Constraints for three-dimensional higher-spin interactions and conformal correlators

Stefan Fredenhagen[1,2,*], Olaf Krüger,[1,†] and Karapet Mkrtchyan[3,‡]

[1]*University of Vienna, Faculty of Physics, Boltzmanngasse 5, 1090 Vienna, Austria*
[2]*Erwin Schrödinger International Institute for Mathematics and Physics,*
Boltzmanngasse 9, 1090 Vienna, Austria
[3]*Max Planck Institute for Gravitational Physics (Albert Einstein Institute),*
Am Mühlenberg 1, 14476 Potsdam, Germany

In the context of higher-spin holography, we compare the classification of cubic interaction vertices for higher-spin gravity theories in three dimensions to the possible three-point correlation functions of conserved higher-spin currents in two-dimensional conformal field theories. In both cases, the allowed structures are governed by triangle inequalities for the involved spins. It is established that higher-order correlators satisfy similar polygon inequalities and that the same inequalities are valid for higher-order continuations of cubic vertices in the three-dimensional higher-spin gravity.

I. INTRODUCTION

Higher-spin (HS) gravity theories [1–3] (for reviews, see, e.g., Refs. [4–9]) provide interesting extensions of gravity for which the holographic duality can in principle be studied perturbatively in contrast to the string theoretic AdS/CFT duality. They are therefore a promising class of gravitational theories for which one can hope to understand the AdS/CFT correspondence on the quantum level and to learn about quantum gravity from the quantum field theory on the boundary. Higher-spin holography (for a review, see Ref. [10]) is to a large extent controlled by an enormous symmetry: the higher-spin gauge symmetry in the bulk, and the corresponding higher-spin global symmetry on the boundary [11]. This symmetry becomes particularly powerful when the boundary theory is two dimensional and the asymptotic symmetry algebra of a higher-spin gauge theory on asymptotically AdS_3 spacetimes becomes an infinite-dimensional $\mathcal{W}$-algebra [12,13] (see also Refs. [14–17]). Studying conformal field theories (CFTs) with such $\mathcal{W}$-algebras as chiral symmetry algebras has led to a concrete proposal for a holographic duality between Prokushkin-Vasiliev higher-spin gauge theory [2]

and limits of minimal model CFTs [18] (for a review, see Ref. [19]).

In all concrete proposals for higher-spin dualities, one of the problems encountered is that there is no standard action known for the higher-spin bulk theory, which makes quantization and thus a quantum analysis of holography difficult. An attempt to construct such an action perturbatively is the Noether-Fronsdal program, which starts out from the free, quadratic Fronsdal action [20] of higher-spin gauge fields and adds interactions order by order while preserving gauge invariance. Successful implementation of this program may provide a classification of all possible higher-spin gauge theories which admit a perturbative formulation. The current status of the Noether-Fronsdal program in arbitrary dimensions is reviewed in Refs. [21,22].

In three dimensions, a Chern-Simons formulation [23] is available for HS gravity without matter (see also Refs. [13,24,25]), allowing an alternative derivation of the action in metric variables. This approach was taken in Refs. [26,27] to directly obtain an action in terms of Fronsdal fields for Chern-Simons spin-3 theories [13].

In three dimensions, the classification of gauge-invariant cubic interaction vertices has recently been completed [21,22]. The purpose of this paper is to establish that this classification matches the structures of three-point functions of conserved currents in a two-dimensional CFT. In that sense, rather than checking the AdS/CFT correspondence in specific examples, we study the structure of general higher-spin theories and general CFTs. We thus extend the known match between cubic bulk vertices and boundary correlators in (boundary) dimensions three and higher [28–36] to two dimensions.

[*]stefan.fredenhagen@univie.ac.at
[†]olaf.krueger@univie.ac.at
[‡]karapet.mkrtchyan@aei.mpg.de

Before we summarize our main results, we briefly want to explain why the analysis in two dimensions is different and not just more or less a corollary of the higher-dimensional case. On the bulk side, the analysis is complicated by the appearance of many dimension-dependent identities (Schouten identities) that are special to three (bulk) dimensions. On the boundary, a particular feature of two-dimensional CFTs is that the conserved currents are chiral or antichiral fields. In higher dimensions, the product of Lorentz-covariant conserved traceless currents

$$\hat{J}_{\mu_1 \ldots \mu_n}(x) = J_{(\mu_1 \ldots \mu_k}(x)\tilde{J}_{\mu_{k+1} \ldots \mu_n)}(x) - (\text{traces}) \quad (1)$$

does not define a conserved current, but in two dimensions the product of two chiral currents is again chiral and hence conserved. Therefore, we have a large freedom of nonlinear redefinitions of the basis fields which can change the correlators. Contrary to higher dimensions, we therefore do not just have to analyze the possible structure of correlation functions from conformal invariance (which is trivial in two dimensions), but we have to analyze what the independent structures are modulo nonlinear changes of the basis.

The analysis of Refs. [21,22] has shown that for three-dimensional higher-spin gauge fields with spins s_1, s_2, $s_3 \geq 2$, there is precisely one parity-even and one parity-odd cubic vertex whenever the spins satisfy triangle inequalities:

$$|s_1 - s_2| < s_3 < s_1 + s_2, \quad (2)$$

and none otherwise. The main results of this paper are as follows:

(1) In any $\mathcal{W}$-algebra, there is a unique basis of quasi-primary fields for which the r-point correlators vanish unless the involved spins $s_1, \ldots, s_r$ satisfy polygon inequalities:

$$s_i < \sum_{j=1, j \neq i}^{r} s_j - r + 3. \quad (3)$$

The name of these inequalities comes from a geometric interpretation: they are equivalent to a statement that r line segments with lengths $s_i - 1$ can form a polygon on a plane. In particular, in this basis the three-point functions obey the same triangle inequalities as the higher-spin cubic vertices in three dimensions.

(2) All higher-order bulk vertices that depend on the cubic vertices, in the sense that they are needed to complete the cubic vertices to a consistent theory, obey the same polygon inequalities (examples of higher-order couplings that are completions of cubic ones are the quartic vertex of Yang-Mills theory and all higher-order vertices in Einstein gravity: they are needed for the consistency of the cubic deformation,

but do not introduce new parameters and necessarily vanish when the coefficient of the cubic coupling goes to zero).

We thus find a perfect match of structures in the bulk and the boundary at this level. Note that at this point we have not implemented the consistency of the gauge algebra in the bulk or the associativity of operator products on the boundary. These will give additional restrictions on both sides of the duality.

The outline of the paper is as follows: We start in Sec. II with a review of the classification of cubic higher-spin vertices in three dimensions. We then give a short review on the structure of correlation functions in two-dimensional CFTs in Sec. III. In the subsequent Sec. IV, we then describe in detail how one can construct a basis for which the correlation functions vanish unless the involved spins satisfy polygon inequalities. This basis necessarily consists of primary fields (except possibly a quasiprimary spin-2 field), and it is uniquely fixed by demanding that the polygon inequalities be obeyed. Motivated by the CFT results, we then show in Sec. V that all dependent higher-order bulk vertices satisfy the same polygon inequalities. We end with a brief discussion.

II. CUBIC COUPLINGS IN THREE DIMENSIONS: REVIEW

The Noether-Fronsdal program is a systematic procedure to construct possible covariant actions for higher-spin gravity theories order by order. In this section, we review the recent results [21,22] on cubic couplings in a three-dimensional theory.

The starting point is the free quadratic Lagrangian:

$$\mathcal{L}^{(2)} = \sum_{k=1}^{N} \mathcal{L}_{s_k}^{(2)}[\varphi^{(k)}], \quad (4)$$

where k is a label counting the fields $\varphi^{(k)}$, and $s_k \geq 1$ is the spin of the kth field. For $s \geq 2$, the $\mathcal{L}_s^{(2)}$ is the spin-s Fronsdal Lagrangian [20], second order in derivatives. For vector fields, we have two different possibilities for the free action: The Maxwell action, which is a particular case of the Fronsdal action for $s = 1$, and an (Abelian) Chern-Simons action of the form $\int A \wedge dA$. The former is equivalent to a scalar, while the latter carries no bulk degrees of freedom (d.o.f.). In a general theory, there may be both types of fields in the spectrum. Here we consider only Chern-Simons vector fields, that similarly to HS fields carry no bulk d.o.f.

A spin $s \geq 1$ field is described by a rank-s symmetric Lorentz tensor $\varphi_{\mu_1 \ldots \mu_s}$. The number N of all fields involved can be arbitrary, or even infinite, as it is in the case of Prokushkin-Vasiliev theory—a theory of HS gravity coupled to matter [2]. We will sketch the idea of the

Noether-Fronsdal program here and briefly discuss what is known so far.

The main assumption is that the theory is given by a local Lagrangian, nonlinear in the fields $\varphi^{(k)}$, which can be constructed order by order in a small parameter g. The latter is usually associated with an overall factor in front of the cubic Lagrangian:

$$\mathcal{L} = \mathcal{L}^{(2)} + g\mathcal{L}^{(3)} + g^2\mathcal{L}^{(4)} + \cdots. \qquad (5)$$

This assumption may be too strong and leave out interesting theories with HS fields. In particular, it is not clear if Prokushkin-Vasiliev theory [2] admits a reformulation of the form (5) in terms of Fronsdal fields $\varphi_{\mu_1\ldots\mu_s}$. It is sufficient for our purposes here to restrict ourselves to strictly local Lagrangian functionals, since our conclusions concern the gauge sector of the three-dimensional HS gravity theories excluding propagating matter.

The Lagrangian (5) should be gauge invariant with respect to gauge transformations of all fields up to total derivative terms. The gauge transformations can be expanded in powers of the fields:

$$\delta\varphi^{(k)} = \delta^0\varphi^{(k)} + g\delta^1\varphi^{(k)} + g^2\delta^2\varphi^{(k)} + \cdots, \qquad (6)$$

where $\delta^n\varphi^{(k)}$ is given as a sum of different terms, linear in one of the gauge parameters and of order n in the number of fields. The zeroth-order gauge transformation is that of a free field given by (we use symmetrization with weight 1)

$$\delta^0\varphi^{(k)}_{\mu_1\ldots\mu_{s_k}} = s_k\nabla_{(\mu_1}\epsilon^{(k)}_{\mu_2\ldots\mu_{s_k})}. \qquad (7)$$

In the following, we will use the shorthand notation $\epsilon^{(k)}$ for the parameter associated with the field $\varphi^{(k)}$ with spin $s_k \geq 1$. The higher-order transformation $\delta^n\varphi^{(k)}$ $(n > 0)$ of a field of spin s_k can contain gauge parameters of all the other fields. The simplest examples of such theories are Yang-Mills theory and gravity with or without matter. There, the full structure of higher-order interactions is fixed by the cubic ones. We assume this is true also for the example under consideration: in the space of all possible nonlinear theories, we will restrict ourselves to those for which the full interaction structure is fixed uniquely by the cubic-order interaction. We will study independent higher-order interactions elsewhere.

The higher-order consistency of the gauge symmetry severely constrains the cubic couplings. We assume that there will remain at least one free parameter. In particular, since we are discussing a theory of gravity, we can assume that the coefficient of the Einstein-Hilbert cubic vertex is nonzero. Hence, we can take its coefficient, the Newton constant, as the parameter g of the expansion, while rescaling all the other coupling constants for cubic vertices

by $g_{k,k',k''} \to gg_{k,k',k''}$ to have an overall factor g in front of the cubic Lagrangian.

The nth-order Lagrangian can be expanded in the basis of vertices of interactions between fields $\varphi^{(k_1)}, \ldots, \varphi^{(k_n)}$:

$$\mathcal{L}^{(n)} = \sum_{k_i,m} g^m_{k_1,\ldots,k_n} V^m_{k_1,\ldots,k_n}(\varphi^{(k_1)}, \ldots, \varphi^{(k_n)}), \qquad (8)$$

where m counts the independent coefficients. Here, the vertices are by convention symmetric in the field labels, i.e., an n-point vertex for n pairwise different fields appears $n!$ times in the sum.

Gauge invariance of the theory given by the Lagrangian (5) under the transformation (6) has to hold order by order. In particular, at cubic order in the fields (first order in the expansion parameter), we get the cubic Noether equations

$$\delta^1\mathcal{L}^{(2)} + \delta^0\mathcal{L}^{(3)} = 0 \text{ up to total derivatives}$$
$$\Rightarrow \delta^0\mathcal{L}^{(3)} \approx 0. \qquad (9)$$

Here, the last equality is up to terms proportional to the free equations of motion and up to total derivatives. This can be understood by considering the contribution of the variation of the free Lagrangian, which is

$$\delta^1\mathcal{L}^{(2)} = \sum_k \delta^1\varphi^{(k)}\mathcal{F}(\varphi^{(k)}) + (\text{total derivatives}) \approx 0, \qquad (10)$$

where $\mathcal{F}(\varphi^{(k)})$ is the free equation of motion for the field $\varphi^{(k)}$.

In the Noether procedure, we often have to consider equivalence classes with respect to $\approx$, and they enter the analysis of cubic vertices in two distinct ways: First, they appear when we consider gauge invariance, as we have just seen. Second, one can add to the cubic vertex any total derivative without changing the action, and any local terms proportional to the free field equations by appropriate quadratic field redefinitions: $\varphi \to \varphi + gO(\varphi^2)$. Given that the free field equations satisfy the lowest-order gauge invariance, $\delta^0\mathcal{F}(\varphi) = 0$, the cubic terms induced by these field redefinitions automatically solve Eq. (9). Fixing this redundancy is essential for parametrizing the independent nontrivial cubic deformations.

Inversely, fixing the freedom to add terms proportional to the free equations of motion to the cubic vertex also fixes the freedom of field redefinitions at quadratic order in the fields. Higher-order field redefinitions are, however, still allowed: at each order $n \geq 2$ in the field variables, $O(\varphi^n)$, we have the freedom to redefine $\varphi \to \varphi + g^{n-1}O(\varphi^n)$ and, correspondingly, the freedom to add terms to the $(n + 1)$th-order Lagrangian $\mathcal{L}^{(n+1)}$ that are proportional to the free field equations.

At cubic order, there is a natural way to fix the freedom of field redefinitions in a Lorentz covariant manner by

ruling out all the terms in the traceless-transverse part of the cubic Lagrangian, where derivatives are contracted to each other. Details can be found in Ref. [21].

Fixing this field redefinition freedom for any triple of fields, we are left with two independent deformations—a parity-even vertex and a parity-odd one—if the spins of the fields satisfy the triangle inequalities

$$|s_{k_2} - s_{k_3}| < s_{k_1} < s_{k_2} + s_{k_3}, \tag{11}$$

and none otherwise [21,22]. Exceptions are the cases with scalar and Maxwell fields, where one has current-type couplings.

Having found the cubic vertices, one can then go back to the Noether equations (9) for the gauge symmetry at cubic order:

$$\delta^1 \mathcal{L}^{(2)} + \delta^0 \mathcal{L}^{(3)} = \text{total derivative.} \tag{12}$$

Explicitly, they can be written as

$$\sum_{k,k',k''} \left(3 \sum_m g^m_{k,k',k''} V^m_{k,k',k''}(\delta^0 \varphi^{(k)}, \varphi^{(k')}, \varphi^{(k'')}) \right.$$
$$\left. + \delta^1_{k,k'} \varphi^{(k'')} \mathcal{F}(\varphi^{(k'')}) \right) = \text{total derivative,} \tag{13}$$

where $\delta^1_{k,k'} \varphi^{(k'')}$ is the gauge variation of the field $\varphi^{(k'')}$, with a parameter $\epsilon^{(k)}$ and linear in the field $\varphi^{(k')}$ (we do not need the explicit form here). The contribution containing a specific gauge parameter $\epsilon^{(k)}$ and two fields with fixed labels k' and k'' has to vanish separately; therefore, we have

$$\delta^1_{k,k'} \varphi^{(k'')} \mathcal{F}(\varphi^{(k'')}) + \delta^1_{k,k''} \varphi^{(k')} \mathcal{F}(\varphi^{(k')})$$
$$+ 6 \sum_m g^m_{k,k',k''} V^m_{k,k',k''}(\delta^0 \varphi^{(k)}, \varphi^{(k')}, \varphi^{(k'')})$$
$$= \text{total derivative.} \tag{14}$$

The contribution of the cubic vertex in Eq. (14) can be compensated by a deformation of the gauge transformation of the form

$$\delta^1_{k,k'} \varphi^{(k'')} = \sum_m g^m_{k,k',k''} d^m_{k,k',k''}(\epsilon^{(k)}, \varphi^{(k')}), \tag{15}$$

where $d^m_{k,k',k''}$ is a differential operator acting on its arguments. Equation (14) shows in particular that the deformations $\delta^1_{k,k'} \varphi^{(k'')}$ of the gauge transformations can be chosen to satisfy the same triangle inequalities (11) as the cubic vertices. Given a vertex $V_{k,k',k''}$, the induced gauge transformations (15) are unique, up to terms that arise from either trivial transformations proportional to free equations, or from field-dependent redefinitions of the gauge parameters. Ignoring such trivial terms, the gauge transformations (15) are always zero when the triangle inequalities

$$|s_k - s_{k'}| < s_{k''} < s_k + s_{k'} \tag{16}$$

are violated. As explained in Ref. [11], the first-order gauge transformations $\delta^1_{k,k'} \varphi^{(k'')}$ form the algebra of global symmetries of the theory for Killing parameters satisfying $\nabla_{(\mu_1} \epsilon^{(k)}_{\mu_2 \dots \mu_{s_k})} = 0$.

To summarize, after fixing the freedom of field redefinitions at cubic level as in Refs. [29,37], there are two vertices—parity-even [21] and parity-odd [22]—for each triple of fields precisely if the spins of the fields satisfy the triangle inequalities (11). This is in striking difference to the situation in higher dimensions, where violations of triangle inequalities are possible, in particular, through Bell-Robinson-type current couplings [38]. Such currents do not define nontrivial couplings in three dimensions due to on-shell triviality of the de Wit–Freedman curvatures of Fronsdal fields in $d = 3$. The precise form of the cubic vertices is not important for the purpose of this article; explicit expressions can be found in Refs. [21,22]. Each of these vertices induces corresponding deformations of the gauge transformations (15). Since the transformations (15) stem from cubic vertices, they also vanish for spins s_k, $s_{k'}$, $s_{k''}$ that violate the triangle inequalities. In Sec. V, we will use this to prove certain properties of higher-order vertices.

III. CORRELATORS IN TWO-DIMENSIONAL CFT: REVIEW

One can compare the above analysis of cubic couplings in a higher-spin gravity theory with the structure of three-point correlation functions in a two-dimensional conformal field theory. As explained in Refs. [21,22], even though the explicit expressions were written down in flat space for simplicity, the classification of cubic vertices is valid for arbitrary Einstein spaces, including anti–de Sitter (AdS) space-times. Via the higher-spin AdS/CFT correspondence, the results on cubic couplings then imply that for three higher-spin currents, dual to the corresponding higher-spin gauge fields, we should find two allowed correlation functions, one parity-even and one parity-odd, whenever the spins satisfy triangle inequalities [Eq. (11)]. Indeed, the conformal invariance constrains the correlators of higher-spin currents to two structures, but with no sign of a triangle inequality.

We will review here the well-known structure of correlation functions in a conformal field theory (for a textbook, see, e.g., Ref. [39]), and we will resolve the apparent mismatch with the triangle inequalities in the subsequent section by a suitable choice of basis.

All fields in a two-dimensional conformal field theory can be obtained from linear combinations of quasiprimary fields and their derivatives. A quasiprimary field transforms in a simple way under global conformal transformations: if we consider the theory on the complex plane (or better, its compactification by including a point at infinity), the global

conformal group is the Möbius group $PSL(2, \mathbb{C})$, and a quasiprimary field ϕ transforms under a conformal transformation $z \mapsto w(z)$ as

$$\phi \mapsto \tilde{\phi}, \qquad \tilde{\phi}(w, \bar{w}) = \left(\frac{dw}{dz}\right)^{-h} \left(\frac{d\bar{w}}{d\bar{z}}\right)^{-\bar{h}} \phi(z, \bar{z}). \quad (17)$$

Here, the real numbers h and $\bar{h}$ are the holomorphic and antiholomorphic conformal weights of the field, respectively. They are related to the scaling dimension Δ and the spin s by

$$\Delta = h + \bar{h}, \quad s = |h - \bar{h}|. \quad (18)$$

Due to the transformation property, the coordinate dependence of a three-point function of three quasiprimary fields is completely fixed:

$$\langle \phi_1(z_1, \bar{z}_1) \phi_2(z_2, \bar{z}_2) \phi_3(z_3, \bar{z}_3) \rangle$$
$$= C z_{12}^{h_3-h_1-h_2} z_{23}^{h_1-h_2-h_3} z_{13}^{h_2-h_1-h_3} \bar{z}_{12}^{\bar{h}_3-\bar{h}_1-\bar{h}_2} \bar{z}_{23}^{\bar{h}_1-\bar{h}_2-\bar{h}_3} \bar{z}_{13}^{\bar{h}_2-\bar{h}_1-\bar{h}_3}, \quad (19)$$

where h_i, $\bar{h}_i$ are the conformal weights of the field ϕ_i, and $z_{ij} = z_i - z_j$. The three-point coefficients C determine the operator product expansion of the fields and thus encode the dynamics of the CFT; all higher-point functions are determined by them.

A spin-s gauge field in the bulk of asymptotically AdS$_3$ space corresponds to a traceless conserved spin-s current $J_{\mu_1...\mu_s}$ of scaling dimension $\Delta = s$ in the CFT. Due to the tracelessness, only two components of J are nonzero, which in the coordinates z, $\bar{z}$ correspond to $J^{(s)} := J_{z...z}$ and $\bar{J}^{(s)} := J_{\bar{z}...\bar{z}}$. These are separately conserved:

$$\partial_{\bar{z}} J^{(s)} = 0, \qquad \partial_z \bar{J}^{(s)} = 0. \quad (20)$$

As for general quasiprimary fields, the coordinate dependence of their three-point correlators is completely fixed—for three holomorphic fields, the form is [see, e.g., Eq. (2.50) of Ref. [39]]

$$\langle J^{(s_1)}(z_1) J^{(s_2)}(z_2) J^{(s_3)}(z_3) \rangle$$
$$= C_{s_1, s_2, s_3} z_{12}^{s_3-s_1-s_2} z_{23}^{s_1-s_2-s_3} z_{13}^{s_2-s_1-s_3}. \quad (21)$$

For every triple of higher-spin currents, there is thus one holomorphic and one antiholomorphic structure for the three-point function, from which one can build one parity-even and one parity-odd structure by taking linear combinations. This matches the results for the classification of cubic vertices, except that there is no restriction on the possible correlators in the form of a triangle inequality: there is no reason that a correlator of three currents with $s_1 \geq s_2 + s_3$ should vanish.

The mismatch can be resolved by using field redefinitions, as we will explain in the following section. The tools that we need there for our explanations are briefly reviewed in the remainder of this section.

The fields in our theory can be realized as operators acting on a Hilbert space that carries a unitary representation of the conformal group. In a two-dimensional conformal field theory, the conformal algebra can be extended to two copies of the Virasoro algebra with generators L_m, $\bar{L}_m$ satisfying

$$[L_m, L_n] = (m - n)L_{m+n} + \frac{c}{12} m(m^2 - 1)\delta_{m+n,0} \quad (22)$$

and the analogous commutation relation for $\bar{L}_m$. Here c is the central charge. In a conformal field theory, there is the operator-state correspondence that relates a field ϕ to the vector $\phi(0, 0)\Omega$, where Ω is the $SL(2, \mathbb{C})$ invariant vacuum vector. When we only consider holomorphic fields $J^{(s)}$, the corresponding subspace $\mathcal{H}$ of the Hilbert space carries a representation of the generators L_m of one copy of the Virasoro algebras.

The holomorphic currents $J^{(s)}$ are operators acting on $\mathcal{H}$, and as holomorphic fields they can be expanded as

$$J^{(s)}(z) = \sum_{n \in \mathbb{Z}} J_n^{(s)} z^{-n-s} \quad (23)$$

in terms of their modes $J_n^{(s)}$. The state corresponding to $J^{(s)}$ is $J_{-s}^{(s)}\Omega \in \mathcal{H}$; modes with mode numbers greater than $-s$ annihilate the vacuum:

$$J_n^{(s)}\Omega = 0 \quad \text{for } n > -s. \quad (24)$$

Due to the transformation properties of the spin-s current, the commutator of the Virasoro mode L_0 with any of the modes is

$$[L_0, J_m^{(s)}] = -m J_m^{(s)}. \quad (25)$$

The mode L_{-1} corresponds to taking the derivative of the field, and its action on the modes is

$$[L_{-1}, J_m^{(s)}] = (1 - s - m) J_{m-1}^{(s)}. \quad (26)$$

We assume that we have real fields, $J^{(s)\dagger} = J^{(s)}$, such that

$$J_n^{(s)\dagger} = J_{-n}^{(s)}. \quad (27)$$

The coefficient C_{s_1, s_2, s_3} in the three-point function (21) can then be computed as an expectation value of some of the modes,

$$C_{s_1,s_2,s_3} = \lim_{z_1 \to \infty} z_1^{2s_1} \langle J^{(s_1)}(z_1) J^{(s_2)}(1) J^{(s_3)}(0) \rangle$$

$$= \langle \Omega, J_{s_1}^{(s_1)} J_{s_3-s_1}^{(s_2)} J_{-s_3}^{(s_3)} \Omega \rangle. \quad (28)$$

The currents form the algebraic structure of a $\mathcal{W}$-algebra: the operator product expansion of two holomorphic currents can be expressed in terms of holomorphic currents and their derivatives; conversely, the commutators of the modes of the currents can be expressed again in terms of modes of conserved currents.

The currents dual to the higher-spin gauge fields in the bulk should form a basis of this $\mathcal{W}$-algebra. In the next section, we will show that there is a specific choice for a basis of any $\mathcal{W}$-algebra such that their correlators satisfy the triangle inequalities that were observed for the higher-spin bulk vertices.

IV. TRIANGLE AND POLYGON INEQUALITIES FOR CFT CORRELATORS

When comparing bulk vertices of higher-spin gauge fields to correlators of currents in the two-dimensional CFT, we have to be aware that in two dimensions, there is the freedom to redefine the CFT currents by adding products or derivatives of currents of lower spin. This is in contrast to the situation in higher dimensions. Fixing this ambiguity corresponds to specifying a concrete basis. In this section, we will show how to choose a basis for the $\mathcal{W}$-algebra of holomorphic currents in the CFT such that the correlators satisfy triangle inequalities for the spins. We will see that in this basis also, the higher-point correlators satisfy analogous polygon inequalities. It is (up to linear transformations) the only basis of quasiprimary fields with this property.

A. The basis

We start with a $\mathcal{W}$-algebra generated by a set of elementary fields $\{J^{(s,i)} | s \in \mathcal{S} \subset \mathbb{N}, 1 \leq i \leq m_s\}$ of definite spin s. The set of all spins that occur is denoted by $\mathcal{S}$, and the number of fields of spin s is denoted by m_s. The $\mathcal{W}$-algebra consists of all chiral fields $J(z)$ that can be obtained by taking linear combinations of normal-ordered products and derivatives of the elementary fields. We assume that the set of elementary fields is minimal in the sense that none of the fields can be obtained from the others by normal-ordered products or derivatives.

We always assume that the $\mathcal{W}$-algebra contains the holomorphic part T of an energy-momentum tensor (but not necessarily as one of the elementary fields),

$$T(z) = \sum_{m \in \mathbb{Z}} L_m z^{-m-2}, \quad (29)$$

whose modes satisfy the commutation relations of the Virasoro algebra [Eq. (22)].

The Hilbert space $\mathcal{H}$ generated by all fields is a direct sum of eigenspaces $\mathcal{H}^{(s)}$ with eigenvalue $s \in \mathbb{N}_0$ of the L_0 operator,

$$\mathcal{H} = \bigoplus_{s=0}^{\infty} \mathcal{H}^{(s)}. \quad (30)$$

The Hilbert space is generated by acting with the modes of the elementary fields on the vacuum vector Ω. The subspace $\mathcal{H}^{(s)}$ is then

$$\mathcal{H}^{(s)} = \left\langle J_{-n_1}^{(s_1,i_1)} \ldots J_{-n_r}^{(s_r,i_r)} \Omega, n_j \geq s_j, \sum_{j=1}^{r} n_j = s \right\rangle, \quad (31)$$

where the angle brackets denote the linear span. We assume that we have chosen a real basis; see Eq. (27).

Inside $\mathcal{H}^{(s)}$, we now consider the subspace $H^{(s)} \subset \mathcal{H}^{(s)}$ that is generated by all modes of basis fields of spin less than s:

$$H^{(s)} = \left\langle J_{-n_1}^{(s_1,i_1)} \ldots J_{-n_r}^{(s_r,i_r)} \Omega, n_j \geq s_j, \sum_{j=1}^{r} n_j = s, s_j < s \right\rangle. \quad (32)$$

The codimension of $H^{(s)}$ in $\mathcal{H}^{(s)}$ is the number m_s of elementary fields of spin s, which by assumption are independent of the products or derivatives of the lower-spin fields. We now consider the orthogonal complement $P^{(s)}$ of $H^{(s)}$ in $\mathcal{H}^{(s)}$,

$$\mathcal{H}^{(s)} = H^{(s)} \oplus P^{(s)}, \quad (33)$$

which is of dimension m_s, and choose a basis $\{\tilde{j}^{(s,1)}, \ldots, \tilde{j}^{(s,m_s)}\}$ of $P^{(s)}$. The corresponding fields $\tilde{J}^{(s,i)}$ constitute the basis we are after.

The space $H^{(s)}$ in terms of fields is the space of all fields that are obtained by taking linear combinations, normal-ordered products, and derivatives of fields with spin lower than s. It is independent of the original choice of basis fields $J^{(s,i)}$, and the construction of the new basis $\tilde{J}^{(s,i)}$ is therefore unique up to linear transformations within the spaces $P^{(s)}$.

To illustrate the construction, we give a simple example. Consider a CFT with a chiral algebra that is generated by a spin-1 current J and the energy-momentum tensor T. At spin 1, there is only one field, so we just take $\tilde{J}^{(1)} = J$. At spin 2, we have three linearly independent fields,

$$T, \quad (JJ), \quad \partial J. \quad (34)$$

The subspace $H^{(2)}$ is spanned by the states corresponding to (JJ) and ∂J. In terms of modes, it can be written as the span of two vectors,

$$H^{(2)} = \langle J_{-1}J_{-1}\Omega, J_{-2}\Omega \rangle. \tag{35}$$

The commutation relations of the modes are (here we choose a specific normalization of the field J)

$$[J_m, J_n] = m\delta_{m+n,0}, [L_m, J_n] = -nJ_{m+n}, \tag{36}$$

and the standard relations for the Virasoro modes L_m. It can be easily checked that the orthogonal complement of $H^{(2)}$ in $\mathcal{H}^{(2)}$ is the one-dimensional space

$$P^{(2)} = \left\langle \left(L_{-2} - \frac{1}{2}J_{-1}J_{-1} \right)\Omega \right\rangle. \tag{37}$$

Therefore we choose the basis field at spin 2 to be

$$\tilde{J}^{(2)} = T - \frac{1}{2}(JJ), \tag{38}$$

i.e., the difference of the energy-momentum tensor and the field obtained by the Sugawara construction for J. Its modes commute with the modes of J,

$$[\tilde{J}_m^{(2)}, J_n] = 0; \tag{39}$$

therefore, in this basis, e.g., the three-point function of fields of spin 1, 1, and 2 vanishes:

$$\langle \tilde{J}^{(2)}\tilde{J}^{(1)}\tilde{J}^{(1)} \rangle = 0. \tag{40}$$

The corresponding statement in the bulk is that there is no cubic coupling of Chern-Simons vector fields [40] to gravity (or any other massless field of spin $s \geq 2$) [22].

More sophisticated examples are provided by the $\mathcal{W}_N$-algebras, which have a basis of fields of spin $2, 3, ..., N$. For spins 3, 4, and 5, the construction just leads to choosing the unique primary field at each spin (we show below that the basis fields $\tilde{J}$ are always primary for spins greater than 2). At spin 6, there are two independent primary fields, and just the requirement to choose a primary basis field does not give a unique procedure. There is one primary field at spin 6 that is constructed out of derivatives and normal-ordered products of fields of lower spins. Our construction is equivalent to choosing a primary field orthogonal to it. Our choice of basis is therefore equivalent to the orthogonality condition that was used, e.g., in Ref. [41] in an analysis of $\mathcal{W}$-algebras generated by fields of spin $2, ..., 6$, or in Ref. [42] in an analysis of the $\mathcal{W}_\infty$-algebra.

B. Correlators

Let us assume that we have chosen a basis as described in the last section, and we denote the basis as $\{J^{(s,i)}\}$, omitting the tilde. We now want to discuss correlators of these fields. Consider the operator product expansion of two fields at the origin evaluated on the vacuum Ω:

$$J^{(s_1,i_1)}(z)J^{(s_2,i_2)}(0)\Omega = \sum_{n\leq s_2} z^{-n-s_1} J_n^{(s_1,i_1)} J_{-s_2}^{(s_2,i_2)}\Omega. \tag{41}$$

For $n > -s_1$, the mode $J_n^{(s_1,i_1)}$ annihilates the vacuum, whereas for $n \leq -s_1$, the mode $J_n^{(s_1,i_1)}$ generates the $(-s_1 - n)$th derivative of $J^{(s_1,i_1)}$ from the vacuum (up to a factor). When we split the sum, we then obtain

$$J^{(s_1,i_1)}(z)J^{(s_2,i_2)}(0)\Omega$$

$$= \sum_{n=-s_1+1}^{s_2} z^{-n-s_1}[J_n^{(s_1,i_1)}, J_{-s_2}^{(s_2,i_2)}]\Omega + \sum_{n\leq -s_1} z^{-n-s_1} J_n^{(s_1,i_1)} J_{-s_2}^{(s_2,i_2)}\Omega$$

$$= \sum_{n=1}^{s_1+s_2} z^{-n}[J_{-s_1+n}^{(s_1,i_1)}, J_{-s_2}^{(s_2,i_2)}]\Omega + \sum_{n\geq 0} \frac{z^n}{n!}(:\partial^n J^{(s_1,i_1)} J^{(s_2,i_2)}:)_{-s_1-s_2-n}\Omega$$

$$\in \bigoplus_{s=0}^{s_1+s_2-1} \mathcal{H}^{(s)} \oplus \bigoplus_{s=s_1+s_2}^{\infty} H^{(s)}. \tag{42}$$

The operator product expansion of the two fields $J^{(s_1,i_1)}$ and $J^{(s_2,i_2)}$ thus only contains fields of spin less than $s_1 + s_2$ and normal-ordered products of (derivatives of) $J^{(s_1,i_1)}$ and $J^{(s_2,i_2)}$. A field $J^{(s,i)}$ with $s \geq s_1 + s_2$ therefore has by construction a vanishing two-point function with any field occurring in the operator product expansion of $J^{(s_1,i_1)}$ and $J^{(s_2,i_2)}$, and hence the corresponding three-point function vanishes:

$$\langle J^{(s_1,i_1)}(z_1)J^{(s_2,i_2)}(z_2)J^{(s,i)}(z_3) \rangle = 0 \quad \text{for } s \geq s_1 + s_2. \tag{43}$$

As the above statement holds for arbitrary spins, the three-point function also vanishes for $s_1 \geq s + s_2$ and $s_2 \geq s + s_1$. Hence, we get a perfect match with the results for the cubic vertices in the bulk: all three-point functions whose spins violate a triangle inequality vanish.

A similar argument leads to constraints also for higher correlation functions. We can compute an r-point function by successively taking operator product expansions—the fields that occur when we take $r - 1$ operator products are fields of spin below $s_1 + \cdots + s_{r-1} - r + 3$, or are normal-ordered products or derivatives of such fields. Hence, the corresponding states are orthogonal to $J^{(s,i)}(0)\Omega$ for $s \geq s_1 + \cdots + s_{r-1} - r + 3$, and the r-point correlation function vanishes in this case. The details of the argument leading to this result are presented in the Appendix.

We conclude that in the specific basis we have chosen, the higher CFT correlators also satisfy polygon inequalities: an r-point correlation function ($r \geq 3$) of fields of spins $s_1, \ldots, s_r$ can be nonzero only if

$$s_k < s_1 + \cdots + s_{k-1} + s_{k+1} \ldots + s_r - r + 3$$
$$\text{for all } k \in \{1, \ldots, r\}, \tag{44}$$

or equivalently, if

$$(s_k - 1) \leq \sum_{\substack{j=1 \\ j \neq k}}^{r} (s_j - 1) \quad \text{for all } k \in \{1, \ldots, r\}. \tag{45}$$

This result suggests that we should be able to make even stronger statements for the classification of bulk vertices. Before we study those in Sec. V, we conclude this section by a short discussion of the specific properties of the basis that we constructed.

C. Primary basis

It follows from the construction of the new basis that it consists of quasiprimary fields. To see this, consider a field $J^{(s,i)}$ and its corresponding state $j^{(s,i)} = J^{(s,i)}(0)\Omega$. To show that the field is quasiprimary, we have to show that $L_1 j^{(s,i)} = 0$. This is equivalent to saying that it is orthogonal to all other states of the same L_0 eigenvalue:

$$\text{for all } v \in \mathcal{H}^{(s-1)}\colon \ \langle v, L_1 j^{(s,i)} \rangle = 0. \tag{46}$$

Let $v \in \mathcal{H}^{(s-1)}$. Then

$$\langle v, L_1 j^{(s,i)} \rangle = \langle L_{-1} v, j^{(s,i)} \rangle. \tag{47}$$

But $L_{-1} v \in H^{(s)}$, because

$$L_{-1} J^{(s_1,i_1)}_{-n_1} \ldots J^{(s_r,i_r)}_{-n_r} \Omega$$
$$= \sum_{j=1}^{r} (1 - s_j + n_j) J^{(s_1,i_1)}_{-n_1} \ldots J^{(s_j,i_j)}_{-n_j-1} \ldots J^{(s_r,i_r)}_{-n_r} \Omega \in H^{(s)}, \tag{48}$$

where all $s_j \leq s - 1$. Finally, as $j^{(s,i)}$ is orthogonal to $H^{(s)}$, we conclude that $\langle L_{-1} v, j^{(s,i)} \rangle = 0$, hence $J^{(s,i)}$ is quasiprimary.

We can perform a similar argument to show that the field is not only quasiprimary, but primary for $s \geq 3$. For that, we have to show that $L_2 j^{(s,i)} = 0$, or, equivalently, that $j^{(s,i)}$ is orthogonal to all states in $L_{-2} \mathcal{H}^{(s-2)}$. Indeed, we have

$$L_{-2} \mathcal{H}^{(s-2)} \subset H^{(s)} \quad \text{for } s \geq 3. \tag{49}$$

This is obvious, because vectors in $\mathcal{H}^{(s-2)}$ are constructed from modes of fields of spin less than or equal to $s - 2$, and applying L_{-2} will result in a vector constructed from modes of fields of spin less than or equal to $s - 2$ (and of spin 2). Thus, it is contained in $H^{(s)}$ for $s \geq 3$.

For the case of spin 2, we cannot argue like that, because $L_{-2}\Omega$ generically is not contained in $H^{(2)}$. There are two cases: if $L_{-2}\Omega \in H^{(2)}$, this means that the energy-momentum tensor is obtained from derivatives or normal-ordered products (as, e.g., in Wess-Zumino-Witten models where the energy momentum tensor is obtained by the Sugawara construction). Then all the elementary spin-2 fields $J^{(2,i)}$ in the basis constructed above are orthogonal to $L_{-2}\Omega$ and therefore primary. If, on the other hand, $L_{-2}\Omega \notin H^{(2)}$, then we can choose one of the spin-2 fields in the basis as the orthogonal projection of $L_{-2}\Omega$ to $P^{(2)}$ (this field is only quasiprimary), and all others can be chosen to be orthogonal to $L_{-2}\Omega$ (they are then primary fields). For example, in the CFT dual of colored AdS$_3$ gravity [24], all of the operators dual to massless spin-2 fields will be primaries except for the one dual to the singlet spin-2 field—the metric graviton, whose dual will be quasiprimary—the stress-energy tensor.

To summarize, we have seen that for any $\mathcal{W}$-algebra, there exists a basis of primary fields (except possibly one quasiprimary spin-2 field) such that the correlators satisfy the polygon inequalities [Eq. (44)]. It is not difficult to see that the basis is already uniquely characterized by this property: there is only one basis (up to linear transformations) of quasiprimary fields that satisfies the polygon inequalities. This can be shown spin by spin: if at any spin s we choose a field $\tilde{J}^{(s)}$ corresponding to a state that is not orthogonal to all states in $H^{(s)}$, there will be some normal-ordered product of (derivatives of) r fields of lower spins (satisfying $s_1 + \cdots + s_r \leq s$) that has a nonvanishing two-point function with $\tilde{J}^{(s)}$. This would imply that the corresponding $(r + 1)$-point function does not vanish—but for $r \geq 2$, this would violate the polygon inequalities, and for $r = 1$ a nonvanishing two-point function with a lower-spin field would be in conflict with the field being quasiprimary. Instead of demanding that the basis fields be quasiprimary, one can impose the "polygon inequalities" [Eq. (44)] also for $r = 2$.

V. POLYGON INEQUALITIES FOR HIGHER-ORDER VERTICES

Motivated by the CFT results, in particular by the inequalities [Eq. (44)] for n-point functions, we want to

see whether in the Noether program we can also make statements about such polygon inequalities for higher-order vertices. The corresponding inequalities for vertices of order n in the fields would be

$$s_i \leq s_1 + s_2 + \cdots + s_{i-1} + s_{i+1} + \cdots + s_n - n + 2$$
$$\text{for } i = 1, \ldots, n. \qquad (50)$$

For cubic vertices ($n = 3$), these inequalities are equivalent to Eq. (11). It is interesting to note that the above inequalities make sense also for $n = 2$. There, $s_1 \leq s_2$ and $s_2 \leq s_1$ imply together $s_1 = s_2$, which is a statement on the diagonal nature of the quadratic action.

To show that the polygon inequalities are indeed satisfied under certain assumptions, we proceed by induction and first prove the following statement:

"Assume that there is a basis of fields, for which the action does not involve any interactions of order $k \leq n$ in fields that violate the polygon inequalities

$$s_i \leq s_1 + s_2 + \cdots + s_{i-1} + s_{i+1} + \cdots + s_k - k + 2$$
$$\text{for } i = 1, 2, \ldots, k, \quad k = 3, 4, \ldots, n \qquad (51)$$

between the spins. Then, the same is true for interactions involving $n + 1$ fields, provided that there are no independent $(n + 1)$th-order deformations that violate these inequalities."

Here, independent deformations of order $n + 1$ are solutions to the Noether equation

$$\delta^0 \mathcal{L}^{(n+1)} \approx 0, \qquad (52)$$

where $\approx$ indicates equality up to the free field equations of motion and total derivatives.

The proof of the statement above is simple—we need to study the Noether equations for gauge invariance:

$$\delta^0 \mathcal{L}^{(n+1)} + \delta^1 \mathcal{L}^{(n)} + \cdots + \delta^{n-2} \mathcal{L}^{(3)} \approx 0. \qquad (53)$$

Assuming that there are no solutions for the homogeneous equation (52) that violate the polygon inequalities (assumption of the statement), we can concentrate on the particular solution of Eq. (53). There, any term of $\delta^0 \mathcal{L}^{(n+1)}$ should be canceled by terms of the form $\delta^i \mathcal{L}^{(n+1-i)}$. One can write any term of the latter variations in the form

$$\delta^i \mathcal{L}^{(n+1-i)} = \sum_{k'} \frac{\delta S^{(n+1-i)}}{\delta \varphi^{k'}} \delta^i \varphi^{k'} + (\text{total derivative}), \quad (54)$$

where $S^{(n+1-i)}$ is the term in the action that is obtained from $\mathcal{L}^{(n+1-i)}$ by integration.

We have seen in Sec. II that the triangle inequalities for the cubic coupling induce the same triangle inequalities for the deformation δ^1 of the gauge transformation (15). Similarly, the polygon inequalities for the lower-order

Lagrangian terms imply analogous inequalities for the gauge transformation deformations δ^i. When we take these inequalities together, we can deduce that the polygon inequalities are satisfied for the spins entering in any variation of the type in Eq. (54).

This can be shown by taking an arbitrary monomial on the right-hand side of Eq. (54), where $\frac{\delta S^{(n+1-i)}}{\delta \varphi^{k'}}$ contains fields, say, $\varphi^{k_1}, \ldots \varphi^{k_{n-i}}$, and $\delta^i \varphi^{k'}$ contains fields, say, $\varphi^{k_{n-i+1}}, \ldots, \varphi^{k_n}$, and a gauge parameter $\epsilon^{k_{n+1}}$. Then the lower-order polygon inequalities (50) for $\mathcal{L}^{(n+1-i)}$ and for $\delta^i \varphi^{k'}$ imply in particular

$$s_{k_1} \leq s_{k_2} + \cdots + s_{k_{n-i}} + s_{k'} - n + i + 1, \qquad (55)$$

$$s_{k'} \leq s_{k_{n-i+1}} + \cdots + s_{k_{n+1}} - i. \qquad (56)$$

Using the second inequality in the first, we obtain

$$s_{k_1} \leq \sum_{i=2}^{n+1} s_{k_i} - n + 1, \qquad (57)$$

a particular instance of the polygon inequalities (50) at order $n + 1$. The analogous inequalities for $k_2, \ldots$ follow in the same way. This concludes the proof.

Our statement above can be rephrased as "Higher-order completions of lower-order vertices satisfying the polygon inequalities (50) do obey the same polygon inequalities."

The classification of cubic vertices in three dimensions [21,22] shows that they obey triangle inequalities. If there are no independent higher couplings (or only ones which satisfy polygon inequalities) in a theory, then we can conclude by induction that the following statement is true: "For a HS gravity theory in three dimensions that is fully determined by its cubic Lagrangian, there exists a basis for which all vertices obey polygon inequalities (50) for spins $s_i \geq 2$."

Note that for the proof, we did not need the explicit forms of the cubic vertices $V_{k,k',k''}^m$ and the gauge transformations $\delta_{k',k''}^1 \varphi^{(k)}$. We only used the fact that the coefficients $g_{k,k',k''}^m$ are zero for three spins $s_k, s_{k'}, s_{k''}$ that violate the triangle inequalities (11).

We exclude scalar and Maxwell fields from the discussion, as their interactions do not respect the polygon inequalities already at the cubic order (but the statement also holds for Chern-Simons vector fields). We also exclude higher-order independent deformations that violate the polygon inequalities in the full nonlinear theory. This is motivated from the CFT, which suggests that no violation of the polygon inequalities occurs. Independent deformations of higher order will be studied elsewhere.

VI. DISCUSSION

We have compared vertices for higher-spin gauge fields in three dimensions with the structure of correlators of conserved currents in a two-dimensional CFT. The classification of cubic vertices is precisely matched by the independent three-point functions in the CFT when one takes into account the possibility of nonlinear basis transformations. The CFT analysis has even revealed restrictions on higher correlators: only those for which the spins satisfy polygon inequalities [Eq. (50)] can be present. This gives strong predictions for the classification of bulk vertices.

On the one hand, we expect that higher-order couplings also respect these polygon inequalities. We checked this explicitly for couplings that are continuations of the cubic couplings, but the CFT result implies that the same is true also for independent higher vertices. We will study this problem elsewhere [43].

On the other hand, the CFT results are full quantum results, so one expects that in the bulk theory—as long as higher-spin gauge invariance is preserved—the polygon inequalities are satisfied even if we take quantum corrections into account. Indeed, it is straightforward to show (by an argument similar to the one used in Sec. V) that in a theory of fields of spin greater or equal to 2, any tree-level Witten diagram built from vertices that satisfy polygon inequalities will vanish if the spins of the external fields do not satisfy the polygon inequalities [Eq. (50)]. Similar conclusions cannot be made for loop diagrams, though.

Whereas from the pure CFT perspective all different bases appear on equal footing, from the holographic perspective the basis we identified seems physically the most sensible choice: it directly reproduces the structure of cubic on-shell vertices. It is interesting to identify the significance of this choice directly in a holographic setting. When we consider as a simple example a Chern-Simons formulation of higher-spin gravity with a gauge algebra $\mathfrak{g} \oplus \mathfrak{g}$, its asymptotic symmetry algebra is a classical $\mathcal{W}$-algebra that is the Drinfeld-Sokolov reduction [45] of the affine Lie algebra $\hat{\mathfrak{g}}_k \oplus \hat{\mathfrak{g}}_k$ (where the level depends on the level of the Chern-Simons theory). The Drinfeld-Sokolov reduction can be performed in different gauges; the gauge fixing determines the basis in which the $\mathcal{W}$-algebra is realized. Performing the reduction in the highest-weight gauge [46], one arrives at a primary basis (except for a quasiprimary spin-2 field). Furthermore, the Poisson brackets (that in the quantum $\mathcal{W}$-algebra turn into commutators which encode the singular operator products of the fields) in this basis are constrained by the representation theory of the embedded $\mathfrak{sl}(2)$ algebra (which corresponds to the gravity part). The Poisson bracket takes the form [15]

$$\{J^{(s_1)}, J^{(s_2)}\} = c_{s_1}\delta_{s_1,s_2} + \begin{pmatrix} \text{fields of spin } s \text{ with} \\ |s_1 - s_2| < s < s_1 + s_2 \end{pmatrix}.$$

(58)

The bound from above on the possible spins appearing in the bracket is satisfied in any basis, but the bound from below is special. If it is still present in the quantum $\mathcal{W}$-algebra, the three-point correlators will satisfy triangle inequalities. The highest-weight gauge in the Drinfeld-Sokolov reduction thus leads to the classical version of the primary basis that we considered in this paper; it seems to be a preferred gauge choice to discuss holography.

The dictionary established here works for the massless fields in the bulk that do not carry propagating d.o.f., and corresponding conserved currents in the dual CFT. The classification of cubic vertices also involving massive fields is not available in full generality in three dimensions except for some examples [50]. There, it is established that the massless limit is well behaved. Therefore, it would be interesting to have a classification with massive fields and see if similar polygon inequalities are governing the interactions, or how such inequalities emerge in the massless limit and how they match with correlators in CFT.

APPENDIX: HIGHER CORRELATORS

In this Appendix, we provide the argument that for $r \geq 2$,

$$J^{(s_1,i_1)}(z_1) \ldots J^{(s_r,i_r)}(z_r)\Omega \in \bigoplus_{s=0}^{S-r+1} \mathcal{H}^{(s)} \oplus \bigoplus_{s=S-r+2}^{\infty} H^{(s)},$$

(A1)

where $S = s_1 + \cdots + s_r$. This can be equivalently stated in terms of modes as

$$J^{(s_1,i_1)}_{-s_1+n_1} \ldots J^{(s_r,i_r)}_{-s_r+n_r}\Omega \in \bigoplus_{s=0}^{S-r+1} \mathcal{H}^{(s)} \oplus \bigoplus_{s=S-r+2}^{\infty} H^{(s)}$$

(A2)

for all $n_k \in \mathbb{Z}$. The idea to prove this is the following: if all n_k's are negative or zero, we immediately see that the vector is in the space generated by normal-ordered products of derivatives of the fields $J^{(s_k,i_k)}$; hence, it is a state of L_0-weight $S - \sum n_k \geq S$ that is generated by fields of spin less than $\max(s_1,\ldots,s_r) < S$. If one or more of the n_k's are positive, we commute them to the right until they annihilate the vacuum vector Ω. Using general structures for the commutation relations (reviewed below), one can see that the spins of the fields that are generated in this process are bounded such that no fields of spin $S - r + 2$ or higher are produced.

We have seen in Sec. IV C that the fields $J^{(s,i)}$ in our basis are quasiprimary. Commutation relations of general holomorphic quasiprimary fields always take the form [see, e.g., Eq. (2.54) of Ref. [39]]

$$[\phi_{(i)m}, \phi_{(j)n}] = \sum_{\substack{k \\ s_k < s_i + s_j}} C_{ij}^k p_{ijk}(m, n) \phi_{(k)m+n}$$

$$+ d_{ij} \delta_{m,-n} \binom{m + s_i - 1}{2s_i - 1}, \qquad (A3)$$

where $\phi_{(i)m}$ denotes the mth mode of a field $\phi_{(i)}$ of spin s_i, and C_{ij}^k are structure constants. The $p_{ijk}(m, n)$ are polynomials in m and n which are completely fixed by the spins s_i, s_j, and s_k. In our case, the basis fields $J^{(s,i)}$ constitute some of the $\phi_{(j)}$, while the remaining ones are obtained by quasiprimary combinations of normal-ordered products and derivatives of the basis fields. The modes $\phi_{(k)m}$ of

such a nonelementary field can be expressed as sums of products of modes of the basis fields, where the sum of the spins of the constituent fields is always lower than or equal to the spin s_k of the field.

Whenever in an expression like (A2) we commute one mode $J^{(s_1,i_1)}$ with another one $J^{(s_2,i_2)}$, the quantity "sum of spins of generators − number of generators" is never increased due to the commutation relations (A3). As on the other hand, the "sum of spins of generators" is decreased in the commutator (A3), the process of commuting generators to the right has to stop at some point. The spin s of a single generator that could remain therefore satisfies

$$s - 1 \leq S - r \Rightarrow s \leq S - r + 1, \qquad (A4)$$

which completes the proof.

[1] M. A. Vasiliev, Consistent equations for interacting gauge fields of all spins in $(3 + 1)$ dimensions, Phys. Lett. B **243**, 378 (1990).

[2] S. F. Prokushkin and M. A. Vasiliev, Higher spin gauge interactions for massive matter fields in 3D AdS space-time, Nucl. Phys. **B545**, 385 (1999).

[3] M. A. Vasiliev, Nonlinear equations for symmetric massless higher spin fields in $(A)dS_d$, Phys. Lett. B **567**, 139 (2003).

[4] X. Bekaert, S. Cnockaert, C. Iazeolla, and M. A. Vasiliev, Nonlinear higher spin theories in various dimensions, arXiv: hep-th/0503128.

[5] A. Campoleoni, Higher spins in $D = 2 + 1$, Subnuclear series **49**, 385 (2013).

[6] V. E. Didenko and E. D. Skvortsov, Elements of Vasiliev theory, arXiv:1401.2975.

[7] R. Rahman and M. Taronna, From higher spins to strings: A primer, arXiv:1512.07932.

[8] C. Sleight, Metric-like methods in higher spin holography, Proc. Sci., Modave2016 (**2017**) 003.

[9] P. Kessel, The very basics of higher-spin theory, Proc. Sci., Modave2016 (**2017**) 001.

[10] S. Giombi, Higher spin-CFT duality, in *New Frontiers in Fields and Strings* (World Scientific, Singapore, 2017), p. 137.

[11] E. Joung and K. Mkrtchyan, Notes on higher-spin algebras: Minimal representations and structure constants, J. High Energy Phys. 05 (2014) 103.

[12] M. Henneaux and S. J. Rey, Nonlinear $\mathcal{W}_\infty$ as asymptotic symmetry of three-dimensional higher spin Anti–de Sitter gravity, J. High Energy Phys. 12 (2010) 007.

[13] A. Campoleoni, S. Fredenhagen, S. Pfenninger, and S. Theisen, Asymptotic symmetries of three-dimensional

gravity coupled to higher-spin fields, J. High Energy Phys. 11 (2010) 007.

[14] M. R. Gaberdiel and T. Hartman, Symmetries of holographic minimal models, J. High Energy Phys. 05 (2011) 031.

[15] A. Campoleoni, S. Fredenhagen, and S. Pfenninger, Asymptotic $\mathcal{W}$-symmetries in three-dimensional higher-spin gauge theories, J. High Energy Phys. 09 (2011) 113.

[16] A. Campoleoni and M. Henneaux, Asymptotic symmetries of three-dimensional higher-spin gravity: The metric approach, J. High Energy Phys. 03 (2015) 143.

[17] E. Joung, J. Kim, J. Kim, and S. J. Rey, Asymptotic symmetries of colored gravity in three dimensions, J. High Energy Phys. 03 (2018) 104.

[18] M. R. Gaberdiel and R. Gopakumar, An AdS$_3$ dual for minimal model CFTs, Phys. Rev. D **83**, 066007 (2011).

[19] M. R. Gaberdiel and R. Gopakumar, Minimal model holography, J. Phys. A **46**, 214002 (2013).

[20] C. Fronsdal, Massless fields with integer spin, Phys. Rev. D **18**, 3624 (1978).

[21] K. Mkrtchyan, Cubic Interactions of Massless Bosonic Fields in Three Dimensions, Phys. Rev. Lett. **120**, 221601 (2018).

[22] P. Kessel and K. Mkrtchyan, Cubic interactions of massless bosonic fields in three dimensions: II. Parity-odd and Chern-Simons vertices, Phys. Rev. D **97**, 106021 (2018).

[23] M. P. Blencowe, A consistent interacting massless higher spin field theory in $D = (2 + 1)$, Classical Quantum Gravity **6**, 443 (1989).

[24] S. Gwak, E. Joung, K. Mkrtchyan, and S. J. Rey, Rainbow valley of colored (Anti) de Sitter gravity in three dimensions, J. High Energy Phys. 04 (2016) 055.

[25] S. Gwak, E. Joung, K. Mkrtchyan, and S. J. Rey, Rainbow vacua of colored higher-spin $(A)dS_3$ gravity, J. High Energy Phys. 05 (2016) 150.

[26] A. Campoleoni, S. Fredenhagen, S. Pfenninger, and S. Theisen, Towards metric-like higher-spin gauge theories in three dimensions, J. Phys. A **46**, 214017 (2013).

[27] S. Fredenhagen and P. Kessel, Metric- and frame-like higher-spin gauge theories in three dimensions, J. Phys. A **48**, 035402 (2015).

[28] R. R. Metsaev, Cubic interaction vertices of massive and massless higher spin fields, Nucl. Phys. **B759**, 147 (2006).

[29] R. Manvelyan, K. Mkrtchyan, and W. Rühl, General trilinear interaction for arbitrary even higher spin gauge fields, Nucl. Phys. **B836**, 204 (2010).

[30] S. Giombi, S. Prakash, and X. Yin, A note on CFT correlators in three dimensions, J. High Energy Phys. 07 (2013) 105.

[31] M. S. Costa, J. Penedones, D. Poland, and S. Rychkov, Spinning conformal correlators, J. High Energy Phys. 11 (2011) 071.

[32] E. Joung and M. Taronna, Cubic interactions of massless higher spins in $(A)dS$: Metric-like approach, Nucl. Phys. **B861**, 145 (2012).

[33] E. Conde, E. Joung, and K. Mkrtchyan, Spinor-helicity three-point amplitudes from local cubic interactions, J. High Energy Phys. 08 (2016) 040.

[34] D. Francia, G. Lo Monaco, and K. Mkrtchyan, Cubic interactions of Maxwell-like higher spins, J. High Energy Phys. 04 (2017) 068.

[35] P. Kravchuk and D. Simmons-Duffin, Counting conformal correlators, J. High Energy Phys. 02 (2018) 096.

[36] C. Sleight and M. Taronna, Spinning Witten diagrams, J. High Energy Phys. 06 (**2017**) 100.

[37] R. Manvelyan, K. Mkrtchyan, and W. Rühl, Direct construction of a cubic self-interaction for higher spin gauge fields, Nucl. Phys. **B844**, 348 (2011).

[38] R. Manvelyan, K. Mkrtchyan, and W. Rühl, Off-shell construction of some trilinear higher spin gauge field interactions, Nucl. Phys. **B826**, 1 (2010).

[39] R. Blumenhagen and E. Plauschinn, Introduction to conformal field theory: With applications to string theory, Lect. Notes Phys. **779**, 1 (2009).

[40] Note that the chiral currents $\tilde{J}^{(1)}$ here correspond through the AdS/CFT dictionary to Chern-Simons vector fields in the bulk. Maxwell vector fields, on the other hand, have current couplings to higher spin fields in the bulk. In general, interactions of Chern-Simons fields respect the triangle inequalities, while those of Maxwell fields violate them.

[41] K. Hornfeck, $\mathcal{W}$-algebras with set of primary fields of dimensions (3, 4, 5) and (3, 4, 5, 6), Nucl. Phys. **B407**, 237 (1993).

[42] T. Procházka, Exploring $\mathcal{W}_\infty$ in the quadratic basis, J. High Energy Phys. 09 (2015) 116.

[43] This is indeed the case. We show in our follow-up paper [44] that there are no independent higher-order vertices with massless higher-spins in three dimensions.

[44] S. Fredenhagen, O. Krüger, and K. Mkrtchyan, Vertex-constraints in 3D higher spin theories, arXiv:1905.00093.

[45] V. G. Drinfeld and V. V. Sokolov, Lie algebras and equations of Korteweg-de Vries type, J. Sov. Math. **30**, 1975 (1985).

[46] There are two other common gauge choices for $\mathfrak{g} = \mathfrak{sl}(N)$: the *u-gauge*, which leads to a particularly simple realization of the $\mathcal{W}$-algebra where the highest nonlinearities are quadratic [47], and the *diagonal gauge* that leads to a free field realization [48], which therefore can be a good starting point to discuss quantization [49].

[47] L. A. Dickey, *Soliton Equations and Hamiltonian Systems* (World Scientific, Singapore, 2003).

[48] J. Balog, L. Feher, L. O'Raifeartaigh, P. Forgacs, and A. Wipf, Toda theory and $\mathcal{W}$ algebra from a gauged WZNW point of view, Ann. Phys. (N.Y.) **203**, 76 (1990).

[49] A. Campoleoni, S. Fredenhagen, and J. Raeymaekers, Quantizing higher-spin gravity in free-field variables, J. High Energy Phys. 02 (**2018**) 126.

[50] I. L. Buchbinder, T. V. Snegirev, and Y. M. Zinoviev, On gravitational interactions for massive higher spins in AdS$_3$, J. Phys. A **46**, 214015 (2013).