SPHERICAL ANALYSIS ON HOMOGENEOUS VECTOR BUNDLES

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Abstract. Given a Lie group G, a compact subgroup K and a representation $\tau \in \widehat{K}$, we assume that the algebra of $\mathrm{End}(V_\tau)$-valued, bi-τ-equivariant, integrable functions on G is commutative. We present the basic facts of the related spherical analysis, putting particular emphasis on the rôle of the algebra of G-invariant differential operators on the homogeneous bundle E_τ over G/K. In particular, we observe that, under the above assumptions, (G,K) is a Gelfand pair and show that the Gelfand spectrum for the triple (G,K,τ) admits homeomorphic embeddings in $\mathbb{C}^n$.

In the second part, we develop in greater detail the spherical analysis for $G = K \ltimes H$ with H nilpotent. In particular, for $H = \mathbb{R}^n$ and $K \subset SO(n)$ and for the Heisenberg group H_n and $K \subset U(n)$, we characterize the representations $\tau \in \widehat{K}$ giving a commutative algebra.

Introduction

Let (G,K) be a Gelfand pair with G a Lie group and K a compact subgroup of it. Recent work has put attention on the fact that the spherical analysis on the bi-K-invariant algebra $L^1(K\backslash G/K)$ gains new interesting aspects from the fact that its Gelfand spectrum Σ, i.e., the space of bounded spherical functions with the compact-open topology, can be naturally embedded into some Euclidean space as a closed set [8].

Such an embedding ρ is defined by choosing a generating k-tuple $(D_1,\dots,D_k)$ in the algebra of G-invariant differential operators on G/K and assigning to the spherical function ϕ on G/K the vector $\rho(\phi) = \big(\lambda_{D_1}(\phi),\dots,\lambda_{D_k}(\phi)\big) \in \mathbb{C}^k$ whose entries $\lambda_{D_j}(\phi)$ are the eigenvalues of ϕ under the D_j's.

This allows to introduce a notion of smoothness for functions defined on Σ, and to pose the problem, classical in Fourier analysis, of relating smoothness of the spherical transform of a given bi-K-invariant function on G with properties of the function itself. This question has been investigated in detail for *nilpotent Gelfand pairs*, in which $G = K \ltimes H$ is a motion group on a nilpotent group H [9, 10, 11].

In this paper we extend the basic framework for such analysis to the spherical transform of type τ, where τ is an irreducible unitary representation of K for which the appropriate commutativity assumptions are satisfied.

The notion of spherical transform of type τ goes back to [15]. In most of the existing literature the accent is on the case where (G,K) is a symmetric pair. For the general case we refer to [25, Ch. 6] and [22, 5].

There are two equivalent ways to introduce the objects of our analysis on a triple (G,K,τ), where G is a Lie group, K a compact subgroup of it and $\tau \in \widehat{K}$ as above.

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In the first (or matrix-valued) picture one considers the homogeneous bundle $E_\tau = G \times_\tau V_\tau$ with basis G/K and linear operators on sections of E_τ commuting with the action of G. The Schwartz kernel theorem implies that, under mild continuity assumptions, any such operator can be represented by convolution with an $\mathrm{End}(V_\tau)$-valued[1] distributional kernel F on G satisfying the identity

$$F(k_1 x k_2) = \tau(k_2^{-1}) F(x) \tau(k_1^{-1}). \tag{0.1}$$

The commutativity condition imposed on τ is that the algebra of $\mathrm{End}(V_\tau)$-valued integrable functions F on G satisfying the above identity is commutative with respect to the convolution defined in (1.4) below.

In the second (scalar-valued) picture one considers the algebra of integrable scalar-valued functions f on G which are K-central and satisfy the identity

$$f * \overline{\chi}_\tau = f,$$

and the requirement on τ is that this algebra be commutative.

We say that (G, K, τ) is a *commutative triple* if either of these conditions is satisfied.

In the first part of the paper we recall definitions and basic facts about commutative triples and analyze in detail the relevant algebras of invariant differential operators in the two pictures, proving that they are finitely generated (Sections 2).

It is proved in [7] that Thomas's characterization [21] of Gelfand pairs admits the appropriate extension to commutative triples (G, K, τ) with (G, K) a symmetric pair. This means that (G, K, τ) is commutative if and only if the algebra $\mathbb{D}(E_\tau)$ of G-invariant differential operators on E_τ is commutative. In Theorem 3.1 we provide a proof of this property which applies to general triples, under the assumption that G/K is connected. Then the spherical functions of a commutative triple coincide with the joint eigenfunctions of these operators which satisfy (0.1) and take unit value at the identity of G.

From this equivalence we derive the following conclusion, which does not seem to appear in the literature: *if (G, K, τ) is a commutative triple, then (G, K) is a Gelfand pair.* In particular, G must be unimodular and the pair (G, K) must fall in the classification of Lie Gelfand pairs [23, 26, 27].

Since our proof is based on the analysis of $\mathbb{D}(E_\tau)$, this result is limited to Lie groups G, K with G/K connected. It would be interesting to know if the same statement holds for commutative triples where G is not a Lie group.

In Section 4 we describe the embeddings of the Gelfand spectrum of the algebra of integrable functions satisfying (0.1) into Euclidean spaces, extending the result in [8] to general τ.

In Section 5 we comment on the special case of a *strong Gelfand pair* (G, K), defined by the condition that (G, K, τ) is commutative for *every* $\tau \in \widehat{K}$. Since (G, K) is a strong Gelfand pair if and only if $(K \ltimes G, K)$ is a Gelfand pair (with K acting on G by inner automorphisms), it is natural to compare the Gelfand spectrum of this pair with those of the individual triples (G, K, τ). We show that the former is the topological disjoint union of the latter ones.

In the second part of the paper we analyze the case where $G = K \ltimes H$ is a motion group on a Lie group H. The sections of E_τ are then identified with the $\mathrm{End}(V_\tau)$-valued functions on H and the algebra of integrable functions satisfying (0.1) with the algebra of $\mathrm{End}(V_\tau)$-valued integrable functions on H which transform according to τ under K:

$$F(k \cdot x) = \tau(k) F(x) \tau(k^{-1}).$$

[1]For V, W finite dimensional complex vector spaces, $\mathrm{Hom}(V, W)$ denotes the space of linear operators from V to W. If $V = W$, we write $\mathrm{End}(V)$ instead of $\mathrm{Hom}(V, V)$.

In this context, the representation theoretical criterion for having a commutative triple can be reduced to conditions on the representations of H rather than of G. At the same time, the spherical functions can be defined directly on H and their properties described without lifting them to G, cf. Sections 6 and 7.

In Section 8 we show that the Gelfand spectrum Σ_1 of the pair (G, K) is identified in a natural way with a quotient of the spectrum Σ_τ of the triple (G, K, τ), and that the quotient map becomes a canonical projection onto a coordinate subspace of $\mathbb{C}^k$ when the two spectra are embedded in a compatible way in complex Euclidean spaces.

From Section 9 on we further restrict ourselves to the case where H is nilpotent. We first extend to this kind of commutative triples the proof in [3] that all bounded spherical functions are of positive type.

In Section 10 we characterize the commutative triples with $H = \mathbb{R}^n$ as those for which τ decomposes without multiplicities when restricted to the stabilizer of any point in $\mathbb{R}^n$, and the commutative triples with H equal to the Heisenberg group H_n as those for which the tensor product of τ and the metaplectic representation restricted to K decomposes without multiplicities.

This allows to easily recognize the known fact that $\big(SO(n) \ltimes \mathbb{R}^n, SO(n)\big)$, $\big(U(n) \ltimes H_n, U(n)\big)$ are strong Gelfand pairs, and in addition to classify the representations τ for which the triples $\big(SU(n) \ltimes \mathbb{C}^n, SU(n), \tau\big)$, $(SU(n) \ltimes H_n, SU(n), \tau)$ are commutative.

In Section 11 we give general formulas for the bounded spherical functions and finally, in Section 12 , we explicitely compute them in the special case $H = \mathbb{R}^n$, $K = SO(n)$ and τ the natural representation on $\mathbb{C}^n$.

1. Commutative triples and spherical functions of type τ

Let G be a locally compact group, K be a compact subgroup of G and τ a finite dimensional unitary representation of K on the space V_τ.

By $C^\infty(G, V_\tau)$ we denote the space of V_τ-valued smooth functions on G and by $C^\infty_\tau(G, V_\tau)$ we denote the subspace of functions u satisfying the identity

$$u(xk) = \tau(k^{-1})u(x), \qquad \forall k \in K. \tag{1.1}$$

Then $C^\infty_\tau(G, V_\tau)$ is naturally identified with the space of smooth sections of the homogeneous bundle $E_\tau = G \times_\tau V_\tau$. Similar notation will be used with C^∞ replaced by other function (or distribution) spaces, like $\mathcal{D}(= C^\infty_c), \mathcal{D}', C^k, L^p$ etc.

It follows from the Schwartz kernel theorem that every linear operator, continuous with the respect to the standard topologies, mapping $\mathcal{D}$-sections of E_τ into $\mathcal{D}'$-sections of E_τ and commuting with the action of G on E_τ, can be represented in a unique way as[2]

$$Tu(x) = \int_G F(y^{-1}x)u(y)dy, \qquad u \in \mathcal{D}_\tau(G, V_\tau), \tag{1.2}$$

with $F \in \mathcal{D}'_{\tau,\tau}\big(G, \mathrm{End}(V_\tau)\big)$, i.e.,

$$F(k_1xk_2) = \tau(k_2^{-1})F(x)\tau(k_1^{-1}). \tag{1.3}$$

Conversely, any $F \in \mathcal{D}'_{\tau,\tau}\big(G, \mathrm{End}(V_\tau)\big)$ defines a continuous G-invariant operator T on $\mathcal{D}$-sections of E_τ via formula (1.2).

[2]The integral notation in (1.2) and the pointwise identity (1.3) must be appropriately interpreted when F is not a function.

In particular, operators $T = T_F$ as in (1.2) with $F \in L^1_{\tau,\tau}\big(G, \mathrm{End}(V_\tau)\big)$ can be composed with each other (e.g., because they are bounded on L^1-sections) and $T_{F_1}T_{F_2} = T_{F_1 * F_2}$, where

$$F_1 * F_2(x) = \int_G F_2(y^{-1}x)F_1(y)dy. \tag{1.4}$$

Definition. *(G, K, τ) is a commutative triple if the algebra $L^1_{\tau,\tau}(G, \mathrm{End}(V_\tau))$ is commutative.*

The following well-known theorem ([25], Vol. II, Page-9, Prop. 6.1.1.6) gives a characterization of commutative triples in terms of representation of G. By $\widehat{G}$ we denote the set of all equivalence classes of irreducible unitary representations of G.

Theorem 1.1. *(G, K, τ) is a commutative triple if and only if, for any $\pi \in \widehat{G}$, the multiplicity of τ in $\pi_{|_K}$ is at most 1.*

To an $\mathrm{End}(V_\tau)$-valued function F we associate the scalar-valued function

$$S_\tau F = d_\tau \mathrm{Tr}\, F. \tag{1.5}$$

The following statement is also well known [25, vol. II].

Proposition 1.2. *An $\mathrm{End}(V_\tau)$-valued function function F satisfies (1.3) if and only if $f = S_\tau F$ is K-central, i.e.,*

$$f(kxk^{-1}) = f(x), \qquad \forall\, k \in K, \tag{1.6}$$

and of type τ, i.e.,

$$f * (d_\tau \bar{\chi}_\tau m_K) := d_\tau \int_K f(xk)\chi_\tau(k)dk = f(x), \tag{1.7}$$

where m_K denotes the normalized Haar measure on K.

Under these assumptions on F and f, S_τ is bijective and the inverse map is given by

$$S_\tau^{-1} f(x) = \int_K \tau(k)f(kx)dk. \tag{1.8}$$

By $L^p_\tau(G)^{\mathrm{int}K}$ we denote the space of K-central L^p functions of type τ. For $p = 1$, $L^1_\tau(G)^{\mathrm{int}K}$ is closed under convolution. It is easy to verify that

$$S_\tau^{-1}(f_1 * f_2) = f_1 * (S_\tau^{-1} f_2) = (S_\tau^{-1} f_1) * (S_\tau^{-1} f_2)\ ,$$

which leads to the following conclusion.

Proposition 1.3. *The map S_τ establishes an algebra isomorphism between $L^1_{\tau,\tau}\big(G, \mathrm{End}(V_\tau)\big)$ and $L^1_\tau(G)^{\mathrm{int}K}$.*

Definition. *Let (G, K, τ) be a commutative triple. A non-trivial function $\Phi \in L^\infty_{\tau,\tau}(G, \mathrm{End}(V_\tau))$ is said to be a τ-spherical function if the map*

$$F \to \widehat{F}(\Phi) := \frac{1}{d_\tau}\int_G \mathrm{Tr}\big[F(x)\Phi(x^{-1})\big]dx = \frac{1}{d_\tau}\mathrm{Tr}\big[F * \Phi(e)\big]$$

is a homomorphism of $L^1_{\tau,\tau}(G, \mathrm{End}(V_\tau))$ into $\mathbb{C}$.

Definition. *Let (G,K,τ) be a commutative triple. A non trivial function $\phi \in L^\infty_\tau(G)^{\mathrm{int}K}$ is said to be a trace τ-spherical function if the map*

$$f \to \widehat{f}(\phi) := \int_G f(x)\phi(x^{-1})dx = f * \phi(e)$$

is a homomorphism of $L^1_\tau(G)^{\mathrm{int}K}$ into $\mathbb{C}$.

Observe that our definition of trace spherical function differs from that in [25] by a factor of d_τ.

The following theorem characterizes the τ-spherical functions in terms of functional equations. We refer to [14] and [5, Thm. 3.6] for the proof.

Theorem 1.4. *For $\Phi \in L^\infty_{\tau,\tau}(G, \mathrm{End}(V_\tau))$ and $\phi = \frac{1}{d_\tau^2} S_\tau(\Phi)$ the following are equivalent:*

(i) *Φ is a τ-spherical function.*
(ii) *ϕ is a trace τ-spherical function.*
(iii) *$\Phi \in L^\infty_{\tau,\tau}(G, \mathrm{End}(V_\tau))$ is nontrivial and satisfies the functional equation*

$$d_\tau \int_K \Phi(xky)\chi_\tau(k)dk = \Phi(y)\Phi(x). \tag{1.9}$$

(iv) *$\phi \in L^\infty_\tau(G)^{\mathrm{int}K}$ is nontrivial and satisfies the functional equation*

$$\int_K \phi(xkyk^{-1})dk = \phi(x)\phi(y). \tag{1.10}$$

A τ-spherical function Φ satisfies $\Phi(e) = I$.

2. Differential operators on homogeneous bundles

Let $\mathbb{D}(G)$ be the algebra of left-invariant differential operator on G. The action of $\mathbb{D}(G)$ on $C^\infty(G)$ induces an action of $\mathbb{D}(G) \otimes \mathrm{End}(V_\tau)$ on $C^\infty(G, V_\tau) = C^\infty(G) \otimes V_\tau$ given by

$$(D \otimes T)u := D(Tu). \tag{2.1}$$

With an abuse of notation, we will write D for the "scalar" operator $D \otimes I$.

The elements of $\mathbb{D}(G) \otimes \mathrm{End}(V_\tau)$ which preserve $C^\infty_\tau(G, V_\tau)$ are the ones which are invariant under all operators $\mu(k)$, $k \in K$ given by

$$\mu(k)(D \otimes T) = D^{\mathrm{Ad}k} \otimes \tau(k^{-1})T\tau(k), \tag{2.2}$$

where, for an automorphism φ of G and $D \in \mathbb{D}(G)$,

$$D^\varphi f = \big(D(f \circ \varphi)\big) \circ \varphi^{-1},$$

cf. [24, p. 120].

Denote by $\big(\mathbb{D}(G) \otimes \mathrm{End}(V_\tau)\big)^K$ the algebra of operators which are invariant under μ in (2.2), and by $\lambda : \mathfrak{S}(\mathfrak{g}) \longrightarrow \mathfrak{U}(\mathfrak{g}) \cong \mathbb{D}(G)$ the symmetrization operator. The following statement is pretty obvious.

Lemma 2.1. *The operator*

$$\Lambda := \lambda \otimes I : \mathfrak{S}(\mathfrak{g}) \otimes \mathrm{End}(V_\tau) \longrightarrow \mathbb{D}(G) \otimes \mathrm{End}(V_\tau)$$

is a linear bijection from the space $\big(\mathfrak{S}(\mathfrak{g}) \otimes \mathrm{End}(V_\tau)\big)^K$ of $\mathrm{End}(V_\tau)$-valued polynomials P on $\mathfrak{g}$ satisfying the identity

$$P \circ \mathrm{Ad}(k) = \tau(k)^{-1}P\tau(k),$$

onto $\big(\mathbb{D}(G)\otimes \mathrm{End}(V_\tau)\big)^K$.

The algebra $\mathbb{D}(E_\tau)$ of G-invariant differential operators acting on smooth sections of the homogeneous bundle E_τ is then the quotient

$$\mathbb{D}(E_\tau)=\big(\mathbb{D}(G)\otimes \mathrm{End}(V_\tau)\big)^K\Big/\big(\mathbb{D}(G)\otimes \mathrm{End}(V_\tau)\big)^K_0\ ,$$

of $\big(\mathbb{D}(G)\otimes \mathrm{End}(V_\tau)\big)^K$ modulo the ideal of those operators which are trivial on $C^\infty_\tau(G,V_\tau)$.

In fact, we have the relations obtained by differentiating (1.1) with respect to k: for $X\in\mathfrak{k}$ and $u\in C^\infty_\tau(G,V_\tau)$,

$$Xu=-d\tau(X)u\ . \tag{2.3}$$

The following statement, proved in [17], essentially says that the full ideal $\big(\mathbb{D}(G)\otimes \mathrm{End}(V_\tau)\big)^K_0$ is generated by the relations (2.3).

Theorem 2.2. *Let $\mathfrak{p}$ be an $\mathrm{Ad}(K)$-invariant complement of $\mathfrak{k}$ in $\mathfrak{g}$. Then Λ is a linear bijection from $\big(\mathfrak{S}(\mathfrak{p})\otimes \mathrm{End}(V_\tau)\big)^K$ onto $\mathbb{D}(E_\tau)$.*

Another consequence of (2.3), combined with the identity $\mathrm{End}(V_\tau)=d\tau\big(\mathfrak{U}(\mathfrak{k})\big)$, is that any $\boldsymbol{D}\in\mathbb{D}(E_\tau)$ acts on $C^\infty_\tau(G,V_\tau)$ in the same way as a (scalar) element of

$$\mathbb{D}_K(G):=\{D\in\mathbb{D}(G):D^{\mathrm{Ad}(k)}=D\}=\{D\in\mathbb{D}(G):D^{R_k}=D\}.$$

To be more precise, for a polynomial $P(x,y)\in\mathfrak{S}(\mathfrak{g})$ on $\mathfrak{g}$ ($x\in\mathfrak{k},y\in\mathfrak{p}$), define the modified symmetrization $\lambda'(P)\in\mathbb{D}(G)$ as

$$\big(\lambda'(P)f\big)(g)=P(\partial_x,\partial_y)_{|x=y=0}f(g\exp y\exp x).$$

It is quite obvious that $\lambda':\mathfrak{S}(\mathfrak{g})\to\mathbb{D}(G)$ is the linear bijection uniquely defined by the requirement that, if $P(x,y)=p(x)q(y)$, then

$$\lambda'(P)=\lambda(q)\lambda(p). \tag{2.4}$$

Moreover, $\lambda'(u)\in\mathbb{D}_K(G)$ if and only if u is $\mathrm{Ad}(K)$-invariant. We set $p^-(x)=p(-x)$.

Corollary 2.3. *For $D=\lambda'\big(\sum p_j(x)q_j(y)\big)\in\mathbb{D}_K(G)$, set*

$$A_\tau(D)=\Lambda\Big(\sum_j q_j\otimes d\tau\big(\lambda(p_j^-)\big)\Big)=\sum_j\lambda(q_j)\otimes d\tau\big(\lambda(p_j^-)\big).$$

Then A_τ is well defined, it maps $\mathbb{D}_K(G)$ linearly onto $\mathbb{D}(E_\tau)$ and $A_\tau(D)u=Du$ for every $u\in C^\infty_\tau(G,V_\tau)$.

The identity $A_\tau(D)F=DF$ also holds for $\mathrm{End}(V_\tau)$-valued smooth functions satisfying the identity $F(xk)=\tau(k)^{-1}F(x)$, in particular for $F\in C_{\tau,\tau}\big(G,\mathrm{End}(V_\tau)\big)$.

Proof. Well defined-ness follows from the linearity of λ and $d\tau$; ontoness follows from Theorem 2.2.

For $u \in C^\infty_\tau(G, V_\tau)$, we have

$$\begin{aligned} Du(g) &= \sum_j q_j(\partial_y)_{|y=0} p_j(\partial_x)_{|x=0} u(g \exp y \exp x) \\ &= \sum_j q_j(\partial_y)_{|y=0} p_j(\partial_x)_{|x=0} \tau\big(\exp(-x)\big) u(g \exp y) \\ &= \sum_j q_j(\partial_y)_{|y=0} d\tau\big(\lambda(p_j^-)\big) u(g \exp y) \\ &= \sum_j d\tau\big(\lambda(p_j^-)\big) \lambda(q_j) u(g) \\ &= A_\tau(D) u(g). \end{aligned}$$

The last part of the statement is an obvious consequence of the fact that the space of smooth $\mathrm{End}(V_\tau)$-valued functions F satisfying the identity $F(xk) = \tau(k)^{-1} F(x)$ can be identified with $C^\infty_\tau(G, V_\tau) \otimes V'_\tau$. □

From the above corollary we can conclude the following :

- $\mathrm{Ker}(A_\tau) = \big\{ D \in \mathbb{D}_K(G) : D\,|_{C^\infty_\tau(G,V_\tau)} \big\} = \Big\{ D \in \mathbb{D}_K(G) : D\,|_{C^\infty_\tau(G)^{\mathrm{int}K}} \Big\}$
- Let S_τ and its inverse S_τ^{-1} be the operators defined in (1.5) and (1.8). For $F \in C^\infty_{\tau,\tau}(G, \mathrm{End}(V_\tau)$, $f \in C^\infty_\tau(G)^{\mathrm{int}K}$ and $D \in \mathbb{D}_K(G)$, Corollary 2.3 gives the identities
$$S_\tau\big(A_\tau(D)F\big) = S_\tau(DF) = D(S_\tau F)\ , \qquad S_\tau^{-1}(Df) = D(S_\tau^{-1} f) = A_\tau(D)(S_\tau^{-1} f).$$

Let $(\mathfrak{S}(\mathfrak{p}) \otimes \mathfrak{S}(\mathfrak{k}))^K$ denotes the $\mathrm{Ad}K$ invariant elements in $\mathfrak{S}(\mathfrak{p} \otimes \mathfrak{S}(\mathfrak{k}) = \mathfrak{S}(\mathfrak{g})$.

Proposition 2.4. *The diagram*

$$\begin{array}{ccc} \big(\mathfrak{S}(\mathfrak{p}) \otimes \mathfrak{S}(\mathfrak{k})\big)^K & \xrightarrow[\lambda']{} & \mathbb{D}_K(G) \\ {\scriptstyle I \otimes (d\tau \circ \lambda)} \Big\downarrow & & \Big\downarrow {\scriptstyle A_\tau} \\ \big(\mathfrak{S}(\mathfrak{p}) \otimes \mathrm{End}(V_\tau)\big)^K & \xrightarrow[\Lambda]{} & \mathbb{D}(E_\tau) \end{array}$$

is commutative, the horizontal arrows indicate bijections and the vertical ones surjections. Moreover, conjugation by S_τ establishes an isomorphism of algebras between $\mathbb{D}(E_\tau)$ and

$$\mathbb{D}_{K,\tau}(G) = \mathbb{D}_K(G) / \ker A_\tau .$$

Corollary 2.5. *$\mathbb{D}(E_\tau)$ and $\mathbb{D}_{K,\tau}(G)$ are finitely generated algebras.*

Proof. It suffices to prove that $\mathbb{D}_{K,\tau}(G)$ is finitely generated. By the Hilbert basis theorem, the space $I(\mathfrak{g})$ of $\mathrm{Ad}(K)$-invariant polynomials on $\mathfrak{g}$ is finitely generated. Let $u_1, \dots, u_m$ be a Hilbert basis, with $\deg u_j = d_j$. Then $\lambda'(u_j u_k) \equiv \lambda'(u_j)\lambda'(u_k)$ modulo elements of degree strictly smaller than $d_j + d_k$. This implies, by induction on the degree, that $\mathbb{D}_K(G)$ is generated by $\lambda'(u_1), \dots, \lambda'(u_m)$.

Being a quotient of $\mathbb{D}_K(G)$, $\mathbb{D}_{K,\tau}(G)$ is also finitely generated. □

3. Characterization of spherical functions as eigenfunctions

Like for the standard case of a pair (G, K) [21, 24], commutativity of convolution algebras is equivalent to commutativity of algebras of invariant differential operators, under the assumption that G/K is connected. We remark that this statement is usually formulated under the stronger assumption that G is connected.

Theorem 3.1. *Consider the following statements:*

(i) (G, K, τ) *is a commutative triple,*
(ii) $\mathbb{D}(E_\tau)$ *is commutative,*
(iii) $D_{K,\tau}(G)$ *is commutative.*

Then (i)⇒(ii)⇔(iii). *If* G/K *is connected, they are all equivalent.*

Proof. The equivalence of (ii) and (iii) follows from Proposition 2.4.

To prove that (i)⇒(iii), let $D \in D_K(G)$, $f \in C^\infty_\tau(G)^{\mathrm{int}K}$, $u \in C^\infty_c(G)^{\mathrm{int}K}$.

With $u_\tau = u * (d_\tau \bar\chi_\tau m_K)$, we have

- $Df \in C^\infty_\tau(G)^{\mathrm{int}K}$,
- $Du_\tau = (Du)_\tau \in C^\infty_\tau(G)^{\mathrm{int}K}$,
- $u * f = u_\tau * f = f * u_\tau = f * u$.

Hence, given $D_1, D_2 \in D_K(G)$,

$$\begin{aligned} u * (D_1D_2f) &= D_1D_2(u * f) = D_1D_2(u_\tau * f)\\ &= D_1D_2(f * u_\tau) = D_1\big(f * (D_2u_\tau)\big)\\ &= D_1\big((D_2u_\tau) * f\big) = (D_2u_\tau) * (D_1f)\\ &= (D_1f) * (D_2u_\tau) = D_2\big((D_1f) * u_\tau\big)\\ &= (D_1f) * (D_2u_\tau) = u_\tau * (D_2D_1f)\\ &= u * (D_2D_1f)\ . \end{aligned}$$

Applying this identity to an approximate identity $\{u_j\} \subset C^\infty_c(G)^{\mathrm{int}K}$, we conclude that $D_1D_2f = D_2D_1f$, hence $D_1D_2 = D_2D_1$ in the quotient algebra $D_{k,\tau}(G)$.

We consider now the opposite implication (iii)⇒(i) assuming first that G is connected.

Let u be any K-central analytic function. The function

$$U(x, y) = \int_K u_\tau(kxk^{-1}y)dk.$$

is analytic in $(x, y) \in G \times G$, K-central in each variable and satisfies the identity $U(k_1, k_2) = U(k_2, k_1)$ for all $k_1, k_2 \in K$.

Let $D_1, D_2 \in \mathbb{D}(G)$. Defining $D^0_i := \int_K D_i^{\mathrm{Ad}K} dk$, for any K-central function g, $D^0_i g$ is K-central. Then, for every $k_1, k_2 \in K$,

$$\begin{aligned} D_{1,x}D_{2,y}U(k_1, k_2) &= D^0_{1,x}D^0_{2,y}U(k_1, k_2)\\ &= \int_K D^0_{1,x}\big(D^0_2u_\tau(kxk^{-1}k_2)\big)\,|_{x=k_1}\, dk\\ &= \int_K D^0_{1,x}\big(D^0_2u_\tau(k^{-1}k_2kx)\big)\,|_{x=k_1}\, dk\\ &= \int_K D^0_1D^0_2u_\tau(k^{-1}k_2kk_1)\big)dk \end{aligned}$$

$$
\begin{aligned}
&= \int_K D_2^0 D_1^0 u_\tau(k^{-1}k_2kk_1)\big)dk \\
&= \int_K D_2^0 D_1^0 u_\tau(kk_1k^{-1}k_2)\big)dk \\
&= D_{2,x}D_{1,y}U(k_2,k_1) \\
&= D_{1,y}D_{2,x}U(k_2,k_1).
\end{aligned}
$$

Denote by G_e the connected component of e in G. By the analyticity of U, it follows that $U(k_1x,k_2y) = U(k_2y,k_1x)$ for all $k_1,k_2 \in K$ and $x,y \in G_e$. Since G/K is connected, every connected component of G contains an element of K. Hence $U(x,y) = U(y,x)$ for all $x,y \in G$.

This implies that, for any $f,g \in L^1_\tau(G)^{\mathrm{int}K}$ and compactly supported,

$$\int_G g * f(x)u(x)dx = \int_G g * f(x)u_\tau(x)dx = \int_G\int_G f(y)g(x)U(x,y)dxdy$$

and similarly

$$\int_G f * g(x)u(x)d = \int_G\int_G g(y)f(x)U(x,y)dxdy = \int_G\int_G g(x)f(y)U(y,x)dxdy.$$

Therefore $\int_G g * f(x)u(x)dx = \int f * g(x)u(x)$ for all K-central analytic functions u. Since analytic integrable functions are dense in $L^1(G)$, the same is true for $L^1(G)^{\mathrm{int}K}$, hence $f * g = g * f$. □

Theorem 3.2. *Let (G,K,τ) be a commutative triple, with G/K connected. For $\Phi \in L^\infty_{\tau,\tau}(G,\mathrm{End}(V_\tau))$ and $\phi = d_\tau \mathrm{Tr}\Phi \in L^\infty_\tau(G)^{\mathrm{int}(K)}$, the following are equivalent :*

(i) *Φ is a τ-spherical function;*
(ii) *Φ is a joint eigenfunction for all $\boldsymbol{D} \in \big(\mathbb{D}(G)\otimes\mathrm{End}(V_\tau)\big)^K$ and $\Phi(e) = I$.*
(iii) *Φ is a joint eigenfunction for all $D \in \mathbb{D}_K(G)$ and $\Phi(e) = 1$.*
(iv) *ϕ is a trace τ-spherical function;*
(v) *ϕ is a joint eigenfunction for all $D \in \mathbb{D}_K(G)$ and $\phi(e) = 1$.*

In particular, both kinds of spherical functions are analytic.

Proof. The equivalence of (ii) and (iii) follows from Corollary 2.3. To complete the proof, it suffices to prove the equivalence of (iv) and (v).

Lifting K-central functions on G to bi-$K^\sharp$-invariant functions on $K \ltimes G$, with $K^\sharp = \mathrm{diag}(K)$, one can see that condition (1.10) is equivalent to the functional equation $\int_K \phi^\sharp(gkg')dk = \phi^\sharp(g)\phi^\sharp(g')$ for the lifted function $\phi^\sharp$.

The proof in [16, Ch. IV, Prop. 2.2] can then be adapted to prove that a non trivial K-central bounded function ϕ satisfies (1.10) if and only if it is an eigenfunction of all $D \in \mathbb{D}_K(G)$.

Analiticity of ϕ follows from the existence of a K-central left-invariant Laplacian on G, and that of Φ by Theorem 1.4. □

Corollary 3.3. *Let (G,K,τ) be a commutative triple with G/K is connected. Then (G,K) is a Gelfand pair. In particular, G is unimodular.*

Proof. By Theorem 3.1, $\mathbb{D}(E_\tau)$ is commutative. Let $I(\mathfrak{p})$ denote the space of all $\mathrm{Ad}(K)$-invariant polynomials on $\mathfrak{p}$. Then clearly $I(\mathfrak{p})\otimes I \subset \big(\mathfrak{S}(\mathfrak{p})\otimes\mathrm{End}(V_\tau)\big)^K$. Therefore, in view of Theorem 2.2, it follows that $\lambda(I(\mathfrak{p}))$ is commutative. But this implies, by [16, Thm. 4.9], that $\mathbb{D}(G/K)$ is commutative. Hence (G,K) is a Gelfand pair. □

4. Gelfand spectrum and embeddings into Euclidean spaces

The following theorem says that the τ-spherical functions are uniquely determined by the set of their eigenvalues.

Theorem 4.1. *Let (G, K, τ) be a commutative triple. Let ϕ_1 and ϕ_2 be two trace τ-spherical function so that $D\phi_i = \mu_i(D)\phi_i$ (i=1,2) for all $D \in \mathbb{D}_K(G)$. If $\mu_1(D) = \mu_2(D)$ for all $D \in \mathbb{D}_K(G)$, then $\phi_1 = \phi_2$.*

Proof. By the given conditions we have, $D\phi_1(e) = D\phi_2(e)$ for all $D \in \mathbb{D}_K(G)$. Now let $D \in \mathbb{D}(G)$. Define $D_0 = \int_K D^{\mathrm{Ad}k} dk$ which clearly belongs to $\mathbb{D}_K(G)$. Since ϕ_i are K-central, it follows that $D_0\phi_i(e) = D\phi_i(e)$. Therefore $D\phi_1(e) = D\phi_2(e)$ for all $D \in \mathbb{D}(G)$. But ϕ_1, ϕ_2 being analytic, they must coincide on whole of G. □

Let Σ_τ be the set of all trace τ-spherical function. By Corollary 2.5, we can fix a finite set $\mathcal{D} = (D_1, \cdots, D_k)$ of generators of $\mathbb{D}_{K,\tau}(G)$. Then the map $\rho_{\mathcal{D}}$ which assigns to a $\phi \in \Sigma_\tau$ the k-tuple

$$\rho_{\mathcal{D}}(\phi) = \big(\lambda_{D_1}(\phi), \cdots, \lambda_{D_k}(\phi)\big) \in \mathbb{C}^k$$

of its eigenvalues is injective, by Proposition 4.1. Assume that Σ_τ is endowed with the weak*-topology.

Theorem 4.2. *On Σ_τ, the weak*-topology and the compact-open topology coincide. The map $\rho_{\mathcal{D}}$ is a homeomorphism from Σ_τ to its image in $\mathbb{C}^k$, and $\rho_{\mathcal{D}}(\Sigma_\tau)$ is closed.*

Proof. Introducing coordinates $t = (t_1, \cdots, t_n)$ on $\mathfrak{g}$ according to the exp map, let $|t|$ an $\mathrm{Ad}(K)$-invariant norm on $\mathfrak{g}$. The operator $\triangle := \lambda(|t|^2)$ is in $\mathbb{D}_K(G)$ and is elliptic (a Laplacian). Being an eigenfunction of $\triangle$, any $\phi \in \Sigma_\tau$ is real-analytic. The rest of the proof goes as in [8, Thm. 10 and Cor. 11]. □

5. Strong Gelfand pairs

Given a pair (G, K), with K a compact subgroup of G, there may be several $\tau \in \hat{K}$ such that (G, K, τ) is a commutative triple. We have already seen that if a nontrivial τ gives a commutative triple and G/K is connected, then also the trivial representation does.

The extreme situation occurs when (G, K, τ) is a commutative triple for all $\tau \in \hat{K}$. When this occurs, one says that (G, K) is a *strong Gelfand pair*.

Lemma 5.1. *The following are equivalent:*

(i) *(G, K) is a strong Gelfand pair,*
(ii) *$(K \ltimes G, K)$ is a Gelfand pair, with K acting on G by inner automorphisms,*
(iii) *$L^1(G)^{\mathrm{int}K}$ is commutative,*
(iv) *(for G connected) $\mathbb{D}_K(G)$ is commutative.*

This follows easily from the identities

$$L^1(G)^{\mathrm{int}K} = \sum_{\tau \in \widehat{K}} L^1_\tau(G)^{\mathrm{int}K} \ , \qquad L^1_\tau(G)^{\mathrm{int}K} * L^1_\sigma(G)^{\mathrm{int}K} = \{0\} \ , \quad \tau \not\sim \sigma \ .$$

Definition. *Let (G,K) be a strong Gelfand pair. A non zero $\phi \in L^\infty(G)^{\mathrm{int}K}$ is said to be a $L^1(G)^{\mathrm{int}K}$-spherical function if the map*

$$f \to \int_G f(g)\phi(g^{-1})dg$$

is an an algebra homomorphism of $L^1(G)^{\mathrm{int}K}$ onto $\mathbb{C}$.

The following statement follows directly from (ii) in Lemma 5.1.

Theorem 5.2. *Let (G,K) be a strong Gelfand pair. The following are equivalent:*

(i) *ϕ is a $L^1(G)^{\mathrm{int}K}$-spherical function,*

(ii) *ϕ is nonzero, bounded and satisfies the functional equation*

$$\int_K \phi(xkyk^{-1})dk = \phi(x)\phi(y), \tag{5.1}$$

(iii) *(if G is connected) $\phi(e) = 1$, ϕ is K-central, analytic and is a joint eigenfunction of all $D \in \mathbb{D}_K(G)$.*

If ϕ is a $L^1(G)^{\mathrm{int}K}$-spherical function, then, by the functional equation (5.1), $\phi(e) = 1 \neq 0$. Therefore, in view of Theorem 5.2 (iii) and Proposition 3.2, we have the following proposition.

Proposition 5.3. *Let (G,K) be a strong Gelfand pair. Let ϕ be a $L^1(G)^{\mathrm{int}K}$-spherical function. Then there is unique $\tau \in \widehat{K}$ such that $\phi \in L^\infty(G)^{\mathrm{int}K}_\tau$. Hence ϕ is an trace τ-spherical function.*

Proof. Since ϕ is nonzero, there is $\tau \in \hat{K}$ such that $\phi_\tau := \phi * (d_\tau \bar{\chi}_\tau m_K)$ is nonzero. By (5.1), we obtain that, for all $x, y \in G$,

$$\phi_\tau(x)\phi(y) = \phi(x)\phi_\tau(y) \ .$$

With $y = e$, this gives that $\phi_\tau = \phi_\tau(e)\phi$. Hence $\phi_\tau(e) \neq 0$, $\phi \in L^\infty(G)^{\mathrm{int}K}_\tau$, and finally $\phi = \phi_\tau$. ☐

Corollary 5.4. *Let Σ denote the spectrum of the Gelfand pair $(K \ltimes G, K)$. Then Σ is the disjoint union of the spectra Σ_τ of the triples (G, K, τ), $\tau \in \hat{K}$. Each Σ_τ is open and closed in Σ.*

Proof. For each $\tau \in \hat{K}$, the map ρ_τ which assigns to a bounded spherical function $\phi \in \Sigma$ the value

$$\rho_\tau(\phi) = \phi * (d_\tau \bar{\chi}_\tau m_K)(e) = \int_K \phi(k)\chi_\tau(k)dk \ ,$$

is continuous on Σ and only takes values 0 or 1. This implies that $\Sigma_\tau = \rho_\tau^{-1}(1)$ is closed. On the other hand, if a sequence of functions $\varphi_n \in \Sigma$ converges to $\phi \in \Sigma_\tau$, then $\rho_\tau(\phi_n)$ must be eventually be equal to 1. Then $\phi_n \in \Sigma_\tau$ eventually. This proves that Σ_τ is open. ☐

6. K-homogeneous bundles over a Lie group H

In this section we consider the special case where $G = K \ltimes H$, H being a Lie group and K a compact group of automorphisms of H, and (τ, V_τ) is an irreducible unitary representation of K. We denote by $k \cdot x$ the action of $k \in K$ on $x \in H$. The product on $K \ltimes H$ is given by

$$(k, x)(k', x') = \big(kk', x(k \cdot x')\big).$$

The sections of E_τ, i.e., the V_τ-valued functions u on G satisfying the identity $u(gk) = \tau(k)^{-1}u(g)$, are naturally identified with V_τ-valued functions u_0 on H, via the map T given by

$$T: \quad u_0(x) \longmapsto u(k,x) = \tau(k)^{-1}u_0(x). \tag{6.1}$$

The action of H on u_0 is given by left translations and that of an element $k \in K$ by

$$k: \quad u_0(x) \longmapsto \tau(k)u_0(k^{-1} \cdot x). \tag{6.2}$$

Similarly, the integral operators on V_τ-valued functions on H commuting with the action of G are given, in analogy with (1.2), by

$$u(x) \longmapsto \int_H F(y^{-1}x)u(y)dy,$$

where $F : H \to \mathrm{End}(V_\tau)$ satisfies the identity

$$F(k \cdot x) = \tau(k)F(x)\tau(k^{-1}). \tag{6.3}$$

This is equivalent to saying that

$$TF(k,x) = \tau(k)^{-1}F(x) \in L^1_{\tau,\tau}(G).$$

Composition of integral operators corresponds to convolution of $F_1, F_2 \in L^1_\tau(H)$, defined as

$$F_1 * F_2(x) = \int_H F_2(y^{-1}x)F_1(y)dy.$$

Then, under this convolution, $L^1_\tau(H)$ becomes an algebra.

Therefore $(K \ltimes H, K, \tau)$ is a commutative triple iff $L^1_\tau(H)$ is commutative. In this case H is unimodular.

As for Gelfand pairs, here too we have a reformulation of Theorem 1.1. Let $\widehat{H}$ be the dual object of H, i.e., the set of equivalence classes $[\pi]$ of irreducible unitary representations π of H. The group K acts on $\widehat{H}$ in the following way: given $k \in K$ and π irreducible and unitary, $\pi^k(x) = \pi(k^{-1} \cdot x)$ defines an irreducible and unitary representation of H, which may or may not be equivalent to π. Note that if $\pi_1 \sim \pi_2$ then $\pi_1^k \sim \pi_2^k$. So we can set $k \cdot [\pi] = [\pi^k]$.

Let K_π be the stabilizer of $[\pi]$, which is clearly compact. For $k \in K_\pi$, there exists a (unique up to a unitary factor) unitary operator $\delta(k)$ on $\mathcal{H}_\pi$ (the Hilbert space where π is realized) which intertwines π with π^k, i.e., $\pi^k(x) = \delta(k^{-1})\pi(x)\delta(k)$ for all $x \in H$. This defines a projective unitary representation δ of K_π on $\mathcal{H}_\pi$. Consider the representation $\delta \otimes (\tau|_{K_\pi})$ of K_π on $\mathcal{H}_\pi \otimes V_\tau$. Since K_π is compact, $\delta \otimes (\tau|_{K_\pi})$ is completely reducible. Then we have the following theorem giving a characterization for (G, K, τ) to be a commutative triple, in analogy with [6].

Theorem 6.1. *$(K \ltimes H, K, \tau)$ is a commutative triple iff for each $\pi \in \widehat{H}$, $\delta \otimes (\tau|_{K_\pi})$ is multiplicity free.*

To prove the Theorem we need the following lemma. Let $\mathcal{L}(\mathcal{H}_\pi)$ denote the set of all bounded operators on $\mathcal{H}_\pi$. For $F \in L^1(H) \otimes \mathrm{End}(V_\tau)$, define $\pi(F) \in \mathcal{L}(\mathcal{H}_\pi) \otimes \mathrm{End}(V_\tau) \cong \mathcal{L}(\mathcal{H}_\pi \otimes V_\tau)$ by

$$\pi(F) = \int_H \pi(x) \otimes F(x)dx.$$

Lemma 6.2. *Let $F \in L^1_\tau(H)$. Then $\pi(F)$ intertwines $\delta \otimes (\tau|_{K_\pi})$ with itself.*

Proof. We have, for $k \in K_\pi$,

$$\begin{aligned}\pi(F)\big(\delta(k)\otimes\tau(k)\big) &= \int_H \big(\pi(x)\delta(k)\big)\otimes\big(F(x)\tau(k)\big)dx\\ &= \int_H \big(\delta(k)\pi^k(x)\big)\otimes\big(\tau(k)F(k^{-1}\cdot x)\big)dx\\ &= \int_H \big(\delta(k)\pi(k^{-1}\cdot x)\big)\otimes\big(\tau(k)F(k^{-1}\cdot x)\big)dx\\ &= \big(\delta(k)\otimes\tau(k)\big)\pi(F). \qquad \square\end{aligned}$$

Now we prove Theorem 6.1. For the proof of 'only if' part, we follow [6], cf. also [3, Thm.3.5]

Proof of Theorem 6.1. Let (G, K, τ) be a commutative triple and let $\pi \in \widehat{H}$. If σ is the multiplier of the associated projective representation δ of K_π, let $\widehat{K_\pi}^\sigma$ denote the set of equivalence classes of unitary irreducible projective representation of K_π with multiplier σ. Then

$$\delta = \bigoplus_{\rho\in\widehat{K_\pi}^\sigma} c(\rho,\delta)\rho,$$

where $c(\rho,\delta)$ is the multiplicity if ρ in δ. For $\rho \in \widehat{K_\pi}^\sigma$, let ρ' be its contragredient representation, with multiplier $\bar\sigma$. Then $R(k,x) := \rho'(k)\otimes\pi(x)\delta(k)$ defines an irreducible linear representation of $K_\pi \ltimes H$ and the induced representation $\mathrm{Ind}_{K_\pi\ltimes H}^{K\ltimes H} R$ is irreducible. Also,

$$(\mathrm{Ind}_{K_\pi\ltimes H}^{K\ltimes H} R)\mid_K \simeq \mathrm{Ind}_{K_\pi}^{K}(R\mid_{K_\pi}) = \mathrm{Ind}_{K_\pi}^{K}(\rho'\otimes\delta).$$

By Frobenius reciprocity,

$$c\big(\tau, (\mathrm{Ind}_{K_\pi\ltimes H}^{K\ltimes H} R)\mid_K\big) = c(\tau\mid_{K_\pi}, \rho'\otimes\delta).$$

Since $(K\ltimes H, K, \tau)$ is a commutative triple, so is $(K\ltimes H, K, \tau')$. Therefore

$$c\big(\tau', (\mathrm{Ind}_{K_\pi\ltimes H}^{K\ltimes H} R)\mid_K\big) = 0 \text{ or } 1,$$

and hence $c(\tau'\mid_{K_\pi}, \rho'\otimes\delta) = 0$ or 1. But

$$c(\tau'\mid_{K_\pi}, \rho'\otimes\delta) = c\big(1, \rho'\otimes\delta\otimes(\tau\mid_{K_\pi})\big) = c\big(\rho, \delta\otimes(\tau\mid_{K_\pi})\big).$$

Hence it follows that $\delta\otimes(\tau\mid_{K_\pi})$ is multiplicity free.

Conversely, assume that the K_π-action on $\mathcal{H}_\pi\otimes V_\tau$ is multiplicity free for all $\pi\in\widehat{H}$. Then it follows from Lemma 6.2 that $\pi(F)$ and $\pi(G)$ commutes whenever $F, G \in L^1_\tau(H)$. Since this is true for all $\pi\in\widehat{H}$, by uniqueness of the Fourier transform we can conclude that $F*G = G*F$. $\square$

Definition. *Let $(K\ltimes H, K, \tau)$ be a commutative triple. A non-trivial function $\Psi\in L^\infty_\tau(H)$ is said to be a τ-spherical function if the map*

$$F \to \widehat{F}(\Psi) := \frac{1}{d_\tau}\int_H \mathrm{Tr}[\Psi(x^{-1})F(x)]dx$$

is a homomorphism of $L^1_\tau(H)$ into $\mathbb{C}$. Here d_τ is the dimension of V_τ.

Recall the definition of T from 6.1.

Theorem 6.3. *The following are equivalent:*

(i) Ψ *is a τ-spherical function on H;*
(ii) $\Phi = T(\Psi)$ *is a τ-spherical function on $K\ltimes H$;*

(iii) $\Psi \in L^\infty_\tau(H)$ *is non-trivial and, for all* $h, h' \in H$*, satisfies the identity*

$$d_\tau \int_K \tau(k^{-1})\Psi\big(h(k\cdot h')\big)\chi_\tau(k)dk = \Psi(h')\Psi(h). \tag{6.4}$$

Proof. An easy calculation shows that $\widehat{T(F)}(\Phi) = \widehat{F}(\Psi)$ for all $F \in L^1_\tau(H)$. This proves the equivalence of (i) and (ii).

In view of Theorem 1.4, it is enough to show that (1.9) is equivalent to (6.4). If $x = (e, h)$ and $y = (e, h')$ then $\Phi(xky) = \Phi(k, hk\cdot h') = \tau(k^{-1})\Psi(hk\cdot h')$. Therefore, putting $x = (e, h)$ and $y = (e, h')$ in (1.9) we get (6.4). Conversely, if $x = (k_1, h_1)$ and $y(k_2, h_2)$ then

$$\begin{aligned}
d_\tau \int_K \Phi(xky)\chi_\tau(k)dk &= d_\tau \int_K \Phi\big(k_1kk_2, h_1(k_1k\cdot h_2)\big)\chi_\tau(k)dk \\
&= d_\tau \int_K \tau(k_1kk_2)^{-1}\Psi\big(h_1(k_1k\cdot h_2)\big)\chi_\tau(k)dk \\
&= d_\tau \int_K \tau(k_1kk_2)^{-1}\tau(k_1)\Psi\big((k_1^{-1}\cdot h_1)(k\cdot h_2)\big)\tau(k_1^{-1})\chi_\tau(k)dk \\
&= \tau(k_2^{-1})\, d_\tau \int_K \tau(k^{-1})\Psi\big((k_1^{-1}\cdot h_1)(k\cdot h_2)\big)\chi_\tau(k)dk\, \tau(k_1^{-1}) \\
&= \tau(k_2^{-1})\Psi(h_2)\Psi(k_1^{-1}\cdot h_1)\tau(k_1^{-1}) = \Phi(y)\Phi(x). \qquad \square
\end{aligned}$$

7. Differential operators on K-homogeneous bundles over H

Once the sections of E_τ have been identified with V_τ-valued functions on H, we can also realize the elements of $\mathbb{D}(E_\tau)$ as differential operators on V_τ-valued functions on H which are left-invariant and commute with the action (6.2) of K.

We denote the algebra of such operators by $\big(\mathbb{D}(H)\otimes \mathrm{End}(V_\tau)\big)^K$.

Keeping Theorem 2.2 in mind, we can choose $\mathfrak{p}$, the $\mathrm{Ad}(K)$-invariant complement of $\mathfrak{k}$ in $\mathfrak{g}$, to be $\mathfrak{h}$, the Lie algebra of H, and define the map

$$\Lambda' = \lambda_{\mathfrak{h}} \otimes I : \big(\mathfrak{S}(\mathfrak{h})\otimes \mathrm{End}(V_\tau)\big)^K \longrightarrow \big(\mathbb{D}(H)\otimes \mathrm{End}(V_\tau)\big)^K.$$

The two maps Λ of Lemma 2.1 and Λ' are conjugate of each other under the map T in (6.1). This gives the following theorem.

Theorem 7.1. *The algebras* $\mathbb{D}(E_\tau)$ *and* $\big(\mathbb{D}(H)\otimes \mathrm{End}(V_\tau)\big)^K$ *are isomorphic. In particular,* $(K \ltimes H, K, \tau)$ *is a commutative triple if and only if* $\big(\mathbb{D}(H)\otimes \mathrm{End}(V_\tau)\big)^K$ *is commutative.*

Corollary 7.2. *Let* $(K \ltimes H, K, \tau)$ *be a commutative triple and* $\Psi \in L^\infty_\tau(H)$*. The following are equivalent:*

(i) Ψ *is a* τ*-spherical function.*

(ii) Ψ *is a joint eigenfunction for all* $\boldsymbol{D} \in \big(\mathbb{D}(H)\otimes \mathrm{End}(V_\tau)\big)^K$.

It must be noticed that we have no analogue of Corollary 2.3 for differential operators on H, i.e., the effective action of $\big(\mathbb{D}(H)\otimes \mathrm{End}(V_\tau)\big)^K$ on C^∞ V_τ-valued functions includes the action of $\mathbb{D}_K(H)$ properly. However, we have the following proposition.

Proposition 7.3. $\big(\mathbb{D}(H)\otimes \mathrm{End}(V_\tau)\big)^K$ *is a finite module over* $\mathbb{D}_K(H)$.

Proof. Given $P \in \big(\mathfrak{S}(\mathfrak{h}) \otimes \mathrm{End}(V_\tau)\big)^K$, consider the characteristic polynomial

$$\det\big(P(x) - \lambda I\big) = \pm\big(\lambda^d + q_1(x)\lambda^{d-1} + \cdots + q_d(x)\big) ,$$

where $q_j \in \mathfrak{S}(\mathfrak{h})^K$ for $j = 1, \dots, d$. It follows from the Cayley-Hamilton theorem that

$$P(x)^d = -q_1(x)P(x)^{d-1} - \cdots - q_d(x)I .$$

Combining this with Corollary 2.5, we obtain the conclusion. □

8. Relations among the spectra Σ_τ and Σ_1

It follows from Corollary 3.3 that, if $(K \ltimes H, K, \tau)$ is a commutative triple, the same is true with τ replaced by the trivial representation, i.e., $(K \ltimes H, K)$ is a Gelfand pair. We establish relations between the two spectra, Σ_τ and Σ_1 respectively, also in terms of their embeddings into complex spaces as introduced in Section 4. We start from the following statement.

Lemma 8.1. *Given an τ-spherical function Ψ, the function $\psi = d_\tau^{-1}\mathrm{Tr}\,\Psi$ is a usual spherical function for the Gelfand pair $(K \ltimes H, K)$. For $D \in \mathbb{D}(H)^K$, the following relation between eigenvalues holds:*

$$\lambda_D(\psi) = \lambda_{D\otimes I}(\Psi) .$$

The map $d_\tau^{-1}\mathrm{Tr}$ is continuous from Σ_τ to Σ_1.

Proof. Since Ψ satisfies (6.3), $\mathrm{Tr}\,\Psi$ is K-invariant. Moreover, $\psi(e) = 1$ and, for $D \in \mathbb{D}(H)^K$,

$$D\psi = \frac{1}{d_\tau}D(\mathrm{Tr}\,\Psi) = \frac{1}{d_\tau}\mathrm{Tr}\,\big((D \otimes I)\Psi\big) = \lambda_{D\otimes I}(\Psi)\psi.$$

Continuity of the map $\Psi \mapsto \psi$ with respect to the topologies of uniform convergence on compact sets is obvious. □

Reformulating the results of Section 3 in terms of $\mathrm{End}(V_\tau)$-valued spherical functions and differential operators, and specializing to the present situation where $G = K \ltimes H$, such embeddings depend on the choice of a finite generating system $\mathcal{D} = \{\boldsymbol{D}_j\} \subset \big(\mathbb{D}(H) \otimes \mathrm{End}(V_\tau)\big)^K$.

It is convenient to consider systems $\mathcal{D}$ formed by a generating system $\mathcal{D}_0 = \{D_1, \dots, D_h\}$ of $\mathbb{D}_K(H)$, completed with $\boldsymbol{D}_{h+1}, \dots, \boldsymbol{D}_d$ generating $\big(\mathbb{D}(H) \otimes \mathrm{End}(V_\tau)\big)^K$ as a $\mathbb{D}_K(H)$-module. Then every element $\boldsymbol{D} \in \big(\mathbb{D}(H) \otimes \mathrm{End}(V_\tau)\big)^K$ can be expressed in the form

$$\boldsymbol{D} = L_0 \otimes I + \sum_{j=h+1}^{d} L_j \boldsymbol{D}_j,$$

with $L_0, L_{h+1}, \dots, L_d \in \mathbb{D}(H)^K$, it follows that, setting $\boldsymbol{D}_j = D_j \otimes I$ for $j = 1, \dots, h$, $\big(\mathbb{D}(H) \otimes \mathrm{End}(V_\tau)\big)^K$ is generated by

$$\mathcal{D} = \{\boldsymbol{D}_1, \dots, \boldsymbol{D}_d\}.$$

Simultaneously, the generating system $\mathcal{D}_0 = \{D_1, \dots, D_h\}$ of $\mathbb{D}_K(G)$ induces an embedding $\rho_{\mathcal{D}_0}$ of the spectrum Σ_1 of the Gelfand pair (G, K) into $\mathbb{C}^h$.

With π_1 denoting the canonical projection of $\mathbb{C}^h \times \mathbb{C}^{d-h}$ onto its first factor, we have the following commutative diagram:

$$\begin{array}{ccc}
\Sigma_\tau & \xrightarrow[\rho_{\mathcal{D}}]{\qquad\qquad\qquad} & \rho_{\mathcal{D}}(\Sigma_\tau) \\
{\scriptstyle d_\tau^{-1}\mathrm{Tr}}\Big\downarrow & & \Big\downarrow{\scriptstyle \pi_1} \\
\Sigma_1 & \xrightarrow[\rho_{\mathcal{D}_0}]{\qquad\qquad\qquad} & \rho_{\mathcal{D}_0}(\Sigma_1)
\end{array}$$

In particular, π_1 maps $\rho_{\mathcal{D}}(\Sigma_\tau)$ into $\rho_{\mathcal{D}_0}(\Sigma_1)$ and, if $\psi = d_\tau^{-1}\mathrm{Tr}\,\Psi$, $\rho_{\mathcal{D}_0}(\psi) = \pi_1 \circ \rho_{\mathcal{D}}(\Psi)$.

9. Spherical functions of positive type

Let V be a finite dimensional Euclidean complex space, i.e., endowed with a positive definite Hermitean product. A continuous $\mathrm{End}(V)$-valued function F is of positive type on a group G if any of the following equivalent conditions holds, for every finite choice of elements $x_1, \dots, x_n \in G$:

(1) for every $v_1, \dots, v_n \in V$,

$$\sum_{j,k} \langle F(x_j x_k^{-1}) v_k, v_j \rangle \geq 0. \tag{9.1}$$

(2) for every $v_1, \dots, v_n \in V$, the matrix

$$\big(\langle F(x_j x_k^{-1}) v_k, v_j \rangle\big)_{jk}$$

is positive semi-definite;

(3) for every $B_1, \dots, B_n \in \mathrm{End}(V)$,

$$\sum_{j,k} B_j^* F(x_j x_k^{-1}) B_k \in \mathrm{End}(V)$$

is positive semi-definite;

(4) for any $E \in C_c^\infty(G, \mathrm{End}(V)))$

$$\int_G \mathrm{Tr}\big[(E^* * E)(x) F(x^{-1})\big]\, dx \geq 0. \tag{9.2}$$

Moreover any measurable function F satisfying condition (4) is continuous and of positive type.

Proposition 9.1. *Let F be of positive type. The following properties hold.*

(i) *$F(e)$ is positive semi-definite;*

(ii) *for every $x \in G$, $F(x^{-1}) = F(x)^*$;*

(iii) *fore every $v \in V$, $\psi_v(x) = \langle F(x)v, v\rangle$ is of positive type;*

(iv) $\ker F(e) \subseteq \ker F(x)$ *and* $\mathrm{im}\, F(x) \subseteq \mathrm{im}\, F(e)$ *for every $x \in G$;*

(v) *for every $x \in G$, $F(x)^* F(x) \leq F(e)^2$.*

Proof. To prove (i), apply (9.1) with $n = 1$.

To prove (ii), apply (9.1) with $n = 2$, $x_1 = x$, $x_2 = e$. Then, for every v_1, v_2,

$$\langle F(e)v_1, v_1\rangle + \langle F(x)v_1, v_2\rangle + \langle F(x^{-1})v_2, v_1\rangle + \langle F(e)v_2, v_2\rangle \geq 0\ . \tag{9.3}$$

Then

$$\mathrm{Im}\big(\langle F(x)v_1, v_2\rangle + \langle F(x^{-1})v_2, v_1\rangle\big) = 0\ ,$$

i.e., $\mathrm{Im}\langle F(x)v_1, v_2\rangle = \mathrm{Im}\langle F(x^{-1})^* v_1, v_2\rangle$. Replacing v_1 by iv_1, we find that also the real parts are equal, and (ii) follows.

(iii) follows from condition (2) with $v_j = v$ for all j.

To prove (iv), apply (9.1) with $n = 2, x_1 = x, x_2 = e, v_1 = v_1, v_2 = \lambda v_2$, where λ is real. Then we get

$$\langle F(e)v_1, v_1\rangle + \lambda^2\langle F(e)v_2, v_2\rangle + \lambda\langle F(x^{-1})v_2, v_1\rangle + \lambda\langle F(x)v_1, v_2\rangle \geq 0 \ .$$

If $v_1 \in \mathrm{Ker}F(e)$,

$$\lambda^2\langle F(e)v_2, v_2\rangle + 2\lambda\mathrm{Re}\langle F(x)v_1, v_2\rangle \geq 0$$

which implies that

$$|\lambda|\langle F(e)v_2, v_2\rangle \geq |\mathrm{Re}\langle F(x)v_1, v_2\rangle|$$

for all real λ. Taking $\lambda \to 0$, we deduce that $\mathrm{Re}\langle F(x)v_1, v_2\rangle = 0$. Replacing $v_2 = iv_2$, we also get that imaginary part of $\langle F(x)v_1, v_2\rangle$ is zero, so that $\langle F(x)v_1, v_2\rangle = 0$. Since this is true for any $v_1 \in \mathrm{Ker}F(e), v_2 \in V$, we conclude that $\mathrm{Ker}F(x) \subset \mathrm{Ker}F(e)$.

(v) is equivalent to the condition $\|F(x)v\| \leq \|F(e)v\|$ for all $v \in V$. By (iv), it is sufficient to take $v \in \big(\mathrm{ker}F(e)\big)^\perp$. So we may assume that $F(e)$ is invertible. In this case, replacing $F(x)$ by $F(e)^{-\frac{1}{2}}F(x)F(e)^{-\frac{1}{2}}$, which remains of positive type, we may assume that $F(e) = I$. Applying (9.3) with v_2 replaced by $e^{i\theta}F(x)v_2$, we have

$$\|v_1\|^2 + 2\mathrm{Re}\big(e^{-i\theta}\langle F(x)v_1, F(x)v_2\rangle\big) + \|F(x)v_2\|^2 \geq 0 \ ,$$

which gives, by the arbitrarity of θ,

$$2\big|\langle v_1, F(x)^*F(x)v_2\rangle\big| \leq \|v_1\|^2 + \|F(x)v_2\|^2 \ .$$

Passing to the supremum over v_1 of unit norm, we obtain the inequality

$$2\|F(x)^*F(x)v_2\| \leq 1 + \|F(x)v_2\|^2 \ .$$

For $\|v_2\| = 1$, this imples that

$$2\|F(x)v_2\|^2 = 2\langle F(x)^*F(x)v_2, v_2\rangle \leq 2\|F(x)^*F(x)v_2\| \leq 1 + \|F(x)v_2\|^2 \ ,$$

whence $\|F(x)v_2\| \leq 1$. □

The proofs of the following result can be found in [20], [25, vol. II p. 15-16 and Remark].

Theorem 9.2. *Let (G, K, τ) be a commutative triple. Given $\pi \in \widehat{G}$ such that $\tau \subset \pi_{|_K}$, (say, with $V_\tau \subset \mathcal{H}_\pi$), the function Φ given by*

$$\langle\Phi(x)u, v\rangle = \langle\pi(x)u, v\rangle \ , \tag{9.4}$$

with $u, v \in V_\tau$, is τ-spherical and of positive type. Conversely, every spherical function of positive type arises in this way.

Consider now the case where $G = K \ltimes H$.

Proposition 9.3. *Let F be an* $\mathrm{End}(V_\tau)$*-valued measurable function on H. Then F is of positive type if and only if $T(F)$, defined according to* (6.1)*, is of positive type on G.*

This leads us to the following description of the $L^1_\tau(H)$-spherical functions of positive type only in terms of irreducible unitary representations of H, instead of representations of $K \ltimes H$.

Fix $\pi \in \widehat{H}$. Let $\mathcal{H}_\pi \otimes V_\tau = \oplus_\alpha W_\alpha(\pi)$ be the multiplicity-free decomposition into irreducible invariant subspaces under the action of $\delta \otimes \tau$, where α runs over an index set $\Lambda = \Lambda(\pi)$. Let $P_\alpha = P_\alpha(\pi)$ denote the orthogonal projection onto $W_\alpha = W_\alpha(\pi)$. If $F \in L^1_\tau(H)$, $\pi(F) \in \mathcal{L}_{K_\pi}(\mathcal{H}_\pi \otimes V_\tau)$

by Lemma 6.2. Therefore $\pi(F) = \oplus_\alpha \widehat{F}(\pi,\alpha)P_\alpha$ for some constants $\widehat{F}(\pi,\alpha)$. Since $\pi(F*G) = \pi(F)\pi(G)$, it follows that, for each α, $F \to \widehat{F}(\pi,\alpha)$ defines a multiplicative linear functional of $L^1_\tau(H)$.

Since for $F \in L^1_\tau(H)$, $\pi(F)$ can be written as

$$\pi(F) = \int_K \int_H \big[I \otimes \tau(k^{-1})\big]\big[\pi(k^{-1}\cdot x) \otimes F(x)\big][I \otimes \tau(k)]dxdk,$$

we have

$$\begin{aligned}\widehat{F}(\pi,\alpha) &= \frac{1}{d_\alpha}\mathrm{Tr}[\pi(F)P_\alpha] \\ &= \frac{1}{d_\alpha}\mathrm{Tr}\Big[\int_K\int_H \big[\pi(k^{-1}\cdot x)\otimes F(x)\big][I\otimes\tau(k)]P_\alpha\big[I\otimes\tau(k^{-1})\big]dxdk\Big],\end{aligned}$$

Defining $\mathrm{Tr}_{\mathcal{H}_\pi}$ as the partial trace of an element of $\mathcal{L}(\mathcal{H}_\pi \otimes V_\tau)$ relative to $\mathcal{H}_\pi$, and setting

$$\Phi_{\pi,\alpha}(x) = \frac{d_\tau}{d_\alpha}\mathrm{Tr}_{\mathcal{H}_\pi}\Big[\int_K \big[\pi(k^{-1}\cdot x)\otimes I\big][I\otimes\tau(k)]P_\alpha\big[I\otimes\tau(k^{-1})\big]dk\Big], \tag{9.5}$$

we then have

$$\begin{aligned}\widehat{F}(\pi,\alpha) &= \frac{1}{d_\alpha}\mathrm{Tr}\Big[\int_H \big[I\otimes F(x)\big]\int_K \big[\pi(k^{-1}\cdot x)\otimes I\big][I\otimes\tau(k)]P_\alpha\big[I\otimes\tau(k^{-1})\big]dkdx\Big] \\ &= \frac{1}{d_\tau}\mathrm{Tr}\Big[\int_H F(x)\Phi_{\pi,\alpha}(x)dx\Big].\end{aligned}$$

This is true for all $F \in L^1_\tau(H)$. Also, note that $\Phi_{\pi,\alpha} \in L^\infty_\tau(H)$. Therefore $\Phi_{\pi,\alpha}$ is a τ-spherical function.

In general, not all bounded spherical functions are of positive type. The following case where the two classes coincide is particularly relevant in view of Vinberg's structure theorem for Gelfand pairs [23, Thm. 5]. The method of proof is taken from [3] and adapted to the nonscalar case.

Theorem 9.4. *Let $(K \ltimes H, K, \tau)$ be a commutative triple, with H nilpotent. Then all bounded τ -spherical functions on H are of positive type.*

Proof. Let Ψ be a bounded τ-spherical function on H i.e. the linear map $\lambda_\Psi : F \to \widehat{F}(\Psi)$ is a non-trivial homomorphism of $L^1_\tau(H)$ onto $\mathbb{C}$. Since H is a locally compact nilpotent lie group, by Corollary 6 in [19], $L^1(H, \mathrm{End}(V_\tau))$ is symmetric Banach $*$-algebra. By Theorem 1 in [18] (page-305), it follows that, there is an irreducible representation $(\pi, \mathcal{H}_\pi)$ of $L^1(H,\mathrm{End}(V_\tau))$ together with a one dimensional subspace H_Ψ of $\mathcal{H}_\pi$ such that $\big(\pi\,|_{L^1_\tau(H)}, H_\Psi\big)$ is equivalent to $(\lambda_\Psi, \mathbb{C})$. Therefore, there is a vector $v_0 \in H_\Psi$ such that $\widehat{F}(\Psi) = \langle \pi(F)v_0, v_0\rangle$ for all $F \in L^1_\tau(H)$. Now, we use Proposition 9.3 to prove that Ψ is of positive type. Let $G \in C^\infty_c(H, \mathrm{End}(V_\tau))$. Define

$$G^\#(x) := \int_K \tau(k^{-1})F(k\cdot x)\tau(k)dk.$$

Then

$$\begin{aligned}\int_H \mathrm{Tr}\big[(G^* * G)(x)\Psi(x^{-1})\big]dx &= \int_H \mathrm{Tr}\big[(G^* * G)^\#(x)\Psi(x^{-1})\big]dx \\ &= \int_H \mathrm{Tr}\big[((G^\#)^* * G^\#)(x)\Psi(x^{-1})\big]dx \\ &= d_\tau\big\langle \pi((G^\#)^* * G^\#)v_0, v_0\big\rangle\end{aligned}$$

$$= \quad d_\tau \langle \pi(G^{\#})v_0, \pi(G^{\#})v_0 \rangle \geq 0.$$

Therefore Ψ is of positive type. □

10. The case where $H = \mathbb{R}^n$ or the Heisenberg group H_n. Commutativity criteria

Let K be a compact subgroup of $O(n)$ acting naturally on $\mathbb{R}^n$, and (τ, V_τ) be an irreducible unitary representation of K.

The following theorem gives a criterion for $(\mathbb{R}^n, K, \tau)$ to be a commutative triple.

Theorem 10.1. *Let $K_x = \{k \in K : k \cdot x = x\}$ be the stabilizer of $x \in \mathbb{R}^n$. Then $(K \ltimes \mathbb{R}^n, K, \tau)$ is a commutative triple if and only if, for all $x \in \mathbb{R}^n$, K_x action on V_τ is multiplicity free.*

The condition "for all $x \in \mathbb{R}^n$" can be replaced by "for generic $x \in \mathbb{R}^n$". Theorem 10.1 can be shown to be a consequence of Theorem 6.1, but it admits a direct proof, based on following lemma. We denote by $\mathrm{End}_{K_{x_0}}(V_\tau)$ the elements of $\mathrm{End}(V_\tau)$ which commute with $\tau_{|K_{x_0}}$.

Lemma 10.2. *Let $x_0 \in \mathbb{R}^n$ and $A \in \mathrm{End}_{K_{x_0}}(V_\tau)$. Then there is a compactly supported function $F \in C_\tau(\mathbb{R}^n)$ such that $F(x_0) = A$.*

Proof. Let S_{x_0} be a slice at x_0. For the existence of slices, cf. [4, Ch. 2, Section 5]. By definition, S_{x_0} is an open and K_{x_0}-invariant neighborhood of 0 in the normal space N_{x_0} to the orbit $K \cdot x_0$ at x_0, such that the K-equivariant map

$$\sigma : K \times_{K_{x_0}} S_{x_0} \to \mathbb{R}^n,$$

given by $\sigma(k, x) = k(x_0 + x)$, is a diffeomorphism of $K \times_{Kx_0} S_{x_0}$ onto the open neighborhood $K(x_0 + S_{x_0})$ of $K \cdot x_0$. Here the notation $K \times_{K_{x_0}} S_{x_0}$ stands for the quotient of $K \times S_{x_0}$ modulo the action of K_{x_0}, i.e., (kk', x) is equivalent to $(k, k'x)$ for all $k' \in K_{x_0}, x \in S_{x_0}$. Also, cf. [11, Corollary 5.3], for every $x \in S_{x_0}$, we have the inclusion $K_x \subset K_{x_0}$, more explicitly $K_x = (K_{x_0})_x$.

Fix a K_{x_0}-invariant scalar-valued function $\psi \in C_c(S_{x_0})$ with $\psi(x_0) = 1$, and define

$$F : K \times_{K_{x_0}} S_{x_0} \to \mathrm{End}(V_\tau)$$

by

$$F(k, x) = \psi(x)\tau(k)A\tau(k^{-1}).$$

This is well defined because, for $k \in K, k' \in K_{x_0}, x \in S_{x_0}$, $F(kk', x) = F(k, k'x)$, since $A \in \mathrm{End}_{K_{x_0}}(V_\tau)$ and ψ is K_{x_0}-invariant. Clearly, F is continuous. Also note that $F(k_1k_2, x) = \tau(k_1)F(k_2, x)\tau(k_1)^{-1}$. □

Proof of Theorem 10.1. For $F \in L^1(\mathbb{R}^n) \otimes \mathrm{End}(V_\tau)$, define its Fourier transform $\mathcal{F}F = \widehat{F} \in C_0(\mathbb{R}^n) \otimes \mathrm{End}(V_\tau)$ via the ordinary Fourier integral. Then

$$\mathcal{F} : L^1_\tau(\mathbb{R}^n) \longrightarrow \big(C_0(\mathbb{R}^n) \otimes \mathrm{End}(V_\tau)\big)^K := (C_0)_\tau(\mathbb{R}^n)\ ,$$

with dense image. Moreover, $\widehat{F_1 * F_2} = \widehat{F}_1\widehat{F}_2$. So, it is enough to show that $(C_0)_\tau(\mathbb{R}^n)$ is commutative under pointwise product if and only if, for each $x \in \mathbb{R}^n$, K_x acts on V_τ without multiplicities.

Assume that, for each x, K_x acts on V_τ without multiplicities. Then $\mathrm{End}_{K_x}(V_\tau)$ is commutative. If $F \in (C_0)_\tau(\mathbb{R}^n)$, $F(k \cdot x) = \tau(k)F(x)\tau(k^{-1})$ for all $k \in K$, which implies that $F(x) \in \mathrm{End}_{K_x}(V_\tau)$. Hence $(C_0)_\tau(\mathbb{R}^n)$ is commutative.

Conversely, suppose that $(C_0)_\tau(\mathbb{R}^n)$ is commutative. Given $x \in \mathbb{R}^n$ and $A, B \in \mathrm{End}_{K_x}(V_\tau)$, by Lemma 10.2 there exist $F, G \in (C_c)_\tau(\mathbb{R}^n)$ with $F(x) = A$, $G(x) = B$. This implies that $AB = BA$. □

Let now H_n be the $(2n+1)$ dimensional Heisenberg group, identified with $\mathbb{C}^n \times \mathbb{R}$ with product

$$(z,t)(z',t') = \left(z + z', t + t' + \frac{1}{2}\langle z, z'\rangle\right) ,$$

where $\langle z, z'\rangle$ denotes the natural Hermitian product on $\mathbb{C}^n$. For $k \in U(n)$,

$$k \cdot (z,t) = (kz, t)$$

is an automorphism of H_n, and, up to conjugation, $U(n)$ is the unique maximal compact and connected subgroup of $\mathrm{Aut}(H_n)$.

Given a compact subgroup K of $U(n)$ and $\tau \in \widehat{K}$, we have the following necessary and sufficient condition for $(K \ltimes H_n, K, \tau)$ to be a commutative triple. By $\mathcal{P}_{\rm hol}(\mathbb{C}^n) = \mathbb{C}[z_1, \dots, z_n]$ we denote the space of holomorphic polynomials on $\mathbb{C}^n$ and by σ be the representation of $U(n)$ on $\mathcal{P}_{\rm hol}(\mathbb{C}^n)$ given by $\sigma(k)P = P \circ k^{-1}$.

Proposition 10.3. *Given a compact subgroup K of $U(n)$ and $\tau \in \widehat{K}$, the triple $(K \ltimes H_n, K, \tau)$ is commutative if and only if $\sigma_{|_K} \otimes \tau$ and $\sigma'_{|_K} \otimes \tau$ decompose without multiplicities.*

Proof. In applying Theorem 6.1 to our case, it suffices to restrict oneself to the infinite-dimensional representations $\pi_\lambda \in \widehat{H}$ associated to nontrivial characters $e^{i\lambda t}$ of the center of H_n.

In the Fock-Bargmann model, the representation space consists of holomorphic functions on $\mathbb{C}^n$ which are square integrable with respect to the measure $e^{-|\lambda||z|^2} dz d\bar{z}$; and polynomials are dense in it. Moreover, the stabilizer K_{π_λ} is the full group K and the representation δ described in the proof of Theorem 6.1 is linear and coincides with $\sigma_{|_K}$ for $\lambda > 0$ and with $\sigma'_{|_K}$ for $\lambda < 0$.

The conclusion is then immediate. □

Corollary 10.4. *The triples $\big(SO(n) \ltimes \mathbb{R}^n, SO(n), \tau\big)$, $\big(U(n) \ltimes H_n, U(n), \tau\big)$ are commutative for every $\tau \in \widehat{K}$.*

For the triples $\big(SU(n) \ltimes \mathbb{C}^n, SU(n), \tau\big)$, $\big(SU(n) \ltimes H_n, SU(n), \tau\big)$ the following are equivalent:

- **(i)** *it is commutative,*
- **(ii)** *the highest weight μ of $\tau \in \widehat{SU(n)}$ is singular, i.e. is annihilated by some root,*
- **(iii)** *condition (6.3) on an integrable $\mathrm{End}(V_\tau)$-valued function F implies the identity $F(e^{i\theta}z) = F(z)$, resp. $F(e^{i\theta}z, t) = F(z, t)$, i.e., the $SU(n)$-action on the bundle extends to $U(n)$.*

Proof. The first part is a restatement of the well-known fact that $\big(SO(n) \ltimes \mathbb{R}^n, SO(n)\big)$, $\big(U(n) \ltimes H_n, U(n)\big)$ are strong Gelfand pairs [27, Thm. 3]. On the other hand, the first case follows directly from Theorem 10.1 applying the branching rules for restrictions of representations of classical groups [13], and the second can be proved by modifying slightly the argument below.

To prove the part of the statement concerning $H = H_n$ and $K = SU(n)$, we disregard the trivial case $n = 1$ and assume that $n \geq 2$. We have

$$\sigma_{|_{SU(n)}} \otimes \tau = \sum_{m=0}^{\infty} \sigma_m \otimes \tau, \tag{10.1}$$

where $\sigma_m(k)$ denotes the restriction of $\sigma_{|_{SU(n)}}(k)$ to the space $\mathcal{P}^m_{\rm hol}(\mathbb{C}^n)$ of holomorphic polynomial homogeneous of degree m.

Let $\lambda_1,\dots,\lambda_{n-1}$ the simple positive roots in decreasing order. The highest weight μ is identified by the $(n-1)$-tuple $a=(a_1,\dots,a_{n-1})$ with $a_j=\lambda_j(\mu)\ge 0$. We then write $\mu=\mu_a$. By Pieri's formula [13],

$$\sigma_m\otimes\mu_a=\sum_{b\in B_m}\mu_b,$$

where B_m is the set of $(n-1)$-tuple $b=(b_1,\dots,b_{n-1})$ with

$$b_j=a_j+c_j-c_{j+1}\ ,\qquad \sum_1^n c_j=m\ ,\qquad 0\le c_{j+1}\le a_j \text{ for } j=1,\dots,n-1.$$

Assume that the same b is obtained from two choices $c_1,\dots,c_n$ with $\sum c_j=m$, and $c'_1,\dots,c'_n$ with $\sum c'_j=m'$. This is equivalent to saying that the differences $c_j-c'_j$ do not depend on j.

If τ is singular, at least one of the a_j is 0 and this forces the equalities $c'_j=c_j$ for every j, $m'=m$. Hence $\sigma_{|_{SU(n)}}\otimes\tau$ splits without multiplicities. Moreover, τ' is also singular and the same conclusion holds for $\sigma'_{|_{SU(n)}}\otimes\tau=(\sigma_{|_{SU(n)}}\otimes\tau')'$.

On the other hand, if τ is regular, then $a_j\ge 1$ for every j, and the choices $m=c_j=0$ and $m'=n$, $c'_j=1$ show that τ is contained both in $\sigma_0\otimes\tau$ and in $\sigma_n\otimes\tau$. This gives the equivalence between (i) and (ii).

Assume now that (i) holds. By Proposition 10.3 and formula (10.1), each irreducible component of $\sigma_{|_{SU(n)}}\otimes\tau$ is contained in $\sigma_m\otimes\tau$. On the other hand, condition (ii) implies that, if χ is any character of $U(1)$ which extends τ from the center of $SU(n)$, for every $F\in L^1_\tau(H_n)$, $\pi_\lambda(F)$ commutes with $\sigma(e^{i\theta}I)\otimes\chi$ for $\lambda>0$, and with $\sigma'(e^{i\theta}I)\otimes\chi$ for $\lambda<0$. Since $\sigma(e^{i\theta}I)=e^{im\theta}I$ on $\mathcal{P}^m_{\rm hol}(\mathbb{C}^n)$, we have that, denoting by $\tilde\tau$ the representation of $U(n)$ given by

$$\tilde\tau(e^{i\theta}k)=\chi(e^{i\theta})\tau(k)\ ,\qquad k\in SU(n),$$

for every $k\in U(n)$, $\pi_\lambda(F)$ commutes with $\sigma\otimes\tilde\tau$ for $\lambda>0$ and with $\sigma'\otimes\tilde\tau$ for $\lambda<0$. Hence $F\in L^1_{\tilde\tau}(H_n)$ and, in particular,

$$F(e^{i\theta}z,t)=\tilde\tau(e^{i\theta}I)F(z,t)\tilde\tau(e^{-i\theta}I)=F(z,t).$$

This argument can be easily reversed to give the opposite implication.

Finally, if $H=\mathbb{C}^n$ and $K=SU(n)$, the stabilizer of any $z\ne 0$ is $SU(n-1)$.

Assume that the highest weight μ of τ is regular, so that $a_j=\lambda_j(\mu)>0$ for every j. Let $\tau^\sharp$ be the extension of τ to U_n with highest weight

$$\mu^\sharp=(\mu_1,\mu_2,\dots,\mu_{n-1},0)=(a_1+\cdots+a_{n-1}\ ,\ a_2+\cdots+a_{n-1}\ ,\ \dots\ ,\ a_{n-1},0)\ .$$

Another application of Pieri's formula, cf. [13, p. 80], shows that $\tau^\sharp_{|_{U(n-1)}}$ contains both representations with highest weights

$$(\mu_1,\mu_2,\dots,\mu_{n-1})\ ,\qquad (\mu_1-1,\mu_2-1,\dots,\mu_{n-1}-1)\ ,$$

which have equivalent restrictions to $SU(n-1)$. By Theorem 10.1, the triple is not commutative for such τ.

Conversely, assume that the highest weight of τ is singular. Given two functions $F_1,F_2\in L^1_\tau(\mathbb{C}^n)$, let $G_j(z,t)=F_j(z)\eta(t)$, $j=1,2$, with $\eta\in L^1(\mathbb{R})$ with integral 1. Then $G_1,G_2\in L^1_\tau(H_n)$, so that $G_1*G_2=G_2*G_1$. But

$$\int_{\mathbb{R}}G_i*G_j(z,t)\,dt=F_i*F_j(z)\ ,$$

hence F_1 and F_2 commute. □

Notice that the implication (i)⇒(iii) holds for every $K \subset U(n)$, with $U(n)$ replaced by $K\,U(1)$.

11. The case where $H = \mathbb{R}^n$ or H_n. Spherical functions

Let $(\mathbb{R}^n, K, \tau)$ be a commutative triple. For fixed $\xi \in \mathbb{R}^n$, let $V_\tau = \oplus_{j=1}^{l(\xi)} V_{j,\xi}$ be the multiplicity-free decomposition of V_τ under the action of K_ξ. Let $P_{j,\xi}$ denote the projection onto $V_{j,\xi}$ with respect to this decomposition, and d_j the dimension of $V_{j,\xi}$. If $F \in L^1_\tau(\mathbb{R}^n)$ then $\widehat{F} \in (C_0)_\tau(\mathbb{R}^n)$, and hence $\widehat{F}(\xi) \in \mathrm{Hom}_{K_\xi}(V_\tau)$. Therefore $\widehat{F}(\xi) = \oplus_{j=1}^{l(\xi)} \widehat{F}_j(\xi) P_{j,\xi}$ for some scalars $\widehat{F}_j(\xi)$. Since $\widehat{F * G} = \widehat{F}\widehat{G}$, it follows that, for each j, $F \to \widehat{F}_j(\xi)$ defines a multiplicative linear functional of $L^1_\tau(\mathbb{R}^n)$. But

$$\begin{aligned} \widehat{F}_j(\xi) &= \frac{1}{d_j}\mathrm{Tr}\big[P_{j,\xi}\widehat{F}(\xi)\big] \\ &= \frac{1}{d_j}\mathrm{Tr}\Big[P_{j,\xi}\int_{\mathbb{R}^n} e^{i\xi\cdot x}F(x)dx\Big] \\ &= \frac{1}{d_j}\int_{\mathbb{R}^n}\mathrm{Tr}\Big[e^{i\xi\cdot x}P_{j,\xi}F(x)\Big]dx \\ &= \frac{1}{d_j}\int_K\int_{\mathbb{R}^n}\mathrm{Tr}\Big[e^{i\xi\cdot(k\cdot x)}P_{j,\xi}F(k\cdot x)\Big]dxdk. \end{aligned}$$

Since $F(k\cdot x) = \tau(k)F(x)\tau(k)^{-1}$ and $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$, we get

$$\begin{aligned} \widehat{F}_j(\xi) &= \frac{1}{d_j}\int_K\int_{\mathbb{R}^n}\mathrm{Tr}\Big[e^{i\xi\cdot(k\cdot x)}\tau(k^{-1})P_{j,\xi}\tau(k)F(x)\Big]dxdk \\ &= \frac{1}{d_\tau}\int_{\mathbb{R}^n}\mathrm{Tr}[\Phi_{\xi,j}(-x)F(x)]dx, \end{aligned} \tag{11.1}$$

where

$$\Phi_{\xi,j}(x) = \frac{d_\tau}{d_j}\int_K e^{-i\xi\cdot(k\cdot x)}\tau(k^{-1})P_{j,\xi}\tau(k)dk. \tag{11.2}$$

Since $\Phi_{\xi,j} \in L^\infty_\tau(\mathbb{R}^n)$, $\Phi_{\xi,j}$ is a bounded τ-spherical function. We call it a *Bessel function of type* τ.

Theorem 11.1.

- **(a)** $\{\Phi_{\xi,j} : \xi \in \mathbb{R}^n, j = 1, 2, \cdots l(\xi)\}$ *is the set of all bounded τ-spherical functions.*
- **(b)** *Two spherical functions $\Phi_{\xi_1,j_1}, \Phi_{\xi_2,j_2}$ coincide if and only if the following conditions hold:*
  - (i) *ξ_1 and ξ_2 lie on the same K-orbit;*
  - (ii) *if $\xi_2 = k\xi_1$, then $V_{j_2,\xi_2} = \tau(k)\big(V_{j_1,\xi_1}\big)$ (this condition is independent of the choice of k).*

Proof. Let Ψ be a bounded spherical function. By Theorem 9.4, Ψ is of positive type. Applying Theorem 9.2 and Proposition 9.3, there is an irreducible unitary representation π of $G = K \ltimes \mathbb{R}^n$ such that $\tau \subset \pi_{|K}$ (say, with $V_\tau \subset \mathcal{H}_\pi$, uniquely determined by the multiplicity-free condition) and

$$\big\langle \Psi(x)u, v\big\rangle = \big\langle \pi(e,x)u, v\big\rangle,$$

for all $u, v \in V_\tau$.

The irreducible unitary representations of G are those induced from irreducible representations of $G_\xi = K_\xi \times \mathbb{R}\xi$ for some $\xi \in \mathbb{R}^n$. By Frobenius reciprocity, for a given $\sigma \in \widehat{K_\xi}$ and $\lambda \in \mathbb{R}$, $\tau \subset \mathrm{Ind}_{G_\xi}^G(\sigma \otimes e^{i\lambda\cdot})$ if and only if $\sigma \subset \tau_{|K_\xi}$. Hence Ψ determines an element $\lambda\xi \in \mathbb{R}^n$, unique up to the action of K, and a unique invariant subspace $V_{j,\xi}$ of V_τ.

The rest of the proof follows from the properties of induced representations. □

Corollary 11.2. *Let $\Phi_{\xi,j}$ be one of the spherical functions in* (11.2) *and* $D \in \big(\mathbb{D}(\mathbb{R}^n)\otimes \mathrm{End}(V_\tau)\big)^K$,

$$D = Q(\partial_x)\ , \qquad Q \in \big(\mathfrak{S}(\mathbb{R}^n)\otimes \mathrm{End}(V_\tau)\big)^K\ .$$

Then the eigenvalue $\lambda_D(\Phi_{\xi,j})$ of $\Phi_{\xi,j}$ under the action of D is the eigenvalue of $Q(-i\xi)$ on $V_{j,\xi}$.

Proof. Since $Q(-i\xi)$ commutes with $\tau_{|K_\xi}$, it is a scalar multiple of the identity, say λI, on $V_{j,\xi}$. Hence $Q(-i\xi)P_{j,\xi} = \lambda P_{j,\xi}$. Therefore,

$$D\big(e^{-i\xi\cdot x}P_{j,\xi}\big) = Q(-i\xi)e^{-i\xi\cdot x}P_{j,\xi} = \lambda e^{-i\xi\cdot x}P_{j,\xi}.$$

By the invariance of D, the same holds for $e^{-i\xi\cdot(k\cdot x)}\tau(k^{-1})P_{j,\xi}\tau(k)$ and all $k\in K$. Hence, by (11.2),

$$D\Phi_{\xi,j} = \lambda\Phi_{\xi,j}. \qquad \square$$

Consider now a commutative triple $(K\ltimes H_n, K, \tau)$ with $K\subseteq U(n)$ and $\tau\in\widehat{K}$. Let

$$\mathcal{P}_{\mathrm{hol}}(\mathbb{C}^n)\otimes V_\tau = \sum_\alpha W_\alpha^+\ , \qquad \mathcal{P}_{\mathrm{hol}}(\mathbb{C}^n)\otimes V_\tau = \sum_\beta W_\beta^-$$

be the decompositions into irreducible K-invariant subspaces under $\sigma_{|_K}\otimes\tau$ and $\sigma'_{|_K}\otimes\tau$ respectively.

Since, for $\lambda\neq 0$, $K_{\pi_\lambda} = K$, the general formula (9.5) for spherical functions gives

$$\begin{aligned}\Phi^+_{\lambda,\alpha}(z,t) &= \frac{d_\tau}{d_\alpha}e^{i\lambda t}\mathrm{Tr}_{\mathcal{P}_{\mathrm{hol}}}\big((\pi_\lambda(z,0)\otimes I)P_\alpha\big)\\ \Phi^-_{\lambda,\beta}(z,t) &= \frac{d_\tau}{d_\beta}e^{i\lambda t}\mathrm{Tr}_{\mathcal{P}_{\mathrm{hol}}}\big((\pi_\lambda(z,0)\otimes I)P_\beta\big)\ ,\end{aligned} \tag{11.3}$$

for $\lambda>0$ and $\lambda<0$ respectively. We call these the *Laguerre-type τ-spherical functions*, because the matrix entries of $\pi_\lambda(z,0)$ contain Laguerre functions in $|z|^2$, cf. [12].

To the Laguerre τ-spherical functions one should add those corresponding to the 1-dimensional representations of H_n (i.e., to $\lambda=0$), which are Bessel functions of type τ.

Theorem 11.3. *The bounded spherical functions for $(K\ltimes H_n, K, \tau)$ are the functions $\Phi^+_{\lambda,\alpha}$ and $\Phi^-_{\lambda,\beta}$ in* (11.3) *and the functions $\Phi^0_{\xi,j}(z,t) = \Phi_{\xi,j}(z)$, where the $\Phi_{\xi,j}$ are the Bessel functions of type τ* (11.2) *for the commutative triple $(K\ltimes\mathbb{C}^n, K,\tau)$.*

12. An example

We give the explicit form of generating differential operators and of the spherical functions in the case

$$H=\mathbb{R}^n\ , \qquad K=SO(n)\ , \qquad V_\tau=\mathbb{C}^n,$$

where τ is the defining n-dimensional representation of $SO(n)$.

Let $\xi\in\mathbb{R}^n\setminus\{0\}$. It $n=2$, $K_\xi=\{I\}$. Therefore V_τ decomposes with multiplicity under the action of K_ξ. Hence, if $n=2$, $(\mathbb{R}^n, K,\tau)$ is not a commutative triple.

On the other hand, for $n \geq 3$ and $\xi = \rho e_1 \neq 0$, $K_\xi \cong SO(n-1)$ and V_τ decomposes under the action of K_ξ as

$$V_\tau = \begin{cases} \mathbb{C}e_1 \oplus \mathbb{C}(e_2 + ie_3) \oplus \mathbb{C}(e_2 - ie_3) = V_{e_1,1} \oplus V_{e_1,2} \oplus V_{e_1,3} & \text{if } n = 3, \\ \mathbb{C}e_1 \oplus e_1^\perp = V_{e_1,1} \oplus V_{e_1,2} & \text{if } n > 3, \end{cases} \tag{12.1}$$

in both cases without multiplicities. Hence for any $n \geq 3$, $(\mathbb{R}^n, K, \tau)$ is a commutative triple.

We describe a system $\mathcal{D}$ of generators of $\big(\mathbb{D}(\mathbb{R}^n) \otimes \mathrm{End}(V_\tau)\big)^K$. To this purpose, we investigate the structure of the algebra of K-invariant elements in $\mathfrak{S}(\mathbb{R}^n) \otimes \mathrm{End}(V_\tau)$.

We denote by σ the representation of K on the space $\mathcal{P}(\mathbb{R}^n, M_{n,n}) \cong \mathfrak{S}(\mathbb{R}^n) \otimes \mathrm{End}(V_\tau)$ of matrix-valued polynomials, given by

$$\big(\sigma(k)P\big)(x) = kP(k^{-1}x)k^{-1}.$$

For a σ-invariant space $W \subset \mathcal{P}(\mathbb{R}^n, M_{n,n})$, let W^τ denote the space of all $\sigma(k)$-fixed elements in W. Note that a polynomial $P : \mathbb{R}^n \to M_{n,n}$ is σ-invariant iff

$$P(k \cdot x) = kP(x)k^{-1} \qquad \forall\, k \in K.$$

Let $\mathcal{I} = \mathbb{C}\big[|x|^2\big]$ be the space of all K-invariant scalar-valued polynomials on $\mathbb{R}^n$, $\mathcal{H}(\mathbb{R}^n)$ the space of all harmonic polynomials, $\mathcal{H}_m(\mathbb{R}^n)$ be the space of all homogeneous harmonic polynomials of degree m. Then

$$\begin{aligned} \big(\mathcal{P}(\mathbb{R}^n, M_{n,n})\big)^\tau &= \mathcal{I}\left(\mathcal{H}(\mathbb{R}^n, M_{n,n})\right)^\tau \\ &= \mathcal{I} \bigoplus_{m=0}^{\infty} \left(\mathcal{H}_m(\mathbb{R}^n, M_{n,n})\right)^\tau \end{aligned} \tag{12.2}$$

Let δ_m and $\pi \sim \tau \otimes \tau'$ denote the natural representations of K on $\mathcal{H}_m(\mathbb{R}^n)$ and $M_{n,n}$ respectively:

$$\delta_m(k)h(x) = h(k^{-1}x)\ , \qquad \pi(k)A = kAk^{-1}\ ,$$

so that the restriction of σ to $\mathcal{H}_m(\mathbb{R}^n, M_{n,n})$ is equivalent to $\delta_m \otimes \pi$.

Under π, $M_{n,n}$ decomposes into irreducibles as:

$$M_{n,n} = \mathbb{C}\, I \oplus M_{n,n}^{\text{skew}} \oplus M_{n,n}^{\text{sym},0},$$

where $M_{n,n}^{\text{skew}}$ is the space of $n \times n$ skew-symmetric matrices, and $M_{n,n}^{\text{sym},0}$ is the space of $n \times n$ symmetric matrices with zero trace. Then, for every m,

$$\big(\mathcal{H}_m(\mathbb{R}^n, M_{n,n})\big)^\tau = \big(\mathcal{H}_m(\mathbb{R}^n, \mathbb{C}\, I)\big)^\tau \bigoplus \big(\mathcal{H}_m(\mathbb{R}^n M_{n,n}^{\text{skew}})\big)^\tau \bigoplus \big(\mathcal{H}_m(\mathbb{R}^n M_{n,n}^{\text{sym},0})\big)^\tau. \tag{12.3}$$

Let us introduce the following elements of $\big(\mathcal{P}(\mathbb{R}^n) \otimes \mathrm{End}(V)\big)^\tau$.

$$I_{3,1}(x) = \begin{pmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{pmatrix} \in \big(\mathcal{H}_1(\mathbb{R}^3, M_{3,3}^{\text{skew}})\big)^\tau,$$

$$I_{n,2}(x) = x\, {}^t x - \frac{1}{n}|x|^2 I \in \big(\mathcal{H}_2(R^n), M_{n,n}^{\text{sym},0}\big)^\tau\big) \qquad n \geq 3.$$

Theorem 12.1. *Let $P \in \big(\mathcal{P}(\mathbb{R}^n, M_{n,n})\big)^\tau$. Then*

$$P(x) = \begin{cases} p_0\big(|x|^2\big) + p_1\big(|x|^2\big) J_{3,1}(x) + p_2\big(|x|^2\big) J_{3,2}(x) & \text{if } n = 3 \\ p_0\big(|x|^2\big) + p_2\big(|x|^2\big) J_{n,2}(x) & \text{if } n > 3 \end{cases}$$

for some $p_0, p_1, p_2 \in \mathcal{I}$.

Proof. Clearly,

$$\big(\mathcal{H}(\mathbb{R}^n, \mathbb{C}\, I)\big)^\tau = \big(\mathcal{H}(\mathbb{R}^n) \cap \mathcal{I}\big) \otimes I = \mathbb{C}\, I.$$

Let $U_1 = M_{n,n}^{\text{skew}}$, $U_2 = M_{n,n}^{\text{sym},0}$. By Schur's lemma it follows that $\big(H_m(\mathbb{R}^n) \otimes U_j\big)^\tau$ contains a non-zero element iff $\delta_m \sim \pi'_{|_{U_j}} \sim \pi_{|_{U_j}}$; and in that case it is unique up to a scalar multiple.

A simple check of dimensions shows that this equivalence can only hold for $j = m = 2$ if $n \geq 3$, and for $j = m = 1$ if $n = 3$. □

Theorem 12.1 implies that $\big(\mathcal{P}_K(\mathbb{R}^n) \otimes \operatorname{End}(V)\big)^\tau$ is generated as an $\mathcal{I}$-module by $I, I_{3,1}, I_{3,2}$ if $n = 3$, and by $I, I_{n,2}$ if $n > 3$.

Corollary 12.2. *$\{I, I_{3,1}\}$ (resp. $\{I, I_{n,2}\}$) is a set of independent generators for the algebra $\big(\mathcal{P}(\mathbb{R}^n) \otimes \operatorname{End}(V)\big)^\tau$ when $n = 3$ (resp. $n > 3$).*

For $F : \mathbb{R}^n \longrightarrow \mathbb{C}^n$, let

$$\boldsymbol{D}F = \begin{cases} I_{3,1}(\partial_x)F = \operatorname{curl} F & \text{if } n = 3 \\ (I_{n,2}(\partial_x) + \frac{1}{n}\Delta I)F = \operatorname{grad}\operatorname{div} F & \text{if } n > 3. \end{cases}$$

Then $\mathcal{D} = \{\Delta I, \boldsymbol{D}\}$ generates the algebra $\big(\mathbb{D}(\mathbb{R}^n) \otimes \operatorname{End}(V_\tau)\big)^K$.

Proof. We just need to mention that, for $n = 3$, $I_{3,2} = |x|^2 I + I_{3,1}^2$ (equivalently, $\operatorname{grad}\operatorname{div} F = \Delta F - \operatorname{curl}\operatorname{curl} F$). Moreover, $I, I_{3,1}$ are algebraically independent when $n = 3$ (resp. $I, I_{n,2}$ when $n > 3$). □

According to Theorem 11.1, the bounded spherical function are given by formula (11.2).

For $\xi = 0$, we have a unique spherical function $\Phi_{0,1}(x) = I$. We can then take $\xi = se_1$ with $s > 0$. By (6.3), it suffices to compute $\Phi_{se_1,j} = \Phi_{s,j}$ for $x = re_1$ with $r > 0$, as we already know that $\Phi_{s,j}(0) = I$.

Factoring the integral in (11.2) modulo $K_\xi = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & h \end{pmatrix} : h \in SO(n-1) \right\}$, we reduce ourselves to an integral over the sphere S^{n-1},

$$\Phi_{s,j}(re_1) = \frac{n}{d_j} \int_{S^{n-1}} e^{isry\cdot e_1} P_{j,y}\, d\sigma_{n-1}(y) \ ,$$

where σ_{n-1} is the normalized surface measure on S^{n-1}, $y = k^{-1}e_1$ and $P_{j,y} = k^{-1}P_{j,e_1}k$.

12.0.1. *The case $n > 3$.*

Referring to the decomposition (12.1), we have

$$P_{1,y} = k^{-1}e_1\, {}^t e_1 k = y\, {}^t y \ , \qquad P_{2,y} = I_n - P_{1,y} = I_n - y\, {}^t y.$$

Hence, for each $s > 0$ we have two spherical functions,

$$\Phi_{s,1}(re_1) = n \int_{S^{n-1}} e^{isry\cdot e_1} y\, {}^t y\, d\sigma(y)$$

$$\Phi_{s,2}(re_1) = \frac{n}{n-1} \int_{S^{n-1}} e^{isry\cdot e_1} (I_n - y\, {}^t y)\, d\sigma(y) = \frac{n}{n-1} \Big(\int_{S^{n-1}} e^{isry\cdot e_1}\, d\sigma(y) \Big) I_n - \frac{1}{n-1} \Phi_{s,1}(re_1) \ .$$

Setting $y = (\cos t, (\sin t)z)$, with $z \in S^{n-2}$,

$$\Phi_{s,1}(re_1) = nc_n^{-1} \int_0^\pi \int_{S^{n-2}} e^{isr\cos t} \begin{pmatrix} \cos^2 t & (\cos t \sin t)\, {}^t z \\ (\cos t \sin t) z & (\sin^2 t) z\, {}^t z \end{pmatrix} d\sigma_{n-2}(z)\, \sin^{n-2} t\, dt,$$

with

$$c_n = \int_0^\pi \sin^{n-2} t\, dt = \pi^{\frac12} \frac{\Gamma(\frac{n-1}{2})}{\Gamma(\frac{n}{2})}.$$

Integration in z annihilates the off-diagonal terms z_j and $z_j z_k$ with $j \neq k$, while $\int_{S^{n-2}} z_j^2 d\sigma_{n-2}(z) = \frac{1}{n-1}$ for symmetry reasons. Hence

$$\begin{aligned}\Phi_{s,1}(re_1) &= nc_n^{-1} \int_0^\pi e^{isr\cos t} \begin{pmatrix} \cos^2 t & 0 \\ 0 & \frac{\sin^2 t}{n-1} I_{n-1}\end{pmatrix} \sin^{n-2} t\, dt \\ &= nc_n^{-1} \int_0^\pi e^{isr\cos t} \sin^{n-2} t\, dt \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \\ &\quad + nc_n^{-1} \int_0^\pi e^{isr\cos t} \sin^{n} t\, dt \begin{pmatrix} -1 & 0 \\ 0 & \frac{1}{n-1} I_{n-1}\end{pmatrix}.\end{aligned}$$

From the identity

$$\int_0^\pi e^{iu\cos t} \sin^k t\, dt = 2^{\frac{k}{2}} \pi^{\frac12} \Gamma\Big(\frac{k+1}{2}\Big) \frac{J_{\frac{k}{2}}(u)}{u^{\frac{k}{2}}}$$

we obtain that

$$\Phi_{s,1}(re_1) = n 2^{\frac{n}{2}-1} \Gamma\Big(\frac{n}{2}\Big) \bigg(\frac{J_{\frac{n}{2}-1}(sr)}{(sr)^{\frac{n}{2}-1}} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \frac{J_{\frac{n}{2}}(sr)}{(sr)^{\frac{n}{2}}} \begin{pmatrix} -(n-1) & 0 \\ 0 & I_{n-1}\end{pmatrix} \bigg).$$

For general $x = rke_1$, we have

$$\begin{aligned} k \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} k^{-1} &= (ke_1)\, {}^t(ke_1) = |x|^{-2} x\, {}^t x, \\ k \begin{pmatrix} 0 & 0 \\ 0 & I_{n-1} \end{pmatrix} k^{-1} &= I_n - |x|^{-2} x\, {}^t x,\end{aligned}$$

so that

$$\Phi_{s,1}(x) = n 2^{\frac{n}{2}-1} \Gamma\Big(\frac{n}{2}\Big) \bigg(\frac{J_{\frac{n}{2}-1}(s|x|)}{s^{\frac{n}{2}-1} |x|^{\frac{n}{2}+1}} x\, {}^t x + \frac{J_{\frac{n}{2}}(s|x|)}{s^{\frac{n}{2}} |x|^{\frac{n}{2}+2}} \big(|x|^2 I_n - n x\, {}^t x \big) \bigg)$$

Similarly,

$$\Phi_{s,2}(re_1) = \frac{n 2^{\frac{n}{2}-1}}{n-1} \Gamma\Big(\frac{n}{2}\Big) \bigg(\frac{J_{\frac{n}{2}-1}(s|x|)}{s^{\frac{n}{2}-1} |x|^{\frac{n}{2}+1}} (n|x|^2 I_n - x\, {}^t x) - \frac{J_{\frac{n}{2}}(s|x|)}{s^{\frac{n}{2}} |x|^{\frac{n}{2}+2}} \big(|x|^2 I_n - n x\, {}^t x \big) \bigg).$$

By Corollary 11.2, the map $\rho_{\mathcal{D}}$ of Section 4 relative to the system $\mathcal{D} = (\Delta I, \mathrm{grad}\,\mathrm{div})$ is given by

$$\rho_{\mathcal{D}}(\Phi_{s,1}) = (-s^2, -s^2)\ , \qquad \rho_{\mathcal{D}}(\Phi_{s,2}) = \big(-s^2, 0 \big).$$

12.0.2. *The case $n = 3$.*

For $y \in S^2$, we have

$$\begin{aligned} P_{1,y} &= y\, {}^t y \\ P_{2,y} &= \frac12 k^{-1} (e_2 + ie_3)\, {}^t(e_2 - ie_3) k = \frac12 \big(I_3 - y\, {}^t y + i I_{3,1}(y) \big) \\ P_{3,y} &= \frac12 k^{-1} (e_2 - ie_3)\, {}^t(e_2 + ie_3) k = \frac12 \big(I_3 - y\, {}^t y - i I_{3,1}(y) \big). \end{aligned}$$

For each $s>0$ we now have three spherical functions,

$$\begin{aligned}\Phi_{s,1}(re_1) &= 3\int_{S^2} e^{isry\cdot e_1} y\,{}^t y\, d\sigma(y)\\ \Phi_{s,2}(re_1) &= \frac{3}{2}\int_{S^2} e^{isry\cdot e_1}\big(I_n - y\,{}^t y + iI_{3,1}(y)\big)\, d\sigma(y)\\ \Phi_{s,3}(re_1) &= \frac{3}{2}\int_{S^2} e^{isry\cdot e_1}\big(I_n - y\,{}^t y - iI_{3,1}(y)\big)\, d\sigma(y).\end{aligned}$$

The only new computation that is needed concerns the integral

$$\mathcal{I}(u) = \int_{S^2} e^{iuy\cdot e_1} I_{3,1}(y)\, d\sigma(y).$$

In the polar coordinates $y = (\cos t, \sin t\cos\varphi, \sin t\sin\varphi)$, this becomes

$$\begin{aligned}\mathcal{I}(u) &= \frac{1}{4\pi}\int_0^{\pi}\int_0^{2\pi} e^{iu\cos t}\begin{pmatrix} 0 & -\sin t\sin\varphi & \sin t\cos\varphi\\ \sin t\sin\varphi & 0 & -\cos t\\ -\sin t\cos\varphi & \cos t & 0\end{pmatrix}\sin t\, d\varphi\, dt\\ &= \frac{1}{2}\Big(\int_0^{\pi} e^{iu\cos t}\cos t\sin t\, dt\Big) I_{3,1}(e_1)\\ &= \sqrt{\frac{\pi}{2}}\frac{J_{\frac{3}{2}}(u)}{u^{\frac{1}{2}}}\, iI_{3,1}(e_1).\end{aligned}$$

Proceeding as for the case $n>3$, we obtain

$$\begin{aligned}\Phi_{s,1}(x) &= 3\sqrt{\frac{\pi}{2}}\Big(\frac{J_{\frac{1}{2}}(s|x|)}{s^{\frac{1}{2}}|x|^{\frac{5}{2}}}x\,{}^t x + \frac{J_{\frac{3}{2}}(s|x|)}{s^{\frac{3}{2}}|x|^{\frac{7}{2}}}\big(|x|^2 I_3 - 3x\,{}^t x\big)\Big)\\ \Phi_{s,2}(x) &= \frac{3}{2}\sqrt{\frac{\pi}{2}}\Big(\frac{J_{\frac{1}{2}}(s|x|)}{s^{\frac{1}{2}}|x|^{\frac{5}{2}}}(3|x|^2 I_3 - x\,{}^t x) - \frac{J_{\frac{3}{2}}(s|x|)}{s^{\frac{3}{2}}|x|^{\frac{7}{2}}}\big(|x|^2 I_3 - 3x\,{}^t x + s|x|^2 I_{3,1}(x)\big)\Big)\\ \Phi_{s,2}(x) &= \frac{3}{2}\sqrt{\frac{\pi}{2}}\Big(\frac{J_{\frac{1}{2}}(s|x|)}{s^{\frac{1}{2}}|x|^{\frac{5}{2}}}(3|x|^2 I_3 - x\,{}^t x) - \frac{J_{\frac{3}{2}}(s|x|)}{s^{\frac{3}{2}}|x|^{\frac{7}{2}}}\big(|x|^2 I_3 - 3x\,{}^t x - s|x|^2 I_{3,1}(x)\big)\Big).\end{aligned}$$

Taking $\mathcal{D} = (\Delta I, \mathrm{curl})$, we have

$$\rho_{\mathcal{D}}(\Phi_{s,1}) = (-s^2, 0)\ , \qquad \rho_{\mathcal{D}}(\Phi_{s,2}) = \big(-s^2, -s\big)\ , \qquad \rho_{\mathcal{D}}(\Phi_{s,3}) = \big(-s^2, s\big).$$

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