On the birational section conjecture with strong birationality assumptions

Giulio Bresciani[1]

Abstract

Let X be a curve over a field k finitely generated over $\mathbb{Q}$ and t an indeterminate. We prove that, if s is a section of $\pi_1(X) \to \mathrm{Gal}(k)$ such that the base change $s_{k(t)}$ is birationally liftable, then s comes from geometry. As a consequence we prove that the section conjecture is equivalent to the cuspidalization of all sections over all finitely generated fields.

Mathematics Subject Classification Primary 11G30 · 14H25 · 14H30 · Secondary 14G05 · 14F35

1 Introduction

Given X a geometrically connected, smooth curve over a field k with absolute Galois group $\mathrm{Gal}(k)$, let $\mathcal{S}_{X/k}$ be the set of sections of $\pi_1(X) \to \mathrm{Gal}(k)$ modulo the action of $\pi_1(X_{\bar{k}})$ by conjugation, these are usually called Galois sections. A Galois section is *geometric* if it is associated with a k-rational point of X and *cuspidal* if it is associated with a k-rational point in the boundary $\bar{X} \setminus X$, where $\bar{X}$ is the smooth completion of X. Grothendieck made the following conjecture, nowadays called the *section conjecture*, in a letter to Faltings [11].

Section Conjecture *For every smooth, geometrically connected hyperbolic curve X over a field k finitely generated over $\mathbb{Q}$, every Galois section of X is either geometric or cuspidal.*

A birational version of the conjecture, called the *birational section conjecture*, has been studied as it seems more approachable [13, 17, 21]. We can define a set $\mathcal{S}_{k(X)/k}$ of sections of $\mathrm{Gal}(k(X)) \to \mathrm{Gal}(k)$ analogously to the above, its elements are called birational Galois sections. A birational section is cuspidal if it comes from a rational point of $\bar{X}$.

✉ G. Bresciani
giulio.bresciani@sns.it

1   CRM Ennio de Giorgi, Collegio Puteano, Office 21, Piazza dei Cavalieri 3, 56126 Pisa, Italy

Birational Section Conjecture *For every smooth, geometrically connected curve X over a field k finitely generated over* $\mathbb{Q}$, *every birational Galois section of X is cuspidal.*

Recall that a Galois section of $\mathcal{S}_{X/k}$ is *birationally liftable* if it's in the image of $\mathcal{S}_{k(X)/k} \to \mathcal{S}_{X/k}$ [21, §2.1.1] (this is slightly different from the definition given in [20, Definition 266]). Here is an alternative formulation of the birational section conjecture, see Lemma 26.

Birational Section Conjecture, alternative formulation *For every smooth, geometrically connected curve X over a field k finitely generated over* $\mathbb{Q}$, *every birationally liftable Galois section of X is either geometric or cuspidal.*

In this note we show that, if we strengthen the birationality assumption, we can obtain the desired result. Namely, we show that if t is an indeterminate and s is a Galois section such that the base change $s_{k(t)}$ is birationally liftable, then s is geometric or cuspidal.

1.1 Known results

J. Koenigsmann [13] proved that the birational section conjecture holds over finite extensions of $\mathbb{Q}_p$. Clearly, one would like to pass from local fields to number fields. Moreover, M. Saïdi and M. Tyler [17] have proved that the birational section conjecture for number fields implies it for finitely generated extensions of $\mathbb{Q}$.

J. Stix [21] obtained partial results about the passage from local fields to number fields, let us describe them. Fix X a smooth, projective curve over a number field k and $s \in \mathcal{S}_{k(X)/k}$ a Galois section of $k(X)$. Using Koenigsmann's results, s induces a point $x_v \in X(k_v)$ for every place v.

Given an open subset $U \subseteq X$, choose some spreading out $\widetilde{U} \to \operatorname{Spec} \mathfrak{o}_{k,S}$ of U, where $\mathfrak{o}_k$ is the ring of integers of k and S is some finite set of places. Let N_U be the set of places of k such that x_v is *not* integral: we have that N_U depends on the choice of $\widetilde{U}$ only up to a finite number of places, hence the Dirichlet density of N_U is well defined.

Stix first proves that, if k is a totally real or imaginary quadratic number field, then N_U is infinite for some open subset $U \subseteq X$ [21, Theorem A]. Secondly he proves that, if N_U has strictly positive Dirichlet density for some open subset $U \subseteq X$, then s comes from a k-rational point of $X \setminus U$ [21, Theorem B]. In order to prove the birational section conjecture for totally real or imaginary quadratic number fields, it is then sufficient to bridge the gap between Stix's two results.

1.2 Our main theorem

We study the passage from local fields to number fields, too, but we use a different approach. We strengthen the birationality assumption: under this strengthened hypothesis, we obtain a complete result.

Definition Let X be a curve over a field k, $s \in \mathcal{S}_{X/k}$ a Galois section and t an indeterminate; consider the base change $s_{k(t)} \in \mathcal{S}_{X_{k(t)}/k(t)}$ of s to $k(t)$ [20, Definition 27]. We say that s is *t-birationally liftable* if $s_{k(t)}$ is birationally liftable.

Theorem A *Let X be a smooth curve over a field k finitely generated over $\mathbb{Q}$. A Galois section of X is geometric or cuspidal if and only if it is t-birationally liftable.*

A common theme in the study of the section conjecture is the attempt to replicate with Galois sections the geometric constructions we make with points, see e.g. [20, Chap. 3]. The idea underlying the proof of Theorem A is, given s a Galois section of $\mathbb{P}^1 \setminus \{0, 1, \infty\}$, to define "the polynomial $t - s$" so that, if s is geometric associated to a k-rational point $x \in k \setminus \{0, 1\}$, then "$t - s$" coincides with the polynomial $t - x$. If s is t-birationally liftable then we are able to define this "polynomial" by using a birational lifting of $s_{k(t)}$ to produce an element of $\mathrm{H}^1(k(t), \widehat{\mathbb{Z}}(1)) = \widehat{k(t)^*}$.

This turns out to be sufficient: if we are over a number field k and, for a place v, the Galois section s_{k_v} is associated with a point $x_v \in k_v \setminus \{0, 1\}$, the fact that the "polynomial $t - s = t - x_v$" is defined over k forces x_v to be k-rational and not to depend on v.

1.3 Consequences for the section conjecture

One of the main reasons for studying the birational section conjecture is to reduce the section conjecture to a *lifting*, or *cuspidalization*, problem.

The section conjecture easily implies the following statement, which is sometimes called *cuspidalization conjecture* in the literature.

Cuspidalization Conjecture *For every smooth, geometrically connected hyperbolic curve X over a field k finitely generated over $\mathbb{Q}$ and every non-empty open subset $U \subset X$, the map $\mathcal{S}_{U/k} \to \mathcal{S}_{X/k}$ is surjective.*

By a limit argument, the cuspidalization conjecture holds if and only if every Galois section of every hyperbolic curve over k is birationally liftable, see Lemma 23.

H. Esnault and P.H. Hai showed that the cuspidalization conjecture reduces the section conjecture to the case of $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ [9, Proposition 7.9]. For $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ one may hope to solve the conjecture with explicit computations, such as the n-nilpotents obstructions introduced by J. Ellenberg and K. Wickelgren [23]. Moreover, the cuspidalization conjecture reduces the section conjecture to the birational section conjecture. Neither the case of $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ nor the birational section conjecture is needed: the section conjecture is equivalent to the cuspidalization conjecture.

Theorem B *Let X be a hyperbolic curve over a field k finitely generated over $\mathbb{Q}$. The following are equivalent.*

- *For every finitely generated extension K/k, every Galois section of X_K is either geometric or cuspidal.*

- *For every finitely generated extension K/k, every Galois section of X_K is birationally liftable.*

As a consequence, the section conjecture is equivalent to the cuspidalization conjecture.

1.4 Consequences for the birational section conjecture

It is well known that it is enough to prove the birational section conjecture for $\mathbb{P}^1 \setminus \{0, 1, \infty\}$. Thanks to Theorem A, it is then enough to show that for every birationally liftable section s of $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ over a number field k the base change $s_{k(t)}$ lifts to $\mathrm{Gal}(k(t)(\mathbb{P}^1))$. We prove that it is enough to find a much simpler lifting, and we manage to reduce the existence of said lifting to a problem of "interpolation of Galois sections", i.e. finding a Galois section with prescribed specializations of some curve over $k(t)$. Let us explain this.

We say that a morphism $X \to V$ is a *family of curves* if it has the form $\bar{X} \setminus D \to V$ where $\bar{X} \to V$ is smooth, proper, with geometrically connected fibers of dimension 1, and $D \subset \bar{X}$ is a divisor finite étale over V. Assume that V is an affine curve over a field of characteristic 0 and let F be a geometric fiber, then the sequence

$$1 \to \pi_1(F) \to \pi_1(X) \to \pi_1(V) \to 1$$

is exact, see Lemma 1 (we stress that we are assuming V affine, otherwise this is false).

We may define a space $\mathcal{S}_{X/V}$ of "global" sections $\pi_1(V) \to \pi_1(X)$ modulo conjugation by $\pi_1(F)$. Let us call these π_1 sections, in analogy with Galois sections. If V is an affine open subset of $\mathbb{P}^1$ over a number field k, a π_1 section r of $X \to V$ is uniquely determined by its specializations $r_v \in \mathcal{S}_{X_v/k}$, $v \in V(k)$, see Lemma 25.

If $z \in \mathcal{S}_{k(\mathbb{P}_1)/k}$ is a birational Galois section of $\mathbb{P}^1$, for every open subset $U \subset \mathbb{P}^1$ denote by $\iota_U(z)$ the image of z in $\mathcal{S}_{U/k}$. Let $\Delta \subset \mathbb{P}^1 \times \mathbb{P}^1$ be the diagonal.

Theorem C *The following are equivalent.*

- *The birational section conjecture holds.*
- *For every number field k, every section $z \in \mathcal{S}_{k(\mathbb{P}_1)/k}$ and every open subset $U \subseteq \mathbb{P}^1$, there exists an open subset $V \subseteq U$ and a π_1 section r of $U \times V \setminus \Delta \to V$ such that the specialization r_v is equal to $\iota_{U \setminus \{v\}}(z)$ for every $v \in V(k)$.*

Since a π_1 section is uniquely determined by its specializations, the section r in the statement of Theorem C is a lifting to $U \times V \setminus \Delta$ of the base change $(\iota_U(z))_V \in \mathcal{S}_{U \times V/V}$.

Notation and conventions Throughout the article, it is tacitly assumed that schemes do not have points of positive characteristic. In particular, fields are of characteristic 0. Curves are smooth and geometrically connected. If X is a curve, we denote by $\bar{X}$ its smooth completion. The letter R always denotes a DVR with fraction field K and residue field k. If A is an abelian group, we denote by $\widehat{A}$ the projective limit $\varprojlim_n A/nA$. We write b.l. (resp. t-b.l.) as an abbreviation for birationally liftable (resp. t-birationally liftable).

2 Étale fundamental gerbes

In addition to the classical language of étale fundamental groups, we use the language of *étale fundamental gerbes* [3, §8, §9], [6, Appendix A]. Étale fundamental gerbes are essentially an alternative point of view on the theory of étale fundamental groups where groups are replaced by a particular type of algebraic stack, i.e. gerbes. Everything that can be done in one language can be translated into the other one and it is actually possible to use both languages at the same time, switching back and forth depending on convenience.

Generally speaking, fundamental groups are better for handling geometric arguments about the original varieties, while fundamental gerbes are better for handling arguments about Galois sections. The reason is that the étale fundamental gerbe, in some sense, *is* the space of Galois sections, but constructed directly without passing through fundamental groups.

Furthermore, fundamental gerbes are naturally base point free. This is particularly helpful since we make a lot of specialization arguments involving different fibers of a single family, and doing so while keeping track of base points would be problematic. With fundamental gerbes, the problem just doesn't exist. Moreover, the theory of non-unique specializations in the classical language requires making non-canonical choices, while this is not the case from the point of view of gerbes.

If X is geometrically connected over a field k, the étale fundamental gerbe $\Pi_{X/k}$ is a profinite étale stack over k with a morphism $X \to \Pi_{X/k}$ universal among morphisms to finite étale stacks over k. The space of Galois sections $\mathcal{S}_{X/k}$ of X is in natural bijection with the set of isomorphism classes of $\Pi_{X/k}(k)$. Furthermore, the étale fundamental gerbe behaves well with respect to base change in characteristic 0 [6, Proposition A.18].

Fundamental gerbes are easier to understand when there is a k-rational point. Recall that, if G is an affine group scheme over k, the classifying stack $\mathcal{B}_k G$ is defined as follows: given a scheme S over k, then $\mathcal{B}_k G(S)$ is the groupoid of G-torsors for the fpqc topology over S, see for instance [16, Definition 8.1.14]. If G is of finite type over k every G-torsor is trivialized by an fppf covering $S' \to S$, namely $S' = T$, and hence the fppf topology is sufficient. In general, e.g. if G is pro-finite, we need to use the fpqc topology, see [3] for details. If X has a rational point $x \in X(k)$, the Galois group $\mathrm{Gal}(\bar{k}/k)$ acts continuously on the étale fundamental group $\pi_1(X_{\bar{k}}, x)$, this allows us to define an étale fundamental group scheme $\underline{\pi}_1(X, x)$ which is a twisted form of $\pi_1(X_{\bar{k}}, x)$. The étale fundamental gerbe identifies naturally with the classifying stack $\mathcal{B}_k \underline{\pi}_1(X, x)$, and the k-rational sections of $\Pi_{X/k}$ correspond to $\underline{\pi}_1(X, x)$-torsors over k.

2.1 Relative fundamental gerbes

In order to do specialization arguments, we need to work with families of curves for which there is a short exact sequence of étale fundamental groups. We identify very restrictive assumptions under which this works, these are sufficient for our purposes.

Lemma 1 *Let k be a field of characteristic 0 and A a ring which is a localization of a 1-dimensional regular domain $\widetilde{A}$ of finite type over k, write $C = \operatorname{Spec} A$. If $X \to C$ is a family of curves and F is a geometric fiber, then*

$$1 \to \pi_1(F) \to \pi_1(X) \to \pi_1(C) \to 1$$

is exact.

Proof Assume first that A is a 1-dimensional regular domain of finite type over k.

If $k = \mathbb{C}$, the sequence of topological fundamental groups is exact since the second homotopy group of C is trivial, and we get an exact sequence of profinite completions by [2, Proposition 5] and [12, Proposition 3.7].

Assume that k is any field of characteristic 0, it is clearly enough to do the case in which k is algebraically closed. In this case, since everything is of finite type there exists an algebraically closed field k' with embeddings $k' \subset k$ and $k' \subset \mathbb{C}$ such that $X \to C$ descends to a family of curves $X' \to C'$ over k'. Since étale fundamental groups are invariant under base change of algebraically closed fields [10, Exposé X, Corollaire 1.8], the statement follows from the case $k = \mathbb{C}$.

If A is a localization of a 1-dimensional regular domain $\widetilde{A}$ of finite type over k, since X is of finite type over $C = \operatorname{Spec} A$ then up to localizing a finite number of elements of $\widetilde{A}$ we may assume that X extends to a family of curves $\widetilde{X} \to \widetilde{C} = \operatorname{Spec} \widetilde{A}$. We have that $\pi_1(C) \simeq \varprojlim_U \pi_1(U)$ where U varies among open subsets of $\widetilde{C}$ containing $C \subset \widetilde{C}$; this follows from the fact that the category of finite étale covers of C is the direct limit of the categories of finite étale covers of U for varying U. Analogously, $\pi_1(X) \simeq \varprojlim_U \pi_1(\widetilde{X}_U)$.

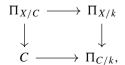For every open subset $C \subset U \subset \widetilde{C}$, by the preceding case we have a short exact sequence

$$1 \to \pi_1(F) \to \pi_1(\widetilde{X}_U) \to \pi_1(U) \to 1,$$

hence we obtain a projective system of short exact sequences where the left term is constant. This implies that

$$\pi_1(X) \simeq \varprojlim_U \pi_1(\widetilde{X}) \times_{\pi_1(\widetilde{C})} \pi_1(U) \simeq \pi_1(\widetilde{X}) \times_{\pi_1(\widetilde{C})} \pi_1(C),$$

and the statement follows. $\qquad\square$

Let $X \to C$ be a family of curves as in Lemma 1. Define the *relative fundamental gerbe* $\Pi_{X/C}$ by the 2-cartesian diagram

$$
\begin{array}{ccc}
\Pi_{X/C} & \longrightarrow & \Pi_{X/k} \\
\downarrow & & \downarrow \\
C & \longrightarrow & \Pi_{C/k},
\end{array}
$$

there is a structural morphism $X \to \Pi_{X/C}$ over C.

If k'/k is a field extension and $c : \operatorname{Spec} k' \to C$ is a point, the induced morphism

$$\Pi_{X_c/k'} \to \Pi_{X/C} \times_C \operatorname{Spec} k'$$

is an isomorphism: using the fact that the étale fundamental gerbe behaves well under base change [6, Proposition A.18], this is a direct consequence of the exactness of the sequence of étale fundamental groups. Because of this, the relative fundamental gerbe is a convenient way of packing the spaces of Galois sections of the fibers without choosing base points. So to speak, it is the "relative family of spaces of Galois sections".

Lemma 2 *The relative fundamental gerbe* $\Pi_{X/C} \to C$ *is a projective limit* $\varprojlim \Phi_i$ *of proper, étale morphisms* $\Phi_i \to C$ *which are gerbes over* C.

Proof Since $\Pi_{X/k}$, $\Pi_{C/k}$ are pro-finite étale, then we may write them as projective limits $\varprojlim_i \Psi_i$, $\varprojlim_k \Lambda_i$ of finite étale gerbes over k with $\Pi_{X/k} \to \Psi_i$, $\Pi_{C/k} \to \Lambda_i$ locally full [4, Definition 3.4, Proposition 3.9]. Up to re-indexing, we may assume that $\Pi_{X/k} \to \Pi_{C/k}$ induces a morphism $\Psi_i \to \Lambda_i$ for every i; the fact that the fibers of $X \to C$ are geometrically connected implies that $\Pi_{X/k} \to \Pi_{C/k}$ is locally full (i.e. the homomorphism of geometric fundamental groups is surjective), which in turn implies that $\Psi_i \to \Lambda_i$ is locally full. The morphism $\Psi_i \to \Lambda_i$ is a proper étale relative gerbe: it is proper étale since Ψ_i, Λ_i are proper étale over k, and it is a relative gerbe by [4, Proposition 3.10]. The statement then follows by defining Φ_i as $\Psi_i \times_{\Lambda_i} C$. $\qquad\square$
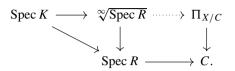
3 Non-unique specializations of ramified Galois sections

Classically, the specialization of a Galois section is defined if the original Galois section is *unramified* [20, §8.2], i.e. when the *ramification homomorphism* is trivial. A notion of specialization exists *always*, as long as we don't require that the specialization is unique. If the section is unramified, this "generalized specialization" is unique and coincides with the classical specialization. While a definition of specialization for ramified sections exists in the literature [20, Corollary 8.9], little else in known beyond the definition itself.

Let R be a DVR with fraction field K and residue field k, the *n-th root stack* $\sqrt[n]{\operatorname{Spec} R}$ of $\operatorname{Spec} R$ is defined for every n [1, Appendix B]. If $\pi \in R$ is a uniformizing parameter, then $\sqrt[n]{\operatorname{Spec} R}$ is isomorphic to the quotient stack $[\operatorname{Spec} R(\sqrt[n]{\pi})/\mu_n]$, but the definition of root stack does not depend on the choice of π. The morphism $\sqrt[n]{\operatorname{Spec} R} \to \operatorname{Spec} R$ is generically an isomorphism, while the closed fiber with the reduced structure is non-canonically isomorphic to the classifying stack $B\mu_n$.

The *infinite root stack* $\sqrt[\infty]{\operatorname{Spec} R}$ is the projective limit $\varprojlim_n \sqrt[n]{\operatorname{Spec} R}$, see [14] for details. Let $c \in \operatorname{Spec} R$ be the closed point and S_c^1 the reduced fiber of $\sqrt[\infty]{\operatorname{Spec} R}$ over c, it is non-canonically isomorphic to the classifying stack $\mathcal{B}_k \widehat{\mathbb{Z}}(1)$ of $\widehat{\mathbb{Z}}(1)$ over k. The notation S_c^1 is meant to be reminiscent of the topological space S^1, which is the classifying space $\mathcal{B}\mathbb{Z}$ of $\mathbb{Z}$. There is a non-canonical isomorphism between the isomorphism classes of $S_c^1(k)$ and $\mathrm{H}^1(k, \widehat{\mathbb{Z}}(1)) = \varprojlim_n k^*/k^{*n} = \widehat{k^*}$.

Recall that a morphism $Y \to X$ of fibered categories over k is *constant* if there exists a factorization $Y \to \operatorname{Spec} k \to X$.

Proposition 3 *Let $X \to C$ be a family of curves as in Lemma* 1, $c \in C$ *a closed point with local ring R and $z : \operatorname{Spec} K \to \Pi_{X/C}$ a generic Galois section, where K is the fraction field of C. Then z extends to a 2-commutative diagram*

$$
\begin{array}{ccc}
\operatorname{Spec} K \longrightarrow \sqrt[\infty]{\operatorname{Spec} R} \dashrightarrow \Pi_{X/C} \\
\searrow \qquad \downarrow \qquad\qquad \downarrow \\
\operatorname{Spec} R \longrightarrow C.
\end{array}
$$

The extension $\sqrt[\infty]{\operatorname{Spec} R} \to \Pi_{X/C}$ is unique up to a unique isomorphism.

We call the induced morphism $S_c^1 \to \Pi_{X_c/k(c)}$ the specializing loop $\gamma_z(c)$ of z at c. An extension $\operatorname{Spec} R \to \Pi_{X/C}$ exists if and only if the specializing loop is constant.

Proof Using Lemma 2, this is a direct consequence of [8, Theorem 3.1]. □

Definition 4 A *specialization* of z at c is any Galois section of $X_c/k(c)$ in the essential image of the specializing loop $S_c^1(k(c)) \to \Pi_{X_c/k(c)}(k(c))$ of z at c evaluated on $k(c)$. A specialization always exists, but in general it might be not unique. If the specializing loop is constant, then the specialization is unique (the converse is false in general).

If we fix any section $s \in S_c^1(k(c))$, the specializing loop $\gamma_z : S_c^1 \to \Pi_{X_c/k_c}$ is constant if and only if the corresponding homomorphism of group schemes $\widehat{\mathbb{Z}}(1) = \underline{\operatorname{Aut}}(s) \to \underline{\operatorname{Aut}}(\gamma_z(s))$ is trivial. Suppose for simplicity that $k(c) = k$. By the characterization given above, the specializing loop of z at c is constant if and only if the specializing loop of $z_{\bar{k}(C)}$ is constant at $c_{\bar{k}}$: having constant specializing loop in the closed fiber X_c only depends on the base change to $\bar{k}$ of the family, nonetheless it encapsulates arithmetic information over k, i.e. uniqueness of the specialization.

Remark 5 A specialization of a ramified section can be constructed in the language of étale fundamental groups as follows (see also [20, Corollary 89]). Let R be a DVR with fraction and residue fields K, k of characteristic 0, denote by R^h a henselianization and K^h the fraction field. We may identify $\operatorname{Gal}(\bar{k}/k) = \pi_1(\operatorname{Spec} R^h)$, and if $X \to \operatorname{Spec} R$ is a family of curves then $\pi_1(X_k) \simeq \pi_1(X_{R^h})$. Since R^h is henselian, the Galois group $\operatorname{Gal}(\bar{K}/K^h)$ coincides with the decomposition subgroup and hence it is an extension of $\operatorname{Gal}(\bar{k}/k)$ by the inertia subgroup.

Let $\pi \in R^h$ be a uniformizing parameter, choose a n-th root of π for every n in a compatible way and let $K^h(\sqrt[\infty]{\pi})$ be the corresponding extension of K^h, we have that $\operatorname{Gal}(\bar{K}/K^h(\sqrt[\infty]{\pi})) \simeq \operatorname{Gal}(\bar{k}/k)$ defines a section of $\operatorname{Gal}(\bar{K}/K^h) \to \pi_1(\operatorname{Spec} R^h) = \operatorname{Gal}(\bar{k}/k)$.

If $\operatorname{Gal}(\bar{K}/K) \to \pi_1(X_K)$ is a generic Galois section, it induces a Galois section $\operatorname{Gal}(\bar{K}/K^h) \to \pi_1(X_{K^h})$ and a specialization is defined by the composition

$$
\operatorname{Gal}(\bar{k}/k) \to \operatorname{Gal}(\bar{K}/K^h) \to \pi_1(X_{K^h}) \to \pi_1(X_{R^h}) \simeq \pi_1(X_k).
$$

Let $I \subset \text{Gal}(\bar{K}/K^h)$ be the inertia subgroup, by composition we have a map $I \to \pi_1(X_{\bar{k}}) \subset \pi_1(X_k)$ which is called *ramification map*: unramified Galois sections are those for which this map is trivial.

If we fix a section of $D \to \text{Gal}(\bar{k}/k)$, the corresponding specialization induces an action of $\text{Gal}(\bar{k}/k)$ on $\pi_1(X_{\bar{k}})$, and the ramification map is Galois-equivariant with respect to the natural Galois action on $I \simeq \widehat{\mathbb{Z}}(1)$. The set of specializations is the image in nonabelian Galois cohomology of the ramification map.

Example 6 Let R be a DVR as in Lemma 1 with fraction field K and valuation $v : K^* \to \mathbb{Z}$, this extends naturally to an homomorphism $\widehat{K^*} \to \widehat{\mathbb{Z}}$ which we still call v. The morphism $\mathbb{G}_m = \mathbb{A}_R^1 \setminus \{0\} \to \text{Spec } R$ is a family of curves.

The section $1 \in \mathbb{G}_m(R)$ gives an identification $\Pi_{\mathbb{G}_m/R} = \mathcal{B}_R\widehat{\mathbb{Z}}(1)$, in particular $\Pi_{\mathbb{G}_m/R}(R) = \widehat{R^*}$ and $\Pi_{\mathbb{G}_{m,K}/K}(K) = \widehat{K^*}$. If $z \in \widehat{K^*}$ is a generic section, by Proposition 3 the specializing loop γ_z is constant if and only if z is in the image of $\widehat{R^*} \to \widehat{K^*}$, or equivalently if and only if $v(z) = 0 \in \widehat{\mathbb{Z}}$.

It can be proved that unramified Galois sections in the sense of [20, Chap. 8] are those for which the specializing loop is constant, though we don't need this fact. If the specializing loop is constant, the specialization is clearly unique; this is coherent with the fact that unramified Galois sections have a canonical specialization.

The converse is false in general: if the residue field is algebraically closed, the specialization is unique since there is only one $\widehat{\mathbb{Z}}(1)$-torsor up to equivalence, but the Galois section might be ramified and the specializing loop not constant. For instance, we may choose $R = \mathbb{C}[x]_{(x)}$ in Example 6, the generic Galois section associated with $x \in \widehat{\mathbb{C}(x)^*} \setminus \widehat{\mathbb{C}[x]_{(x)}^*}$ has non-constant specializing loop but unique specialization.

Still, in arithmetic situations, it is often the case that uniqueness of specialization forces a constant specializing loop.

Lemma 7 *Let T be a torus over a field k with a surjective valuation $v : k^* \to \mathbb{Z}$. Suppose that we have a morphism of gerbes $\gamma : \mathcal{B}_k\widehat{\mathbb{Z}}(1) \to \Pi_{T/k}$. If the image of $\mathcal{B}_k\widehat{\mathbb{Z}}(1)(k)$ in $\Pi_{T/k}(k_v)$ has a finite number of isomorphism classes, then γ is constant.*

Proof Using the image of the preferred k-rational section of $\mathcal{B}_k\widehat{\mathbb{Z}}(1)$ corresponding to the trivial torsor, we may identify $\Pi_{T/k}$ with $\mathcal{B}_k\underline{\pi}_1(T)$, and we reduce to prove the following: if $f : \widehat{\mathbb{Z}}(1) \to \pi_1(T_{\bar{k}})$ is a Galois equivariant homomorphism such that the composition $H^1(k, \widehat{\mathbb{Z}}(1)) \xrightarrow{f} H^1(k, \pi_1(T_{\bar{k}})) \xrightarrow{res} H^1(k_v, \pi_1(T_{\bar{k}}))$ has finite image, then f is trivial.

Choose k'/k a finite extension such that $T_{k'} \simeq \mathbb{G}_m^n$, we get an identification $\pi_1(T_{\bar{k}}) \simeq \widehat{\mathbb{Z}}^n(1)$ of Galois modules over k'. Let v' be an extension of v to k', denote by r the ramification index of v'/v. The valuation v induces an homomorphism $H^1(k_v, \widehat{\mathbb{Z}}(1)) = \varprojlim_i k_v^*/k_v^{*i} \to \widehat{\mathbb{Z}}$, and similarly for $k'_{v'}$; with an abuse of notation, we denote these homomorphisms by v, v'.

We have a commutative diagram

$$
\begin{array}{ccccc}
\mathrm{H}^1(k,\widehat{\mathbb{Z}}(1)) & \xrightarrow{\;res\;} & \mathrm{H}^1(k_v,\widehat{\mathbb{Z}}(1)) & \xrightarrow{\;v\;} & \widehat{\mathbb{Z}} \\
\downarrow{\scriptstyle res\circ\mathrm{H}^1(f)} & & \downarrow{\scriptstyle res} & & \downarrow{\scriptstyle r} \\
& & \mathrm{H}^1(k'_{v'},\widehat{\mathbb{Z}}(1)) & \xrightarrow{\;v'\;} & \widehat{\mathbb{Z}} \\
& & \downarrow{\scriptstyle \mathrm{H}^1(f)} & & \downarrow{\scriptstyle f} \\
\mathrm{H}^1(k_v,\pi_1(T_{\bar{k}})) & \xrightarrow{\;res\;} & \mathrm{H}^1(k'_{v'},\pi_1(T_{\bar{k}})\simeq\widehat{\mathbb{Z}}(1)^n) & \xrightarrow{\;v'\;} & \pi_1(T_{\bar{k}})\simeq\widehat{\mathbb{Z}}^n
\end{array}
$$

where the last identification $\pi_1(T_{\bar{k}})\simeq\widehat{\mathbb{Z}}^n$ ignores the Galois action. The hypothesis implies that the composition

$$
f\circ r\circ v\circ res = v'\circ res\circ\mathrm{H}^1(f):\mathrm{H}^1(k,\widehat{\mathbb{Z}}(1))\to\widehat{\mathbb{Z}}^n
$$

has finite image, hence it is 0 since $\widehat{\mathbb{Z}}$ has no torsion. Notice that, since $v:k^*\to\mathbb{Z}$ is surjective, then the upper horizontal arrow $v\circ res:\mathrm{H}^1(k,\widehat{\mathbb{Z}}(1))=\varprojlim_i k^*/k^{*i}\to\widehat{\mathbb{Z}}$ is surjective as well, and hence $f\circ r=rf=0$. Since $r\neq 0$, we get that $f=0$, as desired. $\qquad\square$

Lemma 8 *Let k be a number field with a finite place v, X a curve over k, $\gamma:\mathcal{B}_k\widehat{\mathbb{Z}}(1)\to\Pi_{X/k}$ a morphism of gerbes. If the image of $\mathcal{B}_k\widehat{\mathbb{Z}}(1)(k)$ in $\Pi_{X/k}(k_v)$ has a finite number of isomorphism classes, then γ is constant.*

Proof If by contradiction γ is not constant, i.e. the associated homomorphism $\widehat{\mathbb{Z}}\to\pi_1(X_{\bar{k}})$ is not trivial, up to replacing X with a finite étale cover we may assume that the composition $\mathcal{B}_k\widehat{\mathbb{Z}}(1)\to\Pi_{X/k}\to\Pi_{X/k}^{\mathrm{ab}}$ is not constant, where $\Pi_{X/k}^{\mathrm{ab}}$ is the abelianized fundamental gerbe, equivalently we may assume that the induced homomorphism $\widehat{\mathbb{Z}}\to\pi_1(X_{\bar{k}})^{\mathrm{ab}}$ is not trivial. We may do so because the map of Galois sections associated with a finite étale cover has finite fibers, hence the hypothesis remains true after passing to the covering.

Let J be the semi-abelian Jacobian of X, it is an extension of an abelian variety A by a torus T and we have a canonical identification $\Pi_{X/k}^{\mathrm{ab}}=\Pi_{J/k}$. We may use the images of preferred section of $\mathcal{B}_k\widehat{\mathbb{Z}}(1)$ to give identifications $\Pi_{J/k}=\mathcal{B}_k\,\mathrm{T}\,J$, $\Pi_{A/k}=\mathcal{B}_k\,\mathrm{T}\,A$ where $\mathrm{T}\,J=\varprojlim_n J[n]$, $\mathrm{T}\,A=\varprojlim_n A[n]$ are the Tate modules; the morphisms of gerbes then correspond to homomorphisms $\widehat{\mathbb{Z}}(1)\to\mathrm{T}\,J$, $\mathrm{T}\,J\to\mathrm{T}\,A$.

By weight reasons, the composition $\widehat{\mathbb{Z}}(1)\to\mathrm{T}\,J\to\mathrm{T}\,A$ is trivial, hence we have a factorization $\widehat{\mathbb{Z}}(1)\to\mathrm{T}\,T\to\mathrm{T}\,J$. We have a short exact sequence

$$
\varprojlim_n A[n](k_v)\to\mathrm{H}^1(k_v,\mathrm{T}\,T)\to\mathrm{H}^1(k_v,\mathrm{T}\,J).
$$

By Mattuck's theorem [15] $A(k_v)$ has finite torsion, thus $\mathrm{H}^1(k_v,\mathrm{T}\,T)\to\mathrm{H}^1(k_v,\mathrm{T}\,J)$ is injective. The hypothesis implies that the image of $\mathrm{H}^1(k,\widehat{\mathbb{Z}}(1))$ in $\mathrm{H}^1(k_v,\mathrm{T}\,J)$ is finite, hence the image in $\mathrm{H}^1(k_v,\mathrm{T}\,T)$ is finite too. By Lemma 7, $\widehat{\mathbb{Z}}(1)\to\mathrm{T}\,T$ is trivial, which is contradiction with the fact that $\mathcal{B}_k\widehat{\mathbb{Z}}(1)\to\Pi_{X/k}^{\mathrm{ab}}$ is not constant. $\qquad\square$

4 Properties of t-birationally liftable Galois sections

In order to work with t-b.l. Galois sections, we need to prove a number of facts about them.

Lemma 9 *Let R be a DVR, $X \to \operatorname{Spec} R$ a family of curves and $p \in X$ a closed point in the closed fiber. There exists a non-empty divisor $D \subset X$ finite étale over R with $p \in D$.*

Proof Let K, k be the fraction and residue fields of R. If p is k-rational, this essentially follows from Bertini's theorem by taking a generic hyperplane section passing through p. If p is not k-rational, though, this is more subtle. We are going to use a Bertini-like argument to reduce to the case of $\mathbb{P}_R^1$, where we are able to make an explicit construction.

By definition, $X \to \operatorname{Spec} R$ has the form $\bar{X} \setminus D_0 \to \operatorname{Spec} R$ with $\bar{X} \to \operatorname{Spec} R$ smooth, projective, with geometrically connected fibers of dimension 1, and $D_0 \subset \bar{X}$ a divisor finite étale over R. Choose a projective embedding $\bar{X} \subset \mathbb{P}_R^n$ and let $p_1, \dots, p_r$ be the geometric points over p.

A generic $(n-2)$-dimensional linear subspace L_k of $\mathbb{P}_k^n$ has the following properties

- $L_k \cap \bar{X}_k = \emptyset$,
- for every i the hyperplane $H_i \subset \mathbb{P}_{\bar{k}}^n$ spanned by L_k and p_i has transversal intersections with $\bar{X}_{\bar{k}}$,
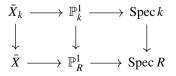- $\forall i : H_i \cap D_0 = \emptyset$.

Let $\bar{q}_0, \bar{q}_1 \in k[t_0, \dots, t_n]$ two k-linear forms which are equations for L_k, choose $q_0, q_1 \in R[t_0, \dots, t_n]$ R-linear forms which lift $\bar{q}_0, \bar{q}_1$ and let $L \subset \mathbb{P}_R^n$ be their vanishing locus, the fact that $L_k \cap \bar{X}_k = \emptyset$ implies that $L \cap \bar{X} = \emptyset$ since everything is proper over R.

The graded homomorphism $R[x_0, x_1] \to R[t_0, \dots, t_n]$, $x_i \mapsto q_i$ defines a morphism

$$\mathbb{P}_R^n \setminus L = \operatorname{Proj} R[t_0, \dots, t_n] \setminus L \to \mathbb{P}_R^1 = \operatorname{Proj} R[x_0, x_1].$$

We have simply constructed the projection centered in L: we can do this even though we don't have a base field.

Since $X \cap L = \emptyset$, we get a morphism $f : \bar{X} \to \mathbb{P}_R^1$ over R with $\bar{X}_k \to \mathbb{P}_k^1$ surjective. The morphism f is flat since it is finite and dominant, and both $\bar{X}$ and $\mathbb{P}_R^1$ are regular. Moreover, f is unramified, and hence étale, at $\operatorname{Spec} k(X_k) \subset X$: notice that the diagram

$$
\begin{array}{ccccc}
\bar{X}_k & \longrightarrow & \mathbb{P}_k^1 & \longrightarrow & \operatorname{Spec} k \\
\downarrow & & \downarrow & & \downarrow \\
\bar{X} & \longrightarrow & \mathbb{P}_R^1 & \longrightarrow & \operatorname{Spec} R
\end{array}
$$

is cartesian, hence $\operatorname{Spec} k(\mathbb{P}^1_k) \times_{\mathbb{P}^1_R} \bar{X} = \operatorname{Spec} k(\bar{X}_k)$ is reduced. By purity, it follows that the branch locus of f is the closure of a finite number of closed points of $\mathbb{P}^1_K$. Let K''/K be some finite extension and $H_{K''} \subset \mathbb{P}^n_{K''}$ a hyperplane containing $L_{K''}$ and corresponding to a branch point in $\mathbb{P}^1_K$; equivalently, $H_{K''}$ does not have transversal intersections with $X_{K''}$. If $H_{k''} \subset \mathbb{P}^n_{k''}$ is a specialization of $H_{K''}$ in some finite extension k'' of k, then $H_{k''}$ does not have transversal intersections with $X_{k''}$, it follows that $H_{k''}$ does not contain p_i for every i. This implies that $H_{k''}$ corresponds to a point of $\mathbb{P}^1(k'')$ different from $f(p)$, hence $f(p)$ is not a branch point for f.

Let $U_0 \subset \mathbb{P}^1_R$ be the open subset where f is étale, i.e. the complement of the image of the ramification locus in $\bar{X}$, and $U = U_0 \setminus f(D_0)$. Since $f(p)$ is not a branch point and $H_i \cap D_0 = \emptyset$ for every i we get that $f(p) \in U$. Furthermore, $f^{-1}(U) \to U$ is finite étale by construction. It is then enough to find a divisor of U containing $f(p)$ and which is finite étale over R.

Since we are in characteristic 0, then $k(p) \simeq k[x]/\bar{q}(x)$ for some irreducible, separable polynomial $\bar{q} \in k[x]$, let $q \in R[x]$ be any lifting of $\bar{q}$. By construction, $R' = R[x]/q(x)$ is finite étale over R and has only one closed point, hence it is a DVR. Let K' be the fraction field of R', there are infinitely many points of $\mathbb{P}^1(K')$ which specialize to $p \in \mathbb{P}^1(k(p))$, hence we may choose one in $U(K')$; denote by $E \subset \mathbb{P}^1_R$ its closure, we have an extension $\operatorname{Spec} R' \to E \subset \mathbb{P}^1_R$. By construction, E is finite and flat over R of degree equal to $[K':K]$. Furthermore, E contains p, which satisfies $[k(p):k] = [K':K]$. Because of this, E contains only p and a generic point contained in U, hence $E \subset U$ and $\operatorname{Spec} R' \to E$ is surjective. Since R' is étale over R we get that E is étale over R as well (we have actually proved that $\operatorname{Spec} R' \to E$ is an isomorphism, though we don't need this).

Notice that if we try to work with R'-sections of $\bar{X}$ we run into problems: it might be that $X(K')$ does not contain points which specialize to p, and if we try to enlarge R' again we might get a semi-local ring with several closed points which we are not able to control simultaneously. Making a Bertini-like argument over $\bar{X}_{R'}$ runs into problems too: while we may find the desired divisor étale over R', there is no guarantee that its image in $\bar{X}$ is étale over R. $\qquad\square$

Corollary 10 *Let R be a DVR, $X \to \operatorname{Spec} R$ a family of curves. There exists a direct system $(D_i)_i$ of divisors finite étale over R such that $X_k \setminus \bigcup_i D_i$ contains only the generic point of X_k.*

Lemma 11 *Specializations of b.l. sections are b.l.*

Proof Let $X \to \operatorname{Spec} R$ be a family of curves with R, K, k as above and $c \in \operatorname{Spec} R$ the closed point. If $z \in \Pi_{X_K/K}(K)$ is a generic, b.l. Galois section, we have a specializing loop $\gamma_z : S^1_c \to \Pi_{X_k/k}$.

Let $D_i \subseteq X$ be a direct system of divisors as in Corollary 10. We have that $X_i = X \setminus D_i$ is a family of curves for every i, write $X_\infty = \varprojlim_i X_i$ and $\Pi_{X_\infty/R} = \varprojlim_i \Pi_{X_i/R}$. The closed fiber of X_∞ is naturally isomorphic to $\operatorname{Spec} k(X_k)$.

Since z is b.l. and $\operatorname{Spec} K(X) \subseteq X_{\infty,K}$, there exists a lifting $z' \in \Pi_{X_{\infty,K}/K}(K)$ of z to $X_{\infty,K}$. By a limit argument, the specializing loops of z' in $\Pi_{X_i/R}$ induce a specializing loop $\gamma_{z'} : S^1_c \to \Pi_{X_{\infty,k}/k} = \Pi_{k(X_k)/k}$, and the composition of $\gamma_{z'}$ with

$\Pi_{k(X_k)/k} \to \Pi_{X_k/k}$ is isomorphic to γ_z. Hence, every specialization of z lifts to $\operatorname{Spec} k(X_k)$. ∎

Corollary 12 *t-b.l. sections are b.l.*

Proof Let X be a curve over a field k and $s \in \Pi_{X/k}(k)$ a t-b.l. Galois section. Write $R = k[t]_{(t)}$, we have that $s_{k(t)}$ is a generic section of $\Pi_{X_R/R}$ and s is a specialization of $s_{k(t)}$. By hypothesis $s_{k(t)}$ is b.l., hence its specialization s is b.l. by Lemma 11. ∎

Lemma 13 *Specializations of t-b.l. sections are t-b.l.*

Proof Let $X \to \operatorname{Spec} R$ be as above, $z \in \Pi_{X_K/K}(K)$ a generic, t-b.l. Galois section, $s \in \Pi_{X_k/k}(k)$ a specialization. Let R' be the local ring of the generic point of the divisor $\mathbb{A}^1_k \subseteq \mathbb{A}^1_R$. We have that R' is a DVR with fraction field $K' = K(t)$ and residue field equal $k' = k(t)$. Consider the family of curves $X_{R'} \to \operatorname{Spec} R'$, then $z_{K(t)} = z_{K'} \in \Pi_{X_{K'}/K'}(K')$ is by hypothesis a b.l. Galois section. We have that $s_{k(t)} = s_{k'}$ is a specialization of $z_{K'}$ and thus it is b.l. by Lemma 11. ∎

Lemma 14 *Let $f : Y \to X$ be a dominant morphism of curves over a field k and $s \in \Pi_{Y/k}(k)$ a Galois section. If s is b.l. then $f(s) \in \Pi_{X/k}(k)$ is b.l. If f is finite étale, the converse holds.*

Proof The first statement is obvious. If f is finite étale, the second statement follows from the fact that $\operatorname{Gal}(k(Y)) = \pi_1(Y) \times_{\pi_1(X)} \operatorname{Gal}(k(X))$. ∎

Corollary 15 *Let $f : Y \to X$ be a dominant morphism of curves over a field k and $s \in \Pi_{Y/k}(k)$ a Galois section. If s is t-b.l. then $f(s) \in \Pi_{X/k}(k)$ is t-b.l. If f is finite étale, the converse holds.* ∎

We say that a curve X is *parabolic* if it has genus 0 and the degree of $\bar{X} \setminus X$ is at most 2. If X is proper of genus 0, then there is a unique Galois section, and it is geometric if and only if $X = \mathbb{P}^1$. If X is parabolic affine, every Galois section of X is cuspidal (the only non-trivial case is settled by [18, Theorem A]). Because of this, we can always assume that X is non-parabolic.

If X is a curve over a number field k and s is a b.l. section, by Koenigsmann's results [13], [21, Proposition 1] for every finite place v of k there exists a local point $x_v \in \bar{X}(k_v)$ such that s_{k_v} is associated with x_v, i.e. if $x_v \in X$ then s_v is the geometric section associated with x_v, otherwise s_v is one of the cuspidal sections of the packet associated with $x_v \in \bar{X} \setminus X$.

If X is non-parabolic, the local point x_v associated with s is unique. To check this, we can pass to an étale neighbourhood Y of genus ≥ 1 and use the injectivity of the section map [20, Proposition 75] on $\bar{Y}_{k_v}$.

Lemma 16 *Let X be a curve over a number field k, $U \subseteq X$ a non-empty open subset and $s \in \Pi_{X/k}(k)$ a t-b.l. Galois section. There exists a t-b.l. section of U which lifts s.*

Proof We may assume that X is non-parabolic and that s is neither geometric nor cuspidal, otherwise this is trivial.

By hypothesis, $s_{k(t)}$ lifts to a b.l. Galois section r_0 of $U_{k(t)}$, we are going to show that r_0 is the base change to $k(t)$ of some Galois section of U. Identify $k(t)$ with the fraction field of $\mathbb{A}^1$ and choose $c \in \mathbb{A}^1$ a closed point with residue field k'.

For every finite place ν of k there is a unique local point $x_\nu \in \bar{X}(k_\nu)$ associated with s. Since s lifts to a b.l. section of U by Corollary 12 and it is not geometric nor cuspidal, the set of places ν such that $x_\nu \in U$ has Dirichlet density 1 by [21, Theorem B], hence we may choose ν with $x_\nu \in U$ and $k' \subset k_\nu$. If $r_{0,c}$ is a specialization of r_0 at c, by construction it is b.l. and lifts s, hence r_{0,c,k_ν} is geometric associated with x_ν. By Lemma 8, the specializing loop $S_c^1 \to \Pi_{U_{k'}/k'}$ of r_0 at c is constant and hence r_0 extends to a section $\mathrm{Spec}\,\mathcal{O}_c \to \Pi_{U \times \mathbb{A}^1/\mathbb{A}^1}$ by Proposition 3.

Since we can do this for every closed point $c \in \mathbb{A}^1$ and we may write $\Pi_{U \times \mathbb{A}^1/\mathbb{A}^1}$ as a product $\Pi_{U/k} \times_k \mathbb{A}^1$, by [5, Corollary A.3] we get that r_0 extends to a section $\tilde{r}_0 : \mathbb{A}^1 \to \Pi_{U \times \mathbb{A}^1/\mathbb{A}^1}$. Since $\Pi_{\mathbb{A}^1/k} = \mathrm{Spec}\,k$, the composition $\mathbb{A}^1 \to \Pi_{U \times \mathbb{A}^1/\mathbb{A}^1} \to \Pi_{U/k}$ factorizes through a section $r : \mathrm{Spec}\,k \to \Pi_{U/k}$ and hence $\tilde{r}_0 = r_{\mathbb{A}^1}$, $r_0 = r_{k(t)}$. This implies that r is a t-b.l. lifting of s. $\qquad\square$

5 The main argument

This section consists of a unique statement. The main theorems will follow easily from it.

Proposition 17 *Let k be a number field, ν a finite place, write $U = \mathbb{A}^1 \setminus \{1, 2\}$. Let $s \in \Pi_{U/k}(k)$ be a b.l. Galois section and $x_\nu \in \mathbb{P}^1(k_\nu)$ the unique associated local point. Let $\delta : \mathrm{Spec}\,k(t) \to \mathbb{A}^1$ be the "diagonal" point, i.e. the generic one, and assume that $s_{k(t)}$ lifts to a section of $\mathrm{Spec}\,k(\mathbb{A}^1) \otimes_k k(t) \setminus \{\delta\}$ (e.g. if s is t-b.l.). Then x_ν is k-rational.*

Proof We may assume that $x_\nu \in U$ since otherwise it is clearly k-rational. Let r_0 be the lifting of $s_{k(t)}$ to $\mathrm{Spec}\,k(\mathbb{A}^1) \otimes_k k(t) \setminus \{\delta\}$, and r the image of r_0 in $U_{k(t)} \setminus \{\delta\}$. We divide the proof in three steps.

Step 1. Specializations of r. We are going to prove that, for every k-rational point $c \in U(k)$ different from x_ν, the specializing loop of r is constant: the idea is to prove that for any specialization r_c of r the base change of r_c to k_ν is the geometric section associated with $x_\nu \in U \setminus \{c\}(k_\nu)$ regardless of the choice of r_c, then we apply Lemma 8.

Write $\Delta \subseteq U \times U$ for the diagonal, consider the family of curves $U \times U \setminus \Delta \to U$ and fix a k-rational point $c \in U(k)$ with $c \neq x_\nu$, we have $(U \times U \setminus \Delta)_c = U \setminus \{c\}$. Consider the scheme $W = \mathrm{Spec}\,\mathcal{O}_c \otimes_k \mathcal{O}_c \setminus \Delta$ over $\mathrm{Spec}\,\mathcal{O}_c$, it contains $\mathrm{Spec}\,k(\mathbb{A}^1) \otimes_k k(t) \setminus \{\delta\}$. The morphism $W \to \mathrm{Spec}\,\mathcal{O}_c$ is a projective limit of families of curves over $\mathcal{O}_c$ (it can be obtained from $U_{\mathcal{O}_c} \to \mathrm{Spec}\,\mathcal{O}_c$ by removing divisors étale over $\mathcal{O}_c$), its generic fiber is $\mathrm{Spec}\,\mathcal{O}_c \otimes_k k(t) \setminus \{\delta\}$ while the special fiber is $\mathrm{Spec}\,\mathcal{O}_c \setminus \{c\} = \mathrm{Spec}\,k(t)$.

Choose $r_c \in \Pi_{U\setminus\{c\}/k}(k)$ any specialization of r. Let r_1 be the image of r_0 in $\operatorname{Spec} \mathcal{O}_c \otimes_k k(t) \setminus \{\delta\}$. The specializing loop $S_c^1 \to \Pi_{U\setminus\{c\}/k}$ of r factorizes through the specializing loop $S_c^1 \to \Pi_{k(t)/k}$ of r_1, hence r_c is b.l. Furthermore, the image of r_c in U is a specialization of $s_{k(t)}$, i.e. it is s.

Since $x_v \in U \setminus \{c\}$, r_c is b.l. and maps to s, we have that the base change r_{c,k_v} is *the* geometric section associated to x_v, regardless of the choice of r_c. By Lemma 8, the specializing loop of r at c is constant.

Step 2. Change of coordinates. The map $y \mapsto t - y$ defines an automorphism $\varphi : \mathbb{A}_{k(t)}^1 \to \mathbb{A}_{k(t)}^1$ with $\varphi(\delta) = 0$. Write $V = \mathbb{A}_{k(t)}^1 \setminus \{\varphi(1), \varphi(2)\}$, we have that φ restricts to an isomorphism $\varphi : U_{k(t)} \to V$. We thus get a Galois section $\varphi(s_{k(t)}) \in \Pi_V(k(t))$ with a lifting $\varphi(r) \in \Pi_{V\setminus\{0\}}(k(t))$. Since $V \setminus \{0\} \subseteq \mathbb{G}_m$, then $\varphi(r)$ induces a section

$$z \in \Pi_{\mathbb{G}_m/k(t)}(k(t)) = \mathcal{B}\widehat{\mathbb{Z}}(1)(k(t)) = \varprojlim_n k(t)^*/k(t)^{*n} = \widehat{k(t)^*},$$

$$z = \lambda \cdot \prod_c q_c^{e_c}$$

where $\lambda \in \widehat{k}^*$, c varies among closed points of $\mathbb{A}^1$, $q_c \in k[t]$ is the monic, irreducible polynomial associated with c and $e_c \in \widehat{\mathbb{Z}}$.[1] For every k-rational point $c \neq x_v, 1, 2$, the fact that the specializing loop of r at c is constant implies that the same holds for $\varphi(r)$ and hence for z, hence $e_c = 0$ by Example 6. Since we can repeat step 1 after base changing to any finite extension of k, we get that $e_c = 0$ for every *closed* point $c \neq x_v, 1, 2$.

If x_v is algebraic over k, let q be its minimal polynomial and e the exponent of q as a factor of z, otherwise $q = 1$ and $e = 0$. We may thus write

$$z = \lambda \cdot q^e \cdot (t-1)^{e_1} \cdot (t-2)^{e_2}.$$

We are going to prove that q has degree 1 and thus x_v is k-rational.

Step 3. Specializations of z. Fix K a finite extension of k which splits q completely (if $q = 1$ choose $K = k$) and let η be an extension of v. In the rest of the proof, we are going to consider many K-rational points $c \in K$ (or $c_i \in K$). We will always tacitly assume that $c \neq 1, 2$ and $q(c) \neq 0$.

If we repeat steps 1 and 2 after base changing to K, for every c the base change to K_η of the unique specialization of $\varphi(r)$ at c is $\varphi(x_\eta)$, hence we obtain the equation

$$\lambda \cdot q(c)^e \cdot (c-1)^{e_1} \cdot (c-2)^{e_2} = c - x_\eta \in \widehat{K_\eta}^*,$$

and by applying $\eta : \widehat{K_\eta}^* \to \widehat{\mathbb{Z}}$ we get

$$\eta(\lambda) + e\eta(q(c)) - \eta(c - x_\eta) + e_1\eta(c-1) + e_2\eta(c-2) = 0 \in \widehat{\mathbb{Z}}.$$

Observe that if $p \in K[t]$ is a polynomial, $c \in K$ an element with $p(c) \neq 0$ and $c_i \neq c$ is a sequence which tends to c in the η-adic topology, then $\eta(p(c_i)) = \eta(p(c))$ is constant for $i \gg 0$ large enough while $\eta(c_i - c)$ is *not* constant.

---

[1] Notice that, while there is an embedding $\widehat{k(t)^*} \subset \widehat{k}^* \cdot \prod_c q_c^{\widehat{\mathbb{Z}}}$, this is not a bijection, since for every n all but finitely many exponents of an element of $\widehat{k(t)^*}$ are multiples of n.

Choose a sequence of K-rational points c_i which tends to 1. Since 1 is not a root of the polynomials q, $t-2$, $t-x_\eta$ we see that for $i \gg 0$ all terms except $e_1\eta(c_i-1)$ in the equation above are constant. It follows that $e_1\eta(c_i-1)$ is constant, too. Since $c_i \xrightarrow{i} 1$ and $c_i \neq 1$, then $\eta(c_i-1)$ is not constant, this implies that $e_1 = 0$. With the same argument we see that $e_2 = 0$, hence

$$\eta(\lambda) + e\eta(q(c)) - \eta(c - x_\eta) = 0 \in \widehat{\mathbb{Z}}.$$

If x_v is transcendental over k and thus $q = 1$, $e = 0$, then $\eta(c - x_\eta) = \eta(\lambda)$ does not depend on c, which is absurd, hence x_v is algebraic over k. Since K splits q, we may write $q(t) = (t - x_\eta) \cdot \prod_j (t - y_j)$,

$$\eta(\lambda) + (e-1)\eta(c - x_\eta) + e\sum_j \eta(c - y_j) = 0 \in \widehat{\mathbb{Z}}.$$

Since we are in characteristic 0 and q is irreducible over k, then $x_v \neq y_j$ for every j and $y_j \neq y_{j'}$ for $j \neq j'$. Using a sequence $c_i \xrightarrow{i} x_\eta$ and the same argument as above we see that $e = 1$ and hence

$$\eta(\lambda) + \sum_j \eta(c - y_j) = 0 \in \widehat{\mathbb{Z}}.$$

If by contradiction x_v is not k-rational and thus $\deg q \geq 2$, using a sequence $c_i \xrightarrow{i} y_j$ we see that $\eta(\lambda) + \sum_j \eta(c - y_j) = 0$ is not constant for $i \gg 0$, which is absurd. $\square$

6 Reduction to number fields

In [17] M. Saïdi and M. Tyler reduced the birational section conjecture to number fields. We are going to do this for the t-birational version, too. A reader only interested in number fields may safely skip this section.

Over number fields, we know that we can lift t-b.l. sections to open subsets thanks to Lemma 16. In order to prove it, we use a theorem of J. Stix [20, Theorem B] which is only available for number fields. Lifting to open subsets is crucial for the reduction to number fields, and we can only do it for number fields: this is a problem. In order to overcome it, we define *quasi-t-b.l.* sections.

Definition 18 Let X be a geometrically connected curve over a field k. A Galois section s of X is quasi-t-b.l. if there is an open subset $U \subseteq X$, a lifting s' of s to U and a dominant map $f : U \to Y$ where Y is a non-parabolic curve such that s' is b.l. and $f(s')$ is t-b.l.

Since t-b.l. sections are b.l. by Corollary 12, it is immediate to check that t-b.l. sections are quasi-t-b.l. If s is a quasi-t-b.l. section of X and $U \subseteq X$ is an open subset, then it is obvious that we can lift s to a quasi-t-b.l. section of U.

Lemma 19 *Let C be a curve with fraction field K, $X \to C$ a family of curves and $s \in \Pi_{X_K/K}(K)$ a generic section which is quasi-t-b.l. There exists a non-empty open subset $U \subseteq C$ such that the specializations of $X|_U \to U$ are quasi-t-b.l.*

Proof Up to shrinking both C and X, we may assume that there exists a family of non-parabolic curves $Y \to C$ and a quasi-finite morphism $f : X \to Y$ over C such that $f(s)$ is t-b.l. The specializations of $f(s)$ are t-b.l. by Lemma 13 and the specializations of s are b.l. by Lemma 11, it follows that they are quasi-t-b.l., too. □

Lemma 20 *Let k be a number field, and assume that t-b.l. sections over k are geometric or cuspidal. Then the same holds for quasi-t-b.l. sections.*

Proof Let X be a curve over k and $s \in \Pi_{X/k}(k)$ a quasi-t-b.l. section. We may assume that there exists a non-parabolic curve Y and a dominant morphism $f : X \to Y$ such that $f(s)$ is t-b.l. Since s is b.l., for every finite place v of k there exists a unique associated point $x_v \in \bar{X}(k_v)$. Since $f(s)$ is t-b.l., there exists a unique k-rational point $y \in \bar{Y}(k)$ associated to $f(s)$, in particular $f(x_v) = y$ for every v. Any b.l. lifting of s to $X \setminus f^{-1}(y)$ is cuspidal thanks to [21, Theorem B], hence s is geometric or cuspidal. □

Let us recall a famous result by A. Tamagawa.

Proposition 21 (Tamagawa) *Let X be a hyperbolic curve over a field k finitely generated over $\mathbb{Q}$ and let $s \in \Pi_{X/k}(k)$ be a Galois section. If s is not geometric nor cuspidal, there exists a curve Y of genus ≥ 2 with $\bar{Y}(k) = \emptyset$, a finite étale morphism $Y \to X$ and a lifting $r \in \Pi_{Y/k}(k)$ of s.*

Proof This is essentially [22, Proposition 2.8 (iv)]. □

Proposition 22 *Assume that t-b.l. sections of curves defined over number fields are geometric or cuspidal. Then quasi-t-b.l. sections of curves over fields of finite type over $\mathbb{Q}$ are geometric or cuspidal.*

Proof We prove this by induction on the transcendence degree n of the base field k over $\mathbb{Q}$. The case $n = 0$ is Lemma 20. Assume $n \geq 1$ and that the statement is proved for fields of transcendence degree $n - 1$, let $h \subseteq k$ be algebraically closed in k and of transcendence degree $n - 1$ over $\mathbb{Q}$. Let X be a curve and $s \in \Pi_{X/k}$ a quasi-t-b.l. section, up to passing to an open subset we may assume that X is hyperbolic. Assume by contradiction that s is not geometric nor cuspidal, thanks to Tamagawa's argument (Proposition 21) and Lemma 14 up to passing to an étale neighbourhood we may assume that $\bar{X}(k) = \emptyset$. Clearly the image in s in $\bar{X}$ is not geometric, so we may replace X with $\bar{X}$ and assume that X is projective.

Choose E any elliptic curve over h, we may find a finite, possibly ramified cover $Y \to X$ with a finite morphism $Y \to E_k$. Let $U \subseteq X$, $V \subseteq Y$ be open subsets such that $V \to U$ is finite étale. Since s is quasi-t-b.l., by applying Lemma 14 to $V \to U$

we may find a finite extension k'/k and a quasi-t-b.l. lifting $r \in \Pi_{Y/k}(k')$ of $s_{k'}$. Thanks to [6, Lemma 6.5, Theorem 7.2], it is enough to prove that r, and thus $s_{k'}$, is geometric. We may thus replace k, X, s with k', $Y_{k'}$, r and assume that there is a finite morphism $X \to E_k$.

Let C be an affine curve over h whose function field is k, up to shrinking C we may extend X to a family of smooth projective curves $\widetilde{X} \to C$ with a finite morphism $\widetilde{X} \to E \times C$ over C. The generic section s extends to a global section $\widetilde{s} : C \to \Pi_{\widetilde{X}/C}$ thanks to [7, Corollary 3.4]. The specializations of $\widetilde{s}$ are geometric thanks to Lemma 19 plus inductive hypothesis. It follows that s is geometric thanks to [7, Definition 4.1, Corollary 4.9]. □

7 Proof of the main theorems

Theorems A, B and C follow rather easily from Proposition 17.

Theorem A *Let X be a smooth curve over a field k finitely generated over $\mathbb{Q}$. A Galois section of X is geometric or cuspidal if and only if it is t-birationally liftable.*

Proof Thanks to Proposition 22, we may assume that k is a number field. Let $s \in \Pi_{X/k}(k)$ be a t-b.l. section, assume by contradiction that s is neither geometric nor cuspidal. By Lemma 16 we may assume that X is hyperbolic. By Tamagawa's argument (Proposition 21) and Corollary 15, up to passing to an étale neighbourhood of s we may assume that X has genus at least 2 and $\bar{X}(k) = \emptyset$. Fix ν any finite place of k and let $x_\nu \in \bar{X}(k_\nu)$ the local point associated with s.

Choose a projective embedding $j : \bar{X} \subseteq \mathbb{P}^n$ such that $j(x_\nu) \in \mathbb{A}^n \subseteq \mathbb{P}^n$ and let $U = \mathbb{A}^n \cap X$, we have that s lifts to a t-b.l. section s' of U thanks to Lemma 16. Since $\bar{X}(k) = \emptyset$ then $j(x_\nu) \in \mathbb{A}^n$ is not k-rational, in particular there exists one coordinate $c : \mathbb{A}^n \to \mathbb{A}^1$ such that $c(j(x_\nu))$ is not k-rational. Up to shrinking U furthermore we may assume that $c(j(U)) \subseteq \mathbb{A}^1 \setminus \{1, 2\}$. Then $c(j(s')) \in \Pi_{\mathbb{A}^1_k \setminus \{1,2\}/k}(k)$ is a t-b.l. section whose base change to k_ν is associated with a non-k-rational local point, which is in contradiction with Proposition 17. □

Lemma 23 *Let X be a curve over a countable field k. The following are equivalent.*

- *For every pair of non-empty open subset $V \subset U \subset X$, the map $\mathcal{S}_{V/k} \to \mathcal{S}_{U/k}$ is surjective.*
- *For every non-empty open subset $U \subset X$, every Galois section of U is birationally liftable.*

Proof The second condition clearly implies the first. Assume that the first holds and let $X' \subset X$ be a non-empty open subset. Since k is countable, then X' has a countable number of open subsets. Since $\mathcal{S}_{k(X')/k} \simeq \varprojlim_{U \subset X'} \mathcal{S}_{U/k}$ [20, Lemma 259], the projective system $(\mathcal{S}_{U/k})_U$ is countable and the transition maps are surjective, then $\mathcal{S}_{k(X')/k} \to \mathcal{S}_{X'/k}$ is surjective as well, i.e. every Galois section of X' is birationally liftable. □

Corollary 24 *The following are equivalent.*

- *The cuspidalization conjecture holds.*
- *For every smooth, geometrically connected hyperbolic curve X over a field k finitely generated over $\mathbb{Q}$, every Galois section of X is birationally liftable.*

Theorem B *Let X be a hyperbolic curve over a field k finitely generated over $\mathbb{Q}$. The following are equivalent.*

- *For every finitely generated extension K/k, every Galois section of X_K is either geometric or cuspidal.*
- *For every finitely generated extension K/k, every Galois section of X_K is birationally liftable.*

As a consequence, the section conjecture is equivalent to the cuspidalization conjecture.

Proof Follows directly from Theorem A and Corollary 24. □

Lemma 25 *Let k be a Hilbertian field, V an open subset of $\mathbb{A}^1$, $X \to V$ a family of curves, $s_1, s_2 \in \Pi_{X/V}(V)$ two sections. If s_1, s_2 have isomorphic specializations at k-rational points of V, then $s_1 \simeq s_2$.*

Proof For the convenience of the reader, we give proofs in both the language of fundamental gerbes and fundamental groups.

With fundamental gerbes Write $\Pi_{X/V} = \varprojlim \Phi_i$ as in Lemma 2, let W_i be the fibered product $V \times_{\Phi_i} V$ with respect to the two morphisms $s_1, s_2 : V \to \Pi_{X/V} \to \Phi_i$, we have that $\underline{\mathrm{Isom}}(s_1, s_2) = V \times_{\Pi_{X/V}} V = \varprojlim_i W_i$.

There is a natural map $W_i = V \times_{\Phi_i} V \to V \times_V V = V$ which coincides with projection on both coordinates, it is a finite étale cover since Φ_i is a proper étale gerbe over V. By hypothesis, $\underline{\mathrm{Isom}}(s_1, s_2)(k) \to V(k)$ is surjective, this implies that $W_i(k) \to V(k)$ is surjective too. Since k is Hilbertian and $W_i \to V$ is finite étale, there is a section $V \to W_i$. Let $W_i(V)$ be the set of sections, it is finite and non-empty. It follows that $\underline{\mathrm{Isom}}(s_1, s_2)(V) = \varprojlim_i W_i(V)$ is non-empty since a projective limit of finite, non-empty sets is non-empty. This gives the desired global isomorphism $s_1 \simeq s_2$.

With fundamental groups For every $v \in V(k)$, choose $r_v : \mathrm{Gal}(\bar{k}/k) \to \pi_1(V)$ a representative of the geometric Galois section associated with V. If $S \subset V(k)$ is not thin in the sense of Serre, the images of the sections r_v for $v \in S$ generate $\pi_1(V)$ topologically: if $H \subset \pi_1(V)$ is an open subgroup containing all of them, the corresponding finite étale covering $C \to V$ is connected and the image of $C(k) \to V(k)$ contains S, hence $C = V$ since S is not thin.

Let F be a geometric fiber of the family, we have that $\pi_1(X)$ is an extension of $\pi_1(V)$ by $\pi_1(F)$. We may write $\pi_1(X)$ as a projective limit $\varprojlim_i G_i$ of pro-finite groups such that G_i is an extension of $\pi_1(V)$ by a finite group H_i and $\pi_1(F) =$

$\varprojlim_i H_i$; this can be done by writing first $\pi_1(X) = \varprojlim_i G_{0,i} = \pi_1(X)/N_i$ as a projective limit of finite groups and then defining $G_i = G_{0,i} \times_{\pi_1(V)/N_i} \pi_1(V)$. Choose representatives $t_1, t_2 : \pi_1(V) \to \pi_1(X)$ of s_1, s_2, and let $t_{1,i}, t_{2,i} : \pi_1(V) \to \pi_1(X) \to G_i$ be the compositions.

For every $v \in V(k)$ and every i, by hypothesis there exists $h \in H_i$ such that $h^{-1}(t_{1,i} \circ r_v)h = t_{2,i} \circ r_v$. Since H_i is finite there exists an $h \in H_i$ such that the set $S_h \subset V(k)$ of points v for which $h^{-1}(t_{1,i} \circ r_v)h = t_{2,i} \circ r_v$ is not thin. Since the homomorphisms r_v for $v \in S_h$ generate $\pi_1(V)$, this implies that $h^{-1}t_{1,i}h = t_{2,i}$. Since a projective limit of finite, non-empty subsets is non-empty, we get an element $g \in \pi_1(F) = \varprojlim_i H_i$ such that $g^{-1}t_1 g = t_2$, i.e. the π_1 sections s_1, s_2 are equal. $\qquad\square$

Let us prove that the two forms of the birational section conjecture given in the introduction are equivalent.

Lemma 26 *The following are equivalent.*

- *The birational section conjecture holds.*
- *For every curve X over a field k finitely generated over $\mathbb{Q}$, every b.l. Galois section of X is either geometric or cuspidal.*

Proof The first condition clearly implies the second. Assume that the second holds, let X be a curve over a field k finitely generated over $\mathbb{Q}$ and let $s \in \mathcal{S}_{k(X)/k}$ be a birational Galois section, we want to show that s is cuspidal. Up to replacing X with an open subset, we can assume that X is hyperbolic.

For every open subset $U \subset X$, denote by $\iota_U(s)$ the image of s in $\mathcal{S}_{U/k}$, by hypothesis it is cuspidal. By [19, Theorem 17 (2)], there exists a unique rational point $x \in \bar{X}(k)$ associated with $\iota_X(s)$, where $\bar{X}$ is the smooth completion of X. By uniqueness, x is associated with $\iota_U(s)$ for every open subset U as well.

For every open subset $U \subset X$, denote by $\mathcal{P}_{U,x} \subset \mathcal{S}_{U/k}$ the packet of cuspidal sections associated with x. We have that $\mathcal{S}_{k(X)/k} \simeq \varprojlim_U \mathcal{S}_{U/k}$ by [20, Lemma 259], and the cuspidal birational sections over x identify naturally with $\varprojlim_U \mathcal{P}_{U,x} \subset \mathcal{S}_{k(X)/k}$. Since $\iota_U(s) \in \mathcal{P}_{U,x}$ for every $U \subset X$ and s is the limit of the sections $\iota_U(s)$, we get that s is cuspidal associated with x too. $\qquad\square$

Theorem C *The following are equivalent.*

- *The birational section conjecture holds.*
- *For every number field k, every section $z \in \mathcal{S}_{k(\mathbb{P}^1)/k}$ and every open subset $U \subseteq \mathbb{P}^1$, there exists an open subset $V \subseteq U$ and a π_1 section r of $U \times V \setminus \Delta \to V$ such that the specialization r_v is equal to $\iota_{U\setminus\{v\}}(z)$ for every $v \in V(k)$.*

Proof Thanks to Lemma 26, we may use the alternative formulation of the birational section conjecture.

Assume that the birational section conjecture holds and let z, U be as in the statement. Up to passing to an open subset, we can assume that U is hyperbolic. By hypothesis, z is associated with a unique rational point $p \in \mathbb{P}^1(k)$. If $p \in U$, choose $V = U \setminus \{p\}$, the morphism $(p, \mathrm{id}_V) : V \to U \times V \setminus \Delta$ defines the desired π_1 section.

If $p \notin U$, choose $V = U$. Since $\iota_U(z)_{k(t)}$ is cuspidal then it lifts to a cuspidal section z' of $U_{k(t)} \setminus \{\delta\}$. Let $v \in V(k)$ be a rational point, since z' is cuspidal it is b.l. and its specializations at v are b.l. too by Lemma 11. By hypothesis, this implies that all the specializations of z' at v are cuspidal liftings of $\iota_U(z)$, and hence coincide with $\iota_{U \setminus \{v\}}(z)$ since there is only one cuspidal lifting of $\iota_U(z)$ by [19, Theorem 17 (2)]. In particular, the specializing loop at v is constant by Lemma 8. By repeating the argument after base changing to every finite extension of k, we get that all the specializing loops of z' at closed points of U are constant. As a consequence, z' extends to the desired section $U \to \Pi_{(U \times U \setminus \Delta)/U}$ by [5, Corollary A.3].

Assume that the second condition holds. Thanks to [17, Theorem C], it is enough to consider number fields. Let k be a number field, X a smooth projective curve over k, $s \in \mathcal{S}_{X/k}$ a b.l. section with lifting $z \in \mathcal{S}_{k(X)/k}$ and v a place of k, we want to prove that s is geometric or cuspidal. With an argument analogous to that of Theorem A, we can reduce to the case in which $X = U$ is an open subset of $\mathbb{P}^1$ and to proving that $x_v \in \mathbb{P}^1$ is k-rational.

Let $V \subseteq U$, $r \in \mathcal{S}_{(U \times V \setminus \Delta)/V}$ be as given by hypothesis. If $U' \subseteq U$ is another open subset and V', r' are given analogously, we can shrink V' and assume that $V' \subseteq V$, then Lemma 25 implies that $r' = r|_{V'}$. Passing to the limit along open subsets of $\mathbb{P}^1$ we get a Galois section of $\operatorname{Spec} k(\mathbb{A}^1) \otimes_k k(t) \setminus \{\delta\}$ which lifts $s_{k(t)}$, hence x_v is k-rational by Proposition 17. $\qquad\square$

Declarations

Competing Interests The author declares no competing interests.

References

1. Abramovich, D., Graber, T., Vistoli, A.: Gromov-Witten theory of Deligne-Mumford stacks. Am. J. Math. **130**(5), 1337–1398 (2008)
2. Anderson, M.P.: Exactness properties of profinite completion functors. Topology **13**(3), 229–239 (1974)
3. Borne, N., Vistoli, A.: The Nori fundamental gerbe of a fibered category. J. Algebraic Geom. **24**, 311–353 (2015)

4. Borne, N., Vistoli, A.: Fundamental gerbes. Algebra Number Theory **13**(3), 531–576 (2019)
5. Bresciani, G.: Essential dimension and pro-finite group schemes. Ann. Sc. Norm. Super. Pisa, Cl. Sci. (5) **22**(4), 1899–1936 (2021)
6. Bresciani, G.: Some implications between Grothendieck's anabelian conjectures. Algebr. Geom. **8**(2), 231–267 (2021)
7. Bresciani, G.: on the section conjecture over fields of finite type (2023). arXiv:1911.03234
8. Bresciani, G., Vistoli, A.: An arithmetic valuative criterion for proper maps of tame algebraic stacks. Manuscr. Math. (2023). https://doi.org/10.1007/s00229-023-01491-6
9. Esnault, H., Hô Hai, P.: Packets in Grothendieck's section conjecture. Adv. Math. **218**(2), 395–416 (2008)
10. Grothendieck, A.: Revêtements étales et groupe fondamental. Lecture Notes in Mathematics, vol. 224. Springer, Berlin (1971). Séminaire de Géométrie Algébrique du Bois Marie 1960–1961 (SGA 1), Dirigé par Alexandre Grothendieck. Augmenté de deux exposés de M. Raynaud
11. Grothendieck, A.: Brief an G. Faltings. In: Geometric Galois Actions, 1. London Math. Soc. Lecture Note Ser., vol. 242, pp. 49–58. Cambridge University Press, Cambridge (1997). With an English translation on pp. 285–293
12. Grunewald, F., Jaikin-Zapirain, A., Zalesskii, P.A.: Cohomological goodness and the profinite completion of Bianchi groups. Duke Math. J. **144**(1), 53–72 (2008)
13. Koenigsmann, J.: On the 'section conjecture' in anabelian geometry. J. Reine Angew. Math. **588**, 221–235 (2005)
14. Mattia, T., Vistoli, A.: Infinite root stacks and quasi-coherent sheaves on logarithmic schemes. Proc. Lond. Math. Soc. (3) **116**(5), 1187–1243 (2018)
15. Mattuck, A.: Abelian varieties over p-adic ground fields. Ann. Math. (2) **62**, 92–119 (1955)
16. Olsson, M.: Algebraic Spaces and Stacks. American Mathematical Society Colloquium Publications, vol. 62. Am. Math. Soc., Providence (2016)
17. Saïdi, M., Tyler, M.: On the birational section conjecture over finitely generated fields. Algebra Number Theory **15**(2), 435–460 (2021)
18. Schmidt, J.: Homotopy rational points of Brauer-Severi varieties (2015). arXiv:1503.08108
19. Stix, J.: On cuspidal sections of algebraic fundamental groups, Galois-Teichmüller theory and arithmetic geometry, Kyoto, Japan, October 25–30, 2010, Tokyo, pp. 519–563. Mathematical Society of Japan (2012). Selected papers based on the presentations at the workshop and conference
20. Stix, J.: Rational Points and Arithmetic of Fundamental Groups. Lecture Notes in Mathematics, vol. 2054. Springer, Berlin (2013)
21. Stix, J.: On the birational section conjecture with local conditions. Invent. Math. **199**(1), 239–265 (2015)
22. Tamagawa, A.: The Grothendieck conjecture for affine curves. Compos. Math. **109**(2), 135–194 (1997)
23. Wickelgren, K.: n-Nilpotent obstructions to π_1 sections of $\mathbb{P}^1 - \{0, 1, \infty\}$ and Massey products. In: Galois-Teichmüller Theory and Arithmetic Geometry. Adv. Stud. Pure Math., vol. 63, pp. 579–600. Math. Soc. Japan, Tokyo (2012)