following subspaces of $L^2\Lambda^{p,q}$ are the same:

(i') $\ker \mathcal{R}_{p-1}^*$*;*
(ii') $\ker \partial^*$*;*

and we call this subspace $(L^2\Lambda^{p,q})_{\partial^*\text{-cl}}$.

For $0 \le p \le n-1$, the following subspaces of $L^2\Lambda^{p,q}$ are the same:

(iii') the range of $\mathcal{R}_p^*\mathcal{R}_p$;
(iv') $\overline{\text{the range of } \mathcal{R}_p^*}$;
(v') $\overline{\partial^*(\mathcal{S}_0\Lambda^{p+1,q})}$*;*

and we call this subspace $(L^2\Lambda^{p,q})_{\partial^*\text{-ex}}$.

Finally, for $1 \le p \le n-1$, $(L^2\Lambda^{p,q})_{\partial^*\text{-cl}} = (L^2\Lambda^{p,q})_{\partial^*\text{-ex}}$.

We also set $(L^2\Lambda^k_H)_{\partial\text{-ex}} = \sum_{p+q=k}(L^2\Lambda^{p,q})_{\partial\text{-ex}}$ etc.

The *antiholomorphic Riesz transforms* $\overline{\mathcal{R}}_q$ and their adjoints $\overline{\mathcal{R}}_q^*$ are defined by conjugating all terms in (4.5) and (4.7) respectively, and replacing p by q. The analogue of formula (10.3) also holds true for all q

$$
\overline{\mathcal{R}}_p\overline{\Box}_q^{\frac{1}{2}} = \overline{\Box}_{q+1}^{\frac{1}{2}}\overline{\mathcal{R}}_q = \bar{\partial} \ . \tag{4.9}
$$

The rest goes in perfect analogy with the holomorphic case.

Definition 4.7. On (p, q)-forms, we also define the operators

$$
\begin{aligned}
&C_p = I - \mathcal{R}_p^*\mathcal{R}_p \ , \qquad \overline{C}_q = I - \overline{\mathcal{R}}_q^*\overline{\mathcal{R}}_q \ , \qquad \text{for} \quad 0 \le p, q \le n-1 \ , \\
&C_n = I = \overline{C}_n.
\end{aligned} \tag{4.10}
$$

Notice that, by (4.8), $C_p = \mathcal{R}_{p-1}\mathcal{R}_{p-1}^*$ for $1 \le p \le n-1$, and similarly $\overline{C}_q = \overline{\mathcal{R}}_{q-1}\overline{\mathcal{R}}_{q-1}^*$ for $1 \le q \le n-1$.

The following statements are obvious in view of Proposition 4.6.

Lemma 4.8. *C_p is the orthogonal projection of $L^2\Lambda^{p,q}$ onto the kernel of ∂, and $\overline{C}_p$ is the orthogonal projection of $L^2\Lambda^{p,q}$ onto the kernel of $\overline{\partial}$,*

Moreover, if $\omega \in \mathcal{S}_0\Lambda^{p,q}$, with $1 \le p \le n-1$, then

$$C_p\omega = 0 \qquad \text{if and only if} \qquad \omega \in \overline{\partial^*(\mathcal{S}_0\Lambda^{p+1,q})} \qquad \text{if and only if} \qquad \partial^*\omega = 0,$$

whereas for $p = 0$,

$$C_0\,\omega = 0 \qquad \text{if and only if} \qquad \omega \in \overline{\partial^*(\mathcal{S}_0\Lambda^{1,q})},$$

and for $p = n$,

$$C_n\,\omega = 0 \qquad \text{if and only if} \qquad \omega = 0.$$

Analogous statements hold for the operators $\overline{C}_q$, if we replace p by q and conjugate all terms. In particular, $C_0 = \mathcal{C}, \overline{C}_0 = \overline{\mathcal{C}}$, and $\partial^*\omega = 0$ whenever $C_p\omega = 0$, and $\overline{\partial}^*\omega = 0$ whenever $\overline{C}_q\omega = 0$.

Given a horizontal k-form $\omega = \sum_{p+q=k}\omega_{pq}$ we finally set

$$C\omega = \sum_{p+q=k} C_p\omega_{pq}\,, \qquad \text{and} \quad \overline{C}\omega = \sum_{p+q=k} \overline{C}_q\omega_{pq}\,. \tag{4.11}$$

5. A decomposition of $L^2\Lambda^k_H$ related to the ∂ and $\bar\partial$ complexes

In this section, we shall work under the *assumption that* $0 \le k \le n$, as this turns out to be more convenient in view of the Lefschetz decomposition described in Prop. 2.1 of [MPR1]. The case where $k > n$ can be reduced to the case $k \le n$ by means of Hodge duality, as will be shown later in Section 8.

Our starting point in the spectral analysis of Δ_k is the decomposition obtained in Proposition 4.5

$$L^2\Lambda^k = (L^2\Lambda^k)_{d\text{-ex}} \oplus (L^2\Lambda^k)_{d^*\text{-cl}}\ . \tag{5.1}$$

Since $d\Delta_{k-1} = \Delta_k d$ for all $k \ge 1$, using the results from [MPR1] for Δ_1, we can lift the decomposition of $L^2\Lambda^1$ into Δ_1-invariant subspaces and the related spectral properties to $(L^2\Lambda^2)_{d\text{-ex}}$. Therefore, inductively we analyse the $(L^2\Lambda^k)_{d\text{-ex}}$-component in the decomposition of $L^2\Lambda^k$ by means of the preceeding step.

Thus, we are led to study the $(L^2\Lambda^k)_{d^*\text{-cl}}$-component in the decomposition of $L^2\Lambda^k$.

By (1.21) we can characterize the d^*-closed forms. Notice that, if $\omega \in \mathcal{S}_0\Lambda^k$, $\omega = \omega_1 + \theta\wedge\omega_2$ with ω_1, ω_2 horizontal, then

$$\omega \in \big(\mathcal{S}_0\Lambda^k\big)_{d^*\text{-cl}} \qquad \text{if and only if} \qquad \omega_2 = T^{-1}d_H^*\omega_1\ .$$

In fact, if $\omega_2 = T^{-1}d_H^*\omega_1$, then the second equation $i(d\theta)\omega_1 - d_H^*\omega_2 = 0$ arising from in (1.21) follows from the first one.

Hence, if we set

$$\Phi(\omega) = \omega + \theta\wedge T^{-1}d_H^*\omega \tag{5.2}$$

we obtain an isomorphism

$$\Phi : \mathcal{S}_0\Lambda^k_H \longrightarrow \big(\mathcal{S}_0\Lambda^k\big)_{d^*\text{-cl}} .$$

Notice that, because of the invariance of θ and the equivariance of d^*_H, Φ commutes with the action of $U(n)$.

Clearly, Δ_k maps a subspace V of $(\mathcal{S}_0\Lambda^k)_{d^*\text{-cl}}$ into itself if and only if the (non-differential, see (5.17) below) operator

$$D_k := \Phi^{-1}\Delta_k\Phi , \tag{5.3}$$

maps $W = \Phi^{-1}V \subset \mathcal{S}_0\Lambda^k_H$ into itself.

For this reason, we begin by decomposing $\mathcal{S}_0\Lambda^k_H$ into orthogonal subspaces which are invariant under D_k and on which D_k takes a simple form.

5.1. **The subspaces.**

The decomposition is based on the following lemma.

Lemma 5.1. *Every $\omega \in \mathcal{S}_0\Lambda^k_H$ decomposes as*

$$\omega = \omega' + \partial\xi + \bar\partial\eta , \tag{5.4}$$

where $\xi, \eta \in \mathcal{S}_0\Lambda^{k-1}_H$, and $\omega' \in \mathcal{S}_0\Lambda^k$ satisfies the condition

$$\partial^*\omega' = \bar\partial^*\omega' = 0 . \tag{5.5}$$

The term ω' is uniquely determined, and we can assume, in addition, that

$$C_{p-1}\xi = \bar C_{q-1}\eta = 0 . \tag{5.6}$$

Notice that, even with the extra assumption (5.6), ξ and η are not uniquely determined.

Proof. Assume that ω is a (p,q)-form. If $p = 0$, we obviously have the decomposition $\omega = \omega' + \bar\partial\eta$, with $\bar\partial^*\omega' = 0$, and $\partial^*\omega' = 0$ holds tivially, since ω' is a $(0,q)$- form. A similar argument applies if $q = 0$.

We therefore assume that $p, q \geq 1$. Consider the homogeneous $U(n)$-equivariant differential operator

$$(\partial\ \bar\partial) : \begin{pmatrix}\xi\\ \eta\end{pmatrix} \mapsto \partial\xi + \bar\partial\eta ,$$

acting from $L^2(\Lambda^{p-1,q}) \oplus L^2(\Lambda^{p,q-1})$ to $L^2\Lambda^{p,q}$ and its adjoint $\begin{pmatrix}\partial^*\\ \bar\partial^*\end{pmatrix}$.

In $L^2\Lambda^{p,q}$, we have

$$\operatorname{ran}(\partial\ \bar\partial) = \operatorname{ran}\partial + \operatorname{ran}\bar\partial , \qquad \ker\begin{pmatrix}\partial^*\\ \bar\partial^*\end{pmatrix} = \ker\partial^* \cap \ker\bar\partial^* ,$$

so that

$$L^2\Lambda^{p,q} = (\ker\partial^* \cap \ker\bar\partial^*) \oplus \overline{(\operatorname{ran}\partial + \operatorname{ran}\bar\partial)} .$$

Moreover,

$$(\partial\ \bar\partial)(\mathcal{S}_0\Lambda^{p-1,q} \oplus \mathcal{S}_0\Lambda^{p,q-1}) = \partial\mathcal{S}_0\Lambda^{p-1,q} + \bar\partial\mathcal{S}_0\Lambda^{p,q-1} ,$$

so that, by Lemma 3.4,

$$\mathcal{S}_0\Lambda^{p,q} = (\ker\partial^* \cap \ker\bar\partial^* \cap \mathcal{S}_0\Lambda^{p,q}) \oplus (\partial\mathcal{S}_0\Lambda^{p-1,q} + \bar\partial\mathcal{S}_0\Lambda^{p,q-1}) \ .$$

This gives the decomposition (5.4). By orthogonality, the two terms ω' and $\partial\xi + \bar\partial\eta$ are uniquely determined. Since C_{p-1} and $\bar C_{q-1}$ preserve $\mathcal{S}_0$-forms, we can replace ξ by $(I - C_{p-1})\xi$ and η by $(I - \bar C_{q-1})\eta$, without changing the equality. Q.E.D.

Observe that the decomposition (5.4), without the extra assumptions on ξ and η, can be iterated, so to obtain in a next step that

$$\begin{aligned}\omega &= \omega' + \partial(\xi' + \partial\alpha_1 + \bar\partial\beta_1) + \bar\partial(\eta' + \partial\alpha_2 + \bar\partial\beta_2)\\ &= \omega' + \partial\xi' + \bar\partial\eta' + \partial\bar\partial\beta_1 + \bar\partial\partial\beta_2 \ ,\end{aligned}$$

where now each of the primed symbols represents a form satisfying (5.5). If ω is a horizontal k-form, the iteration stops after k steps, leaving no "remainder terms".

We are so led to introduce, for each $m \le k$ the spaces of forms

(5.7) $$\omega = \underbrace{\cdots\partial\bar\partial\partial}_{m\text{-terms}}\xi + \underbrace{\cdots\bar\partial\partial\bar\partial}_{m\text{-terms}}\eta \ ,$$

with $\xi, \eta \in \mathcal{S}_0\Lambda_H^{k-m}$ and $\partial^*\xi = \bar\partial^*\xi = \partial^*\eta = \bar\partial^*\eta = 0$.

It is convenient to observe that in a sequence of at least three alternating ∂'s and $\bar\partial$'s, we can replace a product $\partial\bar\partial$ or $\bar\partial\partial$ by $d_H^2 = -T^{-1}e(d\theta)$. Since T^{-1} preserves $\mathcal{S}_0$-forms, the form ω in (5.7) can thus be written as

$$\omega = \begin{cases} e(d\theta)^\ell(\partial\xi + \bar\partial\eta) & \text{if } m = 2\ell+1 \ ,\\ e(d\theta)^\ell(\bar\partial\partial\xi + \partial\bar\partial\eta) & \text{if } m = 2\ell+2 \ .\end{cases}$$

Definition 5.2. We set

$$\begin{aligned} W_0^{p,q} &= \big\{\omega \in \mathcal{S}_0\Lambda^{p,q} : \partial^*\omega = \bar\partial^*\omega = 0\big\} \ ,\\ W_1^{p,q} &= \big\{\omega = \partial\xi + \bar\partial\eta \ : \xi, \eta \in W_0^{p,q}\big\} \ ,\\ W_2^{p,q} &= \big\{\omega = \bar\partial\partial\xi + \partial\bar\partial\eta \ : \ \xi, \eta \in W_0^{p,q}\big\} \,. \end{aligned}$$

For $\ell \in \mathbb{N}$ and $j = 1, 2$, we set

$$W_{j,\ell}^{p,q} = e(d\theta)^\ell W_j^{p,q} \,.$$

We also set

$$W_0^k = \sum_{p+q=k} W_0^{p,q} = \big\{\omega \in \mathcal{S}_0\Lambda_H^k : \partial^*\omega = \bar\partial^*\omega = 0\big\} \ ,$$

and, for $j = 1, 2$ and $\ell \in \mathbb{N}$,

$$W_j^k = \sum_{p+q+j=k} W_j^{p,q}$$

and

$$W_{j,\ell}^k = \sum_{p+q+j+2\ell=k} W_{j,\ell}^{p,q} = e(d\theta)^\ell W_j^{k-2\ell} \,,$$

whenever $k \ge j + 2\ell$.

The symbols $\mathcal{W}_j^k, \mathcal{W}_j^{p,q}$, etc. denote the L^2-closures of the corresponding spaces W_j^k, $W_j^{p,q}$, etc..

We wish to characterize which spaces among the $W_0^{p,q}$ and $W_{j,\ell}^{p,q}$ are non-trivial.

Proposition 5.3. *Let $0 \le k \le n$ and $p+q=k$. Then $W_0^{p,q}$ is trivial if and only if $k=n$ and $1 \le p,q \le n-1$.*

Proof. We show first that $W_0^{p,q}$ is non-trivial for $p+q \le n-1$. In order to do so, it is sufficient to prove that, under this assumption, there is a non-zero $\beta \in \mathcal{P}_1 \otimes \Lambda^{p,q}$, with $\mathcal{P}_1 = \mathrm{span}\,\{w_1,\dots,w_n\}$ as in (2.6), such that

$$\pi_1(\partial^*)\beta = \pi_1(\bar\partial^*)\beta = 0\ .$$

From this it will easily follow from (1.10) and (2.8) that $\pi_\lambda(\partial^*)\beta = \pi_\lambda(\bar\partial^*)\beta = 0$ for every $\lambda > 0$.

Let $\omega \in \Lambda^{p,q}$ be such that $\pi_\lambda(\omega) = \chi(\lambda) P_\beta$, where χ is a smooth cut-off function with compact support in $(0,+\infty)$ and P_β is the orthogonal projection of $\mathcal{F} \otimes \Lambda^{p,q}$ onto $\mathbb{C}\beta$. Then $\omega \in \mathcal{S}_0\Lambda^{p,q}$ and $\partial^*\omega = \bar\partial^*\omega = 0$.

Take

$$\beta = \Big(\sum_{j=1}^{p+1} (-1)^j w_j\, \zeta \wedge \cdots \wedge \widehat{\zeta_j} \wedge \cdots \wedge \zeta_{p+1}\Big) \wedge \bar\zeta^{I'}\ ,$$

where $I' = (p+2,\dots,p+q+1)$. Then, writing $I_{\widehat{j}} = (1,\dots,\widehat{j},\dots,p+1)$ we have

$$\begin{aligned}
\pi_1(\partial^*)(\beta) &= \frac{1}{\sqrt2} \sum_{\ell,J} \sum_{j=1}^{p+1} (-1)^j \varepsilon^{I_{\widehat{j}}}_{\ell J} w_\ell w_j\, \zeta^J \wedge \bar\zeta^{I'} \\
&= \frac{1}{\sqrt2} \sum_{j=1}^{p+1} (-1)^j \Big[\sum_{\ell=1}^{j-1} (-1)^{\ell-1} w_\ell w_j\, \zeta_1 \wedge \cdots \wedge \widehat{\zeta_\ell} \wedge \cdots \wedge \widehat{\zeta_j} \wedge \cdots \wedge \zeta_{p+1} \\
&\qquad\qquad + \sum_{\ell=j+1}^{p+1} (-1)^\ell w_\ell w_j\, \zeta_1 \wedge \cdots \wedge \widehat{\zeta_j} \wedge \cdots \wedge \widehat{\zeta_\ell} \wedge \cdots \wedge \zeta_{p+1} \Big] \wedge \bar\zeta^{I'} \\
&= 0\ .
\end{aligned}$$

Next, since $\pi_1(Z_\ell) = \sqrt2 \partial_{w_\ell}$ we have

$$\pi_1(\bar\partial^*)(\beta) = (-1)^{p+1}\sqrt2 \sum_{\ell,\,J'} \varepsilon^{I'}_{\ell J'} \big(\partial_{w_\ell} w_j\big) \zeta^{I_{\widehat{j}}} \wedge \bar\zeta^{J'} = 0\ ,$$

since $j \notin I'$, so that $\partial_{w_\ell} w_j = 0$.

This shows that $W_0^{p,q} \ne \{0\}$ when $p+q \le n-1$.

Next, consider $W_0^{n,0}$. Take $\beta = \zeta_1 \wedge \cdots \wedge \zeta_n \in \mathcal{P}_0 \otimes \Lambda^{n,0}$. Clearly $\pi_\lambda(\bar\partial^*)\beta = 0$ for every $\lambda \ne 0$, while $\pi_\lambda(\bar\partial^*)\beta = 0$ for $\lambda < 0$ by (1.10) and (2.8). As before, this implies that $W_0^{n,0} \ne \{0\}$.

Finally, consider $W_0^{n-s,s}$, with $1 \le s \le n-1$ and let $\omega \in W_0^{n-s,s}$. Since $\Box$ is injective on this space and $\partial^*\omega = 0$ we have

$$\omega = \partial^*\big(\partial \Box^{-1}\big)\omega =: \partial^*\nu\ .$$

Similarly, since $\bar\partial^*\omega = 0$,

$$\begin{aligned}\omega &= \bar\partial^*\bar\partial\overline{\Box}^{-1}\omega = \bar\partial^*\bar\partial\overline{\Box}^{-1}\partial^*\nu \\ &= \bar\partial^*\bar\partial\partial^*(\overline{\Box}-iT)^{-1}\nu = \bar\partial^*\partial^*\big(-\bar\partial(\overline{\Box}-iT)^{-1}\nu\big)\ ,\end{aligned}$$

i.e.

$$\omega = \bar\partial^*\partial^*\mu\ , \tag{5.8}$$

for some μ. But $\bar\partial^*\omega = 0$ if and only if

$$0 = \partial^*\bar\partial^*(\partial^*\mu) = (\bar\partial^*\partial^* + \partial^*\bar\partial^*)(\partial^*\mu) = Ti(d\theta)(\partial^*\mu)\ .$$

It follows that, if μ is as in (5.8), then $\omega = \bar\partial^*\partial^*\mu \in W_0^{s,n-s}$ if and only if

$$i(d\theta)(\partial^*\mu) = 0\ , \tag{5.9}$$

i.e. $\partial^*\mu \in \ker i(d\theta)$.

Therefore, $\partial^*\mu \in \mathcal{S}_0\Big(\ker_{\Lambda^{s,n-s+1}} i(d\theta)\Big)$. Since $\max\{0, s+(n-s+1)-n\} = \max\{0,1\} = 1 > 0$, according to Prop. 2.1 in in [MPR1], we have $\ker_{\Lambda^{s,n-s+1}} i(d\theta) = \{0\}$, that is, $\partial^*\mu = 0$; hence $\omega = 0$. Q.E.D.

Proposition 5.4. *Assume that $j = 1, 2$. Then the space $W_{j,\ell}^{p,q}$ is non-trivial if and only if $\ell + j + p + q \le n$. In this case, $e(d\theta)^\ell$ is bijective from $W_j^{p,q}$ onto $W_{j,\ell}^{p,q}$.*

Proof. We first prove the "only if" part. Observe that $W_0^{p,q}$ and $W_1^{p,q}$ are in the kernel of $i(d\theta)$, which is immediate from (1.12) and (1.14).

In order to prove the statement for $j = 1$, we set $(\tilde p, \tilde q) = (p+1, q)$ or $(p, q+1)$. By Prop. 2.1 in [MPR1] we know that $e(d\theta)^\ell\big(\ker i(d\theta)_{|_{L^2\Lambda^{\tilde p,\tilde q}}}\big)$ is non-trivial if and only if $\max(0, \tilde p + \tilde q + 2\ell - n) \le \ell$, that is, $\ell \le n - \tilde p - \tilde q$. Since $W_1^{p,q} \subseteq L^2\Lambda^{p+1,q} + L^2\Lambda^{p,q+1}$ it follows that $W_{1,\ell}^{p,q}$ can be non-trivial only when $\ell \le n - p - q - 1$.

To prove that $e(d\theta)^\ell$ is injective on $W_1^{p,q}$ under this condition, we show by induction on ℓ that $e(d\theta)^\ell$ is injective on $\ker i(d\theta)_{|_{\Lambda^{\tilde p,\tilde q}}}$ when $\ell \le n - \tilde p - \tilde q = n - p - q - 1$. The case $\ell = 0$ is trivial. And, by (1.16) we see that for $\ell \ge 1$ when acting on $(\tilde p, \tilde q)$-forms

$$\begin{aligned}[i(d\theta),\, e(d\theta)^\ell] &= \sum_{\nu=0}^{\ell-1} e(d\theta)^\nu [i(d\theta),\, e(d\theta)] e(d\theta)^{\ell-1-\nu} \\ &= \sum_{\nu=0}^{\ell-1} (n - \tilde p - \tilde q - 2\ell + 2 + 2\nu) e(d\theta)^{\ell-1} \\ &= \ell(n - \tilde p - \tilde q - \ell + 1) e(d\theta)^{\ell-1} \\ &= \ell(n - p - q - \ell) e(d\theta)^{\ell-1}\ ,\end{aligned} \tag{5.10}$$

which allows to prove injectivity of $e(d\theta)^\ell$ on $\ker i(d\theta)_{|_{\Lambda^{\tilde p,\tilde q}}}$ from injectivity of $e(d\theta)^{\ell-1}$ under the assumption on ℓ.

We now turn to the case $j=2$, which requires a more refined discussion. Let us set

$$K_2^{p,q} = W_2^{p,q} \cap \ker i(d\theta)_{| L^2\Lambda_H^{p+q+2}}.$$

We claim that $W_2^{p,q}$ decomposes as an orthogonal sum

$$W_2^{p,q} = K_2^{p,q} \oplus e(d\theta) W_0^{p,q}. \tag{5.11}$$

It is obvious by (1.12) that $e(d\theta)W_0^{p,q} \subset W_2^{p,q}$, and clearly the two subspaces on the right-hand side are orthogonal.

Assume that $\omega = \partial\bar\partial\xi + \bar\partial\partial\eta \in W_2^{p,q}$, with $\xi, \eta \in W_0^{p,q}$. Then

$$i(d\theta)\omega = i(\Box\eta - \overline{\Box}\xi) \in W_0^{p,q}.$$

Indeed, by (1.13) and (1.20) we have

$$\begin{aligned} i(d\theta)\big(\partial\bar\partial\xi + \bar\partial\partial\eta\big) &= T^{-1} {d_H^*}^2 \big(\partial\bar\partial\xi + \bar\partial\partial\eta\big) \\ &= T^{-1}\big(-\partial^*\partial\overline{\Box}\xi + \bar\partial^*\Box\bar\partial\xi + \partial^*\overline{\Box}\partial\eta - \bar\partial^*\bar\partial\Box\eta\big) \\ &= T^{-1}\big(-\Box\overline{\Box}\xi + (\Box - iT)\overline{\Box}\xi + (\overline{\Box} + iT)\Box\eta - \overline{\Box}\Box\eta\big) \\ &= i(\Box\eta - \overline{\Box}\xi)\ . \end{aligned}$$

We have seen that $i(d\theta)W_2^{p,q} \subset W_0^{p,q}$, and therefore $\omega \in W_2^{p,q} \cap (e(d\theta)W_0^{p,q})^\perp$ if and only if $\omega \in K_2^{p,q}$. This proves (5.11). Let us set $K_{2,\ell}^{p,q} = e(d\theta)^\ell K_2^{p,q}$. Then

$$W_{2,\ell}^{p,q} = K_{2,\ell}^{p,q} \oplus e(d\theta)^{\ell+1} W_0^{p,q}, \tag{5.12}$$

and this decomposition is again orthogonal.

Indeed, if $\omega \in \ker(i(d\theta))$ and σ is a form orthogonal to ω, then for every $\ell \geq 1$

$$\langle e(d\theta)^\ell\omega,\, e(d\theta)^\ell\sigma\rangle = 0.$$

For, by (5.10) we have

$$i(d\theta)e(d\theta)^\ell\omega = c_\ell e(d\theta)^{\ell-1}\omega\,,$$

which implies, by induction on ℓ, that

$$\langle e(d\theta)^\ell\omega,\, e(d\theta)^\ell\sigma\rangle = \langle i(d\theta)e(d\theta)^\ell\omega, e(d\theta)^{\ell-1}\sigma\rangle = 0\ .$$

In order to prove the statement in the lemma for $j=2$, using the orthogonal decomposition in (5.12) we may now argue as before by means of Prop. 2.1 in [MPR1] in order to see that $W_{2,\ell}^{p,q}$ can be non-trivial only if $\ell \leq n-p-q-2$. Moreover, to verify that $e(d\theta)^\ell$ is injective on $W_2^{p,q}$ under this condition, it suffices to check injectivity on each of the subspaces on the right-hand side of (5.11). But this can be done by the same reasoning that we used for the case $j=1$.

For the "if" part, assume again that $p+q+j+\ell \leq n$ and $j=1,2$. Then, $p+q \leq n-1$, so that $W_0^{p,q} \neq \{0\}$ by Proposition 5.3. Then $W_j^{p,q} \neq \{0\}$, and by the first part $W_{\ell,j}^{p,q} \neq \{0\}$.
Q.E.D.

Lemma 5.5. *For* $\xi \in W_0^{p,q}$,

$$\begin{aligned} \partial^* e(d\theta)^\ell \partial\xi &= e(d\theta)^\ell \Box\xi + i\ell e(d\theta)^{\ell-1}\bar\partial\partial\xi \\ \partial^* e(d\theta)^\ell \bar\partial\xi &= 0 \\ \bar\partial^* e(d\theta)^\ell \bar\partial\xi &= e(d\theta)^\ell \overline{\Box}\xi - i\ell e(d\theta)^{\ell-1}\partial\bar\partial\xi \\ \bar\partial^* e(d\theta)^\ell \partial\xi &= 0 \ , \end{aligned} \tag{5.13}$$

and

$$\begin{aligned} \partial^* e(d\theta)^\ell \bar\partial\partial\xi &= -e(d\theta)^\ell \bar\partial\Box\xi \\ \partial^* e(d\theta)^\ell \partial\bar\partial\xi &= e(d\theta)^\ell \bar\partial\big(\Box\xi - i(\ell+1)T\big)\xi \\ \bar\partial^* e(d\theta)^\ell \partial\bar\partial\xi &= -e(d\theta)^\ell \partial\overline{\Box}\xi \\ \bar\partial^* e(d\theta)^\ell \bar\partial\partial\xi &= e(d\theta)^\ell \partial\big(\overline{\Box}\xi + i(\ell+1)T\big)\xi \ . \end{aligned} \tag{5.14}$$

Proof. Since $[\partial^*, e(d\theta)] = i\bar\partial$ commutes with $e(d\theta)$ (compare (1.14), (1.15)) and similarly for $\bar\partial^*$, we obtain by induction that

$$[\partial^*, e(d\theta)^\ell] = i\ell\bar\partial e(d\theta)^{\ell-1} \ , \qquad [\bar\partial^*, e(d\theta)^\ell] = -i\ell\partial e(d\theta)^{\ell-1} \ . \tag{5.15}$$

We verify the first identity in (5.13), the others being similar and following by invoking also (1.12) and (1.20):

$$\begin{aligned} \partial^* e(d\theta)^\ell \partial\xi &= e(d\theta)^\ell \partial^*\partial\xi + i\ell\bar\partial e(d\theta)^{\ell-1}\partial\xi \\ &= e(d\theta)^\ell \Box\xi + i\ell e(d\theta)^{\ell-1}\bar\partial\partial\xi \ . \end{aligned}$$

Q.E.D.

This immediately gives the following inclusions.

Corollary 5.6. *For* $\ell \geq 1$,

$$\partial^* W_{1,\ell}^{p,q} \subset W_{2,\ell-1}^{p,q} \ , \qquad \partial^* W_{2,\ell}^{p,q} \subset W_{1,\ell}^{p,q} \ ,$$

and similarly for $\bar\partial^*$.

Proposition 5.7. $L^2\Lambda_H^k$ *decomposes as the orthogonal sum*

$$\begin{aligned} L^2\Lambda_H^k &= \sum_{p+q=k}^{\oplus} \mathcal{W}_0^{p,q} \oplus \sum_{\substack{j,\ell,p,q \\ j=1,2 \\ p+q+j+2\ell=k}}^{\oplus} \mathcal{W}_{j,\ell}^{p,q} \\ &= \mathcal{W}_0^k \oplus \sum_{1+2\ell\leq k}^{\oplus} \mathcal{W}_{1,\ell}^k \oplus \sum_{2+2\ell\leq k}^{\oplus} \mathcal{W}_{2,\ell}^k. \end{aligned}$$

We recall that $\mathcal{W}_0^{p,q}$ *is non-trivial for* $p+q \leq n-1$, *and if* $p+q=n$ *for* $pq=0$.

Proof. We have already shown that $\mathcal{S}_0\Lambda_H^k$ is contained in the sum of the subspaces on the right-hand side. It is then sufficient to show that any two $\mathcal{S}_0$-forms belonging to two different subspaces are orthogonal.

It is quite obvious that $W_0^{p,q}$ is orthogonal to $W_0^{p',q'}$ if $(p,q) \neq (p',q')$.

The fact that W_0^k is orthogonal to $W_{j,\ell}^k$ for $j = 1, 2$ is a consequence of the fact that $W_0^k \subset \ker \partial^* \cap \ker \bar\partial^*$, whereas $W_{j,\ell}^k \subset \operatorname{ran} \partial + \operatorname{ran} \bar\partial$.

To prove the remaining orthogonality relations, we shall proceed inductively. For this purpose, it will be convenient to represent the elements of $W_{j,\ell}^k$ in the form (5.7) with $m = j + 2\ell$, and rename, for the purpose of this proof, $W_{j,\ell}^{p,q}$ as $W_m^{p,q}$ if $m = j + 2\ell$. Given $m \geq m' \geq 1$, there are three kinds of scalar products to consider,

$$\langle \underbrace{\partial\bar\partial\cdots}_{m\text{-terms}} \sigma, \underbrace{\partial\bar\partial\cdots}_{m'\text{-terms}} \sigma' \rangle \ , \qquad \langle \underbrace{\bar\partial\partial\cdots}_{m\text{-terms}} \sigma, \underbrace{\partial\bar\partial\cdots}_{m'\text{-terms}} \sigma' \rangle \ , \qquad \langle \underbrace{\bar\partial\partial\cdots}_{m\text{-terms}} \sigma, \underbrace{\bar\partial\partial\cdots}_{m'\text{-terms}} \sigma' \rangle \ ,$$

with $\sigma \in W_0^{p,q}$ and $\sigma' \in W_0^{p',q'}$. In the first case we have

$$\langle \underbrace{\partial\bar\partial\cdots}_{m} \sigma, \underbrace{\partial\bar\partial\cdots}_{m'} \sigma' \rangle = \langle \underbrace{\bar\partial\cdots}_{m-1} \sigma, \partial^* \underbrace{\partial\bar\partial\cdots}_{m'} \sigma' \rangle \ .$$

By Corollary 5.6, this is the scalar product of an element of $W_{m-1}^{p,q}$ with an element of $W_{m'-1}^{p',q'}$.

By induction on m', this shows that $W_m^{p,q} \perp W_{m'}^{p',q'}$ unless $p = p'$, $q = q'$, $m = m'$.

Q.E.D.

We discuss now to what extent the pairs $(\xi, \eta) \in W_0^{p,q} \times W_0^{p,q}$ provide a parametrization of the spaces $W_{j,\ell}^{p,q}$ for $j = 1, 2$.

Lemma 5.8. *Given $\xi \in W_0^{p,q}$, there exists a unique $\xi' \in W_0^{p,q}$ such that $\partial\xi = \partial\xi'$ and $C_p\xi' = 0$. An analogous statement holds for $\bar\partial$ in place of ∂.*

Proof. The case $p = n$ is trivial - here $\xi' = 0$. If $1 \leq p \leq n-1$, then by Lemma 4.8 we have $\xi' = \xi$.

There only remains the case $p = 0$, where $C_0 = \mathcal{C}$ is the orthogonal projection onto the kernel of $\Box$ (which in this case agrees with $\ker \partial$). This is a self-adjoint operator, so that, by Lemma 3.4, $\mathcal{S}_0\Lambda^{0,q} = (\ker \Box \cap \mathcal{S}_0\Lambda^{0,q}) \oplus (\operatorname{ran} \Box \cap \mathcal{S}_0\Lambda^{0,q})$. The commutation relation $\bar\partial^*\Box = (\Box - iT)\bar\partial^*$ from (1.20) then implies that the two subspaces in this decomposition are mapped under $\bar\partial^*$ to $\ker(\Box - iT) \cap \mathcal{S}_0\Lambda^{0,q}$ and $\operatorname{ran}(\Box - iT) \cap \mathcal{S}_0\Lambda^{0,q}$, respectively. This shows that

$$\bar\partial^*\mathcal{C}\xi = P\bar\partial^*\xi = 0 \,,$$

where P denotes the orthogonal projection onto the kernel of $\Box - iT$. Then $\xi' = (I - \mathcal{C})\xi$ has the desired properties. Q.E.D.

Set

$$X^{p,q} = \{\xi \in W_0^{p,q} : C_p\xi = 0\} \ , \quad Y^{p,q} = \{\eta \in W_0^{p,q} : \bar C_q\eta = 0\} \ , \quad Z^{p,q} = X^{p,q} \times Y^{p,q} \ . \tag{5.16}$$

In combination with Proposition 5.4 the previous lemma implies that the spaces $Z^{p,q}$ provide parametrisations for the spaces $W_{j,\ell}^{p,q}$:

Corollary 5.9. *Assume that $j = 1, 2$ and $p + q + j + \ell \leq n$. Then the maps*

$$e(d\theta)^\ell(\partial \quad \bar\partial) : Z^{p,q} \longrightarrow W_{1,\ell}^{p,q}, \quad (\xi, \eta) \mapsto \partial\xi + \bar\partial\eta,$$
$$e(d\theta)^\ell(\bar\partial\partial \quad \partial\bar\partial) : Z^{p,q} \longrightarrow W_{2,\ell}^{p,q}, \quad (\xi, \eta) \mapsto \bar\partial\partial\xi + \partial\bar\partial\eta$$

are bijections. Notice that this applies in particular to the spaces $W^{p,q}_{j,\ell}$ *appearing in the orthogonal decomposition of* $L^2\Lambda^k_H$ *in Proposition 5.7 under the assumption* $k \le n$.

Remark 5.10. Recall that, by Lemma 4.8,

$$X^{p,q} = \begin{cases} W_0^{p,q} & \text{if } 1 \le p \le n-1\,, \\ \{0\} & \text{if } p = n\,, \\ \big\{\xi \in \mathcal{S}_0\Lambda^{0,q} : \mathcal{C}\xi = 0, \bar\partial^*\xi = 0\big\} & \text{if } p = 0\,. \end{cases}$$

By the proof of Lemma 5.8, the latter space is indeed nothing but $(I - \mathcal{C})W_0^{0,q}$.

Analogous statements hold true for $Y^{p,q}$. Finally, notice that the spaces $Z^{p,q}$ are non-trivial if $p + q \le n - 1$.

5.2. The action of Δ_k.

Let Φ be the bijection (5.2) from $\mathcal{S}_0\Lambda^k_H$ onto $(\mathcal{S}_0\Lambda^k)_{d^*\text{-cl}}$, and let $D_k = \Phi^{-1}\Delta_k\Phi$ be the operator in (5.3).

For $\omega \in \mathcal{S}_0\Lambda^k_H$, by (1.22) we have

$$\begin{aligned} D_k\omega &= \big(\Delta_H - T^2 + e(d\theta)i(d\theta)\big)\omega + \big(T^{-1}[d^*_H, e(d\theta)]d^*_H\big)\omega \\ &= \big(\Delta_H - T^2 + T^{-1}d^*_H e(d\theta)d^*_H\big)\omega\ . \end{aligned} \tag{5.17}$$

The following identities are easily derived from (1.12), (1.14) and (1.15):

$$\begin{aligned} &\Box e(d\theta) = e(d\theta)(\Box - iT)\,, \\ &\overline{\Box} e(d\theta) = e(d\theta)(\overline{\Box} + iT)\,, \\ &[\Delta_H,\, e(d\theta)] = 0\,. \end{aligned} \tag{5.18}$$

It follows from (1.19) that, when acting on k-forms,

$$\Box - \overline{\Box} = i(n-k)T\ . \tag{5.19}$$

Lemma 5.11. *The following identities hold*

(i) $D_k e(d\theta) = e(d\theta)(D_{k-2} + n - k + 1)$*;*

(ii) $D_k e(d\theta)^\ell = e(d\theta)^\ell\big(D_{k-2\ell} + \ell(n-k+\ell)\big)$*, for* $\ell \ge 1$.

Proof. By (1.14), (1.15) and (5.17), (5.18), when applied to a horizontal $(k-2)$-form,

$$\begin{aligned} D_k e(d\theta) &= e(d\theta)(\Delta_H - T^2) + T^{-1}\big(d^*_H e(d\theta)\big)^2 \\ &= e(d\theta)(\Delta_H - T^2) + T^{-1}\big(e(d\theta)d^*_H + i(\bar\partial - \partial)\big)^2 \\ &= e(d\theta)D_{k-2} + iT^{-1}e(d\theta)\big(d^*_H(\bar\partial - \partial) + (\bar\partial - \partial)d^*_H\big) - T^{-1}(\bar\partial - \partial)^2 \\ &= e(d\theta)D_{k-2} + iT^{-1}e(d\theta)(\overline{\Box} - \Box) - e(d\theta) \\ &= e(d\theta)(D_{k-2} + n - k + 1)\,. \end{aligned} \tag{5.20}$$

Identity (ii) now follows by induction. Q.E.D.

Proposition 5.12. *The subspaces* $W_0^{p,q}$, $W^{p,q}_{1,\ell}$, $W^{p,q}_{2,\ell}$ *are invariant under the action of* D_k.

Proof. If $\omega \in W_0^{p,q}$, then $d_H^*\omega = 0$ and therefore

$$D_k\omega = (\Delta_H - T^2)\omega = (\Delta_0 + i(q-p)T)\omega\,, \tag{5.21}$$

by (1.19), where Δ_0 denotes the scalar operator $L - T^2$. The last expression shows that $D_k\omega$ is a (p,q)-form, and the previous one that $d_H^* D_k\omega = 0$, by (1.20).

By Lemma 5.11, when $j = 1, 2$, it suffices to take $\ell = 0$.

Take now $\omega \in W_1^k$, $\omega = \partial\xi + \bar\partial\eta$, with $\partial^*\xi = \bar\partial^*\xi = \partial^*\eta = \bar\partial^*\eta = 0$. We have

$$\begin{aligned} D_k\omega &= \big(\Delta_H - T^2 + T^{-1}d_H^* e(d\theta) d_H^*\big)(\partial\xi + \bar\partial\eta) \\ &= \partial(\Delta_H - T^2 + iT)\xi + \bar\partial(\Delta_H - T^2 - iT)\eta + T^{-1}d_H^* e(d\theta) d_H^*(\partial\xi + \bar\partial\eta) \\ &= \partial(\Delta_H - T^2 + iT)\xi + \bar\partial(\Delta_H - T^2 - iT)\eta \\ &\qquad + T^{-1}d_H^* e(d\theta)\Box\xi + T^{-1}d_H^* e(d\theta)\overline{\Box}\eta \\ &= \partial(\Delta_H - T^2 + iT)\xi + \bar\partial(\Delta_H - T^2 - iT)\eta + T^{-1}(i\bar\partial - i\partial)(\Box\xi + \overline{\Box}\eta) \\ &= \partial\big((\Delta_H - T^2 + iT - iT^{-1}\Box)\xi - iT^{-1}\overline{\Box}\eta\big) \\ &\qquad + \bar\partial\big((\Delta_H - T^2 - iT + iT^{-1}\overline{\Box})\eta + iT^{-1}\Box\xi\big)\ . \end{aligned} \tag{5.22}$$

Therefore, $D_k(\partial\xi + \bar\partial\eta) = \partial\xi' + \bar\partial\eta'$, where

$$\begin{aligned} \xi' &= (\Delta_H - T^2 + iT - iT^{-1}\Box)\xi - iT^{-1}\overline{\Box}\eta \\ \eta' &= (\Delta_H - T^2 - iT + iT^{-1}\overline{\Box})\eta + iT^{-1}\Box\xi\ , \end{aligned}$$

that is,

$$\begin{aligned} D_k\begin{pmatrix}\partial & \bar\partial\end{pmatrix} &= \begin{pmatrix}\partial & \bar\partial\end{pmatrix}\begin{pmatrix}\Delta_H - T^2 + iT - iT^{-1}\Box & -iT^{-1}\overline{\Box} \\ iT^{-1}\Box & \Delta_H - T^2 - iT + iT^{-1}\overline{\Box}\end{pmatrix} \\ &= \begin{pmatrix}\partial & \bar\partial\end{pmatrix}\left[(\Delta_H - T^2)I - iT^{-1}\begin{pmatrix}\Box - T^2 & \overline{\Box} \\ -\Box & -\overline{\Box} + T^2\end{pmatrix}\right]. \end{aligned} \tag{5.23}$$

Using the commutation relations (1.20) we see that

$$\partial^*\xi' = \bar\partial^*\xi' = \partial^*\eta' = \bar\partial^*\eta' = 0.$$

Therefore, also W_1^k is D_k-invariant. Moreover, if ξ and η are (p,q)-forms, so are ξ' and η', hence each $W_1^{p,q}$ is D_k-invariant.

Finally, take $\omega \in W_2^k$, $\omega = \bar\partial\partial\xi + \partial\bar\partial\eta$, with $\partial^*\xi = \bar\partial^*\xi = \partial^*\eta = \bar\partial^*\eta = 0$. We first compute

$$\begin{aligned} d_H^* e(d\theta) d_H^* \bar\partial\partial\xi &= d_H^* e(d\theta)(-\bar\partial\Box\xi + \overline{\Box}\partial\xi) \\ &= e(d\theta) d_H^*(-\bar\partial\Box\xi + \overline{\Box}\partial\xi) + i(\bar\partial - \partial)(-\bar\partial\Box\xi + \overline{\Box}\partial\xi) \\ &= e(d\theta)(-\overline{\Box}\Box\xi + \partial^*\overline{\Box}\partial\xi) + i(\partial\bar\partial\Box\xi + \bar\partial\overline{\Box}\partial\xi) \\ &= e(d\theta)\big(-\overline{\Box}\Box\xi + (\overline{\Box} + iT)\Box\xi\big) + i\big(\partial\bar\partial\Box\xi + \bar\partial\partial(\overline{\Box} + iT)\xi\big) \\ &= iTe(d\theta)\Box\xi + i\partial\bar\partial\Box\xi + i\bar\partial\partial(\overline{\Box} + iT)\xi \\ &= i\bar\partial\partial(-\Box + \overline{\Box} + iT)\xi \\ &= (n-k+1)T\bar\partial\partial\xi\ , \end{aligned}$$

by (5.19), since ξ is a $(k-2)$-form. Similarly,

$$d_H^* e(d\theta) d_H^* \partial\bar{\partial}\eta = (n-k+1) T \partial\bar{\partial}\eta \ .$$

Therefore,

$$D_k \omega = (\Delta_H - T^2 + n - k + 1)\omega \ . \tag{5.24}$$

As before, (1.20) implies that $D_k\omega \in W_2^k$, and each subspace $W_2^{p,q}$ is mapped into itself.

Q.E.D.

5.3. **Lifting by Φ.**

Denote by $V_0^{p,q}$, $V_{1,\ell}^{p,q}$, etc., the subspaces $\Phi(W_0^{p,q})$, $\Phi(W_{1,\ell}^{p,q})$, etc., of $(L^2\Lambda^k)_{d^*\text{-cl}}$. We want to show that their closures $\mathcal{V}_0^{p,q}$, $\mathcal{V}_{1,\ell}^{p,q}$, etc. give an orthogonal decomposition of $(L^2\Lambda^k)_{d^*\text{-cl}}$.

In a way, this is not *a priori* obvious, because Φ is not an orthogonal map. The fact that it preserves the orthogonality of the subspaces we are working with is quite peculiar. On the other hand, the reader may have noticed already an instance of this peculiarity in the fact that a non-symmetric operator such as D_k admits a rather fine decomposition into invariant subspaces which are orthogonal.

Proposition 5.13. *For $0 \le k \le n$ we have the orthogonal decompositions*

$$\big(L^2\Lambda^k\big)_{d^*\text{-cl}} = \sum^{\oplus}_{\substack{p+q=k<n \\ p+q=n,\, pq=0}} \mathcal{V}_0^{p,q} \oplus \sum^{\oplus}_{\substack{j,\ell,p,q \\ j=1,2 \\ p+q+j+2\ell=k}} \mathcal{V}_{j,\ell}^{p,q} \ , \tag{5.25}$$

where each of the subspaces $\mathcal{V}_0^{p,q}, \mathcal{V}_{j,\ell}^{p,q}$ is non-trivial and Δ_k-invariant.

Proof. Since Φ is a bijection from $\mathcal{S}_0\Lambda_H^k$ onto $(\mathcal{S}_0\Lambda^k)_{d^*\text{-cl}}$, it follows from Proposition 5.7 that

$$\big(\mathcal{S}_0\Lambda^k\big)_{d^*\text{-cl}} = \sum^{\oplus}_{\substack{p+q=k<n \\ p+q=n,\, pq=0}} V_0^{p,q} \oplus \sum^{\oplus}_{\substack{j,\ell,p,q \\ j=1,2 \\ p+q+j+2\ell=k}} V_{j,\ell}^{p,q} \ .$$

Hence it remains to show that this decomposition is orthogonal. By (5.2), this amounts to proving that

$$d_H^*(W_{j,\ell}^{p,q}) \perp d_H^*(W_{j',\ell'}^{p',q'}) \text{ whenever } W_{j,\ell}^{p,q} \neq W_{j',\ell'}^{p',q'} .$$

This, in turn, is an immediate consequence of Corollary 5.6 and Proposition 5.7. Notice that $d_H^* W_0^{p,q} = \{0\}$.

Q.E.D.

6. Intertwining operators and different scalar forms for Δ_k

Following the decomposition of $L^2\Lambda^k$ described in the previous section, we continue assuming $0 \le k \le n$.

In this section we describe the form that Δ_k attains on each of the subspaces of the decomposition (5.25) of $(L^2\Lambda^k)_{d^*\text{-cl}}$. In particular, we will show that, up to conjugation with invertible operators, Δ_k acts on $V_0^{p,q}$ and on each $V_{2,\ell}^{p,q}$ as a scalar operator. For $V_{1,\ell}^{p,q}$ instead, a further splitting will be necessary in order to reduce Δ_k to a scalar form in a similar way.

In the process, we will also describe the intertwining operators that reduce Δ_k to such scalar forms.

6.1. The case of $V_0^{p,q}$.

The simplest case is the one of $V_0^{p,q}$, because Φ acts on this space as the identity map and we already know by (5.21) that $D_k\big|_{W_0^{p,q}} = \Delta_0 + i(q-p)T$. Hence, in this case Δ_k is itself a scalar operator and we simply have the following

Proposition 6.1. *Let $p+q \le n-1$ or, $pq = 0$ if $p+q = n$. Then, on $V_0^{p,q}$,*

$$\Delta_k = \Delta_0 + i(q-p)T\ . \tag{6.1}$$

When we pass to $j = 1,2$ we want to express Δ_k in terms of the parameters (ξ,η) in the definition of $W_{j,\ell}^{p,q}$ which we can choose from the parameter spaces $Z^{p,q} = X^{p,q} \times Y^{p,q}$.

6.2. The case of $V_{2,\ell}^{p,q}$.

According to Corollary 5.9, we can write

$$W_{2,\ell}^{p,q} = \big\{\omega = e(d\theta)^\ell(\bar\partial\partial\xi + \partial\bar\partial\eta)\ :\ (\xi,\eta) \in Z^{p,q}\big\}. \tag{6.2}$$

Recall from the discussion in Section 4 and the definitions of $X^{p,q}$ and $Y^{p,q}$ (see (5.16)) that $\Box$ is injective when restricted to $X^{p,q}$ and $\overline{\Box}$ is injective when restricted to $Y^{p,q}$.

Proposition 6.2. *Let $A_{2,\ell} = \Phi e(d\theta)^\ell \big(\partial\bar\partial \quad \bar\partial\partial\big) : Z^{p,q} \to V_{2,\ell}^{p,q}$. Then, $A_{2,\ell}$ is injective on $Z^{p,q}$.*

The operator Δ_k restricted to the subspace $V_{2,\ell}^{p,q}$ is given by the following expression:

$$\Delta_k\Big|_{V_{2,\ell}^{p,q}} = A_{2,\ell}\Big(\Delta_0 + i(q-p)T + (\ell+1)(n-k+\ell+1)\Big)A_{2,\ell}^{-1}\,. \tag{6.3}$$

Proof. By Corollary 5.9 it follows at once that $A_{2,\ell}$ is injective on $Z^{p,q}$.

When $k = p+q+2+2\ell$, from Lemma 5.11 we have

$$D_k e(d\theta)^\ell = e(d\theta)^\ell\Big(D_{k-2\ell} + \ell(n-k+\ell)\Big).$$

Moreover, by (5.24) we know that $D_{k-2\ell}$, when acting on $W_2^{p,q}$, is given by $\Delta_0 + i(q-p)T + n - (k-2\ell) + 1$, so that on $W_2^{p,q}$

$$D_k e(d\theta)^\ell = e(d\theta)^\ell\Big(\Delta_0 + i(q-p)T + (\ell+1)(n-k+\ell+1)\Big). \tag{6.4}$$

By the definitions of Φ and $V^{p,q}_{2,\ell}$, and the commutation relations (1.20), this proves (6.3). Q.E.D.

6.3. The case of $V^{p,q}_{1,\ell}$.

We now turn to the case $j = 1$. In this case the situation is quite more involved, as we already observed in the case of 1-forms, see [MPR1]. Let us begin by recalling that according to Corollary 5.9, we can write

$$W^{p,q}_{1,\ell} = \big\{\omega = e(d\theta)^\ell(\partial\xi + \bar\partial\eta) \,:\, (\xi,\eta) \in Z^{p,q}\big\}. \tag{6.5}$$

Consider the subspace $V^{p,q}_{1,\ell} = \Phi(W^{p,q}_{1,\ell})$.

Our next goal will be to formally diagonalize the matrix $\begin{pmatrix} \Box - T^2 & \overline{\Box} \\ -\Box & -\overline{\Box} + T^2 \end{pmatrix}$ appearing in formula (5.23). This matrix operator is acting on column vectors $\begin{pmatrix} \xi \\ \eta \end{pmatrix}$ corresponding to pairs $(\xi,\eta) \in X^{p,q} \times Y^{p,q} = Z^{p,q}$, where $p+q+1+2\ell = k$. We put

$$s := p + q = k - 2\ell - 1\ . \tag{6.6}$$

Notice that $0 \le s \le n-1$.

We define the operator matrix Q acting on $\begin{pmatrix} \xi \\ \eta \end{pmatrix}$ by

$$Q = \begin{pmatrix} -Q^+_- & -Q^-_+ \\ Q^+_+ & Q^-_- \end{pmatrix}\ , \tag{6.7}$$

where, for $\varepsilon, \delta = \pm$, the expression of Q^ε_δ is

$$Q^\varepsilon_\delta = \Gamma + \varepsilon m - \delta i T\ ,$$

where

$$\begin{aligned} m &= \frac{n-s}{2}\ , \\ \Gamma &= \sqrt{\Delta_H - T^2 + m^2}\ . \end{aligned} \tag{6.8}$$

Observe here that the operator $\Delta_H - T^2 + m^2$ satisfies the estimate $\Delta_H - T^2 + m^2 \ge m^2 \ge 1/4$, so that it has a unique positive square root.

The following identities are easily verified:

$$\begin{aligned} Q^+_+ Q^-_- &= 2\Box \\ Q^+_- Q^-_+ &= 2\overline{\Box} \\ Q^+_+ Q^-_+ &= 2\big[\Box - T^2 - iT(m+\Gamma)\big] = 2\big[\overline{\Box} - T^2 + iT(m-\Gamma)\big] \\ Q^-_- Q^+_- &= 2\big[\Box - T^2 - iT(m-\Gamma)\big] = 2\big[\overline{\Box} - T^2 + iT(m+\Gamma)\big]\ , \end{aligned} \tag{6.9}$$

since

$$\Box - imT = \overline{\Box} + imT = \frac{1}{2}\Delta_H. \tag{6.10}$$

Lemma 6.3. *If $p+q\le n-1$, then the following properties hold true:*

(i) *The operator matrix $Q:\mathcal{S}_0\Lambda^{p,q}\times\mathcal{S}_0\Lambda^{p,q}\to\mathcal{S}_0\Lambda^{p,q}\times\mathcal{S}_0\Lambda^{p,q}$ is invertible, with inverse*

$$Q^{-1}=\frac{1}{4iT\Gamma}\begin{pmatrix}-Q_-^- & -Q_+^-\\ Q_+^+ & Q_-^+\end{pmatrix}\ .$$

Moreover, Q maps the subspace $W_0^{p,q}\times W_0^{p,q}$ bijectively onto itself.

(ii) *If $p=0$, then $Q_-^-\mathcal{C}=\mathcal{C}Q_-^-=0$, and if $q=0$, then $Q_+^-\bar{\mathcal{C}}=\bar{\mathcal{C}}Q_+^-=0$.*

Proof. To prove (i), we compute formally the determinant of Q and find by (6.9) that $\det Q=-4iT\Gamma$. The formula for Q^{-1} is now obvious. Notice also that the operators Q_δ^ε leave the space $W_0^{p,q}$ invariant. The remaining statement in (i) is now clear.

As for (ii), notice that if $p=0$ then $\mathcal{C}$ projects onto the kernel of ∂, which coincides with the kernel of $\Box$. And, on $\ker\Box$, by (6.10) we have $\Delta_H=-2i\,mT\ge 0$, so that $\Gamma=\sqrt{-2imT-T^2+m^2}=m-iT$, and hence $Q_-^-=0$ on $\ker\partial$. This implies $\mathcal{C}Q_-^-=Q_-^-\mathcal{C}=0$. The remaining identities in (ii) are proved analogously. Q.E.D.

We set

$$\Xi^{p,q}=X^{p,q}\cap Y^{p,q}=\{\xi\in W_0^{p,q}:C_p\xi=\bar{C}_q\xi=0\}\ ,\qquad \widetilde{Z}^{p,q}=W_0^{p,q}\times\Xi^{p,q}\ . \tag{6.11}$$

Lemma 6.4.

$$\begin{pmatrix}I-C_p & 0\\ 0 & I-\bar{C}_q\end{pmatrix}Q(\widetilde{Z}^{p,q})=Z^{p,q}.$$

Proof. It suffices to show that

$$\begin{pmatrix}I-C_p & 0\\ 0 & I-\bar{C}_q\end{pmatrix}Q(\widetilde{Z}^{p,q})=\begin{pmatrix}I-C_p & 0\\ 0 & I-\bar{C}_q\end{pmatrix}Q(W_0^{p,q}\times W_0^{p,q}),$$

since $Q(W_0^{p,q}\times W_0^{p,q})=W_0^{p,q}\times W_0^{p,q}$.

We have $\Xi^{p,q}=(I-C_p-\bar{C}_q)W_0^{p,q}$, which means that it suffices to show that

$$\begin{pmatrix}I-C_p & 0\\ 0 & I-\bar{C}_q\end{pmatrix}Q\begin{pmatrix}0\\ \eta\end{pmatrix}=\begin{pmatrix}-(I-C_p)Q_+^-\eta\\ (I-\bar{C}_q)Q_-^-\eta\end{pmatrix}$$

is zero for every $\eta=(C_p+\bar{C}_q)\eta'$. This follows from the identities

$$(I-C_p)(C_p+\bar{C}_q)=\bar{C}_q\ ,\qquad (I-\bar{C}_q)(C_p+\bar{C}_q)=C_p\ ,$$

and from (ii) of Lemma 6.3. Q.E.D.

Lemma 6.5. *Let*

$$G=\begin{pmatrix}\Box-T^2 & \overline{\Box}\\ -\Box & -\overline{\Box}+T^2\end{pmatrix} \tag{6.12}$$

be the matrix appearing in formula (5.23)*. Then, $-iT^{-1}G$ admits the diagonalization*

$$-iT^{-1}G=Q\begin{pmatrix}m+\Gamma & 0\\ 0 & m-\Gamma\end{pmatrix}Q^{-1}\ .$$

Proof. In order to formally compute the eigenvalues $\lambda_\pm$ of $-iTG$, observe that the characteristic equation for G is

$$\tau^2 - \tau\big[\Box - \overline{\Box}\big] + T^2\big[\Box + \overline{\Box} - T^2\big] = 0 \ ,$$

which has roots

$$\tau_\pm = imT \pm \sqrt{-T^2\big[\Box + \overline{\Box} - T^2 + m^2\big]} = iT(m \pm \Gamma) \ .$$

Therefore,

$$G - \tau_\pm I = \begin{pmatrix} \Box - T^2 - iT(m\pm\Gamma) & \overline{\Box} \\ -\Box & -\overline{\Box} + T^2 - iT(m\pm\Gamma) \end{pmatrix} \ ,$$

and, by (6.9),

$$\begin{aligned} G - \tau_+ I &= \frac{1}{2}\begin{pmatrix} Q_+^+ Q_+^- & Q_-^+ Q_+^- \\ -Q_+^+ Q_-^- & -Q_-^+ Q_-^- \end{pmatrix} \\ &= \frac{1}{2}\begin{pmatrix} Q_+^- \\ -Q_-^- \end{pmatrix}\begin{pmatrix} Q_+^+ & Q_-^+ \end{pmatrix} \ , \end{aligned}$$

and analogously,

$$G - \tau_- I = \frac{1}{2}\begin{pmatrix} Q_-^+ \\ -Q_+^+ \end{pmatrix}\begin{pmatrix} Q_-^- & Q_+^- \end{pmatrix} \ .$$

These equations show that eigenvectors of G of eigenvalues $\tau_\pm$ are given, respectively, by

$$Q^+ = \begin{pmatrix} -Q_-^+ \\ Q_+^+ \end{pmatrix} \ , \qquad Q^- = \begin{pmatrix} -Q_+^- \\ Q_-^- \end{pmatrix} \ , \tag{6.13}$$

so that

$$Q = \big(Q^+ | Q^-\big)$$

is indeed a matrix which formally diagonalizes $-iT^{-1}G$ as claimed. Q.E.D.

Recall now from (6.5) and the definition of $V_{1,\ell}^{p,q}$ that if we define the operator $A_{1,\ell} : (W_0^{p,q})^2 \to L^2\Lambda^k$ as

$$\begin{aligned} A_{1,\ell}\begin{pmatrix} \xi \\ \eta \end{pmatrix} &:= \Phi e(d\theta)^\ell \begin{pmatrix} \partial & \bar\partial \end{pmatrix}\begin{pmatrix} \xi \\ \eta \end{pmatrix} \\ &= \begin{pmatrix} I \\ T^{-1} d_H^* \end{pmatrix} e(d\theta)^\ell \begin{pmatrix} \partial & \bar\partial \end{pmatrix}\begin{pmatrix} \xi \\ \eta \end{pmatrix} \ , \end{aligned} \tag{6.14}$$

then $A_{1,\ell}(Z^{p,q}) = V_{1,\ell}^{p,q}$. Observe also that Lemma 6.4 shows that we may realize $Z^{p,q}$ in this identity as the space $Q(\widetilde{Z}^{p,q})$ and use $\widetilde{Z}^{p,q}$ as a parameter space for $V_{1,\ell}^{p,q}$. This has the advantage of reducing the operator D_k in (5.23) to diagonal form.

We therefore define the modified intertwining operator $\mathcal{A}_{1,\ell}$ by

$$\mathcal{A}_{1,\ell} := A_{1,\ell} Q_{\big|\widetilde{Z}^{p,q}} : \widetilde{Z}^{p,q} \to V_{1,\ell}^{p,q} \ . \tag{6.15}$$

By (5.23), Lemma 5.11 and Lemma 6.5, we have

$$\Delta_k \mathcal{A}_{1,\ell} = \mathcal{A}_{1,\ell}\Big(\Delta_h - T^2 + \ell(n-k+\ell) + \begin{pmatrix} m+\Gamma & 0 \\ 0 & m-\Gamma \end{pmatrix}\Big). \tag{6.16}$$

This suggests to further introduce the operators $\mathcal{A}^{\pm}_{1,\ell}$ acting by

$$\mathcal{A}^+_{1,\ell}\xi = \mathcal{A}_{1,\ell}\begin{pmatrix}\xi\\0\end{pmatrix} = A_{1,\ell}Q^+\xi\ , \quad \mathcal{A}^-_{1,\ell}\eta = \mathcal{A}_{1,\ell}\begin{pmatrix}0\\\eta\end{pmatrix} = A_{1,\ell}Q^-\eta\ , \tag{6.17}$$

with $Q^{\pm}$ as in (6.13). The following proposition is then immediate.

Proposition 6.6. *The space $V^{p,q}_{1,\ell}$ decomposes as the direct sum*

$$V^{p,q}_{1,\ell} = \mathcal{A}^+_{1,\ell}(W^{p,q}_0) + \mathcal{A}^-_{1,\ell}(\Xi^{p,q})\ .$$

Moreover, the linear mappings

$$\mathcal{A}^+_{1,\ell} : W^{p,q}_0 \to \mathcal{A}^+_{1,\ell}(W^{p,q}_0), \quad \mathcal{A}^-_{1,\ell} : \Xi^{p,q} \to \mathcal{A}^-_{1,\ell}(\Xi^{p,q})$$

are bijective, and the following identities hold, on $W^{p,q}_0$ and $\Xi^{p,q}$ respectively:

$$\begin{aligned}(\mathcal{A}^+_{1,\ell})^{-1}\Delta_k\mathcal{A}^+_{1,\ell} &= L - T^2 + i(q-p)T + \ell(n-k+\ell) + m\\ &\qquad + \sqrt{L - T^2 + i(q-p)T + m^2}\ ,\\ (\mathcal{A}^-_{1,\ell})^{-1}\Delta_k\mathcal{A}^-_{1,\ell} &= L - T^2 + i(q-p)T + \ell(n-k+\ell) + m\\ &\qquad - \sqrt{L - T^2 + i(q-p)T + m^2}\ .\end{aligned} \tag{6.18}$$

Define

$$V^{p,q,+}_{1,\ell} = \mathcal{A}^+_{1,\ell}(W^{p,q}_0)\ , \quad V^{p,q,-}_{1,\ell} = \mathcal{A}^-_{1,\ell}(\Xi^{p,q}). \tag{6.19}$$

It should be stressed that up to this point we have not yet shown that the subspaces $V^{p,q,+}_{1,\ell}$ and $V^{p,q,-}_{1,\ell}$ are mutually orthogonal. This fact will be a consequence of the analysis of the intertwining operators of the next section, see Lemma 7.6.

7. Unitary intertwining operators and projections

The intertwining operators for Δ_k that we have defined in the previous section where non-unitary and unbounded. In order to verify that the forms to which Δ_k, when restricted to the subspaces $V^{p,q}_0, V^{p,q,\pm}_{1,\ell}$ and $V^{p,q}_{2,\ell}$, had been reduced on the corresponding parameter spaces by means of the formulas (6.1), (6.18) and (6.3) are indeed describing the spectral theory of Δ on these subspaces, we need to replace the previous intertwining operators by unitary ones. Our next tasks will therefore be the following ones:

(1) replace these intertwining operators with unitary ones;
(2) determine the orthogonal projections from $L^2\Lambda^k$ onto $\mathcal{V}^{p,q}_0, \mathcal{V}^{p,q,\pm}_{1,\ell}$ and $\mathcal{V}^{p,q}_{2,\ell}$, the L^2-closures of the invariant subspaces $V^{p,q}_0, V^{p,q,\pm}_{1,\ell}$ and $V^{p,q}_{2,\ell}$.

These two tasks can be accomplished simultaneously by making use of the polar decomposition of the intertwining operators.

We shall repeatedly use the following basic fact from spectral theory (compare [RS] for the case $H = K$).

Proposition 7.1. *Let H, K be Hilbert spaces and $A : \operatorname{dom} A \subset H \to K$ be a densely defined, closed operator. Then there exist a positive self-adjoint operator $|A| : \operatorname{dom} A \subset H \to H$, with $\operatorname{dom} |A| = \operatorname{dom} A$, and a partial isometry $U : H \to K$ with $\ker U = \ker A$ and $\operatorname{ran} U = \overline{\operatorname{ran} A}$, so that $A = U|A|$. $|A|$ and U are uniquely determined by these properties together with the additional condition $\ker |A| = \ker A$.*

Moreover, $|A| = \sqrt{A^*A}$, U^*U is the orthogonal projection from H onto $(\ker A)^\perp = \overline{\operatorname{ran} A^*}$, and UU^* is the orthogonal projection from K onto $\overline{\operatorname{ran} A} = (\ker A^*)^\perp$.

In order to pass from a possibly unbounded intertwining operator to a unitary one, we also need the following general principle.

Proposition 7.2. *Let H_1, H_2 be Hilbert spaces and let $\mathcal{D}_1 \subset H_1$, $\mathcal{D}_2 \subset H_2$ be dense subspaces. Assume that for $j = 1, 2$, $S_j : \operatorname{dom} S_j \subset H_j \to H_j$ is a self-adjoint operator on H_j for which $\mathcal{D}_j$ is a core such that $S_j(\mathcal{D}_j) \subset \mathcal{D}_j$. Moreover, let $A : \operatorname{dom} A \subset H_1 \to H_2$ be a closed operator such that the following properties hold true:*

(i) *$\mathcal{D}_1 \subset \operatorname{dom} A$ and $A(\mathcal{D}_1) \subseteq \mathcal{D}_2$;*
(ii) *A intertwines S_1 and S_2 on the core $\mathcal{D}_1$, i.e.,*

$$AS_1\xi = S_2A\xi \qquad \text{for all } \xi \in \mathcal{D}_1\,. \tag{7.1}$$

Consider the polar decomposition $A = U|A|$ from Proposition7.1, where $|A| = \sqrt{A^*A}$, and where $U : H_1 \to H_2$ is a partial isometry, and assume furthermore that $\mathcal{D}_1 \subset \operatorname{dom} |A|$, and that

(iii) *$|A|(\mathcal{D}_1) = \mathcal{D}_1$;*
(iv) *the commutation relation*

$$S_1|A|\xi = |A|S_1\xi \qquad \text{for all } \xi \in \mathcal{D}_1 \tag{7.2}$$

holds true on the core $\mathcal{D}_1$.

Then, also U intertwines S_1 and S_2 on the core $\mathcal{D}_1$, i.e., $U(\mathcal{D}_1) = A(\mathcal{D}_1) \subset \mathcal{D}_2$, and

$$US_1\xi = S_2U\xi \qquad \text{for all } \xi \in \mathcal{D}_1\,. \tag{7.3}$$

Moreover, we have $\overline{\operatorname{ran} A} = \overline{A(\mathcal{D}_1)} = U(H_1)$, $\ker A = \ker |A| = \ker U$, and $P := UU^*$ is the orthogonal projection from H_2 onto $\overline{A(\mathcal{D}_1)}$.

Let us finally denote by $S_2^r = S_2\big|_{\overline{A(\mathcal{D}_1)}}$ the restriction of S_2 to $\overline{A(\mathcal{D}_1)}$, with domain $\operatorname{dom} S_2^r :=$ $\operatorname{dom} S_2 \cap \overline{A(\mathcal{D}_1)}$. If we assume in addition that

(v) $\ker |A| = \{0\}$;
(vi) $(I - iS_1)^{-1}(\mathcal{D}_1) \subseteq \mathcal{D}_1$;
(vii) $P(\mathcal{D}_2) = A(\mathcal{D}_1)$,

then U is injective, and we even have that $U(\operatorname{dom} S_1) = \operatorname{dom} S_2^r$, and

$$S_2^r = US_1U^{-1} \quad \text{on} \quad \operatorname{dom} S_2^r.$$

Proof. Let us re-write (7.1) as

$$U|A|S_1\xi = S_2U|A|\xi \qquad \text{for all } \xi \in \mathcal{D}_1\,.$$

Applying (7.2), we find that

$$US_1(|A|\xi) = S_2U(|A|\xi) \qquad \text{for all } \xi \in \mathcal{D}_1\,,$$

which implies (7.3) because of (iii). Note that $U(\mathcal{D}_1) = A(\mathcal{D}_1) \subset \mathcal{D}_2$ in view of (iii).

Since A is closed and $\mathcal{D}_1$ is a core for A, we have $\overline{\operatorname{ran} A} = \overline{A(\mathcal{D}_1)}$, and the remaining statements about $\overline{\operatorname{ran} A}$, $\ker A$ and UU^* are obvious by Proposition 7.1.

If we assume in addition that (v) and (vi) hold true, then clearly U is injective. Moreover, (7.3) implies that

$$U(I - iS_1)\xi = (I - iS_2)U\xi \qquad \text{for all } \xi \in \mathcal{D}_1.$$

Since $U(\mathcal{D}_1) \subseteq \mathcal{D}_1$, by (vi) we then obtain that $U(I-iS_1)^{-1}\xi = (I-iS_2)^{-1}U\xi$ for every $\xi \in \mathcal{D}_1$, hence

$$U(I - iS_1)^{-1} = (I - iS_2)^{-1}U \tag{7.4}$$

on H_1. Noticing that $\operatorname{dom} S_j = \operatorname{ran}(I - iS_j)^{-1}$, (7.4) implies that $U(\operatorname{dom} S_1) \subseteq \operatorname{dom} S_2$, so that $U(\operatorname{dom} S_1) \subseteq \operatorname{dom} S_2^r$, and that (7.3) holds true even for every $\xi \in \operatorname{dom} S_1$:

If $x = (I - iS_1)^{-1}y \in \operatorname{dom} S_1$ (with $y \in H_1$), then $Ux = (I - iS_2)^{-1}Uy \in \operatorname{dom} S_2$, and

$$(I - iS_2)Ux = Uy = U(I - iS_1)x.$$

It therefore only remains to show that $\operatorname{dom} S_2^r \subseteq U(\operatorname{dom} S_1)$.

To this end, we first observe that, because of (vii) and (7.1),

$$S_2P(\mathcal{D}_2) \subseteq S_2A(\mathcal{D}_1) \subseteq AS_1(\mathcal{D}_1) \subseteq A(\mathcal{D}_1) = P(\mathcal{D}_2).$$

Since S_2 is self-adjoint, this implies

$$S_2P(\mathcal{D}_2) \subseteq P(\mathcal{D}_2) \quad \text{and} \quad (S_2(I-P)(\mathcal{D}_2) \subseteq (I-P)(H_2).$$

Assume now that $x \in \operatorname{dom} S_2 \cap \operatorname{ran} U$. Then $x = Uy$ for some unique $y \in H_1$. Choose a sequence $\{x_n\}_n$ in $\mathcal{D}_2$ such that

$$x_n \to x \quad \text{and} \quad S_2x_n \to S_2x.$$

Since $S_2x_n = S_2(Px_n) + S_2((I-P)x_n))$, where the components in this decomposition lie in mutually orthogonal spaces, we see that there is some $z = Uw \in P(\mathcal{D}_2) \subset U(H_2)$ such that

$$Px_n \to x = Uy \quad \text{and} \quad S_2(Px_n) \to z = Uw.$$

We can write Px_n in a unique way as $Px_n = Uy_n$, with $y_n \in \mathcal{D}_1$, since $U(\mathcal{D}_1) = A(\mathcal{D}_1) = P(\mathcal{D}_2)$. Since U is isometric on H_1, we then must have that $y_n \to y$. Moreover, by (7.3), $US_1y_n = S_2Uy_n = S_2(Px_n) \to z$, so that $S_1y_n \to w$. This shows that $y \in \operatorname{dom} S_1$, hence $x = Uy \in U(\operatorname{dom} S_1)$.

Q.E.D.

Remark 7.3. If we do not require that the crucial commutation relation in (iv) is satisfied, but that in addition to the conditions (i) to (iii) the natural assumptions $\mathcal{D}_2 \subset \operatorname{dom} A^*$ and $A^*(\mathcal{D}_2) \subset \mathcal{D}_1$ hold true, then one can conclude that

$$S_1|A|^2\xi = |A|^2S_1\xi \qquad \text{for all } \xi \in \mathcal{D}_1\,. \tag{7.5}$$

Indeed, then for $\xi \in \mathcal{D}_1$ and $\eta \in \mathcal{D}_2$, (7.1) and (i) imply that $\langle \xi, S_1 A^* \eta \rangle = \langle \xi, A^* S_2 \eta \rangle$, hence

$$S_1 A^* \eta = A^* S_2 \eta \qquad \text{for all } \eta \in \mathcal{D}_2 .$$

Combining this with (7.1), we obtain $S_1 A^* A \xi = A^* A S_1$ for every $\xi \in \mathcal{D}_1$, which verifies (7.5).

One might hope that (7.2) would follow from (7.5) by means of general spectral theory. However, this hope is destroyed by a classical example due to Nelson (cf. [RS]), which shows that condition (7.5) will in general not suffice to conclude that the operators S_1 and $|A|^2$ commute, in the sense that their respective spectral resolutions commute. This, however, would be needed in order to derive (7.2).

However, in our applications, S_1 will turn out to be a scalar operator on the Heisenberg group, and A a positive square matrix whose entries are scalar operators too, so that (7.2) will easily follow from formula (12.7) for the square root of such a matrix.

In the sequel, by $P_{H_1} : H \to H_1$ we shall denote the orthogonal projection from the Hilbert space H onto its closed subspace H_1.

In our later applications of Proposition 7.1, the next observation will often facilitate the computation of the corresponding operators A^*A.

Lemma 7.4. *Let H, K be Hilbert spaces and $H_1 \subseteq H$ and $K_1 \subseteq K$ be closed subspaces. Let $A : \operatorname{dom} A \subset H \to K$ be a densely defined, closed operator, and assume that $\mathcal{D} \subset \operatorname{dom} A$ is a core for A. Assume furthermore that $\mathcal{D}_1 := \mathcal{D} \cap H_1$ is dense in H_1 and that $\operatorname{dom} A_1 := \operatorname{dom} A \cap H_1$ is mapped under A into K_1, so that the operator $A_1 : \operatorname{dom} A_1 \subset H_1 \to K_1$, given by restricting A to $\operatorname{dom} A_1 := \operatorname{dom} A \cap H_1$, is densely defined and closed.*

Under these conditions, also A^* is densely defined, and $\operatorname{dom} A^* \cap K_1 \subset \operatorname{dom} A_1^*$. We shall further assume that $\mathcal{E} \subset K$ is a subspace of $\operatorname{dom} A^*$ such that $A(\mathcal{D}) \subset \mathcal{E}$ and $A^*(\mathcal{E}) \subset \mathcal{D}$ (so that, in particular, $\mathcal{E}_1 := \mathcal{E} \cap K_1$ is contained in $\operatorname{dom} A_1^*$). Then we have

$$A_1^* A_1 \xi = P_{H_1} A^* A \xi \qquad \textit{for all } \xi \in \mathcal{D}_1 .$$

In particular, if we know that A^*A maps $\mathcal{D}_1$ into H_1, then $A_1^* A_1 \xi = A^* A \xi$ for every $\xi \in \mathcal{D}_1$.

Proof. Since $A(\mathcal{D}_1) \subset \mathcal{E}_1$, it suffices to prove that $A_1^* = P_{H_1} A^*$ on $\mathcal{E}_1$. But, if $x \in \mathcal{D}_1 \subset \operatorname{dom} A_1, \xi \in \mathcal{E}_1 \subset \operatorname{dom} A_1^*$, then

$$\langle x, A_1^* \xi \rangle = \langle A_1 x, \xi \rangle = \langle A x, \xi \rangle = \langle x, A^* \xi \rangle = \langle x, P_{H_1} A^* \xi \rangle .$$

This implies that $A_1^* \xi = P_{H_1} A^* \xi$, since $\mathcal{D}_1$ is dense in H_1. Q.E.D.

7.1. A unitary intertwining operator for $V_0^{p,q}$.

We recall from the preceding discussion that the intertwining operator on $\mathcal{V}_0^{p,q}$ is Φ, which reduces to the identity on this space. Hence, this case is trivial.

7.2. Unitary intertwining operators for $V^{p,q,\pm}_{1,\ell}$.

Our next goal is to replace the intertwining operators $\mathcal{A}^{\pm}_{1,\ell}$ from Proposition 6.6 by unitary ones. Recall from Proposition 6.6 and (6.17) that

$$\mathcal{A}^{\pm}_{1,\ell} = A_{1,\ell}Q^{\pm} , \qquad \operatorname{dom}\mathcal{A}^{+}_{1,\ell} = W^{p,q}_0 , \quad \operatorname{dom}\mathcal{A}^{-}_{1,\ell} = \Xi^{p,q} ,$$

where, according to (6.14),

$$A_{1,\ell} = \begin{pmatrix} e(d\theta)^\ell\partial & e(d\theta)^\ell\bar\partial \\ i\ell T^{-1}e(d\theta)^{\ell-1}\bar\partial\partial + T^{-1}e(d\theta)^\ell\Box & -i\ell T^{-1}e(d\theta)^{\ell-1}\partial\bar\partial + T^{-1}e(d\theta)^\ell\overline{\Box} \end{pmatrix} . \tag{7.6}$$

According to Proposition 7.2, we seek to define unitary intertwining operators $U^{\pm}_{1,\ell}$ by defining

$$U^{\pm}_{1,\ell} := U^{p,q,\pm}_{1,\ell} := \mathcal{A}^{\pm}_{1,\ell}\Big((\mathcal{A}^{\pm}_{1,\ell})^*\mathcal{A}^{\pm}_{1,\ell}\Big)^{-\frac12}, \tag{7.7}$$

which are expected to be isometries from the closed subspaces $\overline{W^{p,q}_0} = \mathcal{W}^{p,q}$, resp. $\overline{\Xi^{p,q}}$, onto their ranges $\mathcal{V}^{p,q,\pm}_{1,\ell}$. Recall, however, that we have not shown yet that the latter spaces are mutually orthogonal; this will in fact follow easily from the subsequent discussions.

Now, since

$$(\mathcal{A}^{\pm}_{1,\ell})^*\mathcal{A}^{\pm}_{1,\ell} = Q^{\pm *}(A^*_{1,\ell}A_{1,\ell})Q^{\pm},$$

we shall begin by computing $A^*_{1,\ell}A_{1,\ell}$.

Subsequently, we will compute the product $Q^*A^*_{1,\ell}A_{1,\ell}Q$, showing, in particular, that it is a diagonal matrix. The diagonal terms will give the explicit forms of $(\mathcal{A}^{\pm}_{1,\ell})^*\mathcal{A}^{\pm}_{1,\ell}$, whereas the vanishing of the off-diagonal terms will prove the orthogonality of the spaces $\mathcal{V}^{p,q,\pm}_{1,\ell}$. Since these computations are tedious and unenlightening, we shall only state here the relevant identities, postponing their proofs to the Appendix.

Let us set, for $s + j \le n$,

$$c_{s,j} = \frac{j!(n-s)!}{(n-s-j)!} . \tag{7.8}$$

Lemma 7.5. *We have that* $A^*_{1,\ell}A_{1,\ell} = -c_{s+1,\ell}T^{-2}N$, *where*

$$N = \begin{pmatrix} \Box(\Box - i\ell T - T^2) & \Box\overline{\Box} \\ \Box\overline{\Box} & \overline{\Box}(\overline{\Box} + i\ell T - T^2) \end{pmatrix} . \tag{7.9}$$

Lemma 7.6. *Let* $R = -T^{-2}Q^*NQ$ *on* $(W^{p,q}_0)^2$. *Then*

$$R = \begin{pmatrix} R_{11} & 0 \\ 0 & R_{22} \end{pmatrix} ,$$

where

$$\begin{aligned} R_{11} = (\Gamma+m)^2\big(\Delta_H + 2m(2m-\ell)\big) + 2(\Gamma+m)\big((2m-\ell)\Delta_H - 2mT^2\big) \\ + \Delta_H(\Delta_H - T^2) + 2m\ell T^2 , \end{aligned}$$

maps $W_0^{p,q}$ bijectively onto itself, and

$$R_{22} = (\Gamma - m)^2\big(\Delta_H + 2m(2m-\ell)\big) - 2(\Gamma - m)\big((2m-\ell)\Delta_H - 2mT^2\big) + \Delta_H(\Delta_H - T^2) + 2m\ell T^2 \ ,$$

maps $\Xi^{p,q}$ bijectively onto itself, and is zero on $C_p W_0^{p,q} \oplus \bar C_q W_0^{p,q}$, the orthogonal complement of $\Xi^{p,q}$ in $W_0^{p,q}$.

Proof. The proof of the formulas for the components of R is postponed to the Appendix. Given these formulas, we prove here the mapping properties of R_{11} and R_{22}.

On $W_0^{p,q}$, R_{11} acts as a symmetric scalar operator. Since $\Delta_H = L + i(q-p)T$, Γ and $-T^2$ are positive operators, we have

$$\begin{aligned} R_{11} &\geq (\Gamma + m)^2\big(\Delta_H + 2m(2m-\ell)\big) - 4m^2T^2 + 2m\ell T^2 \\ &= (\Gamma + m)^2\big(\Delta_H + 2m(2m-\ell)\big) - 2m(2m-\ell)T^2 \\ &\geq 2m^3(2m-\ell) > 0 \ . \end{aligned}$$

It follows that the operators $(R_{11})_{\lambda,\sigma}$ in (3.1) also satisfy the same inequality from below, and hence are invertible. Applying Lemma 3.2 (ii), we obtain that R_{11} admits an inverse $R_{11}^{-1} : \mathcal{S}_0 \longrightarrow \mathcal{S}_0$.

We tensor with $\Lambda^{p,q}$ and restrict R_{11}^{-1} to $W_0^{p,q}$. By (1.20), the composition $\partial^* R_{11}$ can be expressed as $R'_{11}\partial^*$, with R'_{11} differing from R_{11} in that Δ_H is replaced by $\Delta_H - iT$ (also in the expression of Γ), and similarly for $\bar\partial^* R_{11}$, $\partial^* R_{11}^{-1}$ and $\bar\partial^* R_{11}^{-1}$. Therefore, R_{11} maps $W_0^{p,q}$ bijectively onto itself.

As to R_{22}, we first observe that

$$\begin{aligned} R_{11}R_{22} &= \det R \\ &= T^{-4}(\det Q)^2 \det N \\ &= T^{-4}(-4iT\Gamma)^2(-T^2)\big(\Delta_H - T^2 + \ell(2m-\ell)\big)\Box\overline{\Box} \\ &= 16(\Delta_H - T^2 + m^2)\big(\Delta_H - T^2 + \ell(2m-\ell)\big)\Box\overline{\Box} \ , \end{aligned} \tag{7.10}$$

so that

$$R_{22} = 16(\Delta_H - T^2 + m^2)\big(\Delta_H - T^2 + \ell(2m-\ell)\big)\Box\overline{\Box}R_{11}^{-1} \ . \tag{7.11}$$

Moreover, by the injectivity of R_{11},

$$\ker R_{22} = \ker R_{11}R_{22} = \ker \Box \oplus \ker \overline{\Box} \ .$$

In order to repeat the same argument used above for R_{11}, we start from the operator $\widetilde{R}_{22} = R_{22} + \delta_{p,0}\mathcal{C} + \delta_{q,0}\bar{\mathcal{C}}$ (with δ denoting the Kronecker symbol) acting on scalar-valued functions. By (7.11), $\widetilde{R}_{22}$ in invertible on $\mathcal{S}_0$ and, after tensoring and restricting, it is also invertible on $W_0^{p,q}$. For $\xi \in \Xi^{p,q}$,

$$\widetilde{R}_{22}^{-1}R_{22}\xi = \widetilde{R}_{22}^{-1}\widetilde{R}_{22}\xi = \xi \ .$$

The conclusion now follows at once. Q.E.D.

Corollary 7.7. *We have that*

$$\mathcal{A}_{1,\ell}^*\mathcal{A}_{1,\ell} = c_{s+1,\ell} R_{|_{\widetilde{Z}^{p,q}}} \ .$$

In particular,

$$(\mathcal{A}_{1,\ell}^+)^*\mathcal{A}_{1,\ell}^+ = c_{s+1,\ell}R_{11} \ , \qquad (\mathcal{A}_{1,\ell}^-)^*\mathcal{A}_{1,\ell}^- = c_{s+1,\ell}R_{22|_{\Xi^{p,q}}} \ ,$$

and the subspaces $\mathcal{V}_{1,\ell}^{p,q,+}$ *and* $\mathcal{V}_{1,\ell}^{p,q,-}$ *are orthogonal.*

Proof. Obviously, R maps the subspace $\widetilde{Z}^{p,q}$ of $(W_0^{p,q})^2$ into itself, so that the identities follow from Lemma 7.6 and Lemma 7.4. The first statements are obvious. And, since the matrix Q^*NQ is diagonal, so is $\mathcal{A}_{1,\ell}^*\mathcal{A}_{1,\ell}$. Thus, the map $\mathcal{A}_{1,\ell}$ preserves the orthogonality of the coordinate subspaces $W_0^{p,q}\times\{0\}$ and $\{0\}\times\Xi_0^{p,q}$. Q.E.D.

Let us finally compute $U_{1,\ell}^{\pm}$ more explicitly. To this end, notice that if we combine the column-vectors of operators $U_{1,\ell}^{\pm}$ to form a square matrix, then

$$U_{1,\ell} := \begin{pmatrix} U_{1,\ell}^+ & U_{1,\ell}^- \end{pmatrix} = \mathcal{A}_{1,\ell}(\mathcal{A}_{1,\ell}^*\mathcal{A}_{1,\ell})^{-\frac12}. \tag{7.12}$$

Recall that we have set $s = p+q$ and $m = (n-s)/2$.

Proposition 7.8. *We have that*

$$U_{1,\ell} = c_{s+1,\ell}^{-\frac12}\, e(d\theta)^{\ell-1}\begin{pmatrix} S_{11} & S_{12}\\ S_{21} & S_{22}\end{pmatrix}\begin{pmatrix}\Sigma_{11} & 0\\ 0 & \Sigma_{22}\end{pmatrix} \ , \tag{7.13}$$

where

$$\begin{aligned} S_{11} &= e(d\theta)(-\partial Q_-^+ + \bar\partial Q_+^+)\\ S_{12} &= e(d\theta)(-\partial Q_+^- + \bar\partial Q_-^-)\\ S_{21} &= \ell(\bar\partial\partial - \partial\bar\partial) - ie(d\theta)\big[\Delta_H + (\Gamma+m)(2m-\ell)\big]\\ S_{22} &= -\ell(\bar\partial\partial - \partial\bar\partial) + ie(d\theta)\big[\Delta_H - (\Gamma-m)(2m-\ell)\big] \end{aligned} \tag{7.14}$$

and Σ_{11}, Σ_{22} *are given by*

$$\begin{aligned} \Sigma_{11} = R_{11}^{-\frac12} &= \Big[(\Gamma+m)^2\big(\Delta_H + 2m(2m-\ell)\big) + 2(\Gamma+m)\big((2m-\ell)\Delta_H - 2mT^2\big)\\ &\qquad + \Delta_H(\Delta_H - T^2) + 2m\ell T^2\Big]^{-\frac12}\\ \Sigma_{22} = R_{22}^{-\frac12} &= \Big[(\Gamma-m)^2\big(\Delta_H + 2m(2m-\ell)\big) - 2(\Gamma-m)\big((2m-\ell)\Delta_H - 2mT^2\big)\\ &\qquad + \Delta_H(\Delta_H - T^2) + 2m\ell T^2\Big]^{-\frac12} \ . \end{aligned} \tag{7.15}$$

Proof. From Corollary 7.7, we have

$$(\mathcal{A}_{1,\ell}^*\mathcal{A}_{1,\ell})^{-\frac12} = c_{s+1,\ell}^{-\frac12} R^{-\frac12} \ ,$$

so that

$$U_{1,\ell} = A_{1,\ell} Q(\mathcal{A}_{1,\ell}^* \mathcal{A}_{1,\ell})^{-\frac12} = c_{s+1,\ell}^{-\frac12} A_{1,\ell} Q R^{-\frac12}$$
$$= c_{s+1,\ell}^{-\frac12} \begin{pmatrix} I \\ T^{-1} d_H^* \end{pmatrix} e(d\theta)^\ell \begin{pmatrix} \partial & \bar\partial \end{pmatrix} \begin{pmatrix} -Q_-^+ & -Q_+^- \\ Q_+^+ & Q_-^- \end{pmatrix} \begin{pmatrix} (R^+)^{-\frac12} & 0 \\ 0 & (R^-)^{-\frac12} \end{pmatrix} .$$

We verify that the factor T^{-1} in the second row is going to disappear. From Lemma 5.5, we have

$$T^{-1} d_H^* e(d\theta)^\ell (\partial \ \ \bar\partial) = T^{-1} e(d\theta)^\ell \begin{pmatrix} \Box & \overline{\Box} \end{pmatrix} + i\ell T^{-1} e(d\theta)^{\ell-1} \begin{pmatrix} \bar\partial\partial & -\partial\bar\partial \end{pmatrix} .$$

Let us define the matrix S by requiring that

$$e(d\theta)^{\ell-1} S = \begin{pmatrix} I \\ T^{-1} d_H^* \end{pmatrix} e(d\theta)^\ell \begin{pmatrix} \partial & \bar\partial \end{pmatrix} \begin{pmatrix} -Q_-^+ & -Q_+^- \\ Q_+^+ & Q_-^- \end{pmatrix} .$$

Then

$$S = \begin{pmatrix} e(d\theta)\partial & e(d\theta)\bar\partial \\ i\ell T^{-1}\bar\partial\partial + T^{-1} e(d\theta)\Box & -i\ell T^{-1}\partial\bar\partial + T^{-1} e(d\theta)\overline{\Box} \end{pmatrix} \begin{pmatrix} -Q_-^+ & -Q_+^- \\ Q_+^+ & Q_-^- \end{pmatrix} .$$

In particular,

$$S_{11} = e(d\theta)(-\partial Q_-^+ + \bar\partial Q_+^+)$$
$$S_{12} = e(d\theta)(-\partial Q_+^- + \bar\partial Q_-^-) .$$

Moreover,

$$\begin{aligned} S_{21} &= \big[i\ell T^{-1}\bar\partial\partial + T^{-1} e(d\theta)\Box\big](-Q_-^+) + \big[-i\ell T^{-1}\partial\bar\partial + T^{-1} e(d\theta)\overline{\Box}\big] Q_+^+ \\ &= \ell(\bar\partial\partial - \partial\bar\partial) - ie(d\theta)\Delta_H - (\Gamma+m)\big[i\ell T^{-1}(\bar\partial\partial + \partial\bar\partial) + T^{-1} e(d\theta)(\Box - \overline{\Box})\big] \\ &= \ell(\bar\partial\partial - \partial\bar\partial) - ie(d\theta)\big[\Delta_H + (\Gamma+m)(2m-\ell)\big] . \end{aligned}$$

Finally, a similar computation shows that

$$S_{22} = -\ell(\bar\partial\partial - \partial\bar\partial) + ie(d\theta)\big[\Delta_H - (\Gamma-m)(2m-\ell)\big] ,$$

as we claimed.

In order to conclude the proof, it suffice to notice that $\Sigma_{jj} = R_{jj}^{-\frac12}$, $j = 1,2$, where R_{jj} are given in Lemma 7.6. Q.E.D.

We wish now to apply Proposition 7.2 to $\mathcal{A}_{1,\ell}^\pm$. We restrict ourselves to $\mathcal{A}_{1,\ell}^-$, the other case being simpler.

We set

$$\mathcal{D}_1 = \Xi^{p,q}, \quad H_1 = \overline{\Xi^{p,q}},$$
$$\mathcal{D}_2 = \mathcal{S}_0 \Lambda^k, \quad H_2 = L^2\Lambda^k,$$
$$S_1 = D^-, \quad S_2 = \Delta_k,$$

where

$$D^\pm := L - T^2 + i(q-p)T + \ell(n-k+\ell) + m \pm \sqrt{L - T^2 + i(q-p)T + m^2}, \tag{7.16}$$

and denote by A the closure of $\mathcal{A}_{1,\ell}^-$. The commutation relation (7.1) is then satisfied because of (6.18). Moreover, clearly $S_2(\mathcal{D}_2) \subset \mathcal{D}_2$.

Notice also that A maps $\mathcal{D}_1$ bijectively onto $V_{1,\ell}^{p,q,-} \subseteq \mathcal{D}_2$.

Next, according to Corollary 7.7, A^*A is a positive scalar operator, and so is S_1. But then also $|A| = \sqrt{A^*A} = \sqrt{c_{s+1,\ell}}\, R_{22}^{\frac{1}{2}}$ is a scalar operator, hence commutes with S_1, so that condition (iv) in Proposition 7.2 is satisfied too.

Conditions (iii) and (v) of Proposition 7.2 follow from Lemma 7.6 and condition (vi) is obvious.

Finally, our explicit formulas for $U = U_{1,\ell}^-$ in Proposition 7.8 show that here U maps the space $\Xi^{p,q}$ into $\mathcal{S}_0\Lambda^k$, so that U^* maps $\mathcal{S}_0\Lambda^k$ into $\Xi^{p,q}$, and we see that $P(\mathcal{D}_2) = P(\mathcal{S}_0\Lambda^k) = U(\Xi^{p,q}) = A\Big(|A|^{-1}(\Xi^{p,q})\Big) = A(\Xi^{p,q}) = A(\mathcal{D}_1)$. This shows that also condition (vii) is satisfied. Q.E.D.

In the same way, we see that all the hypotheses of Proposition 7.2 are satisfied by $U_{1,\ell}^-$, and as a consequence we obtain

Proposition 7.9. *$U_{1,\ell}^\pm$ defined by (7.7) maps $W_0^{p,q}$, respectively $\Xi^{p,q}$, onto $V_{1,\ell}^{p,q,\pm}$ and intertwines $D^\pm$ with Δ_k on the core.*

Moreover, $U_{1,\ell}^+ : \mathcal{W}_0^{p,q} \to L^2\Lambda^k$ and $U_{1,\ell}^- : \overline{\Xi^{p,q}} \to L^2\Lambda^k$ are linear isometries onto their ranges $\mathcal{V}_{1,\ell}^{p,q,+}$ and $\mathcal{V}_{1,\ell}^{p,q,-}$, respectively, which intertwine D^+ resp. D^- with the restriction of Δ_k to $\mathcal{V}_{1,\ell}^{p,q,\pm}$, i.e.,

$$\Delta_k\big|_{\mathcal{V}_{1,\ell}^{p,q,\pm}} = U_{1,\ell}^\pm\, D^\pm\, (U_{1,\ell}^\pm)^{-1} \quad on \quad \mathrm{dom}\,\Delta_k\big|_{\mathcal{V}_{1,\ell}^{p,q,\pm}} . \tag{7.17}$$

Here, $(U_{1,\ell}^\pm)^{-1}$ denotes the inverse of $U_{1,\ell}^\pm$ when viewed as an operator into its range $\mathcal{V}_{1,\ell}^{p,q,\pm}$.

Finally, if we regard of $U_{1,\ell}^\pm$ as an operator mapping into $L^2\Lambda^k$, then $P_{1,\ell}^\pm := P_{1,\ell}^{p,q,\pm} := U_{1,\ell}^\pm(U_{1,\ell}^\pm)^*$ is the orthogonal projection from $L^2\Lambda^k$ onto $\mathcal{V}_{1,\ell}^{p,q,\pm}$.

7.3. A unitary intertwining operator for $V_{2,\ell}^{p,q}$.

We next wish to replace the intertwining operator $\mathcal{A}_{2,\ell}$ from Proposition 6.2 by a unitary one, denoted by $U_{2,\ell} = U_{2,\ell}^{p,q}$, which, according to Proposition 7.2, should be given by $A_{2,\ell}(A_{2,\ell}^* A_{2,\ell})^{-\frac{1}{2}}$. In fact, it will be convenient to modify this expression introducing the unitary central factor $\sigma(T) = i^{-1}T/|T|$.

Recall that the non-unitary intertwining operator $\mathcal{A}_{2,\ell}$ from $Z^{p,q}$ to $\mathcal{V}_{2,\ell}^{p,q}$ is

$$\begin{aligned} A_{2,\ell} &= \Phi e(d\theta)^\ell \begin{pmatrix} \bar\partial\partial & \partial\bar\partial \end{pmatrix} = \begin{pmatrix} I \\ T^{-1}d_H^* \end{pmatrix} e(d\theta)^\ell \begin{pmatrix} \bar\partial\partial & \partial\bar\partial \end{pmatrix} \\ &= \begin{pmatrix} e(d\theta)^\ell\bar\partial\partial & e(d\theta)^\ell\partial\bar\partial \\ T^{-1}d_H^* e(d\theta)^\ell\bar\partial\partial & T^{-1}d_H^* e(d\theta)^\ell\partial\bar\partial \end{pmatrix} . \end{aligned} \tag{7.18}$$

Since $A_{2,\ell}$ acts on $Z^{p,q}$, the identities in Lemma 5.5 in combination with (1.20) imply that

$$d_H^* e(d\theta)^\ell \bar\partial\partial = e(d\theta)^\ell \big[(\overline{\Box} + i\ell T)\partial - (\Box + iT)\bar\partial\big] .$$

Analogously,

$$d_H^* e(d\theta)^\ell \partial\bar\partial = e(d\theta)^\ell \big[(\Box - i\ell T)\bar\partial - (\overline{\Box} - iT)\partial\big] .$$

Therefore,

$$A_{2,\ell} = e(d\theta)^\ell \begin{pmatrix} \bar\partial\partial & \partial\bar\partial \\ T^{-1}\big[(\overline{\Box}+i\ell T)\partial - (\Box + iT)\bar\partial\big] & T^{-1}\big[(\Box - i\ell T)\bar\partial - (\overline{\Box}-iT)\partial\big] \end{pmatrix} . \tag{7.19}$$

Lemma 7.10. *We have*

(i)

$$A_{2,\ell}^*A_{2,\ell} = -c_{s+1,\ell}T^{-2}E =: -c_{s+1,\ell}T^{-2}\begin{pmatrix} E_{11} & E_{12} \\ E_{21} & E_{22}\end{pmatrix} ,$$

where

$$\begin{aligned} E_{11} &= \Box\overline{\Box}(\Delta_H - T^2) + i(\ell+1)T\Box\big[\Delta_H - T^2 - i(n-s-\ell-1)T\big] , \\ E_{12} &= E_{21} = -\Box\overline{\Box}(\Delta_H - T^2) , \\ E_{22} &= \Box\overline{\Box}(\Delta_H - T^2) - i(\ell+1)T\overline{\Box}\big[\Delta_H - T^2 + i(n-s-\ell-1)T\big] ; \end{aligned} \tag{7.20}$$

(ii)

$$\big(A_{2,\ell}^*A_{2,\ell}\big)^{\frac12} = \frac{\sqrt{c_{s+1,\ell}}}{|T|\sqrt{\Delta'}}\Big[E - T^2\sqrt{c\Box\overline{\Box}\Delta'\Delta''}\, I\Big],$$

with E as above.

Moreover, $\big(A_{2,\ell}^*A_{2,\ell}\big)^{\frac12}$ maps $Z^{p,q}$ bijectively onto itself, and, on $Z^{p,q}$,

$$\big(A_{2,\ell}^*A_{2,\ell}\big)^{-\frac12} = \frac{1}{\sqrt{c_{s+1,\ell}\, c}|T|\sqrt{\Box\overline{\Box}\Delta'\Delta''}}\tilde M ,$$

where $\tilde M$ and Δ' are given by

$$\tilde M = \Box\overline{\Box}(\Delta_H - T^2)\begin{pmatrix} 1 & 1 \\ 1 & 1\end{pmatrix} + \begin{pmatrix} M_{11} & 0 \\ 0 & M_{22}\end{pmatrix} ,$$

with

$$\begin{aligned} M_{11} &= -i(\ell+1)T\overline{\Box}\big(\Delta_H - T^2\big) - T^2\Big(c\overline{\Box} + \sqrt{c\Box\overline{\Box}\Delta''}\Big) , \\ M_{22} &= i(\ell+1)T\Box\big(\Delta_H - T^2\big) - T^2\Big(c\Box + \sqrt{c\Box\overline{\Box}\Delta''}\Big) , \end{aligned} \tag{7.21}$$

$$\begin{aligned} \Delta' &:= \Big(2\Box\overline{\Box} - (\ell+1)^2T^2\Big)(\Delta_H - T^2) - T^2\Big(-cT^2 + 2\sqrt{c\Box\overline{\Box}\Delta''}\Big) , \\ \Delta'' &:= \Delta_H - T^2 + c , \end{aligned} \tag{7.22}$$

and

$$c := (\ell+1)(n-s-\ell-1) .$$

Proof. The proof of the formulas is postponed to the Appendix, where we also prove the identity

$$\det E = cT^4\Box\overline{\Box}(\Delta_H - T^2 + c) . \tag{7.23}$$

Hence we only prove here that $\big(A_{2,\ell}^*A_{2,\ell}\big)^{\frac12}$ maps $Z^{p,q}$ bijectively onto itself, assuming the validity of (7.20) and (7.23).

We can factor E as

$$E = -T^2 \begin{pmatrix} \Box & 0 \\ 0 & \overline{\Box} \end{pmatrix} E'$$

where

$$\det E' = c\Delta'' \geq c^2 > 0 \ .$$

Applying Lemma 3.2 as in the proof of Lemma 7.6, we can conclude that the operator

$$\tilde{E} = -T^2 \begin{pmatrix} \Box + \delta_{p,0}\mathcal{C} & 0 \\ 0 & \overline{\Box} + \delta_{q,0}\bar{\mathcal{C}} \end{pmatrix} E'$$

maps bijectively $(W_0^{p,q})^2$ onto itself. Restricting to $Z^{p,q}$, we obtain the conclusion. Q.E.D.

Some cancellations occur when we proceed to computing the matrix product $A_{2,\ell}\tilde{M}$, as the next lemma shows.

Lemma 7.11. *We have that*

$$A_{2,\ell}\tilde{M} =: e(d\theta)^\ell T P = e(d\theta)^\ell T \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix} \ ,$$

where

(7.24)
$$\begin{aligned} P_{11} = \overline{P_{12}} &= -\bar{\partial}\partial\Big[i(\ell+1)\overline{\Box}\big(\Delta_H - T^2\big) + T\big(c\overline{\Box} + \sqrt{c\Box\overline{\Box}\Delta''}\big)\Big] - e(d\theta)\Box\overline{\Box}(\Delta_H - T^2) \ , \\ P_{21} = \overline{P_{22}} &= -\partial\Big[c\overline{\Box}(\Delta_H - T^2) + (\overline{\Box} + i(\ell+1)T)(c\overline{\Box} + \sqrt{c\Box\overline{\Box}\Delta''})\Big] + \bar{\partial}\Box(c\overline{\Box} + \sqrt{c\Box\overline{\Box}\Delta''}) \ . \end{aligned}$$

Proof. Let $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$ denote the matrix on the right hand side of (7.19), and set $P = T^{-1}A\tilde{M}$. Then

$$\begin{aligned} A\tilde{M} = &\begin{pmatrix} \partial\bar{\partial} + \bar{\partial}\partial & \partial\bar{\partial} + \bar{\partial}\partial \\ i(\ell+1)(\partial - \bar{\partial}) & -i(\ell+1)(\bar{\partial} - \partial) \end{pmatrix} \Box\overline{\Box}(\Delta_H - T^2) \\ &+ \begin{pmatrix} \bar{\partial}\partial M_{11} & \partial\bar{\partial} M_{22} \\ T^{-1}\big[(\overline{\Box} + i\ell T)\partial - (\Box + iT)\bar{\partial}\big] M_{11} & T^{-1}\big[(\Box - i\ell T)\bar{\partial} - (\overline{\Box} - iT)\partial\big] M_{22}, \end{pmatrix} \end{aligned}$$

where by (1.12) $\partial\bar{\partial} + \bar{\partial}\partial = -Te(d\theta)$. This implies that

$$P_{11} = -\bar{\partial}\partial\Big[i(\ell+1)\overline{\Box}\big(\Delta_H - T^2\big) + T\big(c\overline{\Box} + \sqrt{c\Box\overline{\Box}\Delta''}\big)\Big] - e(d\theta)\Box\overline{\Box}(\Delta_H - T^2)$$

and $P_{12} = \overline{P_{11}}$, which proves the statements about P_{11} and P_{12}, and

$$\begin{aligned} P_{21} = &\, iT^{-1}(\ell+1)(\partial - \bar{\partial})\Box\overline{\Box}(\Delta_H - T^2) \\ &+ T^{-2}\Big[(\overline{\Box} + i\ell T)\partial - (\Box + iT)\bar{\partial}\Big]\Big[-i(\ell+1)T\overline{\Box}\big(\Delta_H - T^2\big) - T^2\big(c\overline{\Box} + \sqrt{c\Box\overline{\Box}\Delta''}\big)\Big] \end{aligned}$$

where $P_{22} = \overline{P_{21}}$.

Next, using (1.20) and the identity (5.19), with $s=p+q$ in place of k, we have

$$
\begin{aligned}
P_{21} &= iT^{-1}(\ell+1)(\partial-\bar\partial)\square\overline{\square}(\Delta_H-T^2)\\
&\qquad + T^{-1}\Big[\partial(\overline{\square}+i(\ell+1)T)-\bar\partial\square\Big]\Big[-i(\ell+1)\overline{\square}\big(\Delta_H-T^2\big)-T\big(c\overline{\square}+\sqrt{c\square\overline{\square}\Delta''}\big)\Big]\\
&= \Big[iT^{-1}(\partial-\bar\partial)\square-iT^{-1}\partial\overline{\square}+(\ell+1)\partial+iT^{-1}\bar\partial\square\Big](\ell+1)\overline{\square}(\Delta_H-T^2)\\
&\qquad - \Big[\partial(\overline{\square}+i(\ell+1)T)-\bar\partial\square\Big]\big(c\overline{\square}+\sqrt{c\square\overline{\square}\Delta''}\big)\\
&= -c\partial\overline{\square}(\Delta_H-T^2)-\Big[\partial(\overline{\square}+i(\ell+1)T)-\bar\partial\square\Big]\big(c\overline{\square}+\sqrt{c\square\overline{\square}\Delta''}\big)\\
&= -\partial\Big[c\overline{\square}(\Delta_H-T^2)+(\overline{\square}+i(\ell+1)T)(c\overline{\square}+\sqrt{c\square\overline{\square}\Delta''})\Big]+\bar\partial\square(c\overline{\square}+\sqrt{c\square\overline{\square}\Delta''})\ .
\end{aligned}
$$

This proves the lemma.

Q.E.D.

From the previous results we immediately get an explicit formula for $U_{2,\ell}$, at least when $p\neq 0$ and $q\neq 0$. However, if $p=0$ or $q=0$, our formulas, when properly interpreted, persist, and we obtain the following result:

Recall that if $p=0$, then $X^{p,q}=(I-\mathcal{C})X^{p,q}$, and if $q=0$, then $Y^{p,q}=(I-\overline{\mathcal{C}})Y^{p,q}$. Let us correspondingly put

$$
\square_r=\begin{cases}\square & \text{if } p\geq 1,\\ \square' & \text{if } p=0,\end{cases}\qquad \overline{\square}_r=\begin{cases}\overline{\square} & \text{if } q\geq 1,\\ \overline{\square}' & \text{if } q=0,\end{cases}
$$

so that $\square_r$ is always invertible on $X^{p,q}$, and $\overline{\square}_r$ on $Y^{p,q}$.

Proposition 7.12. *The operator $U_{2,\ell}$, which acts on $Z^{p,q}$, is given by*

$$
U_{2,\ell}=\frac{e(d\theta)^\ell}{\sqrt{c_{s+1,\ell}\,c}}\,H\,\frac{1}{\sqrt{\Delta'\Delta''}}\ , \tag{7.25}
$$

where the operator matrix $H=\begin{pmatrix}H_{11} & H_{12}\\ H_{21} & H_{22}\end{pmatrix}$ *is defined by*

$$
\begin{aligned}
H_{11}=\overline{H_{12}} &= -\bar{\mathcal{R}}\mathcal{R}(\overline{\square}+iT)^{\frac12}\Big[i(\ell+1)\overline{\square}^{\frac12}\big(\Delta_H-T^2\big)+T\big(c\overline{\square}^{\frac12}+\square^{\frac12}\sqrt{c\Delta''}\big)\Big]\\
&\qquad - e(d\theta)\square^{\frac12}\overline{\square}^{\frac12}(\Delta_H-T^2)\ ,\\
H_{21}=\overline{H_{22}} &= -\mathcal{R}\Big[c\overline{\square}^{\frac12}(\Delta_H-T^2)+(\overline{\square}+i(\ell+1)T)(c\overline{\square}^{\frac12}+\square^{\frac12}\sqrt{c\Delta''})\Big]\\
&\qquad + \bar{\mathcal{R}}(c\overline{\square}\square^{\frac12}+\square\overline{\square}^{\frac12}\sqrt{c\Delta''})\ ,
\end{aligned}\tag{7.26}
$$

and where Δ',Δ'' and c are given by Lemma 7.10.

Finally, we have the following analogue of Proposition 7.9.

Proposition 7.13. *The operator $U_{2,\ell}$ in Proposition 7.12 maps the space $Z^{p,q}$ onto $V_{2,\ell}^{p,q}$ and intertwines $D := \Delta_0 + i(q-p)T + (\ell+1)(n-k+\ell+1)$ with Δ_k on the core. Moreover $U_{2,\ell} : \overline{Z^{p,q}} \to L^2\Lambda^k$ is a linear isometry onto $\mathcal{V}_{2,\ell}^{p,q}$ which intertwines D with the restriction of Δ_k to $\mathcal{V}_{2,\ell}^{p,q}$, i.e.,*

$$\Delta_k\big|_{\mathcal{V}_{2,\ell}^{p,q}} = U_{2,\ell}\, D\, (U_{2,\ell})^{-1} \quad \text{on} \quad \operatorname{dom} \Delta_k\big|_{\mathcal{V}_{2,\ell}^{p,q}} . \tag{7.27}$$

Here, $(U_{2,\ell})^{-1}$ denotes the inverse of $U_{2,\ell}$ when viewed as an operator into its range $\mathcal{V}_{2,\ell}^{p,q}$.

Finally, if we regard of $U_{2,\ell}$ as an operator mapping into $L^2\Lambda^k$, then $P_{2,\ell} := P_{2,\ell}^{p,q} := U_{2,\ell}(U_{2,\ell})^*$ is the orthogonal projection from $L^2\Lambda^k$ onto $\mathcal{V}_{2,\ell}^{p,q}$.

Proof. This will follow by applying Proposition 7.2 to $\mathcal{A}_{2,\ell}$. To this end, we set

$$\mathcal{D}_1 = Z^{p,q}, \quad H_1 = \overline{Z^{p,q}},$$
$$\mathcal{D}_2 = \mathcal{S}_0\Lambda^k, \quad H_2 = L^2\Lambda^k,$$
$$S_1 = D, \quad S_2 = \Delta_k,$$

and denote by A the closure of $\mathcal{A}_{2,\ell}$ on $Z^{p,q}$. The commutation relation (7.1) is then satisfied because of (6.3). Moreover, clearly $S_2(\mathcal{D}_2) \subset \mathcal{D}_2$, and A maps $\mathcal{D}_1$ bijectively onto $V_{2,\ell}^{p,q} \subseteq \mathcal{D}_2$.

Next, according to Lemma 7.10, A^*A is a positive matrix with scalar operator entries, and $S_1 = S_1 I$ is a scalar operator. But then also $|A| = \sqrt{A^*A}$ is a matrix with scalar operator entries, hence commutes with $S_1 I$, so that condition (iv) in Proposition 7.2 is satisfied too.

In order to verify conditions (iii), (v) and (vi), we can make use of the joint spectral theory of L and $i^{-1}T$ described in Section 9. Indeed, it is immediate by means of the spectral decomposition of S_1 that (vi) is satisfied.

Moreover, $|A|$ maps $Z^{p,q}$ into itself; this can be verified as follows:

The formula for $|A| = \left(A_{2,\ell}^* A_{2,\ell}\right)^{\frac{1}{2}}$ in Lemma 7.10 shows that it suffices to prove that the operator matrix E maps $Z^{p,q}$ into itself. This in return will be verified if we can show that E_{12} maps $Y^{p,q}$ into $X^{p,q}$, and E_{21} maps $X^{p,q}$ into $Y^{p,q}$. But, according to Lemma 12.3 in the Appendix, $\Box\overline{\Box}$ maps $W_0^{p,q}$ into $\Xi^{p,q}$, so that the latter claims are immediate.

And, the formula for $A_{2,\ell}^* A_{2,\ell}$ in Lemma 7.10 in combination with Lemma 10.9 and Plancherel's theorem shows that $A_{2,\ell}^* A_{2,\ell} = |A|^2$ has a trivial kernel in L^2, and then the same applies to $|A|$, which proves (v).

Finally, our explicit formulas for $U = U_{2,\ell}$ in Proposition 7.12 show that U maps the space $Z^{p,q}$ into $\mathcal{S}_0\Lambda^k$, so that U^* maps $\mathcal{S}_0\Lambda^k$ into $Z^{p,q}$, and we see that $P(\mathcal{D}_2) = P(\mathcal{S}_0\Lambda^k) = U(Z^{p,q}) = A\Big(|A|^{-1}(Z^{p,q})\Big) = A(Z^{p,q}) = A(\mathcal{D}_1)$, where $P = UU^*$. This shows that also condition (vii) is satisfied, which concludes the proof of Proposition 7.13.

Q.E.D.

8. Decomposition of $L^2\Lambda^k$

We are now in the position to completely describe the orthogonal decomposition of $L^2\Lambda^k$ into Δ_k-invariant subspaces and the unitary intertwining operators that reduce Δ_k into scalar form.

Theorem 8.1. *Let $0 \le k \le n$. Then $L^2\Lambda^k$ admits the orthogonal decomposition*

$$L^2\Lambda^k = \sum^{\oplus}_{\substack{p+q=k<n\\ p+q=n,\,pq=0}} \mathcal{V}_0^{p,q} \oplus \sum^{\oplus}_{\varepsilon=\pm}\ \sum^{\oplus}_{p+q+2\ell=k-1} \mathcal{V}_{1,\ell}^{p,q,\varepsilon} \oplus \sum^{\oplus}_{p+q+2\ell=k-2} \mathcal{V}_{2,\ell}^{p,q}$$
$$\oplus \sum^{\oplus}_{p+q=k-1} R\,\mathcal{V}_0^{p,q} \oplus \sum^{\oplus}_{\varepsilon=\pm}\ \sum^{\oplus}_{p+q+2\ell=k-2} R\,\mathcal{V}_{1,\ell}^{p,q,\varepsilon} \oplus \sum^{\oplus}_{p+q+2\ell=k-3} R\,\mathcal{V}_{2,\ell}^{p,q}\ , \tag{8.1}$$

where $R = R_{k-1}$ denotes the Riesz transform.

Proof. This follows immediately from (5.1), Proposition 5.13 and Proposition 4.5, since, according to Lemma 4.4, $R_{k-1}R_{k-2} = 0$. Q.E.D.

SUBSPACE	SCALAR FORM	INTERTWINING (WITH DOMAIN)	ORTHOGONAL PROJECTION
$\mathcal{V}_0^{p,q}$ ($p+q=k<n$, or $pq=0$ if $p+q=n$)	$\Delta_0 + i(q-p)T$	I ($\mathcal{V}_0^{p,q}$)	$I - RR^* - \sum P_{1,\ell}^{p,q,\pm} - \sum P_{2,\ell}^{p,q}$
$R\mathcal{V}_0^{p,q}$ ($p+q=k-1$)	$\Delta_0 + i(q-p)T$	R ($\mathcal{V}_0^{p,q}$)	$RR^* - \sum RP_{1,\ell}^{p,q,\pm}R^* - \sum RP_{2,\ell}^{p,q}R^*$
$\mathcal{V}_{1,\ell}^{p,q,\pm}$ ($p+q=k-2\ell-1$)	$\Delta_0 + i(q-p)T + \frac{n-p-q}{2} + \ell(n-p-q+\ell) \pm\sqrt{\Delta_0 + i(q-p)T + \left(\frac{n-p-q}{2}\right)^2}$	$U_{1,\ell}^{p,q,\pm}$ ($\mathcal{W}_0^{p,q}$, resp. $\overline{\Xi^{p,q}}$)	$P_{1,\ell}^{p,q,\pm}$ (Prop. 7.9)
$R\,\mathcal{V}_{1,\ell}^{p,q,\pm}$ ($p+q=k-2\ell-2$)	$\Delta_0 + i(q-p)T + \frac{n-p-q}{2} + \ell(n-p-q+\ell) \pm\sqrt{\Delta_0 + i(q-p)T + \left(\frac{n-p-q}{2}\right)^2}$	$RU_{1,\ell}^{p,q,\pm}$ ($\mathcal{W}_0^{p,q}$, resp. $\overline{\Xi^{p,q}}$)	$R\,P_{1,\ell}^{p,q,\pm}R^*$
$\mathcal{V}_{2,\ell}^{p,q}$ ($p+q=k-2\ell-2$)	$\Delta_0 + i(q-p)T + (\ell+1)(n-k+\ell+1)$	$U_{2,\ell}^{p,q}$ ($\overline{Z^{p,q}}$)	$P_{2,\ell}^{p,q}$ (Prop. 7.12)
$R\,\mathcal{V}_{2,\ell}^{p,q}$ ($p+q=k-2\ell-3$)	$\Delta_0 + i(q-p)T + (\ell+1)(n-k+\ell+2)$	$RU_{2,\ell}^{p,q}$ ($\overline{Z^{p,q}}$)	$R\,P_{2,\ell}^{p,q}R^*$

Table 1. Components of $L^2\Lambda^k$, $0 \le k \le n$

The Hodge Laplacian Δ_k leaves all the subspaces in this decomposition invariant, and we have seen that, after applying the unitary intertwining operators derived in the previous sections, it will assume a scalar form on each of the corresponding parameter spaces.

In Table 1, we list these subspaces, the corresponding scalar forms of Δ_k, the associated unitary intertwining operators as well as the orthogonal projections onto these subspaces.

By J, we denote the inclusion the operator of a given subspace into $L^2\Lambda^k$.

8.1. **The $*$-Hodge operator and the case $n < k \leq 2n+1$.**

We now remove the condition $0 \leq k \leq n$ and prove a decomposition theorem for $L^2\Lambda^k$ also in the case $n < k \leq 2n+1$.

We are going to use the $*$-Hodge operator defined on an arbitrary Riemannian d-manifold M, acting for each point $m \in M$ as a linear mapping

$$* : \Lambda^k_m \to \Lambda^{d-k}_m,$$

where Λ^k_m denotes the k-th exterior product of the dual of the tangent space at m. It will be viewed also as a linear mapping acting on forms on M. For its definition and basic properties we refer to [Ra]. We summarize the main properties in the following statement.

Proposition 8.2. *The $*$-Hodge operator is almost involutive, i.e., , $*(*\omega) = (-1)^{k(d-k)}\omega$, and the following properties hold true:*

(1) *for $\omega_1, \omega_2 \in L^2\Lambda^k(M)$,*

$$\int_M \omega_1 \wedge *\overline{\omega_2} = \langle \omega_1, \omega_2 \rangle_{L^2\Lambda^k} \ ;$$

(2) *as a mapping $* : L^2\Lambda^k \to L^2\Lambda^{d-k}$, the operator $*$ is unitary;*

(3) $d^* = - * d *$*;*

(4) $*\Delta_k = \Delta_{d-k}*$.

In our situation, $M = H_n$ and $d = 2n+1$.

It follows from property (4) above that a subspace $\mathcal{V} \subseteq L^2\Lambda^k$ is Δ_k-invariant if and only if $*\mathcal{V} \subseteq L^2\Lambda^{d-k}$ is Δ_{d-k}-invariant. Thus, we wish to describe the Δ_k-invariant subspaces of $L^2\Lambda^k$, when $n < k \leq 2n+1$.

We denote by Λ^k_V the space of *vertical k-forms,* that is, the forms $\omega = \theta \wedge \omega_2$, with $\omega_2 \in \Lambda^{k-1}_H$, and by $\mu = \theta \wedge d\theta \wedge \cdots \wedge d\theta$ the volume element on H_n. Similarly, $\mu_H = d\theta \wedge \cdots \wedge d\theta$ will denote the corresponding volume element on the horizontal structure. In the same way as the $*$-Hodge operator on H_n is determined by the relations $\sigma \wedge *\overline{\omega} = \langle \sigma, \omega \rangle \mu$ for all $\sigma, \omega \in \Lambda^k$, we can introduce the $*$-Hodge operator $*_H$ acting on the horizontal structure, by requiring that $\sigma \wedge *_H\overline{\omega} = \langle \sigma, \omega \rangle \mu_H$ for all $\sigma, \omega \in \Lambda^k_H$.

The following results are easy consequences of these defining relations.

Lemma 8.3. Let $\omega \in \mathcal{S}_0\Lambda^k_H$. Then the following hold true:

(i) if we put $\omega' := (-1)^k *_H \omega$, then $*\omega = \theta \wedge \omega'$;

(ii) $*_H\omega = *(\theta \wedge \omega)$.

We set

$$\overset{*}{W}{}_0^{r,s} = \big\{\omega' \in \mathcal{S}_0\Lambda^{r,s} :\ \partial\omega' = \bar\partial\omega' = 0\big\} \tag{8.2}$$

and define

$$\begin{aligned} Z_0^{r,s} &= \big\{\omega = \theta\wedge\omega' \in \mathcal{S}_0\Lambda_V^k :\ \omega' \in \overset{*}{W}{}_0^{r,s}\ \big\}\ , \\ Z_1^{r,s} &= \big\{\omega = \theta\wedge\omega' \in \mathcal{S}_0\Lambda_V^k :\ \omega' = \partial^*\sigma + \bar\partial^*\tau,\ \sigma,\tau \in \overset{*}{W}{}_0^{r,s}\ \big\}\ , \\ Z_2^{r,s} &= \big\{\omega = \theta\wedge\omega' \in \mathcal{S}_0\Lambda_V^k :\ \omega' = \bar\partial^*\partial^*\sigma + \partial^*\bar\partial^*\tau,\ \sigma,\tau \in \overset{*}{W}{}_0^{r,s}\ \big\}\ . \end{aligned} \tag{8.3}$$

We also set

$$Z_{j,\ell}^{r,s} = i(d\theta)^\ell Z_1^{r,s}\ , \qquad\qquad j = 1,2. \tag{8.4}$$

Notice that $Z_0^{r,s}$ is a subspace of $\mathcal{S}_0\Lambda_V^k$, where $k = r+s+1$, $Z_1^{r,s} \subseteq \mathcal{S}_0\Lambda_V^k$ with $k = r+s$, and $Z_2^{r,s} \subseteq \mathcal{S}_0\Lambda_V^k$ with $k = r+s-1$. Therefore, $Z_{j,\ell}^{r,s} \subseteq \mathcal{S}_0\Lambda_V^k$ where $k = r+s+1-j-2\ell$, $j = 1,2$.

Observe also that from (1.21) it follows that $\omega \in \mathcal{S}_0\Lambda^k$, $\omega = \omega_1 + \theta\wedge\omega_2$ is d-closed if and only if $\omega_1 = T^{-1}d_H\omega_2$.

The mapping $\overset{*}{\Phi}$: $\mathcal{S}_0\Lambda_V^k \to (\mathcal{S}_0\Lambda^{d-k})_{d\text{-cl}}$ defined by

$$\overset{*}{\Phi}\ (\theta\wedge\omega') = T^{-1}d_H\omega' + \theta\wedge\omega'\ ,$$

where $\omega' \in \mathcal{S}_0\Lambda_H^k$, is an isomorphism.

Lemma 8.4. *The following properties hold true:*

(i) $*(L^2\Lambda^k)_{d\text{-ex}} = (L^2\Lambda^{d-k})_{d^*\text{-ex}}$ *and* $*(L^2\Lambda^k)_{d^*\text{-cl}} = (L^2\Lambda^{d-k})_{d\text{-cl}}$*;*

(ii) *if* $\omega \in \mathcal{S}_0\Lambda_H^k$*, then* $*\Phi(\omega) = \overset{*}{\Phi}\ (*\omega)$.

Moreover, if for given p,q *we put* $r = n-q$ *and* $s = n-p$*, then*

(iii) $*(W_0^{p,q}) = Z_0^{r,s}$*, hence* $*(V_0^{p,q}) = \overset{*}{\Phi}\ (Z_0^{r,s})$*;*

(iv) $*(W_{1,\ell}^{p,q}) = Z_{1,\ell}^{r,s}$*, hence* $*(V_{1,\ell}^{p,q}) = \overset{*}{\Phi}\ (Z_{1,\ell}^{r,s})$*;*

(v) $*(W_{2,\ell}^{p,q}) = Z_{2,\ell}^{r,s}$*, hence* $*(V_{2,\ell}^{p,q}) = \overset{*}{\Phi}\ (Z_{2,\ell}^{r,s})$*.*

Finally, the spaces $Z_{j,\ell}^{r,s}$*,* $j = 1,2$ *are non-trivial, and* $Z_0^{r,s}$ *are non-trivial if and only if* $r+s > n$ *or, if* $r+s = n$*,* $rs = 0$*.*

Proof. Property (i) follows from Proposition 8.2 (3).

If $\omega \in \Lambda^k$, we shall put

$$\omega' := (-1)^k *_H \omega,$$

so that according to Lemma 8.3, $*\omega = \theta\wedge\omega'$.

Then

$$\begin{aligned} *\Phi(\omega) &= *\omega + *\big(\theta\wedge T^{-1}d_H^*\omega\big) = *\omega + T^{-1} *_H d_H^*\omega \\ &= *\omega + (-1)^{(2n-k+1)(k-1)+1}T^{-1}d_H *_H \omega \\ &= \theta\wedge\omega' + T^{-1}d_H\omega'\ , \end{aligned}$$

which proves (ii).

Using Lemma 8.3, the fact that (on horizontal forms) $\partial^* = -*_H \partial *_H$ and the analogous formula for $\bar\partial^*$, for $\omega \in W_0^{p,q}$ we obtain

$$*\omega = \theta \wedge (-1)^{p+q} *_H \omega = \theta \wedge \omega',$$

where here $\omega' \in \mathcal{S}_0\Lambda^{r,s}$ and $\partial\omega' = \bar\partial\omega' = 0$.

This shows that $*(W_0^{p,q}) \subset Z_0^{r,s}$, and in a similar way one proves that $*(Z_0^{r,s}) \subset W_0^{p,q}$. Combining this with (ii), we obtain (iii).

Next, if $\omega = \partial\xi + \bar\partial\eta$, with $\xi, \eta \in W_0^{p,q}$, then

$$\begin{aligned} *\omega &= (-1)^k \theta \wedge *_H(\partial\xi + \bar\partial\eta) \\ &= \theta \wedge (\partial^* *_H \xi + \bar\partial^* *_H \eta) =: \theta \wedge (\partial^*\sigma + \bar\partial^*\tau) \end{aligned} \tag{8.5}$$

where $\sigma, \tau \in \overset{*}{W}_0^{r,s}$, hence $\theta \wedge (\partial^*\sigma + \bar\partial^*\tau) \in Z_1^{r,s}$. This shows that $*(W_1^{p,q}) \subset Z_1^{r,s}$, and in a similar way one proves that $*(Z_1^{r,s}) \subset W_1^{p,q}$, and we obtain (iv) in the case $\ell = 0$.

For the general case, we observe that, for all test forms ω and σ,

$$\begin{aligned} \int \sigma \wedge *_H\overline{i(d\theta)\omega} &= \langle \sigma,\, i(d\theta)\omega\rangle = \langle e(d\theta)\sigma,\, \omega\rangle \\ &= \int e(d\theta)\sigma \wedge *_H\overline{\omega} = \int \sigma \wedge \overline{e(d\theta)(*_H\omega)}\ . \end{aligned}$$

It follows that

$$*_H i(d\theta) = e(d\theta)*_H\ , \qquad \text{and} \qquad *_H\, e(d\theta) = i(d\theta) *_H\ .$$

Hence, if $\omega = \partial\xi + \bar\partial\eta$, with $\xi, \eta \in W_0^{p,q}$,

$$\begin{aligned} *\Phi\big(e(d\theta)^\ell\omega\big) &= \overset{*}{\Phi}\big(* e(d\theta)^\ell\omega\big) \\ &= (-1)^k\theta \wedge *_H e(d\theta)^\ell\omega + (-1)^k T^{-1} d_H *_H (e(d\theta)^\ell\omega) \\ &= (-1)^k\theta \wedge i(d\theta)^\ell *_H \omega + (-1)^k T^{-1} d_H i(d\theta)^\ell *_H \omega \\ &= \overset{*}{\Phi}\big(i(d\theta)^\ell\theta \wedge \omega'\big) \end{aligned}$$

where $\theta \wedge \omega' = \theta \wedge (\partial^*\sigma + \bar\partial^*\tau) \in Z_1^{r,s}$, with σ, τ as in (8.5). This shows that $*(W_{1,\ell}^{p,q}) \subset Z_{1,\ell}^{r,s}$, and in a similar way one proves that $*(Z_{1,\ell}^{r,s}) \subset W_{1,\ell}^{p,q}$, and we obtain (iv).

The proof of (v) follows along the same lines and is therefore omitted.

The proof about the non-triviality of these subspaces follows from Propositions 5.3 and 5.4. Q.E.D.

Definition 8.5. When $r + s \geq n$ we set

$$Y_0^{r,s} = \overset{*}{\Phi}(Z_0^{r,s}) = Z_0^{r,s}\ , \quad Y_{1,\ell}^{r,s,\pm} = \overset{*}{\Phi}(Z_{1,\ell}^{r,s,\pm})\ , \quad Y_{2,\ell}^{r,s} = \overset{*}{\Phi}(Z_{2,\ell}^{r,s})\ ,$$

and denote by $\Upsilon_0^{r,s}, \Upsilon_{1,\ell}^{r,s,\pm}$ respectively $\Upsilon_{2,\ell}^{r,s}$ the closures of these spaces in $L^2\Lambda^k$.

Let us finally observe that, in view of Lemmas 4.4 and 8.2, the Riesz transforms on H_n satisfy

$$*R_{2n-k}(\omega) = -R^*_{k+1}(*\omega).$$

Then, from Theorem 8.1, Lemma 8.4 and Proposition 8.2 we immediately obtain the following decomposition of $L^2\Lambda^k$ into Δ_k-invariant subspaces when $n < k \leq 2n+1$.

Theorem 8.6. *Let $n < k \leq 2n+1$. Then $L^2\Lambda^k$ admits the orthogonal decomposition*

$$(8.6)\quad L^2\Lambda^k = \sum^{\oplus}_{\substack{r+s=k-1>n\\ r+s=n,\, rs=0}} \Upsilon^{r,s}_0 \oplus \sum^{\oplus}_{\varepsilon=\pm}\ \sum^{\oplus}_{r+s-2\ell=k} \Upsilon^{r,s,\varepsilon}_{1,\ell} \oplus \sum^{\oplus}_{r+s-2\ell=k+1} \Upsilon^{r,s}_{2,\ell}$$
$$\oplus \sum^{\oplus}_{r+s=k} R^*\,\Upsilon^{r,s}_0 \oplus \sum^{\oplus}_{\varepsilon=\pm}\ \sum^{\oplus}_{r+s-2\ell=k+1} R^*\,\Upsilon^{r,s,\varepsilon}_{1,\ell} \oplus \sum^{\oplus}_{r+s-2\ell=k+2} R^*\,\Upsilon^{r,s}_{2,\ell}\ ,$$

where $R^* = R^*_{k+1}$.

Moreover, since the $*$-Hodge operator transform the subspaces in this decomposition into the corresponding subspaces in the decomposition given by Theorem 8.1, with $p := n-s$ and $q := n-r$, the unitary intertwining operators which transform Δ_k on each of these subspaces into scalar forms are simply given by those from Table 1 at the end of Section 8, composed on the right hand side by the $*$-Hodge operator, and similar remarks apply to the orthogonal projections and scalar forms.

9. L^p-MULTIPLIERS

The decomposition of $L^2\Lambda^k$ presented in the previous sections, together with the description of the action of Δ_k on the various subspaces, can be used for the L^p- functional calculus of Δ_k. For this purpose, we are going to show that $L^p\Lambda^k$ admits the same decomposition when $1 < p < \infty$. Concretely, this means proving that the orthogonal projections on the various invariant subspaces and the intertwining operators that reduce Δ_k to scalar forms are L^p-bounded.

9.1. **The multiplier theorem.**

The joint spectrum of L and $i^{-1}T$ is the *Heisenberg fan* $F \subset \mathbb{R}^2$ defined as follows. If

$$\ell_{k,\pm} = \{(\lambda,\xi) : \xi = \pm(n+2k)\lambda, \lambda \in \mathbb{R}^*_+\}\ ,$$

then

$$F = \overline{\bigcup_{k\in\mathbb{N}} (\ell_{k,+} \cup \ell_{k,-})}\ .$$

The variable λ corresponds to $i^{-1}T$ and ξ to L, i.e., calling $dE(\lambda,\xi)$ the spectral measure on F, then

$$i^{-1}T = \int_F \lambda\, dE(\lambda,\xi)\ , \qquad L = \int_F \xi\, dE(\lambda,\xi)\ .$$

If m is any bounded, continuous function on $\mathbb{R} \times \mathbb{R}_+^*$, we can then define the associated multiplier operator $m(i^{-1}T, L)$ by

$$m(i^{-1}T, L) := \int_F m(\lambda, \xi)\, dE(\lambda, \xi),$$

which is clearly bounded on $L^2(H_n)$.

It follows from Plancherel's formula that the spectral measure of the vertical half-line $\{(0, \xi) : \xi \geq 0\} \subset F$ is zero. A spectral multiplier is therefore a function $m(\lambda, \xi)$ on F whose restriction to each ℓ_k is measurable w.r. to $d\lambda$ for every k.

We shall use the following results from [MRS1, MRS2] concerning L^p-boundedness of spectral multipliers, see also Section 5 in [MPR1].

Given $\rho, \sigma > 0$, we say that a measurable function $f(\lambda, \xi)$ is in the mixed Sobolev space $L^2_{\rho,\sigma} = L^2_{\rho,\sigma}(\mathbb{R}^2)$ if

$$\begin{aligned} \|f\|^2_{L^2_{\rho,\sigma}} &:= \int_{\mathbb{R}^2} (1 + |\xi'|)^{2\rho} (1 + |\lambda'| + |\xi'|)^{2\sigma} |\hat{f}(\lambda', \xi')|^2 \, d\lambda' \, d\xi' \\ &= c \|(1 + |\partial_\xi|)^\rho (1 + |\partial_\lambda| + |\partial_\xi|)^\sigma f\|_2^2 < \infty \ . \end{aligned} \tag{9.1}$$

Let $\eta_0 \in C_0^\infty(\mathbb{R})$ be a non-trivial, non-negative, smooth bump function supported in $\mathbb{R}_+^* := (0, \infty)$, put $\eta_1(x) := \eta_0(x) + \eta_0(-x)$ and set $\chi := \eta_1 \otimes \eta_0$. If $f(\lambda, \xi)$ is a continuous, bounded function on $\mathbb{R} \times \mathbb{R}_+^*$, then we put $f^r(\lambda, \xi) = f(r_1\lambda, r_2\xi)$, $r = (r_1, r_2) \in (\mathbb{R}_+^*)^2$, and say that f lies in $L^2_{\rho,\sigma,\mathrm{sloc}}\big(\mathbb{R} \times \mathbb{R}_+^*\big)$ if for every $r = (r_1, r_2) \in (\mathbb{R}_+^*)^2$, the function $f^r\chi$ lies in $L^2_{\rho,\sigma}$ and

$$\|f\|_{L^2_{\rho,\sigma,\mathrm{sloc}}} := \sup_r \|f^r \chi\|_{L^2_{\rho,\sigma}} < \infty \ . \tag{9.2}$$

Definition 9.1. A function m satisfying (9.2) is called a *Marcinkiewicz multiplier of class* (ρ, σ). A *smooth Marcinkiewicz multiplier* is a Marcinkiewicz multiplier of every class (ρ, σ), i.e., satisfying the pointwise estimates

$$\big|\partial_\lambda^j \partial_\xi^k m(\lambda, \xi)\big| \leq C_{jk} |\lambda|^{-j} |\xi|^{-k} \ , \tag{9.3}$$

for every j, k.

Theorem 9.2. ([MRS2]) *Let m be a Marcinkiewicz multiplier of class (ρ, σ) for some $\rho > n$ and $\sigma > \frac{1}{2}$. Then $m(i^{-1}T, L)$ is bounded on $L^p(H_n)$ for $1 < p < \infty$, with norm controlled by $\|m\|_{L^2_{\rho,\sigma,\mathrm{sloc}}}$.*

9.2. Some classes of multipliers.

We introduce the classes $\Psi_\tau^{\rho,\sigma}$ of (possibly unbounded) smooth multipliers, in terms of which we will understand the behavior of the projections and intertwining operators presenteded in the previous sections.

These classes are defined by pointwise estimates on all derivatives, in analogy to (9.3), which must be satisfied on some open angle $\Gamma_{n-\varepsilon} := \{(\lambda, \xi) \in \mathbb{R}^2 : \xi > (n - \varepsilon)|\lambda|\}$ containing the Heisenberg fan F taken away the origin.

Definition 9.3. *We say that* $m \in \Psi_\tau^{\rho,\sigma}$ $(\rho,\sigma,\tau \in \mathbb{R})$ *if*

$$\big|\partial_\lambda^j \partial_\xi^k m(\lambda,\xi)\big| \lesssim \begin{cases} \xi^{\tau-j-k} & \text{for } \xi \le 1 \\ (\xi+\lambda^2)^{\rho-\frac{j}{2}}\xi^{\sigma-k} & \text{for } \xi > 1 \ . \end{cases} \tag{9.4}$$

for every $j,k \in \mathbb{N}$. *We also say that* $m \in {}^*\Psi_\tau^{\rho,\sigma}$ *if* $m \in \Psi_\tau^{\rho,\sigma}$ *and, moreover,*

$$m(\lambda,\xi) \gtrsim \begin{cases} \xi^\tau & \text{for } \xi < 1 \\ (\xi+\lambda^2)^\rho \xi^\sigma & \text{for } \xi > 1 \ . \end{cases} \tag{9.5}$$

Prototypes are given by the smooth functions m such that

$$m(\lambda,\xi) = \begin{cases} (\xi + p\lambda + a\lambda^2)^\tau & \text{for } \xi < 1 \\ (\xi+\lambda^2)^\rho(\xi+q\lambda)^\sigma & \text{for } \xi > 2 \ , \end{cases}$$

with $|p|, |q| < n$. The following properties are easy to prove.

Lemma 9.4. *The classes* $\Psi_\tau^{\rho,\sigma}$ *satisfy the following properties:*

(i) $\partial_\lambda \Psi_\tau^{\rho,\sigma} \subset \Psi_{\tau-1}^{\rho-\frac12,\sigma}$, $\partial_\xi \Psi_\tau^{\rho,\sigma} \subset \Psi_{\tau-1}^{\rho,\sigma-1}$;
(ii) $\Psi_\tau^{\rho,\sigma}\Psi_{\tau'}^{\rho',\sigma'} \subset \Psi_{\tau+\tau'}^{\rho+\rho',\sigma+\sigma'}$;
(iii) *if* $m \in {}^*\Psi_\tau^{\rho,\sigma}$ *and then* $m^s \in \Psi_{s\tau}^{s\rho,s\sigma}$ *for every* $s \in \mathbb{R}$ *(for* $s \in \mathbb{N}$, $m \in \Psi_\tau^{\rho,\sigma}$ *is sufficient);*
(iv) *if* $\rho+\sigma \le \rho'+\sigma'$, $2\rho+\sigma \le 2\rho'+\sigma'$ *and* $\tau \ge \tau'$, *then* $\Psi_\tau^{\rho,\sigma} \subset \Psi_{\tau'}^{\rho',\sigma'}$.
(v) *In particular, if* $\rho+\sigma \le 0$, $2\rho+\sigma \le 0$ *and* $\tau \ge 0$, *then* $\Psi_\tau^{\rho,\sigma} \subset \Psi_0^{0,0}$, *and* $\Psi_\tau^{\rho,\sigma}$ *consists of Marcinkiewicz multipliers.*

Remark 9.5.

(i) Observe that if χ is a smooth cut-off function on $\mathbb{R}$, compactly supported on $\mathbb{R}\setminus\{0\}$ and with $0 \le \chi \le 1$, then $\eta = \chi(\xi/|\lambda|)$ and $1-\eta$ are in $\Psi_0^{0,0}$. By Lemma 9.4 (ii), multiplication by η or $1-\eta$ preserves the classes $\Psi_\tau^{\rho,\sigma}$. This property provides a certain amount of flexibility, of which we give two examples.
(ii) If we are given a multiplier m, which satisfies the inequalities (9.4), but is only defined on an angle Γ leaving out a finite number of half-lines $\ell_{k,\pm}$ of F, we can easily extend m to a multiplier in $\Psi_\tau^{\rho,\sigma}$ which vanishes identically on the missing lines.
(iii) Property (iii) in Lemma 9.4 also applies to the situation where $s > 0$, (9.5) only holds on an angle omitting a finite number of half-lines in F, and m vanishes identically on these half-lines.

We denote by the same symbol $\Psi_\tau^{\rho,\sigma}$ the class of operators defined by the multipliers in this class. For notational convenience, we shall often use the same symbol to denote an operator $M \in \Psi_\tau^{\rho,\sigma}$ and (a convenient choice of) its multiplier $M(\lambda,\xi)$.

10. Decomposition of $L^p\Lambda^k$ and boundedness of the Riesz transforms

Since the letter p is already used to denote degrees of differential forms, the summability exponent will be denoted by r.

If $\mathcal{V}$ is any of the spaces $\mathcal{V}_0^{p,q}$, $\mathcal{V}_{1,\ell}^{p,q}$, $\Upsilon_{1,\ell}^{p,q}$, etc., by ${}^r\mathcal{V}$ we shall denote the closure of this space in $L^r\Lambda^k$. Our goal will be to prove the following theorem, whose parts (i) and (ii) extend Theorems 8.1 and 5.13.

Theorem 10.1. *Let $1 < r < \infty$.*

(i)$_{\rm k}$ *For $0 \le k \le n$, $L^r\Lambda^k$ admits the direct sum decomposition*

$$L^r\Lambda^k = \sideset{}{^\oplus}\sum_{\substack{p+q=k<n\\ p+q=n,\, pq=0}} {}^r\mathcal{V}_0^{p,q} \oplus \sideset{}{^\oplus}\sum_{\varepsilon=\pm}\ \sideset{}{^\oplus}\sum_{p+q+2\ell=k-1} {}^r\mathcal{V}_{1,\ell}^{p,q,\varepsilon} \oplus \sideset{}{^\oplus}\sum_{p+q+2\ell=k-2} {}^r\mathcal{V}_{2,\ell}^{p,q}$$
$$\oplus \sideset{}{^\oplus}\sum_{p+q=k-1} R_{k-1}{}^r\mathcal{V}_0^{p,q} \oplus \sideset{}{^\oplus}\sum_{\varepsilon=\pm}\ \sideset{}{^\oplus}\sum_{p+q+2\ell=k-2} R_{k-1}{}^r\mathcal{V}_{1,\ell}^{p,q,\varepsilon} \oplus \sideset{}{^\oplus}\sum_{p+q+2\ell=k-3} R_{k-1}{}^r\mathcal{V}_{2,\ell}^{p,q}\ , \tag{10.1}$$

where $R_{k-1} = d\Delta_{k-1}^{-\frac{1}{2}}$ is the Riesz transform;

(ii)$_{\rm k}$ *For $n+1 \le k \le 2n+1$, $L^r\Lambda^k$ admits the direct sum decomposition*

$$L^2\Lambda^k = \sideset{}{^\oplus}\sum_{\substack{r+s=k-1>n\\ r+s=n,\, rs=0}} {}^r\Upsilon_0^{r,s} \oplus \sideset{}{^\oplus}\sum_{\varepsilon=\pm}\ \sideset{}{^\oplus}\sum_{r+s-2\ell=k} {}^r\Upsilon_{1,\ell}^{r,s,\varepsilon} \oplus \sideset{}{^\oplus}\sum_{r+s-2\ell=k+1} {}^r\Upsilon_{2,\ell}^{r,s}$$
$$\oplus \sideset{}{^\oplus}\sum_{r+s=k} R^{*r}\Upsilon_0^{r,s} \oplus \sideset{}{^\oplus}\sum_{\varepsilon=\pm}\ \sideset{}{^\oplus}\sum_{r+s-2\ell=k+1} R^{*r}\Upsilon_{1,\ell}^{r,s,\varepsilon} \oplus \sideset{}{^\oplus}\sum_{r+s-2\ell=k+2} R^{*r}\Upsilon_{2,\ell}^{r,s}\ , \tag{10.2}$$

where $R^* = R^*_{k+1}$.

(iii)$_{\rm k}$ *For $0 \le k \le 2n$, the Riesz transform R_k is bounded from $L^r\Lambda^k$ to $L^r\Lambda^{k+1}$.*

By L^r-boundedness of the $*$-Hodge operator, we can restrict ourselves to the case $0 \le k \le n$.

The proof is based on the following lemma.

Lemma 10.2. *Let $U = \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix}$ denote any of the operators $U_{1,\ell}^{p,q}$ in (7.13) or $U_{2,\ell}^{p,q}$ in (7.25). Then each component U_{ij} of U consists of a multiplier operator in $\Psi_0^{0,0}$, possibly composed with powers of $e(d\theta)$ and the holomorphic and antiholomorphic Riesz transforms $\mathcal{R}$, $\bar{\mathcal{R}}$.*

In particular, for $1 < r < \infty$, all these operators are L^r-bounded on the spaces of differential forms of the appropriate (bi-)degrees.

This lemma will be proved in the last part of this section. Taking it for granted, we give the proof of the theorem.

Proof of Theorem 10.1. We prove the two parts of the theorem simultaneously, via the inductive steps $\big((\mathrm{i})_{k-1}+(\mathrm{ii})_{k-1}\big) \Longrightarrow (\mathrm{i})_k \Longrightarrow (\mathrm{ii})_k$. The statement $(\mathrm{i})_0$ is trivial, and $(\mathrm{ii})_0$ and $(\mathrm{i})_1$ are proved in [MPR1].

Assume that $(\mathrm{i})_{k-1}$ and $(\mathrm{ii})_{k-1}$ hold, and consider anyone of the orthogonal projections in the last column of Table 1. This is a product (or a sum of products) of factors, each of which can be either R_{k-1}, or its adjoint R^*_{k-1}, or $P=UU^*$, U being one of the operators in Lemma 10.2. Then $(\mathrm{i})_k$ follows easily.

We prove now the implication $(\mathrm{i})_k \Longrightarrow (\mathrm{ii})_k$. Factoring

$$R_k = d\Delta_k^{-\frac12} = R_0\Delta_0^{\frac12}\Delta_k^{-\frac12}\ ,$$

and using $(\mathrm{ii})_0$, it suffices to prove the boundedness of $\Delta_0^{\frac12}\Delta_k^{-\frac12}$ on $L^r\Lambda^k$.

Referring to the decomposition (10.1), we disregard the d-exact components of $L^r\Lambda^k$ (i.e., those with R_{k-1}), on which $R_k=0$, and adopt the simplified notation

$$(L^r\Lambda^k)_{d^*-\mathrm{cl}} = \sum_\beta{}^{\oplus}\ {}^r\mathcal{V}_\beta\ .$$

Denote by $U_\beta : {}^rZ_\beta \longrightarrow {}^r\mathcal{V}_\beta$ the L^r-closure of the unitary intertwining operator in Table 1, with ${}^rZ_\beta$ denoting the L^r-closure of the appropriate space $X^{p,q}$, $Y^{p,q}$ or $Z^{p,q}$ in (5.16). Let $P_\beta = U_\beta U_\beta^*$ be the projection of $L^r\Lambda^k$ onto ${}^r\mathcal{V}_\beta$.

Decomposing $\omega\in L^r\Lambda^k$ as

$$\omega = \sum_\beta \omega_\beta = \sum_\beta U_\beta\sigma_\beta\ ,$$

with $\sigma_\beta \in {}^rZ_\beta$, we have

$$\Delta_0^{\frac12}\Delta_k^{-\frac12}\omega = \sum_\beta \Delta_0^{\frac12}U_\beta D_\beta^{-\frac12}\sigma_\beta\ ,$$

where $D_\beta = U_\beta^*\Delta_k U_\beta$ is the scalar operator appearing in (6.1), (6.3), (6.18). Explicitely,

$$D_\beta = \begin{cases} \Delta_0 + i(q-p)T & \text{if } {}^r\mathcal{V}_\beta = {}^r\mathcal{V}_0^{p,q} \\ \Delta_0 + i(q-p)T + \ell(n-k+\ell) + m \pm \sqrt{\Delta_0 + i(q-p)T + m^2} & \\ \qquad\qquad\qquad\qquad \big(m=\frac{n-p-q}{2}\big) & \text{if } {}^r\mathcal{V}_\beta = {}^r\mathcal{V}_{1,\ell}^{p,q,\pm} \\ \Delta_0 + i(q-p)T + (\ell+1)(n-k+\ell+1) & \text{if } {}^r\mathcal{V}_\beta = {}^r\mathcal{V}_{2,\ell}^{p,q}\ . \end{cases}$$

Denote by m_β be the spectral multiplier of D_β. Then, for each of the above cases,

$$m_\beta = \begin{cases} \xi + \lambda^2 + (p-q)\lambda & \in {}^*\Psi_1^{1,0} \\ \xi + \lambda^2 + (p-q)\lambda + \ell(n-k+\ell) + m + \sqrt{\xi + \lambda^2 + (p-q)\lambda + m^2} & \in {}^*\Psi_0^{1,0} \\ \xi + \lambda^2 + (p-q)\lambda + \ell(n-k+\ell) + m - \sqrt{\xi + \lambda^2 + (p-q)\lambda + m^2} & \\ \qquad\qquad \in {}^*\Psi_1^{1,0} \text{ if } \ell = 0\ ,\ {}^*\Psi_0^{1,0} \text{ otherwise} & \\ \xi + \lambda^2 + (p-q)\lambda + (\ell+1)(n-k+\ell+1) & \in {}^*\Psi_0^{1,0}\ , \end{cases}$$

respectively. By Lemma 9.4 (iii), $D_\beta^{-\frac12}$ is in $\Psi_{-\frac12}^{-\frac12,0}$ or in $\Psi_0^{-\frac12,0}$, depending on the case. Combining together Lemma 10.2, Lemma 9.4 (ii) and (v), and the fact that the multiplier $\xi+\lambda^2$ of Δ_0 is in ${}^*\Psi_1^{1,0}$, we conclude that the composition $\Delta_0^{\frac12}U_\beta D_\beta^{-\frac12}$ has all its components in $\Psi_0^{0,0}$.

Therefore,

$$\begin{aligned}\|\Delta_0^{\frac12}\Delta_k^{-\frac12}\omega\|_r &\le \sum_\beta \|\Delta_0^{\frac12}U_\beta D_\beta^{-\frac12}\sigma_\beta\|_r\\ &\le C\sum_\beta\|\sigma_\beta\|_r\\ &\le C\|\omega\|_r\ .\end{aligned}$$

Q.E.D.

10.1. L^p- boundedness of the intertwining operators $U_{1,\ell}^\pm$.

Our next goal will be to prove

Proposition 10.3. *Assume that $p+q+1+2\ell\le n$ and $1<r<\infty$. Then there is a constant C_r so that*

$$\begin{aligned}\|U_{1,\ell}^+\xi\|_{L^r}\le C_r\|\xi\|_{L^r}\qquad &\text{for every } \xi\in W_0^{p,q},\\ \|U_{1,\ell}^-\eta\|_{L^r}\le C_r\|\eta\|_{L^r}\qquad &\text{for every } \eta\in \Xi^{p,q}.\end{aligned}$$

Proof. According to Proposition 7.8, we have to prove the L^r-boundedness of the operators:

(i) $\partial Q_-^+\Sigma_{11}$, $\bar\partial Q_+^+\Sigma_{11}$,
(ii) $(\bar\partial\partial-\partial\bar\partial)\Sigma_{11}$, when $\ell\ge1$,
(iii) $\big[\Delta_H+(2m-\ell)(\Gamma+m)\big]\Sigma_{11}$,

defined on $W_0^{p,q}$, with $p+q+2\ell+1\le n$ in (i) and (iii), and $p+q+2\ell\le n$ in (ii), and of the operators:

(i') $\partial Q_+^-\Sigma_{22}$, $\bar\partial Q_-^-\Sigma_{22}$,
(ii') $(\bar\partial\partial-\partial\bar\partial)\Sigma_{22}$, when $\ell\ge1$,
(iii') $\big[\Delta_H-(2m-\ell)(\Gamma-m)\big]\Sigma_{22}$.

defined on $\Xi^{p,q}$, with $p+q+2\ell+1\le n$ in (i') and (iii'), and $p+q+2\ell\le n$ in (ii').

Recall that if $p=0$, then $\partial=\partial(I-\mathcal{C})$, and if $q=0$, then $\bar\partial=\bar\partial(I-\bar{\mathcal{C}})$, so that, putting again

$$\Box_r=\begin{cases}\Box, & \text{if } p\ge1,\\ \Box', & \text{if } p=0,\end{cases}\qquad \overline{\Box}_r=\begin{cases}\overline{\Box}, & \text{if } q\ge1,\\ \overline{\Box}', & \text{if } q=0,\end{cases}$$

we have

$$\partial=\mathcal{R}\Box^{\frac12},\ \bar\partial=\overline{\mathcal{R}}\,\overline{\Box}^{\frac12}, \tag{10.3}$$

where $\mathcal{R}$ and $\overline{\mathcal{R}}$ are the holomorphic and antiholomorphic Riesz transforms of (4.5), which are known to be Calderón-Zygmund type singular integral operators, and consequently are L^r-bounded for $1 < r < \infty$.

Moreover, observe that $\bar\partial\partial - \partial\bar\partial = 2\bar\partial\partial + Te(d\theta)$. Since this term appears only when $\ell \geq 1$ and $p + q + 2\ell \leq n$, we have $p + q \leq n - 2$, which easily implies that the operator $\overline{\Box} + iT$ is injective on its domain in $L^2\Lambda^{p,q}$, so that we can factorize

$$\bar\partial\partial = \bar{\mathcal{R}}\overline{\Box}^{\frac{1}{2}}\partial = \bar{\mathcal{R}}\partial(\overline{\Box} + iT)^{\frac{1}{2}} = \bar{\mathcal{R}}\mathcal{R}\,(\overline{\Box} + iT)^{\frac{1}{2}}\Box^{\frac{1}{2}} \text{ on } W^{p,q}, \tag{10.4}$$

since, on the core, $\overline{\Box}\partial = \partial(\overline{\Box} + iT)$, hence $\overline{\Box}^{\frac{1}{2}}\partial = \partial(\overline{\Box} + iT)^{\frac{1}{2}}$.

Observe also that $\Xi^{p,q} = (I - C_p - \bar C_q)(W_0^{p,q})$.

Thus it will suffice to prove that the following scalar operators are in $\Psi_0^{0,0}$:

(I) $\Box^{\frac{1}{2}}Q_-^+\Sigma_{11}$, $\overline{\Box}^{\frac{1}{2}}Q_+^+\Sigma_{11}$, $\big[\Delta_H + (2m-\ell)(\Gamma + m)\big]\Sigma_{11}$, for $\ell \geq 0$;
(II) $(\overline{\Box} + iT)^{\frac{1}{2}}\Box^{\frac{1}{2}}\Sigma_{11}$, $i^{-1}T\Sigma_{11}$, for $\ell \geq 1$;

(I') $\Box^{\frac{1}{2}}Q_+^-\Sigma_{22}$, $\overline{\Box}^{\frac{1}{2}}Q_-^-\Sigma_{22}$, $\big[\Delta_H - (2m-\ell)(\Gamma - m)\big]\Sigma_{22}$, for $\ell \geq 0$;
(II') $(\overline{\Box} + iT)^{\frac{1}{2}}\Box^{\frac{1}{2}}\Sigma_{22}$, $i^{-1}T\Sigma_{22}$, for $\ell \geq 1$.

This will be a direct consequence of the following Lemmas 10.4, 10.5, 10.6, on the basis of Lemma 9.4. Q.E.D.

Observe that $m = (n-p-q)/2 \geq 1/2,\ 2m - \ell \geq 1$ in (I) and (I'), and $m \geq 1,\ 2m - \ell \geq 1$ in (II) and (II').

Lemma 10.4. *Assume that $p + q + 1 \leq n$. Then the following hold true:*

(a) $i^{-1}T \in \Psi_1^{\frac{1}{2},0}$;
(b) $\Box^{\frac{1}{2}}, \overline{\Box}^{\frac{1}{2}} \in \Psi_{\frac{1}{2}}^{0,\frac{1}{2}}$, *and* $(\overline{\Box} + iT)^{\frac{1}{2}} \in \Psi_{\frac{1}{2}}^{0,\frac{1}{2}}$*;*
(c) $\Delta_H \in \Psi_1^{0,1} \subset \Psi_1^{1,0}$, $\Delta_H - aT^2 \in \Psi_1^{1,0}$ *for every* $a \in \mathbb{C}$, *and* $(\Delta_H - T^2 + c)^\alpha \in \Psi_0^{\alpha,0}$ *for every* $c > 0$.

Proof. (a) is obvious.

As for (b), note that
$(2\Box)^{\frac{1}{2}}(\lambda,\xi) = (\xi - (n-2p)\lambda)^{\frac{1}{2}}$. We have

$$\xi - (n-2p)\lambda \sim \xi\ ,$$

on an angle containing the whole fan if $p \geq 1$, and, if $p = 0$, on an angle avoiding just the half-line $\xi = n\lambda$, $\lambda > 0$. By Lemma 9.4 (iii), $\Box^{\frac{1}{2}} \in \Psi_{\frac{1}{2}}^{0,\frac{1}{2}}$, and a similar argument applies to $\overline{\Box}^{\frac{1}{2}}$ and $(\overline{\Box} + iT)^{\frac{1}{2}}$. Remark 9.5 (iii) must be used for $\Box^{\frac{1}{2}}$ when $p = 0$ and for $(\overline{\Box} + iT)^{\frac{1}{2}}$ when $q = n - 1$.

Moreover, $(2(\overline{\Box} + iT))^{\frac{1}{2}}(\lambda,\xi) = (\xi + (n-2q-2)\lambda)^{\frac{1}{2}}$, where $p + q + 2 \leq n$, hence $2(q+1) \leq 2n-2$, i.e., $|n - 2(q+1)| \leq n - 2$. This implies that $(\overline{\Box} + iT)^{\frac{1}{2}} \in \Psi_{\frac{1}{2}}^{0,\frac{1}{2}}$.

Finally,

$$\Delta_H(\lambda,\xi) = \xi + (p-q)\lambda \sim \xi \ , \tag{10.5}$$

on an angle containing the full fan, which shows that $\Delta_H \in \Psi_1^{0,1}$. In combination with (a) and Lemma 9.4, this easily yields (c). Q.E.D.

According to (10.5), the quantity $\xi + (p-q)\lambda$, which we will also denote by $\tilde{\xi}$, is comparable to ξ. We then set

$$\Gamma(\lambda,\xi) = (\tilde{\xi} + \lambda^2 + m^2)^{\frac12} \ . \tag{10.6}$$

Let R_{11} be as in Lemma 7.6, so that, according to (7.15), $\Sigma_{11} = R_{11}^{-\frac12}$.

Lemma 10.5. *For $p+q+2\ell+1 \le n$, the following hold true:*

(a) $\Gamma + m\,,\ Q_+^+\,,\ Q_-^+ \in {}^*\Psi_0^{\frac12,0}$;

(b) $R_{11} \in {}^*\Psi_0^{1,1}$, *consequently* $\Sigma_{11} = R_{11}^{-\frac12} \in \Psi_0^{-\frac12,-\frac12}$

Proof. We have

$$\Gamma(\lambda,\xi) = (\tilde{\xi} + \lambda^2 + m^2)^{\frac12},$$

and, since $\tilde{\xi} \sim \xi$, this shows that $\Gamma \in {}^*\Psi_0^{\frac12,0}$. Then (a) follows easily.

As for R_{11}, recall that

$$\begin{aligned} R_{11} &= (\Gamma+m)^2\big(\Delta_H + 2m(2m-\ell)\big) + 2(\Gamma+m)\big((2m-\ell)\Delta_H - 2mT^2\big) \\ &\quad + \Delta_H(\Delta_H - T^2) + 2m\ell T^2 \ . \end{aligned} \tag{10.7}$$

By Lemma 9.4, and in view of what has been shown already, we find that

$$R_{11} \in \Psi_0^{\frac12,0}\Psi_0^{\frac12,0}\Psi_0^{0,1} + \Psi_0^{\frac12,0}\Psi_1^{1,0} + \Psi_1^{0,1}\Psi_1^{1,0} + \Psi_2^{1,0} \subseteq \Psi_0^{1,1}.$$

Moreover, since here $\Gamma \ge 0$, we have

$$\begin{aligned} R_{11} &\ge \Gamma^2(\Delta_H + 2m(2m-\ell)) + 2m(-2mT^2) + 2m\ell T^2 \\ &= (\tilde{\xi} + \lambda^2 + m^2)(\tilde{\xi} + 2m(2m-\ell)) + 2m(2m-\ell)|T|^2 \\ &\gtrsim (\xi + \lambda^2 + 1)(\xi + 1), \end{aligned}$$

which shows that also the estimates from below for R_{11} hold true, so that $R_{11}^{-\frac12} \in \Psi_0^{-\frac12,-\frac12}$. This concludes the proof of (b). Q.E.D.

Lemma 10.6. *For $p+q+2\ell+1 \le n$, the following hold true:*

(a) $\Gamma - m\,,\ Q_+^-\,,\ Q_-^- \in \Psi_1^{\frac12,0}$;

(b) $R_{22} \in \Psi_2^{1,1}$ *and* $\Sigma_{22}(I - C_p - \bar{C}_q) = R_{22}^{-\frac12}(I - C_p - \bar{C}_q) \in \Psi_{-1}^{-\frac12,-\frac12}$;

Proof. We have

$$\Gamma(\lambda,\xi) - m = (\tilde{\xi} + \lambda^2)\big(\Gamma(\lambda,\xi)+m\big)^{-1} \in \Psi_1^{1,0}\Psi_0^{-\frac12,0} \subset \Psi_1^{\frac12,0} .$$

By (7.11), on $W_0^{p,q}$ we have the identity

$$R_{22} = 16(\Delta_H - T^2 + m^2)\big(\Delta_H - T^2 + \ell(2m-\ell)\big)\Box\overline{\Box}R_{11}^{-1} ,$$

where

$$(\Delta_H - T^2 + m^2)\big(\Delta_H - T^2 + \ell(2m-\ell)\big)\Box\overline{\Box} \in \begin{cases} \Psi_2^{2,2} & \text{if } \ell \neq 0 \\ \Psi_3^{2,2} & \text{if } \ell = 0 \end{cases} \subset \Psi_2^{2,2} . \tag{10.8}$$

Applying Lemma 10.5 (b), we obtain that $R_{22} \in \Psi_2^{1,1}$.

To prove the last part of the statement, observe that the presence of the factor $I - C_p - \bar{C}_q$ allows us, on the basis of Remark 9.5 (ii), to restrict, if necessary, our analysis to an angle omitting one of the external half-lines of F, where the multipliers of $\Box$ and $\overline{\Box}$ are non-zero, and their reciprocal satisfy (9.5) with $\rho = 0$ and $\sigma, \tau = -1$. Each of remaining factors in (10.8) is in ${}^*\Psi_0^{1,0}$, and this, together with Lemma 10.5 (b), gives the conclusion.

Q.E.D.

10.2. L^p- **boundedness of the intertwining operators** $U_{2,\ell}$.

We next turn to the intertwining operator $U_{2,\ell}$. Our goal will be to prove

Proposition 10.7. *Assume that $p + q + 2 + 2\ell \le n$ and $1 < r < \infty$. Then there is a constant C_r so that*

$$\|U_{2,\ell}\,(\xi,\eta)\|_{L^r} \le C_r\|(\xi,\eta)\|_{L^r} \qquad \textit{for every } (\xi,\eta) \in Z^{p,q}.$$

In view of the explicit expression for $U_{2,\ell}$ in Proposition 7.12, it will suffice to prove that the operators $\frac{H_{11}}{\sqrt{\Delta'\Delta''}}$ and $\frac{H_{21}}{\sqrt{\Delta'\Delta''}}$ are L^r-bounded on $X^{p,q}$, and the operators $\frac{H_{12}}{\sqrt{\Delta'\Delta''}}$ and $\frac{H_{22}}{\sqrt{\Delta'\Delta''}}$ on $Y^{p,q}$ (notice the the multiplier $\sigma(T)$ corresponds essentially to the Hilbert transform along the center of the Heisenberg group, which is L^r-bounded).

We shall prove the estimates on $X^{p,q}$ only, since the estimates on $Y^{p,q}$ follow along the same lines.

Using again the factorizations (10.3), (10.4) by means of Riesz transforms, we see that we are reduced to estimating the following scalar operators on $X^{p,q}$ with respect to the L^r- norm:

$$\text{(III)} \qquad \frac{(\overline{\Box}+iT)^{\frac12}\overline{\Box}^{\frac12}(\Delta_H-T^2)}{\sqrt{\Delta'\Delta''}}, \quad \frac{T(\overline{\Box}+iT)^{\frac12}\overline{\Box}^{\frac12}}{\sqrt{\Delta'\Delta''}}, \quad \frac{\Box_r^{\frac12}\overline{\Box}^{\frac12}(\Delta_H-T^2)}{\sqrt{\Delta'\Delta''}},$$
$$\frac{T\Box_r^{\frac12}(\overline{\Box}+iT)^{\frac12}}{\sqrt{\Delta'}},$$
$$\text{(IV)} \qquad \frac{\overline{\Box}^{\frac12}(\Delta_H-T^2)}{\sqrt{\Delta'\Delta''}}, \quad \frac{(\overline{\Box}+i(\ell+1)T)\overline{\Box}^{\frac12}}{\sqrt{\Delta'\Delta''}}, \quad \frac{\Box_r^{\frac12}\overline{\Box}}{\sqrt{\Delta'\Delta''}}, \quad \frac{\Box_r\overline{\Box}^{\frac12}}{\sqrt{\Delta'}},$$
$$\frac{(\overline{\Box}+i(\ell+1)T)\Box_r^{\frac12}}{\sqrt{\Delta'}}\,.$$

Lemma 10.8. *Let Δ',Δ'' be as in Lemma 7.10. Then, the following properties hold:*

(a) $(\Delta'')^\alpha\in\Psi_0^{\alpha,0}$ *for every $\alpha\in\mathbb{R}$;*

(b) *if $q\geq 1$, then $(\Delta')^\alpha\in\Psi_{3\alpha}^{\alpha,2\alpha}$ for every $\alpha\in\mathbb{R}$;*

(b) *if $q\geq 1$, then $\big(\Delta'\Delta''\big)^{-\frac12}\in\Psi_{-\frac32}^{-1,-1}$.*

Proof. (a) is immediate from Lemma 10.4 (c).

As for (b), we first recall that $c>0$. Moreover, Δ' has multiplier

$$\Big[\frac12\big(\xi-(n-2p)\lambda\big)\big(\xi+(n-2q)\lambda\big)+(\ell+1)^2\lambda^2\Big]\big(\xi+(p-q)\lambda+\lambda^2\big)$$
$$+\lambda^2\Big[c\lambda^2+\sqrt{c\big(\xi-(n-2p)\lambda\big)\big(\xi+(n-2q)\lambda\big)\big(\xi+(p-q)\lambda+\lambda^2+c\big)}\,.$$

Here, $\xi=(n+2k)\lambda,\ k\in\mathbb{N}$, and $k\geq1$, if $\lambda>0$ and $p=0$, since we are acting on $X^{p,q}$. Since we are also assuming that $q\geq1$, this shows that

$$\xi-(n-2p)\lambda\sim\xi, \qquad \xi+(n-2q)\lambda\sim\xi\,. \tag{10.9}$$

By means of Lemma 10.4 and Lemma 9.4, we thus easily see that

$$\Delta'\in\Psi_1^{0,1}\Psi_1^{0,1}\Psi_1^{1,0}+\Psi_2^{1,0}\big(\Psi_2^{1,0}+\Psi_{\frac32}^{\frac12,1}\big)\subseteq\Psi_3^{1,2}+\Psi_4^{2,0}+\Psi_{\frac72}^{\frac32,1}\subseteq\Psi_3^{1,2}\,.$$

Moreover, the inverse estimate (9.5) holds true for $\rho=1,\sigma=2$ and $\tau=3$ because of (10.9), which yields (b). Finally, (c) is a direct consequence of (a) and (b). Q.E.D.

The lemmata 10.8 and 10.4 now easily imply that

$$\text{(III)}\qquad \frac{(\overline{\Box}+iT)^{\frac12}\overline{\Box}^{\frac12}(\Delta_H-T^2)}{\sqrt{\Delta'\Delta''}}\in\Psi^{0,0}_{\frac12},\quad \frac{T(\overline{\Box}+iT)^{\frac12}\overline{\Box}^{\frac12}}{\sqrt{\Delta'\Delta''}}\in\Psi^{-\frac12,0}_{\frac12},\quad \frac{\Box_r^{\frac12}\overline{\Box}^{\frac12}(\Delta_H-T^2)}{\sqrt{\Delta'\Delta''}}\in\Psi^{0,0}_{\frac12},$$
$$\frac{T\Box_r^{\frac12}(\overline{\Box}+iT)^{\frac12}}{\sqrt{\Delta'}}\in\Psi^{0,0}_{\frac12};$$
$$\text{(IV)}\qquad \frac{\overline{\Box}^{\frac12}(\Delta_H-T^2)}{\sqrt{\Delta'\Delta''}}\in\Psi_0^{0,-\frac12},\quad \frac{(\overline{\Box}+i(\ell+1)T)\overline{\Box}^{\frac12}}{\sqrt{\Delta'\Delta''}}\in\Psi_0^{-1,\frac12},\quad \frac{\Box_r^{\frac12}\overline{\Box}}{\sqrt{\Delta'\Delta''}}\in\Psi_0^{-1,\frac12},$$
$$\frac{\Box_r\overline{\Box}^{\frac12}}{\sqrt{\Delta'}}\in\Psi_0^{-\frac12,\frac12},\quad \frac{(\overline{\Box}+i(\ell+1)T)\Box_r^{\frac12}}{\sqrt{\Delta'}}\in\Psi_0^{-\frac12,\frac12}.$$

All these classes are contained in $\Psi_0^{0,0}$, so that all these operators are L^r-bounded Marcinkiewic type operators, for $1<r<\infty$. This proves Proposition 10.7 when $q\geq 1$.

The situation is slightly more complicated when $q=0$. The problem is that the second relation in (10.9) will fail to be true in this case on the ray

$$\rho := \{(\lambda,\xi) : \lambda < 0 \text{ and } \xi = n|\lambda|\} \subseteq F$$

of the Heisenberg fan, on which the multiplier of $\overline{\Box}$ will vanish identically. If we remove this ray, the preceding arguments remain valid and we get L^r- boundedness of the restrictions of our operators to the orthogonal complement of the kernel of $\overline{\Box}$, i.e., on $(I-\bar{\mathcal{C}})(X^{p,q})$. So, what remains is the restriction on $\bar{\mathcal{C}}(X^{p,q})$. This corresponds to the restrictions of our multipliers to the ray ρ. However, all of the multipliers listed in (III) and (IV) which contain a factor $\overline{\Box}$ or $\overline{\Box}^{\frac12}$ vanish identically on this ray, so what remains are the operators

$$\frac{T\Box_r^{\frac12}(\overline{\Box}+iT)^{\frac12}}{\sqrt{\Delta'}} \text{ and } \frac{(\overline{\Box}+i(\ell+1)T)\Box_r^{\frac12}}{\sqrt{\Delta'}}.$$

On the ray ρ, the multipliers of these operators are given, up to multiplicative constants, by

$$\mu_1 = \frac{\lambda^2}{\sqrt{\Big(c+(\ell+1)^2\Big)\lambda^4+(\ell+1)^2(n-p)|\lambda|^3}}$$

and

$$\mu_2 = \frac{|\lambda|^{\frac32}}{\sqrt{\Big(c+(\ell+1)^2\Big)\lambda^4+(\ell+1)^2(n-p)|\lambda|^3}}.$$

It is easy to see that these are Mihlin–Hörmander multipliers in λ, so that $\dfrac{T\Box_r^{\frac12}(\overline{\Box}+iT)^{\frac12}}{\sqrt{\Delta'}}\bar{\mathcal{C}}$ and $\dfrac{(\overline{\Box}+i(\ell+1)T)\Box_r^{\frac12}}{\sqrt{\Delta'}}\bar{\mathcal{C}}$ are compositions of Calderón-Zygmund operators acting in the central variable of the Heisenberg group with the singular integral operator $\bar{\mathcal{C}}$, which shows that they are L^r-bounded, for $1 < r < \infty$, too.

This completes the proof of Proposition 10.7.

Finally, let us denote by $\rho^\pm$ the rays

$$\rho^\pm := \{(\lambda,\xi) : \xi = \pm\lambda, \lambda > 0 \subseteq F$$

of the Heisenberg fan F, and define for $(\lambda,\xi) \in F$ the spaces

$$X^{p,q}(\lambda,\xi) := \begin{cases} \{0\}, & \text{if } p = 0 \text{ and } (\lambda,\xi) \in \rho^+, \\ \mathbb{C}, & \text{if } p = 0 \text{ and } (\lambda,\xi) \notin \rho^+, \text{ or if } p > 0\,, \end{cases}$$

$$Y^{p,q}(\lambda,\xi) := \begin{cases} \{0\}, & \text{if } q = 0 \text{ and } (\lambda,\xi) \in \rho^-, \\ \mathbb{C}, & \text{if } q = 0 \text{ and } (\lambda,\xi) \notin \rho^-, \text{ or if } q > 0\,, \end{cases}$$

and

$$Z^{p,q}(\lambda,\xi) = \Big\{ \begin{pmatrix} \mu \\ \nu \end{pmatrix} : \mu \in X^{p,q}(\lambda,\xi), \nu \in Y^{p,q}(\lambda,\xi) \Big\}.$$

Lemma 10.9. *For $(\lambda,\xi) \in F$ let $E(\lambda,\xi) = \begin{pmatrix} E_{11}(\lambda,\xi) & E_{12}(\lambda,\xi) \\ E_{21}(\lambda,\xi) & E_{22}(\lambda,\xi) \end{pmatrix}$, where the E_{ij} are given in Lemma 7.10. Then, when viewed as a linear mapping from the space $Z^{p,q}(\lambda,\xi)$ into itself, $E(\lambda,\xi)$ is invertible for almost every (λ,ξ) with respect to the Plancherel measure on F.*

Proof. When $(\lambda,\xi) \in F \setminus (\rho^+ \cup \rho^-)$, then $\square(\lambda,\xi) \neq 0, \overline{\square}(\lambda,\xi) \neq 0$, and since, according to (7.23), $\det E = cT^4 \square\overline{\square}\Delta''$, the claim is immediate.

Assume next that $(\lambda,\xi) \in \rho^+$. Then, if $p > 0$, we can argue as before. So, assume that $p = 0$. In this case, $\square(\lambda,\xi) = 0$, $Z^{p,q}(\lambda,\xi) = \Big\{ \begin{pmatrix} 0 \\ \nu \end{pmatrix} : \nu \in \mathbb{C} \Big\}$, and $E(\lambda,\xi) = \begin{pmatrix} 0 & 0 \\ 0 & E_{22}(\lambda,\xi) \end{pmatrix}$, where $E_{22}(\lambda,\xi) = -i(\ell+1)(n-q)\lambda(\lambda - n + \ell + 2)$. Since $p + q + 2\ell + 2 \leq n$, the factor $(n-q)$ is non-zero, and the claim follows.

Finally, the case where $(\lambda,\xi) \in \rho^-$ can be dealt with in a very similar way. Q.E.D.

11. Applications

11.1. Multipliers of Δ_k.

We are in a position now to extend Theorem 6.8 of [MPR1] to forms of any degree. A function μ definied on the positive half-line is a Mihlin–Hörmander multiplier of class $\rho > 0$ if, given a smooth function χ supported on $[\frac{1}{2}, 4]$ and equal to 1 on $[1,2]$,

$$\|\mu\|_{\rho,\mathrm{sloc}} := \sup_{t>0} \big\|\mu(t\cdot)\chi\big\|_{L^2_\rho} < \infty\ .$$

Theorem 11.1. *Let $m : \mathbb{R} \to \mathbb{C}$ be a bounded, continuous function in $L^2_{\rho,\mathrm{sloc}}(\mathbb{R})$ for some $\rho > (2n+1)/2$. Then, for every $k = 0,\dots,2n+1$, the operator $m(\Delta_k)$ is bounded on $L^p(H_n)\Lambda^k$ for $1 < p < \infty$, with norm controlled by $\|m\|_{\rho,\mathrm{sloc}}$.*

The proof follows the same lines as in [MPR1].

11.2. **Exact L^p-forms.**

As a corollary to Theorem 10.1 and its proof, we can derive the following extension of Lemma 4.2 in [MPR1].

Lemma 11.2. *Let r be such that $1/2 - 1/r = 1/(2n+2)$. If $\omega \in L^2\Lambda^k$ is such that $\omega = du$ in the distributional sense for some $u \in \mathcal{D}'\Lambda^{k-1}$, then there is some $v \in L^r\Lambda^{k-1}$ such that $\omega = dv$ in the sense of distributions. Moreover, $\omega \in R_{k-1}(L^2\Lambda^{k-1})$.*

Proof. Define

$$v := L^{-\frac{1}{2}}(L^{\frac{1}{2}}\Delta_{k-1}^{-\frac{1}{2}})R_{k-1}^*\omega.$$

We have seen that the operator $\Delta_0^{\frac{1}{2}}\Delta_{k-1}^{-\frac{1}{2}}$ is L^p-bounded for $1 < p < \infty$, which implies that the same is true for $L^{\frac{1}{2}}\Delta_{k-1}^{-\frac{1}{2}} = (L^{\frac{1}{2}}\Delta_0^{-\frac{1}{2}})(\Delta_0^{\frac{1}{2}}\Delta_{k-1}^{-\frac{1}{2}})$. As in the proof of Lemma 4.2 in [MPR1], we can thus conclude that $v \in L^r\Lambda^{k-1}$. And, if $\xi \in \mathcal{S}\Lambda^k$, then

$$\langle dv, \xi\rangle = \langle \omega, R_{k-1}(\Delta_{k-1}^{-\frac{1}{2}}L^{\frac{1}{2}})L^{-\frac{1}{2}}d^*\xi\rangle = \langle \omega, R_{k-1}R_{k-1}^*\xi\rangle = \langle R_{k-1}R_{k-1}^*\omega, \xi\rangle,$$

so that $dv = R_{k-1}R_{k-1}^*\omega \in L^2\Lambda^k$. By Lemma 4.4, this implies that

$$\omega = dv + R_k^*R_k\omega,$$

and by the same lemma $R_k\omega = \Delta_{k+1}^{-\frac{1}{2}}d\omega$, where $d\omega = d^2u = 0$ in the sense of distributions. This implies that $\omega = dv$, and thus also that $\omega \in R_{k-1}(L^2\Lambda^{k-1})$. Q.E.D.

Corollary 11.3. *If $\omega \in L^2\Lambda^k$, then $\omega \in R_{k-1}(L^2\Lambda^{k-1})$ if and only if there is some $u \in \mathcal{D}'\Lambda^{k-1}$ such that $\omega = du$ in the sense of distributions.*

Proof. One implication is immediate by Lemma 11.2. To prove the converse implication, let us assume that $\omega \in R_{k-1}(L^2\Lambda^{k-1})$. Then, according to Lemma 4.4 and Proposition 4.5, $\omega = R_{k-1}R_{k-1}^*\omega$. Moreover, if we define v as in the proof of Lemma 11.2, then $v \in L^r\Lambda^{k-1}$ and $dv = R_{k-1}R_{k-1}^*\omega$, hence $dv = \omega$. We may thus choose $u = v$. Q.E.D.

11.3. **The Dirac operator.**

Let us denote by $\Lambda = \sum_{k=0}^{2n+1}{}^{\oplus}\,\Lambda^k$ the Grassmann algebra of $\mathfrak{h}_n^*$, and by $L^p\Lambda = L^p(H_n)\Lambda = \sum_{k=0}^{2n+1}{}^{\oplus}\,L^p\Lambda^k, \mathcal{S}\Lambda$ etc. the space of L^p-section, $\mathcal{S}$-sections etc. of the corresponding bundle over H_n.

The *Dirac operator* acting on $\mathcal{S}\Lambda$ is given by

$$D := d + d^* \ . \tag{11.1}$$

Notice that $D^2 = \Delta$ on dom$\,(\Delta)$, that is, the Dirac operator D and the Hodge Laplacian Δ commute as differential operators.

However, in order to reduce the spectral theory of D to that one of Δ we need to show that D and Δ *strongly commute*, in the sense that the all spectral projections in the spectral decompositions of D and Δ commute.

Proposition 11.4. *We have that $\overline{D}^2 = \Delta$. In particular, D and Δ strongly commute.*

Proof. Recall from the previous section that the Riesz transform $R = d\Delta^{-\frac{1}{2}}$ and its adjoint $R^* = \Delta^{-\frac{1}{2}}d^* = d^*\Delta^{-\frac{1}{2}}$ are L^p-bounded for $1 < p < \infty$. Let us put

$$P_\pm := \frac{1}{\sqrt{2}}\Big(I \pm D\Delta^{-\frac{1}{2}}\Big) = \frac{1}{\sqrt{2}}\Big(I \pm (R + R^*)\Big).$$

One easily verifies that $P_\pm^2 = P_\pm$ and $P_\pm^* = P_\pm$, so that P_+ and P_- are orthogonal projections, which are in fact L^p-bounded for $1 < p < \infty$. Moreover,

$$DP_\pm = \pm\Delta^{\frac{1}{2}}P_\pm, \tag{11.2}$$

i.e.,

$$D = \Delta^{\frac{1}{2}}P_+ - \Delta^{\frac{1}{2}}P_- \ .$$

Let $\Delta = \int_0^{+\infty} \lambda\, dE(\lambda)$, so that

$$\Delta^{\frac{1}{2}} = \int_0^{+\infty} \sqrt{\lambda}\, dE(\lambda) = \int_0^{+\infty} s\, d\tilde{E}(s) \ ,$$

where $\tilde{E}$ denotes the image of the spectral measure E under the mapping $\lambda \mapsto \sqrt{\lambda}$. Therefore,

$$D = \Delta^{\frac{1}{2}}P_+ - \Delta^{\frac{1}{2}}P_- \quad = \int_0^{+\infty} s\, d(\tilde{E}P_+)(s) - \int_0^{+\infty} s\, d(\tilde{E}_s P_-)(s)$$

Now, if $A \subset \mathbb{R}$ is a Borel set, let us put

$$F(A) := \tilde{E}(A_+)P_+ + \tilde{E}(-A_-)P_- \ , \tag{11.3}$$

where $A_+ := A \cap [0, +\infty)$ and $A_- := A \cap (-\infty, 0)$. Then F is a spectral measure on $\mathbb{R}$, and

$$D = \int_{-\infty}^{+\infty} s\, dF(s) \qquad \text{on } \ \mathcal{S}\Lambda \ .$$

Indeed, notice that, since the operators R, R^* are bounded and commute with Δ on the core, they also commute with the spectral projections $\tilde{E}(B)$, i.e.,

$$\tilde{E}(B)P_\pm = P_\pm\tilde{E}(B) \ . \tag{11.4}$$

Moreover, we clearly have

$$F(A) = F(A_+) + F(A_-) \ ,$$

and $P_+P_- = P_-P_+ = 0$. This implies that $F(A)$ is an orthogonal projection, and that F is a spectral measure on $\mathbb{R}$.

We set

$$\tilde{D} := \int_{-\infty}^{+\infty} s\, dF(s) \ , \tag{11.5}$$

as a closed operator, and claim that indeed $\tilde{D} = \overline{D}$.

To verify this, denote by D_0 the restriction of D to $\mathcal{S}_0\Lambda$. Since $D = \tilde{D}$ on $\mathcal{S}\Lambda$, we then have $\overline{D_0} \subset \overline{D} \subset \tilde{D}$, and clearly

$$\tilde{D}^2 = \int_{-\infty}^{+\infty} s^2 \, dF(s) = \int_0^{+\infty} \lambda \, dE(\lambda) = \Delta \ .$$

Thus, it remains to show that $\tilde{D} \subset \overline{D_0}$.

Let $\xi \in \operatorname{dom} \tilde{D}$. It suffices to assume that there is an interval $I = [a, b]$, with $0 < a < b$, so that $F(\mathbb{R} \setminus K)\xi = 0$, where $K = I \cup (-I)$; hence $\tilde{D}\xi = \int_K s \, dF(s)\xi$.

Let φ be a smooth cut-off function, even, identically 1 on K and with support contained in $K' = I' \cup (-I')$, where $I' = [a', b']$ with $0 < a' < a < b < b'$. Then,

$$\Delta\xi = \tilde{D}^2\xi = \int_K s^2 \, dF(s)\xi = \int_K s^2\varphi(s) \, dF(s)\xi = \int_0^\infty \lambda\psi(\lambda) \, dE(\lambda)\xi \ ,$$

where ψ is a smooth function with compact support in ${K'}^2 = \{s^2 : s \in K'\}$.

Hence, if $Q := \psi(\Delta) = \int_0^\infty \psi(\lambda) dE(\lambda)$, we have that $\Delta\xi = \Delta Q\xi$. It is clear that Q is given by right-convolution with a Schwartz function, and that Q is $U(n)$-equivariant, hence it preserves the core $\mathcal{S}_0\Lambda$ for Δ.

Choose a sequence $\{\xi_n\} \subset \mathcal{S}_0\Lambda$ such that $\xi_n \to \xi$ and $\Delta\xi_n \to \Delta\xi$.

Then $\{Q\xi_n\} \subset \mathcal{S}_0$, $Q\xi_n \to Q\xi$ and $\Delta Q\xi_n \to \Delta\xi$. Therefore, we may assume that $\xi_n = Q\xi_n$ and $\xi = Q\xi$. Then

$$\begin{aligned}
\Delta^{\frac{1}{2}}\xi = \Delta^{\frac{1}{2}}Q\xi &= \int_0^\infty \lambda^{\frac{1}{2}}\psi(\lambda) \, dE(\lambda)\xi \\
&= \lim_{n\to+\infty} \int_0^\infty \lambda^{\frac{1}{2}}\psi(\lambda) \, dE(\lambda)\xi_n \\
&= \lim_{n\to+\infty} \int_0^\infty s\varphi(s) \, dF(s)\xi_n \ .
\end{aligned}$$

Hence, by (11.3) it follows that

$$\tilde{D}\xi = \lim_{n\to+\infty} \tilde{D}\xi_n = \lim_{n\to+\infty} D_0\xi_n \ .$$

This implies that $\mathcal{S}_0\Lambda$ is a core also for $\tilde{D}$; hence $\tilde{D} = \overline{D_0}$. We have thus seen that

$$\overline{D} = \int_{-\infty}^{+\infty} s \, dF(s) \ .$$

By (11.3) and (11.4) $F(A)$ and $E(B)$ commute; hence $\overline{D}$ and Δ strongly commute. Q.E.D.

Moreover, if m is a bounded, Borel measurable spectral multiplier of $\mathbb{R}$, then

$$m(D) = m(\Delta^{\frac{1}{2}})P_+ + m(-\Delta^{\frac{1}{2}})P_-. \tag{11.6}$$

As an immediate consequence of Theorem 11.1 and Theorem 10.1 we therefore obtain

Corollary 11.5. *Let $m : \mathbb{R} \to \mathbb{C}$ be a bounded, continuous function in $L^2_{\rho,\mathrm{sloc}}(\mathbb{R})$ for some $\rho > (2n+1)/2$. Then $m(D)$ is bounded on $L^p(H_n)\Lambda$ for $1 < p < \infty$, with norm controlled by $\|m\|_{\rho,\mathrm{sloc}}$.*

12. Appendix

In this final section we collect some technical facts and proofs that we have previously set aside.

We need some preliminary computations.

Recall first from Lemma 5.5 that

$$[\partial^*, e(d\theta)^\ell] = i\ell\bar\partial e(d\theta)^{\ell-1} , \qquad [\bar\partial^*, e(d\theta)^\ell] = -i\ell\partial e(d\theta)^{\ell-1} . \tag{12.1}$$

Taking adjoints, this implies

$$[\partial, i(d\theta)^\ell] = i\ell\bar\partial^* i(d\theta)^{\ell-1} , \qquad [\bar\partial, i(d\theta)^\ell] = -i\ell\partial^* i(d\theta)^{\ell-1} . \tag{12.2}$$

Lemma 12.1. *If $\sigma \in \ker i(d\theta) \subset \Lambda^s_H$ and $s + 2j \leq n$, then*

$$i(d\theta)^j e(d\theta)^j \sigma = c_{s,j}\sigma ,$$

where the coefficients $c_{s,j}$ are defined in (7.8), *i.e.,*

$$c_{s,j} = \frac{j!(n-s)!}{(n-s-j)!} .$$

Moreover, the following relations hold:

$$j^2 c_{s+1,j-1} = c_{s,j} - c_{s+1,j}, \qquad c_{s,j}(n-s-j) = c_{s+1,j}(n-s), \qquad jc_{s+1,j-1}(n-s-j) = c_{s+1,j} .$$

Proof. Use formula (2.8) in [MPR1] to compute $i(d\theta)e(d\theta)^j\sigma$. Observe that ω_j there corresponds to our σ and the value $k = p + q$ there is our $s + 2j$. Therefore,

$$i(d\theta)e(d\theta)^j\sigma = j(n-s-j+1)e(d\theta)^{j-1}\sigma .$$

Consequentely,

$$i(d\theta)^j e(d\theta)^j\sigma = j(n-s-j+1)i(d\theta)^{j-1}e(d\theta)^{j-1}\sigma ,$$

and the statement follows inductively. Q.E.D.

If $\xi \in W^s_0$, then $\xi, \partial\xi, \bar\partial\xi$ are in $\ker i(d\theta)$, and consequently we see that

$$i(d\theta)^j e(d\theta)^j \xi = c_{s+1,j}\xi , \quad i(d\theta)^j e(d\theta)^j \partial\xi = c_{s+1,j}\partial\xi , \quad i(d\theta)^j e(d\theta)^j \bar\partial\xi = c_{s+1,j}\bar\partial\xi . \tag{12.3}$$

This is not necessarily the case for $\partial\bar\partial\xi$, $\bar\partial\partial\xi$. We therefore need some more computations to simplify the expressions

$$\partial^*\bar\partial^* i(d\theta)^\ell e(d\theta)^\ell \bar\partial\partial , \ i(d\theta)^\ell e(d\theta)^{\ell-1}\bar\partial\partial , \quad \text{etc. .}$$

Lemma 12.2. *For $\xi \in W^s_0$,*

$$\begin{aligned}
\partial^*\bar\partial^* i(d\theta)^j e(d\theta)^j \bar\partial\partial\xi &= c_{s+1,j}\big(\overline{\Box} + i(j+1)T\big)\Box\xi , \\
\bar\partial^*\partial^* i(d\theta)^j e(d\theta)^j \partial\bar\partial\xi &= c_{s+1,j}\big(\Box - i(j+1)T\big)\overline{\Box}\xi , \\
\partial^*\bar\partial^* i(d\theta)^j e(d\theta)^j \partial\bar\partial\xi = \bar\partial^*\partial^* i(d\theta)^j e(d\theta)^j \bar\partial\partial\xi &= -c_{s+1,j}\overline{\Box}\Box\xi .
\end{aligned}$$

Proof. We have, by (12.2)

$$\begin{aligned}\partial^*\bar\partial^* i(d\theta)^j e(d\theta)^j \bar\partial\partial\xi &= \partial^*\bar\partial^*[i(d\theta)^j,\bar\partial]e(d\theta)^j\partial\xi + \partial^*\bar\partial^*\bar\partial i(d\theta)^j e(d\theta)^j\partial\xi\\ &= ij\partial^*\bar\partial^*\partial^* i(d\theta)^{j-1}e(d\theta)^j\partial\xi + c_{s+1,j}\partial^*\bar\partial^*\bar\partial\partial\xi\\ &= ijT\partial^* i(d\theta)^j e(d\theta)^j\partial\xi + c_{s+1,j}\partial^*\bar\partial^*\bar\partial\partial\xi\\ &= c_{s+1,j}\partial^*(\overline{\Box}+ijT)\partial\xi\\ &= c_{s+1,j}\big(\overline{\Box}+i(j+1)T\big)\Box\xi\ .\end{aligned}$$

The second identity follows in a similar way. Finally, we have

$$\begin{aligned}\partial^*\bar\partial^* i(d\theta)^j e(d\theta)^j \partial\bar\partial\xi &= \partial^*\bar\partial^*[i(d\theta)^j,\partial]e(d\theta)^j\bar\partial\xi + \partial^*\bar\partial^*\partial i(d\theta)^j e(d\theta)^j\bar\partial\xi\\ &= c_{s+1,j}\partial^*\bar\partial^*\partial\bar\partial\xi\\ &= -c_{s+1,j}\partial^*\partial\bar\partial^*\bar\partial\xi\\ &= -c_{s+1,j}\overline{\Box}\Box\xi\ .\end{aligned}$$

This proves the lemma. Q.E.D.

Lemma 12.3. *The following properties hold true.*

(i) *For all p,q, $\Box W_0^{p,q}=X^{p,q}$ and $\overline{\Box}W_0^{p,q}=Y^{p,q}$.*

(ii) *In particular, $\Box$ maps $X^{p,q}$ into itself, and $\overline{\Box}$ maps $Y^{p,q}$ into itself. We therefore set*

$$\Box_X=\Box_{|_{X^{p,q}}}:X^{p,q}\to X^{p,q}\,,\qquad \overline{\Box}_Y=\overline{\Box}_{|_{Y^{p,q}}}:Y^{p,q}\to Y^{p,q}\,. \tag{12.4}$$

Then $\Box_X^{-1}$ and $\overline{\Box}_Y^{-1}$ are well defined on $X^{p,q}$ and $Y^{p,q}$, respectively.

Proof. (ii) is immediate from (i). To prove (i), we verify the statements concerning the spaces $X^{p,q}$, the discussion of the spaces $Y^{p,q}$ being similar. Observe first that $\Box$ leaves $W_0^{p,q}$ invariant because of (1.20).

Thus, in view of Remark 5.10, if $1\le p\le n-1$, then $X^{p,q}=W_0^{p,q}$, and we are done. And, if $p=0$ and $\xi\in W_0^{0,q}$, then $\Box\xi=\partial^*\partial\xi$, so that by Lemma 4.8 $C_0\Box\xi=0$, hence $\Box\xi\in X^{p,q}$. Finally, if $p=n$ and $\xi\in W_0^{n,q}$, then $\Box\xi=\partial\partial^*\xi=0$. Q.E.D.

Proof of Lemma 7.5. Define $B_{1,\ell}$ to be the unbounded operator from $L^2\Lambda^{p,q}$ to $(L^2\Lambda^k)^2$ defined by the matrix on the right-hand side of (7.6), with core $\mathcal{S}_0\Lambda^{p,q}$, and set

$$B_{1,\ell}^*B_{1,\ell}{}_{|_{(W_0^{p,q})^2}}=:B=\begin{pmatrix}B_{11}&B_{12}\\B_{21}&B_{22}\end{pmatrix}\ . \tag{12.5}$$

We will use Lemma 7.4 to see that

$$A_{1,\ell}^*A_{1,\ell}=B_{1,\ell}^*B_{1,\ell}{}_{|_{(W_0^{p,q})^2}}\ .$$

Then, the matrix entries of B are:

$$
\begin{aligned}
B_{11} &= \partial^* i(d\theta)^\ell e(d\theta)^\ell \partial \\
&\qquad + \Big(i\ell T^{-1}\partial^*\bar\partial^* i(d\theta)^{\ell-1} - T^{-1}\Box i(d\theta)^\ell \Big)\Big(i\ell T^{-1} e(d\theta)^{\ell-1}\bar\partial\partial + T^{-1}e(d\theta)^\ell \Box \Big) \\
B_{22} &= \bar\partial^* i(d\theta)^\ell e(d\theta)^\ell \bar\partial \\
&\qquad + \Big(-i\ell T^{-1}\bar\partial^*\partial^* i(d\theta)^{\ell-1} - T^{-1}\overline{\Box} i(d\theta)^\ell \Big)\Big(-i\ell T^{-1} e(d\theta)^{\ell-1}\partial\bar\partial + T^{-1}e(d\theta)^\ell \overline{\Box} \Big) \\
B_{12} &= B_{21}^* = \partial^* i(d\theta)^\ell e(d\theta)^\ell \bar\partial \\
&\qquad - \Big(i\ell T^{-1}\partial^*\bar\partial^* i(d\theta)^{\ell-1} - T^{-1}\Box i(d\theta)^\ell \Big)\Big(i\ell T^{-1} e(d\theta)^{\ell-1}\partial\bar\partial - T^{-1}e(d\theta)^\ell \overline{\Box} \Big) \,.
\end{aligned}
$$

By (12.3) and Lemma 12.2 we have that

$$
\begin{aligned}
B_{11} &= c_{s+1,\ell}\Box + T^{-2}\Big[-\ell^2\partial^*\bar\partial^* i(d\theta)^{\ell-1}e(d\theta)^{\ell-1}\bar\partial\partial + i\ell\partial^*\bar\partial^* i(d\theta)^{\ell-1}e(d\theta)^\ell\Box \\
&\qquad - i\ell\Box i(d\theta)^\ell e(d\theta)^{\ell-1}\bar\partial\partial - \Box i(d\theta)^\ell e(d\theta)^\ell \Box\Big] \\
&= c_{s+1,\ell}\Box + T^{-2}\Big[-\ell^2 c_{s+1,\ell-1}(\overline{\Box}+i\ell T)\Box \\
&\qquad - i\ell T^{-1}\partial^*\bar\partial^* i(d\theta)^{\ell-1}e(d\theta)^{\ell-1}(\partial\bar\partial+\bar\partial\partial)\Box \\
&\qquad - i\ell T^{-1}\Box(\partial^*\bar\partial^*+\bar\partial^*\partial^*)i(d\theta)^{\ell-1}e(d\theta)^{\ell-1}\bar\partial\partial - c_{s,\ell}\Box^2\Big] \,.
\end{aligned}
\tag{12.6}
$$

Now notice that by Lemma 12.2

$$
\begin{aligned}
\partial^*\bar\partial^* i(d\theta)^{\ell-1}e(d\theta)^{\ell-1}(\partial\bar\partial+\bar\partial\partial) &= -c_{s+1,\ell-1}\overline{\Box}\Box + c_{s+1,\ell-1}(\overline{\Box}+i\ell T)\Box \\
&= i\ell c_{s+1,\ell-1}T\Box \,,
\end{aligned}
$$

and, by taking adjoints,

$$
(\partial^*\bar\partial^*+\bar\partial^*\partial^*)i(d\theta)^{\ell-1}e(d\theta)^{\ell-1}\bar\partial\partial = i\ell c_{s+1,\ell-1}T\Box \,.
$$

Therefore, substituting into (12.6) and applying the identities from Lemma 12.1, we obtain that

$$
\begin{aligned}
B_{11} &= c_{s+1,\ell}\Box + T^{-2}\Big[-\ell^2 c_{s+1,\ell-1}(\overline{\Box}+i\ell T)\Box + 2\ell^2 c_{s+1,\ell-1}\Box^2 - c_{s,\ell}\Box^2\Big] \\
&= T^{-2}\Box\Big[c_{s+1,\ell}T^2 + \ell^2 c_{s+1,\ell-1}(\Box-\overline{\Box}-i\ell T) + (-c_{s,\ell}+\ell^2 c_{s-1,\ell-1})\Box\Big] \\
&= T^{-2}\Box\Big[c_{s+1,\ell}T^2 + i\ell^2 c_{s+1,\ell-1}(n-s-\ell)T - c_{s+1,\ell}\Box\Big] \\
&= c_{s+1,\ell}T^{-2}\Box\Big[T^2 + i\ell T - \Box\Big] \,.
\end{aligned}
$$

By conjugation, we also get

$$
B_{22} = c_{s+1,\ell}T^{-2}\overline{\Box}\Big[T^2 - i\ell T - \overline{\Box}\Big] \,.
$$

Finally, arguing in a similar way, we find that

$$\begin{aligned}
B_{12} &= T^{-2}\Big[\ell^2\partial^*\bar\partial^* i(d\theta)^{\ell-1}e(d\theta)^{\ell-1}\partial\bar\partial - c_{s,\ell}\Box\overline{\Box} + i\ell\partial^*\bar\partial^* i(d\theta)^{\ell-1}e(d\theta)^{\ell}\overline{\Box} \\
&\qquad + i\ell\Box i(d\theta)^{\ell}e(d\theta)^{\ell-1}\partial\bar\partial\Big] \\
&= T^{-2}\Big[-(\ell^2 c_{s+1,\ell-1} + c_{s,\ell})\Box\overline{\Box} - i\ell T^{-1}\partial^*\bar\partial^* i(d\theta)^{\ell-1}e(d\theta)^{\ell-1}(\partial\bar\partial + \bar\partial\partial)\overline{\Box} \\
&\qquad\quad + i\ell T^{-1}\Box(\partial^*\bar\partial^* + \bar\partial^*\partial^*)i(d\theta)^{\ell-1}e(d\theta)^{\ell-1}\partial\bar\partial\Big] \\
&= T^{-2}\Box\overline{\Box}\Big[-\ell^2 c_{s+1,\ell-1} - c_{s,\ell} + i\ell c_{s+1,\ell-1}T^{-1}\overline{\Box} - i\ell c_{s+1,\ell-1}T^{-1}(\overline{\Box} + i\ell T) \\
&\qquad - i\ell c_{s+1,\ell-1}T^{-1}\Box + i\ell c_{s+1,\ell-1}T^{-1}(\Box - i\ell T)\Big] \\
&= T^{-2}\Box\overline{\Box}\Big[-\ell^2 c_{s+1,\ell-1} - c_{s,\ell} + 2\ell^2 c_{s+1,\ell-1}\Big] \\
&= -c_{s+1,\ell}T^{-2}\Box\overline{\Box}\ .
\end{aligned}$$

These computations show that

$$B = B^*_{1,\ell}B_{1,\ell}|_{(W_0^{p,q})^2} = -c_{s+1,\ell}T^{-2}\begin{pmatrix}\Box(\Box - i\ell T - T^2) & \Box\overline{\Box} \\ \Box\overline{\Box} & \overline{\Box}(\overline{\Box} + i\ell T - T^2)\end{pmatrix}\ .$$

It is obvious that B maps $(W_0^{p,q})^2$ into itself, so that, by Lemma 7.4, $A^*_{1,\ell}A_{1,\ell} = B^*_{1,\ell}B_{1,\ell}|_{(W_0^{p,q})^2}$. This proves the lemma. Q.E.D.

Completion of the proof of Lemma 7.6. Since Q maps the subspace $(W_0^{p,q})^2$ into itself, we have that

$$\begin{aligned}
R &= \begin{pmatrix}-Q^+_- & Q^+_+ \\ -Q^-_+ & Q^-_-\end{pmatrix}\begin{pmatrix}\Box(\Box - i\ell T - T^2) & \Box\overline{\Box} \\ \Box\overline{\Box} & \overline{\Box}(\overline{\Box} + i\ell T - T^2)\end{pmatrix}\begin{pmatrix}-Q^+_- & -Q^-_+ \\ Q^+_+ & Q^-_-\end{pmatrix} \\
&= \begin{pmatrix}R_{11} & R_{12} \\ R_{21} & R_{22}\end{pmatrix}\ .
\end{aligned}$$

We compute first R_{11}. Using the identity $\square - \overline{\square} = 2imT$ and (6.9), we have

$$
\begin{aligned}
-T^2 R_{11} &= (Q^*NQ)_{11} \\
&= \big(-Q^+_-\square(\square - i\ell T - T^2) + Q^+_+\square\overline{\square}\big)(-Q^+_-) + \big(-Q^+_-\square\overline{\square} + Q^+_+\overline{\square}(\overline{\square} + i\ell T - T^2)\big)Q^+_+ \\
&= (Q^+_-)^2\square(\square - i\ell T - T^2) - 2Q^+_+Q^+_-\square\overline{\square} + (Q^+_+)^2\overline{\square}(\overline{\square} + i\ell T - T^2) \\
&= (\Gamma + m + iT)^2\square(\square - i\ell T - T^2) - 2(\Gamma + m - iT)(\Gamma + m + iT)\square\overline{\square} \\
&\qquad + (\Gamma + m - iT)^2\overline{\square}(\overline{\square} + i\ell T - T^2) \\
&= (\Gamma + m)^2\Big[\square(\square - i\ell T - T^2) - 2\square\overline{\square} + \overline{\square}(\overline{\square} + i\ell T - T^2)\Big] \\
&\qquad + 2iT(\Gamma + m)\Big[\square(\square - i\ell T - T^2) - \overline{\square}(\overline{\square} + i\ell T - T^2)\Big] \\
&\qquad - T^2\Big[\square(\square - i\ell T - T^2) + 2\square\overline{\square} + \overline{\square}(\overline{\square} + i\ell T - T^2)\Big] \\
&= (\Gamma + m)^2\Big[(\square - \overline{\square})^2 - i\ell T(\square - \overline{\square}) - T^2(\square + \overline{\square})\Big] \\
&\qquad + 2iT(\Gamma + m)\Big[(\square + \overline{\square})(\square - \overline{\square}) - i\ell T(\square + \overline{\square}) - T^2(\square - \overline{\square})\Big] \\
&\qquad - T^2\Big[(\square + \overline{\square})^2 - i\ell T(\square - \overline{\square}) - T^2(\square + \overline{\square})\Big] \\
&= -T^2\Big[(\Gamma + m)^2\big(\Delta_H + 2m(2m - \ell)\big) + 2(\Gamma + m)\big((2m - \ell)\Delta_H - 2mT^2\big) \\
&\qquad + \big(\Delta_H(\Delta_H - T^2) + 2m\ell T^2\big)\Big]\ .
\end{aligned}
$$

In the same way one finds that

$$
\begin{aligned}
-T^2 R_{22} &= (Q^*NQ)_{22} \\
&= -T^2\Big[(\Gamma - m)^2\big(\Delta_H + 2m(2m - \ell)\big) - 2(\Gamma - m)\big((2m - \ell)\Delta_H - 2mT^2\big) \\
&\qquad + \big(\Delta_H(\Delta_H - T^2) + 2m\ell T^2\big)\Big]\ .
\end{aligned}
$$

Finally, using the formulas in (6.9) we see that

$$
\begin{aligned}
-T^2 R_{12} = -T^2 R_{21} &= (Q^*NQ)_{12} \\
&= \big(-Q^+_-\square(\square - i\ell T - T^2) + Q^+_+\square\overline{\square}\big)(-Q^-_+) \\
&\qquad + \big(-Q^+_-\square\overline{\square} + Q^+_+\overline{\square}(\overline{\square} + i\ell T - T^2)\big)Q^-_- \\
&= Q^+_-Q^-_+\square(\square - i\ell T - T^2) - Q^+_+Q^-_+\square\overline{\square} - Q^+_-Q^-_-\square\overline{\square} \\
&\qquad + Q^+_+Q^-_-\overline{\square}(\overline{\square} + i\ell T - T^2) \\
&= 2\square\overline{\square}(\square - i\ell T - T^2) - (\Delta_H - 2T^2 - 2iT\Gamma)\square\overline{\square} \\
&\qquad - (\Delta_H - 2T^2 + 2iT\Gamma)\square\overline{\square} + 2\square\overline{\square}(\overline{\square} + i\ell T - T^2) \\
&= 0\ .
\end{aligned}
$$

Q.E.D.

Proof of Lemma 7.10 (i). In order to compute $A^*_{2,\ell}A_{2,\ell}$ we apply again Lemma 7.4. We first define $B_{2,\ell}$ as the term on the right hand side of (7.19) acting as an unbounded operator from $\big(L^2\Lambda^{p,q}\big)^2$ to $\big(L^2\Lambda^{p,q}\big)^2$, with core $\big(\mathcal{S}_0\Lambda^{p,q}\big)^2$, and then we compute $B := B^*_{2,\ell}B_{2,\ell}{}_{|_{(W_0^{p,q})^2}}$. If we then show that B maps $Z^{p,q}$ into itself, the equality $A^*_{2,\ell}A_{2,\ell} = B_{|_{Z^{p,q}}}$, which in turn equals $-c_{s+1,\ell}T^{-2}E$, will follow.

We have that

$$
\begin{aligned}
B^*_{2,\ell}B_{2,\ell} = &\begin{pmatrix} \partial^*\bar\partial^* & -T^{-1}\big[\partial^*(\overline{\Box}+i\ell T)-\bar\partial^*(\Box+iT)\big] \\ \bar\partial^*\partial^* & -T^{-1}\big[\bar\partial^*(\Box-i\ell T)-\partial^*(\overline{\Box}-iT)\big]\end{pmatrix} \\
&\times i(d\theta)^\ell e(d\theta)^\ell \begin{pmatrix} \bar\partial\partial & \partial\bar\partial \\ T^{-1}\big[(\overline{\Box}+i\ell T)\partial-(\Box+iT)\bar\partial\big] & T^{-1}\big[(\Box-i\ell T)\bar\partial-(\overline{\Box}-iT)\partial\big]\end{pmatrix} .
\end{aligned}
$$

Using Lemmas 12.2 and 12.1, and recalling that we are acting on elements in $W_0^{p,q}$, we see that the matrix entries of $B^*_{2,\ell}B_{2,\ell}$ are

$$
\begin{aligned}
B_{11} &= \partial^*\bar\partial^* i(d\theta)^\ell e(d\theta)^\ell \bar\partial\partial \\
&\qquad - T^{-2}\big[\partial^*(\overline{\Box}+i\ell T)-\bar\partial^*(\Box+iT)\big] i(d\theta)^\ell e(d\theta)^\ell \big[(\overline{\Box}+i\ell T)\partial-(\Box+iT)\bar\partial\big] \\
&= c_{s+1,\ell}\big[\overline{\Box}+i(\ell+1)T\big]\Box \\
&\qquad - c_{s+1,\ell}T^{-2}\big[\partial^*(\overline{\Box}+i\ell T)-\bar\partial^*(\Box+iT)\big]\big[(\overline{\Box}+i\ell T)\partial-(\Box+iT)\bar\partial\big] \\
&= c_{s+1,\ell}\big[\overline{\Box}+i(\ell+1)T\big]\Box \\
&\qquad - c_{s+1,\ell}T^{-2}\big[(\overline{\Box}+i(\ell+1)T)\partial^*-\Box\bar\partial^*\big]\big[(\overline{\Box}+i\ell T)\partial-(\Box+iT)\bar\partial\big] \\
&= c_{s+1,\ell}T^{-2}\Big[\big[\overline{\Box}+i(\ell+1)T\big]\Box T^2-(\overline{\Box}+i(\ell+1)T)^2\Box-\Box^2\overline{\Box}\Big] \\
&= c_{s+1,\ell}T^{-2}\Box\Big[\big[\overline{\Box}+i(\ell+1)T\big]T^2-\big(\overline{\Box}+i(\ell+1)T\big)^2-\Box\overline{\Box}\Big] .
\end{aligned}
$$

From this it follows that

$$
\begin{aligned}
E_{11} &= -\Box\Big[\big(\overline{\Box}+i(\ell+1)T\big)T^2-\big(\overline{\Box}+i(\ell+1)T\big)^2-\Box\overline{\Box}\Big] \\
&= \Box\overline{\Box}(\Delta_H-T^2)+i(\ell+1)T\Box\big[2\overline{\Box}-T^2+i(\ell+1)T\big] \\
&= \Box\overline{\Box}(\Delta_H-T^2)+i(\ell+1)T\Box\big[\Delta_H-T^2-i(n-s-\ell-1)T\big] ,
\end{aligned}
$$

where we have used the the equality (1.19).

Thus, the statement for E_{11} follows. The term E_{22} is its complex conjugate and thus it follows as well.

Finally, we compute E_{12}, and hence E_{21} too. We have that

$$
\begin{aligned}
B_{12} &= \partial^*\bar\partial^* i(d\theta)^\ell e(d\theta)^\ell \partial\bar\partial \\
&\qquad - T^{-2}\Big[\partial^*(\overline{\Box}+i\ell T)-\bar\partial^*(\Box+iT)\Big] i(d\theta)^\ell e(d\theta)^\ell \Big[(\Box-i\ell T)\bar\partial-(\overline{\Box}-iT)\partial\Big] .
\end{aligned}
$$

Therefore, using Lemmas 12.2 and 12.1,

$$\begin{aligned}
B_{12} &= -c_{s+1,\ell}\Box\overline{\Box} + c_{s+1,\ell}T^{-2}\Big[(\overline{\Box}+i(\ell+1)T)\partial^*(\overline{\Box}-iT)\partial + \Box\partial^*(\Box - i\ell T)\bar\partial\Big] \\
&= -c_{s+1,\ell}\Box\overline{\Box} + c_{s+1,\ell}T^{-2}\Big[\big(\overline{\Box}+i(\ell+1)T\big)\Box\overline{\Box} + \big(\Box - i(\ell+1)T\big)\Box\overline{\Box}\Big] \\
&= c_{s+1,\ell}T^{-2}\Box\overline{\Box}(\Delta_H - T^2)\ .
\end{aligned}$$

Recalling that $B_{12} = -c_{s+1,\ell}T^{-2}E_{12}$, the assertion follows.

It is now easy to check that B maps $Z^{p,q}$ into itself. For, suppose that $\xi \in X^{p,q}$ and $\eta \in Y^{p,q}$, and put $\sigma = B_{11}\xi + B_{12}\eta$. Observe that both B_{11} and B_{22} factor as $B_{1j} = \Box D_{1j},\ j = 1,2$, where D_{1j} leaves $W_0^{p,q}$ invariant. Therefore Lemma 12.3 shows that $\sigma \in X^{p,q}$. In a similar way, one shows that $B_{21}\xi + B_{22}\eta \in Y^{p,q,}$ which concludes the proof. Q.E.D.

To compute the square root of a matrix, we shall make use of the following formula, which is an application of Cayley-Hamilton's theorem: For a positive definite 2×2 matrix A we have that

$$A^{\frac12} = \frac{A + \sqrt{\det A}\, I}{\sqrt{\operatorname{tr} A + 2\sqrt{\det A}}}\ . \tag{12.7}$$

Proof of Lemma 7.10 (ii). We have

$$(A^*_{2,\ell}A_{2,\ell})^{\frac12} = \frac{\sqrt{c_{s+1,\ell}}}{|T|}E^{\frac12}, \quad (A^*_{2,\ell}A_{2,\ell})^{-\frac12} = \frac{|T|}{\sqrt{c_{s+1,\ell}}}E^{-\frac12},$$

where the matrix E is as in Lemma 7.10.

Then, recalling that we set $c = (\ell+1)(n-s-\ell-1)$,

$$\begin{aligned}
\operatorname{tr} E &= 2\Box\overline{\Box}(\Delta_H - T^2) + i(\ell+1)T(\Box - \overline{\Box})(\Delta_H - T^2) + cT^2\Delta_H \\
&= 2\Box\overline{\Box}(\Delta_H - T^2) - (\ell+1)(n-s)T^2(\Delta_H - T^2) + cT^2\Delta_H \\
&= \big[2\Box\overline{\Box} - (\ell+1)^2T^2\big](\Delta_H - T^2) + cT^4\ .
\end{aligned}$$

Moreover, using (1.19) and recalling that the operators are acting on s-forms, we have

$$
\begin{aligned}
\det E = \Box\overline{\Box}(\Delta_H - T^2)\Big[& i(\ell+1)T\Box(\Delta_H - T^2 - i(n-s-\ell-1)T) \\
& - i(\ell+1)T\overline{\Box}(\Delta_H - T^2 + i(n-s-\ell-1)T)\Big] \\
& + (\ell+1)^2T^2\Box\overline{\Box}\Big[(\Delta_H - T^2)^2 + (n-s-\ell-1)^2T^2\Big] \\
= \Box\overline{\Box}(\Delta_H - T^2)\Big[& i(\ell+1)T(\Delta_H - T^2)(\Box - \overline{\Box}) + cT^2\Delta_H\Big] \\
& + (\ell+1)^2T^2\Box\overline{\Box}\Big[(\Delta_H - T^2)^2 + (n-s-\ell-1)^2T^2\Big] \\
= \Box\overline{\Box}(\Delta_H - T^2)T^2\Big[& -(\ell+1)(n-s)(\Delta_H - T^2) + c\Delta_H\Big] \\
& + (\ell+1)^2T^2\Box\overline{\Box}\Big[(\Delta_H - T^2)^2 + (n-s-\ell-1)^2T^2\Big] \\
= c\Box\overline{\Box}T^4\Big[& \Delta_H - T^2 + c\Big] \\
= c\Box\overline{\Box}T^4 & \Delta'' \,.
\end{aligned}
$$

This implies in particular that

$$
\begin{aligned}
\Delta' &:= \operatorname{tr} E + 2\sqrt{\det E} \\
&= \big[2\Box\overline{\Box} - (\ell+1)^2T^2\big](\Delta_H - T^2) + cT^4 + 2\sqrt{c\Box\overline{\Box}T^4\Delta''} \\
&= \big[2\Box\overline{\Box} - (\ell+1)^2T^2\big](\Delta_H - T^2) - T^2\Big(-cT^2 + 2\sqrt{c\Box\overline{\Box}\Delta''}\Big) \,.
\end{aligned}
$$

The formula for $(A_{2,\ell}^* A_{2,\ell})^{\frac12}$ in Lemma 7.10 (ii) is now immediate.

And, if we write

$$
E^{-1} = \frac{1}{\det E} E^{(\mathrm{co})}
$$

then we have

$$
\begin{aligned}
(A_{2,\ell}^* A_{2,\ell})^{-\frac12} &= \frac{|T|}{\sqrt{c_{s+1,\ell}}\sqrt{\det E}} E^{(\mathrm{co})\,\frac12} \\
&= \frac{|T|}{\sqrt{c_{s+1,\ell}}\sqrt{\det E}} \frac{1}{\sqrt{\operatorname{tr} E^{(\mathrm{co})} + 2\sqrt{\det E^{(\mathrm{co})}}}}\big(E^{(\mathrm{co})} + \sqrt{\det E^{(\mathrm{co})}} I\big) \\
&= \frac{|T|}{\sqrt{c_{s+1,\ell}}\sqrt{\det E}} \frac{1}{\sqrt{\operatorname{tr} E + 2\sqrt{\det E}}}\big(E^{(\mathrm{co})} + \sqrt{\det E} I\big) \\
&= \frac{1}{\sqrt{c_{s+1,\ell}}|T|\sqrt{c\Box\overline{\Box}\Delta''}} \frac{1}{\sqrt{\Delta'}}\big(E^{(\mathrm{co})} + \sqrt{\det E} I\big) \,.
\end{aligned}
$$

The result for $(A^*_{2,\ell}A_{2,\ell})^{-\frac{1}{2}}$ now follows easily, since

$$\begin{aligned}
&E^{(\mathrm{co})} + \sqrt{\det E}\, I \\
&= \square\overline{\square}(\Delta_H - T^2)\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \\
&\qquad + \begin{pmatrix} \overline{\square}\big[-i(\ell+1)T\big(\Delta_H - T^2\big) - cT^2\big] & 0 \\ 0 & \square\big[i(\ell+1)T\big(\Delta_H - T^2\big) - cT^2\big] \end{pmatrix} \\
&\qquad\quad - T^2\sqrt{c\square\overline{\square}\Delta''}\, I \\
&= \square\overline{\square}(\Delta_H - T^2)\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + \begin{pmatrix} M_{11} & \\ & M_{22} \end{pmatrix} ,
\end{aligned}$$

where M_{11} and M_{22} are as claimed in (7.21). Q.E.D.

References

[A] G. Alexopoulos, L^p Fourier multipliers on Riemannian symmetric spaces of the noncompact type, *Proc. Amer. math. Soc.*, **120** (1994), 973-979.

[An] J.-Ph. Anker, Spectral multipliers on Lie groups of polynomial growth, *Ann. Math.*, **132** (1990), 597-628.

[AnLoh] J.-Ph. Anker, N. Lohoué, Multiplicateurs sur certains espaces symétriques, *Amer. J. Math.*, **108** (1986), 1303-1353.

[ACDH] P. Auscher, T. Coulhon, X. Duong, S. Hofmann, Riesz transforms on manifolds and heat kernel regularity, *Ann. Sci. Ec. Norm. Sup. Paris*, **37** (2004), 911-957.

[Ch] M. Christ, L^p bounds for spectral multipliers on nilpotent groups, *Trans. Amer. Math. Soc.*, **328** (1991), 73-81.

[ChM] M. Christ ans D. Müller, On L^p spectral multipliers for a solvable Lie group, *Geom. Funct. Anal.*, **6** (1996), 860-876.

[ClS] J.-L. Clerc, E. M. Stein, L^p multipliers for noncompact symmetric spaces, *Proc. Nat. Acad. Sci. USA* **71** (1974), 3911-3912.

[CD] Th. Coulhon, X. T. Duong, Riesz transform and related inequalities on noncompact Riemannian manifolds, *Comm. Pure Appl. Math.*, **56** (2003), 1728-1751.

[CMZ] Th. Coulhon, D. Müller, J. Zienkiewicz, About Riesz transforms on the Heisenberg groups, *Math. Ann.*, **305** (1996), 369-379.

[CoKSi] M. Cowling, O. Klima, and A. Sikora, Spectral multipliers for the Kohn sublaplacian on the sphere in $\mathbb{C}^n$, *Trans. Amer. Math. Soc.*, **363** (2011), 611-631.

[F] G. B. Folland, *Harmonic Analysis in Phase Space*, Ann. Math. Studies 122, Princeton Univ. Press, Princeton 1989.

[FS] G. B. Folland, E. M. Stein, Estimates for the $\bar\partial_b$ complex and analysis on the Heisenberg group, *Comm. Pure Appl. Math.* **27** (1974), 429-522.

[GSj] G. Gaudry, P. Sjögren, Singular integrals on the complex affine group, *Coll. Math.* **75** (1998), 133-148.

[He] W. Hebisch, Multiplier theorem on generalized Heisenberg groups, *Colloq. Math.* **65** (1993), 231-239.

[HeZ] W. Hebish, J. Zienkiewicz, Multiplier theorems on generalized Heisenberg groups, II, *Coll. Math.* **69** (1995), 29-36.

[Hu] A. Hulanicki, A functional calculus for Rockland operators on nilpotent Lie groups, *Studia Math.* **78** (1984), 253-266.

[Ka] T. Kato, *Perturbation theory for linear operators*, Springer-Verlag New York, Inc., New York 1966.

[Li] Xiang-Dong Li, On the strong L^p-Hodge decomposition over complete Riemannian manifolds, *J. Funct. Anal.* **257** (2009), 3617-3646.

[Loh] Lohoué N., Estimation des projecteurs de De Rham-Hodge de certaines variétés riemanniennes non compactes, unpublished manuscript, 1984.

[Loh2] ———, Transformées de Riesz et fonctions sommables, *Amer. J. Math.* **114** (1992), 875-922.

[LohMu] Lohoué N., S. Mustapha, Sur les transformées de Riesz sur les groupes de Lie moyennables et sur certains espaces homogènes, *Can. J. Math.* **50** (1998), 1090-1104.

[Lot] J. Lott, Heat kernels on covering spaces and topological invariants, *J. Diff. Geom.*, **35** (1992), 471-510.

[LuM] J. Ludwig, D. Müller, Sub-Laplacians of holomorphic L^p-type on rank one AN -groups and related solvable groups, *J. Funct. Anal.*, **170** (2000), 366-427.

[LuMS] J. Ludwig, D. Müller, S. Souaifi, Holomorphic-type for sub-Laplacians on connected Lie groups, *J. Funct. Anal.*, **255** (2008), 1297-1338.

[Lü] W. Lück, *L^2-Invariants: Theory and Applications to Geometry and K-Theory*, Springer-Verlag, Berlin 2002.

[Mar] A. Martini, Analysis of joint spectral multipliers on Lie groups of polynomial growth, arXiv:1010.1186v2.

[MauMe] G. Mauceri, S. Meda, Vector-valued multipliers on stratified groups, *Rev. Math. Iberoam.* **6** (1990), 141-154.

[MPR1] D. Müller, M. M. Peloso, F. Ricci, L^p-spectral multipliers for the Hodge Laplacian acting on 1-forms on the Heisenberg group, *Geom. Funct. Anal.* **17** (2007), 852-886.

[MPR2] ———, Eigenvalues of the Hodge Laplacian on a quotient of the Heisenberg group, *Collect. Math.* Vol Extra (2006), 327-342.
[MRS1] D. Müller, F. Ricci, E. M. Stein, Marcinkiewicz multipliers and multi-parameter structure on Heisenberg(-type) groups, I, *Inv. Math.* **119** (1995), 199-233.
[MRS2] ———, Marcinkiewicz multipliers and multi-parameter structure on Heisenberg(-type) groups, II, *Math. Z.*, **221** (1996), 267-291.
[MS] D. Müller, E. M. Stein, On spectral multipliers for Heisenberg and related groups, *J. Math. Pures Appl.* **73** (1994), 413440.
[Ra] M. Range, *Holomorphic functions and integral representations in several complex variables*, Graduate Texts in Mathematics, Springer-Verlag, Berlin 1986.
[RS] M. Reed, B. Simon, *Methods of Modern Mathematical Physics, I, Functional Analysis*, Acad. Press, New York 1980.
[Ru1] M. Rumin, Formes différentielles sur les variétés de contact, *J. Diff. Geom.*, **39** (1994), 281-330.
[R2] ———, Sub-Riemannian limit of the differential form spectrum of contact manifolds, *GAFA*, **10** (2000), 407-452.
[Si] A. Sikora, Riesz transform, Gaussian bounds and the method of wave equation, *J. Funct. Anal.* **52** (1983), 48-79.
[St] R. Strichartz, Analysis of the Laplacian on the complete Riemannian manifold, *Math. Z.* **247** (2004), 643662.
[Ta] M. Taylor, L^p-estimates on functions of the Laplace operator, *Duke Math. J.* **58** (1989), 773-793.
[Th] S. Thangavelu, *Harmonic analysis on the Heisenberg group*, Progress in Mathematics, Birkhäuser, Berlin 1998.
[W] A. Weil, *Introduction à l'étude des variétés kählériennes*, Hermann, 1958.

Mathematisches Seminar, Christian-Albrechts-Universität zu Kiel, Ludewig-Meyn-Strasse 4, D-24098 Kiel, Germany
E-mail address: mueller@math.uni-kiel.de
URL: http://analysis.math.uni-kiel.de/mueller/

Dipartimento di Matematica, Università degli Studi di Milano, Via C. Saldini 50, 20133 Milano, Italy
E-mail address: marco.peloso@unimi.it
URL: http://users.mat.unimi.it/~peloso/

Scuola Normale Superiore, Piazza dei Cavalieri 7, 56126 Pisa, Italy
E-mail address: fricci@sns.it
URL: http://www.math.sns.it/HomePages/Ricci/