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Equivariant algebraic concordance of strongly invertible knots

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Abstract

By considering a particular type of invariant Seifert surfaces we define a homomorphism Φ from the (topological) equivariant concordance group of directed strongly invertible knots $\widetilde{\mathcal{C}}$ to a new equivariant algebraic concordance group $\widetilde{\mathcal{G}}^{\mathbb{Z}}$. We prove that Φ lifts both Miller and Powell's equivariant algebraic concordance homomorphism (*J. Lond. Math. Soc.* (2023), no. 107, 2025-2053) and Alfieri and Boyle's equivariant signature (*Michigan Math. J. 1* (2023), no. 1, 1–17). Moreover, we provide a partial result on the isomorphism type of $\widetilde{\mathcal{G}}^{\mathbb{Z}}$ and obtain a new obstruction to equivariant sliceness, which can be viewed as an equivariant Fox–Milnor condition. We define new equivariant signatures and using these we obtain novel lower bounds on the equivariant slice genus. Finally, we show that Φ can obstruct equivariant sliceness for knots with Alexander polynomial one.

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1 | INTRODUCTION

A knot $K \subset S^3$ is said to be *invertible* if there is an orientation-preserving homeomorphism ρ of S^3 such that $\rho(K) = K$ and ρ reverses the orientation on K. If such a homeomorphism can be taken to be a locally linear involution, we say that K is *strongly invertible*. Kawauchi [16, Lemma 1] proved that the two notions agree for hyperbolic knots, however, there exist examples of invertible knots that are not strongly invertible, see [12, Section 5].

As such an involution for a strongly invertible knot has always a nonempty fixed-point set, we know that it is always conjugated to an element of SO(4) by the solution of the Smith conjecture [3]. As a consequence, we can think of a strongly invertible knot as a knot that is invariant under a π-rotation around some unknotted axis in S^3. The knot intersects the axis in two points, by which the axis is separated into two half-axes.

In [25], Sakuma defined a notion of *direction* on a strongly invertible knot, which consists of an orientation of the axis of the involution together with the choice of one of the half-axes. Using this additional structure he was able to define unambiguously an operation of *equivariant connected sum* between *directed strongly invertible knots*.

As strongly invertible knots are naturally equipped with an involution, it is natural to ask whether a strongly invertible slice knot is also *equivariantly slice*, that is, it bounds a locally flat slice disk in B^4 which is invariant under a locally linear extension of the involution. Similarly to the classical case, this leads to the definition of the *equivariant concordance group* $\widetilde{\mathcal{C}}$ as the set of classes of directed strongly invertible knots up to an appropriate definition of *equivariant concordance*. Recently, in [6] we proved that $\widetilde{\mathcal{C}}$ is not abelian†, which is in stark contrast with the classical concordance group.

Several authors [1, 2, 5, 7, 23, 25] have found invariants and obstructions for the equivariant concordance of strongly invertible knots. In particular, in [5] the authors define several invariants for *smooth* equivariant concordance using knot Floer homology. Using the lower bounds on the *smooth equivariant slice genus* $\widetilde{g}_4^s$ provided by these invariants, they construct the first examples of strongly invertible knots with $\widetilde{g}_4^s(K) - g_4^s(K)$ arbitrarily large, where $g_4^s(K)$ is the classical smooth slice genus, answering [2, Question 1.1].

Miller and Powell, in [23] introduce a notion of *equivariant algebraic concordance*, by studying the action of the strong inversion on the Blanchfield pairing on the Alexander module of a strongly invertible knot. In this way, they define a homomorphism

$$\Psi : \widetilde{\mathcal{C}} \longrightarrow \widetilde{\mathcal{AC}}$$

from the equivariant concordance group to an *equivariant algebraic concordance group* $\widetilde{\mathcal{AC}}$ of *equivariant Blanchfield pairings*. From the equivariant Blanchfield pairings, they obtain new lower bounds on the equivariant slice genus, and they provide examples of genus one slice knots with arbitrarily large *topological* equivariant slice genus $\widetilde{g}_4$.

In [1, 2], the authors define an equivariant version of the classical knot signature for directed strongly invertible knots, obtaining a group homomorphism

$$\widetilde{\sigma} : \widetilde{\mathcal{C}} \longrightarrow \mathbb{Z}.$$

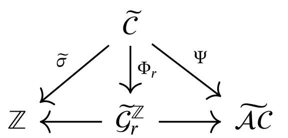
In this paper, we define a notion of *equivariant algebraic concordance* for directed strongly invertible knots, analogous to Levine's algebraic concordance [17, 18], by considering a particular *type* of *invariant Seifert surfaces*. In Theorem 4.7, we construct a homomorphism Φ from the equivariant concordance group $\widetilde{\mathcal{C}}$ to an *equivariant algebraic concordance group* $\widetilde{\mathcal{G}}^{\mathbb{Z}}$ of *equivariant Seifert systems*.

† The main result [6, Theorem 1.1.] of the paper relies on Donaldson's diagonalization theorem, which is inherently a smooth result. However, in [6, section 3.1] we give an alternate proof that $\widetilde{\mathcal{C}}$ is not abelian, relying on twisted Alexander polynomials, which can be adapted in the topological category.

The homomorphism Φ is able to detect valuable information about equivariant concordance. For instance, it can be used to obstruct equivariant sliceness of knots with Alexander polynomial one, see Example 4.13. However, it seems a hard task to give a complete description of the group structure of $\widetilde{\mathcal{G}}^{\mathbb{Z}}$. In Subsection 4.2, we introduce a more manageable quotient $\widetilde{\mathcal{G}}^{\mathbb{Z}}_r$ of this group that we call *reduced equivariant algebraic concordance group* and we denote by $\Phi_r : \widetilde{\mathcal{C}} \longrightarrow \widetilde{\mathcal{G}}^{\mathbb{Z}}_r$ the homomorphism obtained by composition.

The main result of this paper is the following theorem, which sums up the results of Theorems 4.12 and 5.13.

Theorem 1.1. *The homomorphism Ψ and the equivariant signature $\widetilde{\sigma}$ factor through $\widetilde{\mathcal{G}}^{\mathbb{Z}}_r$, that is, they fit in the following commutative diagram.*



Moreover, in Theorem 6.11 we obtain a partial result on the structure of $\widetilde{\mathcal{G}}^{\mathbb{Z}}$, determining the isomorphism type of $\widetilde{\mathcal{G}}^{\mathbb{Q}}_r$ which is defined similarly to $\widetilde{\mathcal{G}}^{\mathbb{Z}}_r$, by using *rational* equivariant Seifert forms. In particular, we have a natural inclusion $\widetilde{\mathcal{G}}^{\mathbb{Z}}_r \hookrightarrow \widetilde{\mathcal{G}}^{\mathbb{Q}}_r$ (see Lemma 6.1) and the following theorem holds.

Theorem (6.11). *The isomorphism type of the reduced rational equivariant algebraic concordance group is given by*

$$\widetilde{\mathcal{G}}^{\mathbb{Q}}_r \cong \mathbb{Z}^{\infty} \oplus \mathbb{Z}/2\mathbb{Z}^{\infty} \oplus \mathbb{Z}/4\mathbb{Z}^{\infty} \oplus \mathbb{Z}/8\mathbb{Z}^{\infty}.$$

The same arguments used in the proof of Theorem 6.11 fail to be easily adapted to study the (unreduced) group $\widetilde{\mathcal{G}}^{\mathbb{Z}}$. Therefore, we propose the following open problem that we would like to address in the future.

Open Question A. Determine the full isomorphism type of the equivariant algebraic concordance group $\widetilde{\mathcal{G}}^{\mathbb{Z}}$.

As pointed out in Remark 6.12 the composite map $\widetilde{\mathcal{C}} \longrightarrow \widetilde{\mathcal{G}}^{\mathbb{Q}}_r$ is not surjective, as $\widetilde{\mathcal{G}}^{\mathbb{Z}}_r$ is a proper subgroup. Hence, we ask the following question.

Open Question B. Determine the image of $\widetilde{\mathcal{C}}$ in $\widetilde{\mathcal{G}}^{\mathbb{Q}}_r$. In particular, does there exist a directed strongly invertible knot K whose image has order 8 in $\widetilde{\mathcal{G}}^{\mathbb{Q}}_r$?

As a consequence of the investigation on the group structure of $\widetilde{\mathcal{G}}^{\mathbb{Q}}_r$, in Theorem 6.13 we get the following obstruction to equivariant sliceness, which can be seen as an *equivariant Fox–Milnor condition*.

Theorem (6.13). *Let K be a strongly invertible knot and let $\Delta_K(t)$ be its Alexander polynomial, normalized so that $\Delta_K(t) = \Delta_K(t^{-1})$ and $\Delta_K(1) = 1$. If K is equivariantly slice then $\Delta_K(t)$ is a square.*

Theorem 6.13 is especially fascinating due to its intriguing resemblance to a result of Hartley and Kawauchi [14] which states that if a knot K is *strongly positive amphichiral* then $\Delta_K(t)$ is a square. While at the moment we are not able to formulate a precise conjecture, it would be interesting to understand better the relation between being equivariantly slice and strongly positive amphichiral for a strongly invertible knot.

The results of Section 6 suggest naturally the definition of new *equivariant signatures* $\{\widetilde{\sigma}_\lambda\}_{\lambda\in\mathbb{R}}$ and *equivariant signature jumps* $\{\widetilde{J}_\lambda\}_{\lambda\in\mathbb{R}}$, which are homomorphisms

$$\widetilde{\sigma}_\lambda,\ \widetilde{J}_\lambda : \widetilde{\mathcal{C}} \longrightarrow \mathbb{Z}$$

that we introduce in Section 7. In Proposition 7.3, we clarify the relation between $\widetilde{\sigma}_\lambda$, the Levine–Tristram signatures σ_ω proving the following.

Proposition (7.3). *Let K be a directed strongly invertible knot. Then for any $\lambda \leqslant 0$ and $\omega \in S^1$ such that $\lambda(\omega-1)^2 = (\omega+1)^2$ we have*

$$\widetilde{\sigma}_\lambda(K) = \sigma_\omega(K),$$

Although the equivariant signatures coincide with the Levine–Tristram signatures for $\lambda \leqslant 0$, for positive values of λ we actually get a new invariant of equivariant concordance (see Remark 7.11). The main result of Section 7 is Theorem 7.7, which gives a new lower bound on the equivariant slice genus $\widetilde{g}_4$ of a strongly invertible knot.

Theorem (7.7). *Given a directed strongly invertible knot K, for every $\lambda > 0$, $\lambda \neq 1$ we have*

$$\widetilde{g}_4(K) \geqslant \frac{|\widetilde{J}_\lambda(K)|}{4}.$$

Theorem 7.7 can be used to obtain new examples of strongly invertible knots with $\widetilde{g}_4(K) - g_4(K)$ arbitrarily large (see Remark A.4), where $g_4(K)$ is the classical (topological) slice genus.

Organization of the paper

In Section 2, we briefly recall some notions and results on equivariant concordance and on algebraic concordance, and we introduce the definition of *n-butterfly link*, which is a generalization of the *butterfly link* [2]. Section 3 contains some results on the extension and transversality of equivariant maps, that are used in the next section. In Section 4, we use Proposition 4.3 to motivate the definition of the equivariant algebraic concordance group $\widetilde{\mathcal{G}}^{\mathbb{Z}}$. In Section 5, we define a homomorphism from of $\widetilde{\mathcal{G}}^{\mathbb{Z}}$ to an equivariant version $\widetilde{W}(\mathbb{Q})$ of the Witt group of $\mathbb{Q}$, and we show that the equivariant signature [1] factors through $\widetilde{W}(\mathbb{Q})$. Section 6 is dedicated to studying the group structure of $\widetilde{\mathcal{G}}_r^{\mathbb{Q}}$, which is a simpler variant of $\widetilde{\mathcal{G}}^{\mathbb{Z}}$. Using these results, we introduce in Section 7 a new *equivariant signature* function $\widetilde{\sigma}_\lambda$ and we describe how it can be used to obtain lower bounds on the equivariant slice genus.

Finally, in the Appendix, we provide a table of examples of application of Theorem 7.7 on a family of 2-bridge knots with at most 12 crossings.

Conventions

We work in the topological category (see [8, 23] for details) unless otherwise specified. More precisely, we will implicitly consider:

- maps between manifolds to be continuous,
- submanifolds to be (properly) locally flat embedded,
- group actions (specifically involutions) on manifolds to be locally linear.

Throughout the paper, we will refer to and use some results appearing in [2, section 4] and [7]. Although in this paper such results are stated in the smooth category, we want to remark that the proofs can be adapted to work in the topological category.

2 | PRELIMINARIES

2.1 | Directed strongly invertible knots

We recall the definition of *directed strongly invertible knots* and *equivariant concordance group* following [2, 25].

Let (K, ρ) be a strongly invertible knot. By the resolution of the Smith conjecture [3], we know that ρ acts on S^3 as a rotation around the *axis* $\mathrm{Fix}(\rho)$, which is an unknotted S^1. As the restriction of ρ on K is orientation-reversing, the fixed axis intersects K in two points, which separate $\mathrm{Fix}(\rho)$ in two so-called *half-axes*.

Definition 2.1. A *direction* h on a strongly invertible knot (K, ρ) is the choice of one of the half-axes h and an orientation on $\mathrm{Fix}(\rho)$. We say that (K, ρ, h) is a *directed strongly invertible knot*.

Definition 2.2. We say that two directed strongly invertible knots (K_i, ρ_i, h_i), $i = 0, 1$ are *equivariantly isotopic* if there exists an orientation-preserving homeomorphism $\varphi : S^3 \longrightarrow S^3$ such that:

- $\varphi(K_0) = K_1$,
- $\varphi \circ \rho_0 = \rho_1 \circ \varphi$,
- $\varphi(h_0) = h_1$, preserving the chosen orientations on h_0 and h_1.

We will often omit to specify the choice of strong inversion and direction when it is not strictly necessary to specify them.

Remark 2.3. A direction on (K, ρ) induces an ordering on $K \cap \mathrm{Fix}(\rho)$: we say that the *first fixed point* of K is the initial point of the chosen half-axis, while the final point is the *second fixed point*.

Definition 2.4. Let K and J, be two directed strongly invertible knots. Their *equivariant connected sum* $K \widetilde{\#} J$ is the directed strongly invertible knot obtained by cutting K at its second fixed point and J at its first fixed point, gluing the two knots and axes equivariantly in a way that is compatible with the orientations on the axes, and choosing the half-axis of the sum to be the union of the half-axes of the two components, as depicted in Figure 1.

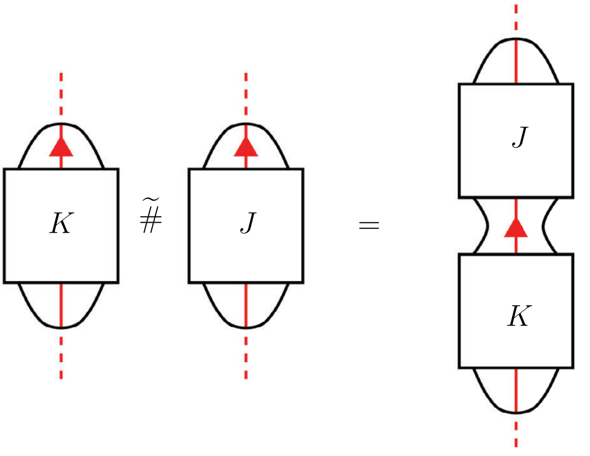


FIGURE 1 The equivariant connected sum of K and J. The vertical axis (colored red) is the axis of the strong inversion. The chosen half-axis is the solid one.

Definition 2.5. Let (K, ρ, h) be a directed strongly invertible knot. We define

- the *mirror* of (K, ρ, h) by $mK = (mK, \rho, h)$,
- the *axis-inverse* of (K, ρ, h) by $iK = (K, \rho, -h)$, where $-h$ is the direction given by the half-axis h with the opposite orientation,
- the *antipode* of (K, ρ, h) by $aK = (K, \rho, h')$, where h' is the direction given by the oriented half-axis complementary to h.

Definition 2.6. Let (K, ρ) be a strongly invertible knot. We say that K is *equivariantly slice* if there exists a locally flat slice disk $D \subset B^4$ for K, invariant with respect to a locally linear involution of B^4 extending ρ. We define the *equivariant slice genus* of (K, ρ) as

$$\widetilde{g}_4(K) = \min_{\Sigma} \operatorname{genus}(\Sigma),$$

where Σ ranges among the orientable locally flat surfaces in B^4 with boundary K, invariant under an involution extending ρ.

Definition 2.7. We say that two directed strongly invertible knots (K_i, ρ_i, h_i), $i = 0, 1$ are *equivariantly concordant* if there exists a locally flat properly embedded annulus $C \cong S^1 \times I \subset S^3 \times I$, invariant with respect to some locally linear involution ρ of $S^3 \times I$ such that:

- $\partial(S^3 \times I, C) = (S^3, K_0) \sqcup -(S^3, K_1)$,
- ρ is in an extension of the strong inversion $\rho_0 \sqcup \rho_1$ on $S^3 \times 0 \sqcup S^3 \times 1$,
- the orientations of h_0 and $-h_1$ induce the same orientation on the annulus (see [23, Remark 2.12]), $\operatorname{Fix}(\rho)$, and h_0 and h_1 are contained in the same component of $\operatorname{Fix}(\rho) \setminus C$.

The operation of equivariant connected sum induces a group structure on the set $\widetilde{\mathcal{C}}$ of classes of directed strongly invertible knots up to equivariant concordance. The class of the unknot gives the group identity, while the inverse of K can be represented by $K^{-1} := m(i(K))$.

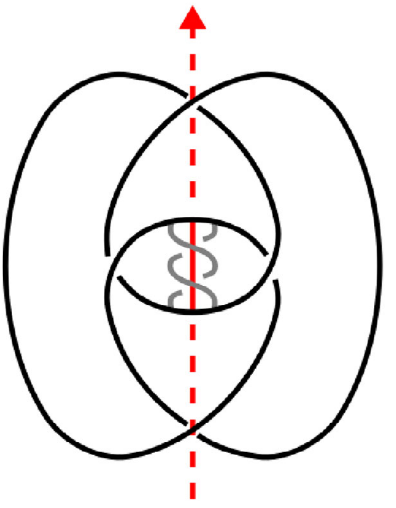


FIGURE 2 The band move (in gray) that produces the 0-butterfly link of 4_1^+.

Notice that, while the direction is essential to define an equivariant connected sum, K is equivariantly slice if and only if iK or aK is so. Therefore, we can consider the mirror, axis-inverse, and antipode as involutive maps from $\widetilde{C}$ to itself.

2.2 | Butterfly links

In [2], Boyle and Issa associate with a directed strongly invertible knot the so-called *butterfly link*. Using this link they construct several invariants. In this section, we recall some of the invariants defined in [2]. Additionally, we introduce the definition of *n-butterfly link* of a directly strongly invertible knot, which is important in the following sections. The n-butterfly link is a generalization of the butterfly link and it coincides with the definition in [2] for $n = 0$.

Definition 2.8. Let (K, ρ, h) be a directed strongly invertible knot. Take an equivariant band B, parallel to the preferred half-axis h, which attaches to K at the two fixed points. Perform a band move on K along B such that the result is a 2-component link. The linking number between the components of such a link depends on the number of twists of B (see, for example, Figure 2). Observe that $\partial B \setminus K$ consists of two arcs parallel to h, which we orient as h. The arcs lie in different components of the link and we consider on each component the orientation induced from the respective arc. The *n-butterfly link* $L_b^n(K)$, is the 2-components 2-periodic link (i.e., the involution ρ exchanges its components) obtained from such a band move on K so that the linking number between its components is n.

Recall that a *semi-orientation* on a link L is the choice of an orientation on each component of L, up to reversing the orientation on all components simultaneously.

Definition 2.9. Define $\widehat{L}_b^n(K)$ to be the n-butterfly link of K endowed with the opposite semi-orientation. Observe that the semi-orientation on $\widehat{L}_b^n(K)$ makes the band move along B coherent with the unique semi-orientation on K.

With a slight abuse of notation, we will also call $\widehat{L}_b^n(K)$ the n-butterfly link of K.

Remark 2.10. Notice that the linking number between the components of $\widehat{L}_b^n(K)$, taken with respect to the chosen semi-orientation, is $-n$.

Definition 2.11. Let (L_i, ρ_i), $i = 0, 1$, be two 2-component 2-periodic links. We say that (L_0, ρ_0) and (L_1, ρ_1) are *equivariantly concordant* if they bound two disjoint properly embedded locally flat annuli in $S^3 \times I$, which are invariant with respect to some locally linear involution of $S^3 \times I$ extending $\rho_0 \sqcup \rho_1$.

Proposition 2.12. *Let (K_i, ρ_i, h_i), $i = 0, 1$, be two equivariantly concordant directed strongly invertible knots. Then, $L_b^n(K_0)$ (resp., $\widehat{L}_b^n(K_0)$) is equivariantly concordant to $L_b^n(K_1)$ (resp., $\widehat{L}_b^n(K_1)$).*

Proof. The proof is identical to the proof of [7, Proposition 2.6]. □

Remark 2.13. It follows from the proposition above that if K is equivariantly slice then also $\widehat{L}_b^0(K)$ is *equivariantly slice*, that is, it bounds two disjoint equivariant disks in B^4. On the other hand, as $\widehat{L}_b^0(K)$ is obtained by an equivariant band move from K (which can be seen as a genus 0 equivariant cobordism) we have that if $\widehat{L}_b(K)$ is equivariantly slice then so is K.

Despite Remark 2.13, it is not true in general that if $L_b^0(K)$ is equivariantly concordant to $L_b^0(J)$ then K is equivariantly concordant to J.

Definition 2.14 [2]. Let K be a directed strongly invertible knot. Define

- $\mathfrak{b}(K)$ to be the knot given by one component of $L_b^0(K)$,
- $\mathfrak{qb}(K)$ to be the knot $L_b^0(K)/\rho$ in $S^3/\rho \cong S^3$.

As proven in [2], we have that $\mathfrak{b}$ induces a group homomorphism

$$\mathfrak{b} : \widetilde{C} \longrightarrow C,$$

where $\widetilde{C}$ is the classical (topological) knot concordance group.

Definition 2.15. Given an oriented knot K, its *double* $\mathfrak{r}(K)$ is the directed strongly invertible knot given by $K\#r(K)$, with the involution ρ that exchanges K and $r(K)$ (the π-rotation around the vertical axis in Figure 3). The direction on $\mathfrak{r}(K)$ is given as follows: the connected sum can be performed by a suitable band move along some band B, in gray in the figure, in such a way that $\mathrm{Fix}(\rho) \cap B$ is the half-axis h. We orient h as the portion of B lying on K (note that h is parallel to $B \cap K$).

As proven by Boyle and Issa [2], $\mathfrak{r}$ defines a homomorphism

$$\mathfrak{r} : C \longrightarrow \widetilde{C}.$$

It is immediate from the definitions that given an oriented knot K, the 0-butterfly link of $\mathfrak{r}(K)$ is given by two split copies of K (see again [2] for details). Therefore, the composition $\mathfrak{b}\circ\mathfrak{r} : C \longrightarrow C$ is the identity homomorphism.

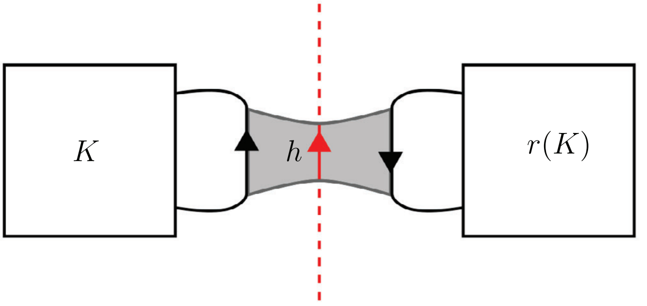


FIGURE 3 The directed strongly invertible knot $\mathfrak{r}(K)$. The chosen half-axis is the solid one.

Hence, we get that $\mathfrak{b}$ is surjective, $\mathfrak{r}$ is injective, and that $\mathfrak{r}(\mathcal{C})$ is a copy of the classical concordance group, contained in the center of $\widetilde{\mathcal{C}}$ (as noted in [25]). As a consequence, we observe the following corollary.

Corollary 2.16. *The equivariant concordance group splits as*

$$\widetilde{\mathcal{C}} = \ker(\mathfrak{b}) \oplus \mathfrak{r}(\mathcal{C}).$$

2.3 | Algebraic concordance

In [17, 18], Levine defined a surjective homomorphism from the classical concordance group to a Witt group of Seifert forms, called the *algebraic concordance group*, which is given by

$$\varphi : \mathcal{C} \longrightarrow \mathcal{G}^{\mathbb{Z}}$$
$$[K] \longmapsto [\theta_F],$$

where F is a Seifert surface for K and θ_F is the Seifert form on $H_1(F, \mathbb{Z})$.

By taking the symmetrization of the Seifert form one obtains a group homomorphism

$$\mathcal{G}^{\mathbb{Z}} \longrightarrow W(\mathbb{Q})$$
$$[A] \longmapsto [A + A^t],$$

where $W(\mathbb{Q})$ is the Witt group of nondegenerate symmetric forms on finite-dimensional $\mathbb{Q}$-vector spaces. Denote by $\varphi_W : \mathcal{C} \longrightarrow W(\mathbb{Q})$ the composition. Clearly by composing φ_W with the signature homorphism $\sigma : W(\mathbb{Q}) \longrightarrow \mathbb{Z}$ one get the knot signature $\sigma(K)$.

Given a (possibly nonorientable) spanning surface F for a link L, Gordon and Litherland [10] defined a bilinear form

$$\mathcal{G}_F : H_1(F, \mathbb{Z}) \times H_1(F, \mathbb{Z}) \longrightarrow \mathbb{Z}$$

$$(a, b) \longmapsto \mathrm{lk}(\widetilde{a}, b)$$

given by the linking number of b with a pushed off F "in both directions simultaneously". This form is bilinear and symmetric and if F is oriented it coincides with the symmetrization of the Seifert form.

In [10], Gordon and Litherland proved that it is possible to compute the signature of a knot from the Gordon–Litherland form of any spanning surface, by introducing a corrective term. We briefly recall some of the notation used in [10] and we observe in Proposition 2.19 how the results of Gordon and Litherland allow us to compute not only the signature of a knot K but the whole Witt class $\varphi_W(K)$, using any spanning surface. This fact is presumably known to the experts but we could not find it in the literature.

Definition 2.17. Let F be a spanning surface for a knot K and let K^F be a longitude of K which misses F. The *relative Euler number of F* is defined as

$$e(F) = -\operatorname{lk}(K, K^F),$$

where K and K^F are coherently oriented.

Observe that as K^F and F are disjoint, $[K] = 0 \in H_1(S^3 \setminus K^F, \mathbb{Z}/2\mathbb{Z})$. Hence, $e(F)$ is always an even integer.

Definition 2.18. Let F_1, F_2 be two surfaces in S^3 with $\partial F_1 = \partial F_2$ and suppose that there exists a 3-ball $B^3 = B^1 \times B^2 \subset S^3 \setminus \partial F_i$ such that

- $F_1 \cap B^3 = \partial B^1 \times B^2$,
- $F_2 \cap B^3 = B^2 \times \partial B^1$,
- $F_1 \setminus B^3 = F_2 \setminus B^3$.

In this situation, we say that F_2 is obtained from F_1 by a *1-handle move*.

Proposition 2.19. *Let F be spanning surface for K, and A a matrix representing Gordon–Litherland form $\mathcal{G}_F$ on $H_1(F)$. Then, the Witt class of K is represented by*

$$\begin{pmatrix} A & 0 \\ 0 & \varepsilon Id \end{pmatrix},$$

where $\varepsilon = \operatorname{sign}(e(F))$ and the Id block has size $n \times n$ with $n = |e(F)|/2$.

Proof. Let G be a Seifert surface for K (in particular, $e(G) = 0$). By [10, Theorem 11], we can obtain G from F by a finite sequence of the following moves (and their inverses):

- ambient isotopy,
- 1-handle moves,
- addition of a small half-twisted band at the boundary.

It is not difficult to check that the first two moves do not change the Witt class of the Gordon–Litherland form. By attaching a half-twisted band the Gordon–Litherland form and the relative Euler number change as

$$A \longrightarrow \begin{pmatrix} A & 0 \\ 0 & \varepsilon \end{pmatrix},$$

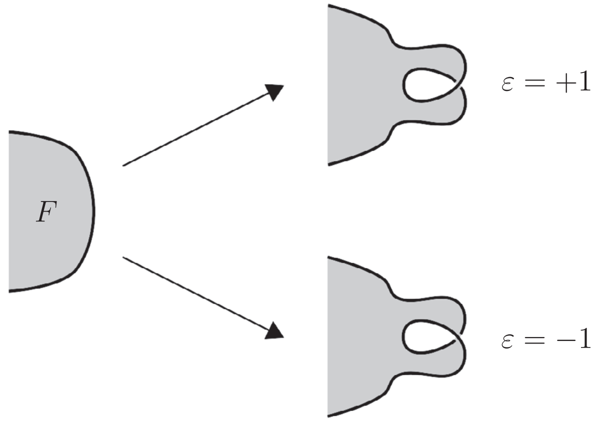


FIGURE 4 The addition of a half-twisted band.

$$e(F) \longmapsto e(F) - 2\varepsilon,$$

where $\varepsilon = \pm 1$ depends on the twist of the band, as in Figure 4. As the matrix $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ is metabolic, if we attach two bands with opposite half-twists, the overall move leaves the Witt class unchanged. The conclusion follows by observing that, up to algebraic cancellation, one has to attach $n = |e(F)|/2$ bands with the same half-twist. □

3 | EXTENSION AND TRANSVERSALITY OF EQUIVARIANT MAPS

In this section, we show some results on the extension and transversality of equivariant maps. We use these results to prove Lemma 3.2, which is fundamental for the constructions in Section 4.

Let X be a connected manifold with boundary, such that the inclusion of ∂X in X induces an isomorphism $H^1(X, \mathbb{Z}) \longrightarrow H^1(\partial X, \mathbb{Z})$. As S^1 is a $K(\mathbb{Z}, 1)$, every map $\partial X \longrightarrow S^1$ can be extended to a map $X \longrightarrow S^1$, which is unique up to homotopy.

Consider now the $\mathbb{Z}/2\mathbb{Z}$-action on S^1 given by

$$\iota : S^1 \longrightarrow S^1$$

$$z \longmapsto \bar{z},$$

and suppose that X is endowed with a $\mathbb{Z}/2\mathbb{Z}$-action, generated by $\rho : X \longrightarrow X$.

Lemma 3.1. *Let $(X, \partial X, \rho)$ be as above. Let $f : (\partial X, \rho) \longrightarrow (S^1, \iota)$ be an equivariant map, that is, $f = \iota \circ f \circ \rho$, and suppose there exists $x_0 \in \partial X$ such that $f(x_0) = 1$. Then f admits an equivariant extension $F : (X, \rho) \longrightarrow (S^1, \iota)$.*

Proof. Let $G : X \longrightarrow S^1$ be a (possibly nonequivariant) extension of f, and define $H : X \longrightarrow S^1$ as $H = G \cdot (\iota \circ G \circ \rho)$, where $\cdot$ is the group operation on S^1. By construction H is equivariant. As f

is equivariant, we have that $\iota \circ G \circ \rho$ is another extension of f, and hence that it is homotopic to G. Therefore, the induced maps are the same.

$$G_* = (\iota \circ G \circ \rho)_* : H_1(X, \mathbb{Z}) \longrightarrow H_1(S^1, \mathbb{Z}) = \mathbb{Z}.$$

It follows that $H_* = G_* + (\iota \circ G \circ \rho)_* = 2G_*$ and then H can be lifted to the twofold covering.

$$\begin{array}{ccccc} & & S^1 & & z \\ & \overset{F}{\nearrow} & \downarrow & & \downarrow \\ X & \xrightarrow{H} & S^1 & & z^2 \end{array}$$

Choose the lift F such that $F(x_0) = 1$. Observe that as f is equivariant, $H_{|\partial X} = f \cdot f$, then $F_{|\partial X} = f$. Therefore, we only need to prove that F is equivariant. Notice that $F\circ\rho$ is a lift of $H\circ\rho = \iota\circ H$. Therefore, we have that $(F \cdot F \cdot (F\circ\rho) \cdot (F\circ\rho)) = H \cdot \iota H \equiv 1$ and hence $F \cdot (F\circ\rho) \equiv \pm 1$. As $F(x_0) = f(x_0) = 1$ and f is equivariant, we have that $F(x_0) \cdot F(\rho(x_0)) = 1$ and as X is connected $F \cdot (F\circ\rho) \equiv 1$, that is, $F = \iota\circ F\circ\rho$. □

Lemma 3.2. *Let $\rho : B^4 \longrightarrow B^4$ be an orientation preserving, locally linear involution, with fixed-point set homeomorphic to a 2-disk D. Let $F \subset S^3$ and $\Sigma \subset B^4$ be two oriented locally flat surfaces with $L = \partial F = \partial\Sigma$. Suppose that both F and Σ are ρ-invariant and that ρ reverses their orientations. If both F and Σ are disjoint from* Fix(ρ)*, then there exists a ρ-invariant, oriented, compact, locally flat 3-manifold $M \subset B^4$, disjoint from* Fix(ρ) *and such that $\partial M = F \cup \Sigma$.*

Proof. Let $N(D)$ and $N(\Sigma)$ be equivariant, closed tubular neighborhoods of D and Σ in B^4. Let $X = B^4 \setminus (\operatorname{int} N(D) \cup \operatorname{int} N(\Sigma))$.

Let Y be the complement in S^3 of an equivariant tubular neighborhood of $\partial F \cup \partial D$. Let $N \cong F \times [-1, 1]$ be and equivariant tubular neighborhood of F in Y. Observe that the restriction of ρ acts on N as

$$\rho : N \longrightarrow N$$
$$(x, t) \longmapsto (\rho(x), -t).$$

Let $\varphi : \mathbb{R} \longrightarrow \mathbb{R}$ be a smooth, odd map such that $\varphi' \geqslant 0$ and

$$\varphi(x) = \begin{cases} x \text{ for } |x| \leqslant 1/2 \\ \operatorname{sign}(x) \text{ for } |x| \geqslant 2/3 \end{cases}.$$

Then we can define an equivariant map

$$f : (N, \rho) \longrightarrow (S^1, \iota)$$
$$(x, t) \longmapsto e^{\pi i \varphi(t)}$$

and we can extend it to Y by setting f to be -1 outside N.

Such f is topologically transverse to $1 \in S^1$ in the sense of [8, Definition 10.7] and $f^{-1}(1)$ is given by the union of F and a nearby copy of Σ. Using Lemma 3.1 we can extend f to an equivariant map

$$f : (X, \rho) \longrightarrow (S^1, \iota),$$

which in turn gives us the equivariant map

$$\begin{aligned} \mathrm{id}_X \times f : (X, \rho) &\longrightarrow (X \times S^1, \rho \times \iota) \\ x &\longmapsto (x, f(x)). \end{aligned}$$

Consider now the quotient spaces of X and $X \times S^1$ by the respective involutions. We have the following commutative diagram

$$\begin{array}{ccc} X \times S^1 & \longrightarrow & X \\ \big\downarrow q & & \big\downarrow p \\ (X \times S^1)/(\rho \times \iota) & \longrightarrow & X/\rho \end{array}$$

where the vertical maps are twofold regular covering maps, Observe that the map $\mathrm{id}_X \times f$ induces a map between the quotients

$$\overline{f} : X/\rho \longrightarrow (X \times S^1)/(\rho \times \iota).$$

By construction $\overline{f}_{|\partial(X/\rho)}$ is topologically transverse to $(\partial X \times 1)/(\rho \times \iota)$. According to [8, Theorems 10.3 and 10.8], $\overline{f}$ is homotopic relative to the boundary to a map $\overline{g}$ transverse to $(X \times 1)/(\rho \times \iota)$. As $\overline{g}$ is homotopic to $\overline{f}$, we can lift $\overline{g}$ to an equivariant map

$$g : (X, \rho) \longrightarrow (X \times S^1, \rho \times \iota)$$

with $g_{|\partial X} = \mathrm{id}_{\partial X} \times f$ and g topologically transverse to $X \times 1$. Finally, $M = g^{-1}(X \times 1)$ is an equivariant, compact, orientable, locally flat 3-dimensional submanifold of X with $\partial M = F \cup \Sigma$. □

4 | EQUIVARIANT ALGEBRAIC CONCORDANCE

In this section, we define an equivariant algebraic concordance group $\widetilde{\mathcal{G}}^{\mathbb{Z}}$ and a homomorphism $\Phi : \widetilde{\mathcal{C}} \longrightarrow \widetilde{\mathcal{G}}^{\mathbb{Z}}$. We compare $\widetilde{\mathcal{G}}^{\mathbb{Z}}$ with the equivariant algebraic concordance group $\widetilde{\mathcal{AC}}$ defined in [23]. Finally, we use $\widetilde{\mathcal{G}}^{\mathbb{Z}}$ to obtain a lower bound on the equivariant slice genus of a strongly invertible knot.

4.1 | Equivariant Seifert systems

Definition 4.1. Let (K, ρ, h) be a directed strongly invertible knot. An *invariant Seifert surface of type n* for K is a connected, orientable surface $F \subset S^3$ such that:

- F is ρ-invariant, that is, $\rho(F) = F$,
- $h = \mathrm{Fix}(\rho) \cap F$,
- the surface $\widehat{F}$ obtained from F by equivariantly cutting along h is a ρ-invariant Seifert surface for $\widehat{L}_b^n(K)$.

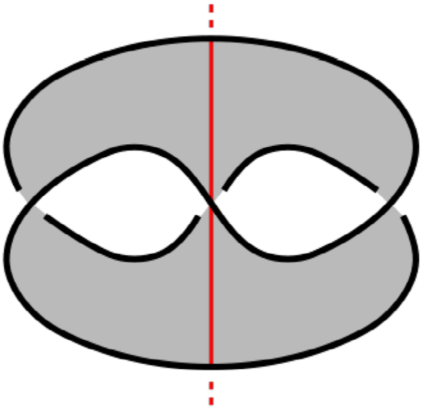


FIGURE 5 The invariant Seifert surface G for the unknot.

Proposition 4.2. *For any directed strongly invertible knot K and every $n \in \mathbb{Z}$ there exists an invariant Seifert surface of type n.*

Proof. From [13], we know that for any (K, ρ, h) there exists a ρ-invariant Seifert surface F such that $\mathrm{Fix}(\rho) \cap F = h$. Cutting F along h we obtain a (possibly disconnected) orientable surface $\widetilde{F}$ and the linking number between the components of $\partial\widetilde{F}$ would not be generally $-n$. Now let G be the equivariant Seifert surface for the unknot described in Figure 5.

Observe that by cutting G along the fixed-point set we obtain a Seifert surface for a link with linking number $+1$ between its components. In other words, G is an invariant Seifert surface of type -1 for the unknot. Therefore, by taking the equivariant connected sum of F with an appropriate number of copies of G and/or its mirror image, we easily get an invariant Seifert surface of type n for K. □

As a consequence of Lemma 3.2, we have the following proposition.

Proposition 4.3. *Let (K, ρ, h) be a directed strongly invertible knot and F be an equivariant Seifert surface for $\widehat{L}_b^n(K)$. Suppose that $\widehat{L}_b^n(K)$ bounds an orientable surface $\Sigma \subset B^4$ invariant under an involution of B^4 extending ρ (which we still denote by ρ). Assume that ρ has no fixed point on $\Sigma \cup F$. Denote by g_F and g_Σ the genus of F and Σ, respectively. Then there exists a ρ_*-invariant submodule $H \subset H_1(F, \mathbb{Z})$ such that:*

- rank $H \geqslant g_F - g_\Sigma$ *if Σ is connected and* rank $H \geqslant g_F - g_\Sigma + 1$ *if Σ is not connected,*
- *the Seifert form of F vanishes on H,*
- *for every $\alpha \in H$, the linking number between α and the fixed axis is zero.*

Proof. By Lemma 3.2, there exists a ρ-invariant oriented 3-manifold $M \subset B^4$, such that $\partial M = F \cup \Sigma$ and $M \cap \mathrm{Fix}(\rho) = \emptyset$.

Denote by V the kernel of $H_1(\partial M, \mathbb{Q}) \longrightarrow H_1(M, \mathbb{Q})$. It is easy to see that $2 \cdot \dim V = \dim H_1(\partial M, \mathbb{Q}) = \mathrm{genus}(\partial M)$, by standard duality argument (half-lives, half-dies principle) or by computing the Euler characteristic of the exact sequence of the couple $(M, \partial M)$.

Suppose now that Σ is connected. Then it is easy to see that $\mathrm{genus}(\partial M) = g_F + g_\Sigma + 1$ and that the map induced by the inclusion $i_* : H_1(F, \mathbb{Q}) \longrightarrow H_1(\partial M, \mathbb{Q})$ is injective. As $\dim V = g_F + g_\Sigma +$

1 and $\dim H_1(F,\mathbb{Q}) = 2g_F + 1$, we have that the preimage W of V in $H_1(F,\mathbb{Q})$ has dimension at least $g_F - g_\Sigma$.

Suppose now that Σ is not connected. Then $\mathrm{genus}(\partial M) = g_F + g_\Sigma$ and the map induced by the inclusion $i_* : H_1(F,\mathbb{Q}) \longrightarrow H_1(\partial M,\mathbb{Q})$ has kernel of dimension 1. As $\dim V = g_F + g_\Sigma$ and $\dim H_1(F,\mathbb{Q}) = g_F + 1$, we have that the preimage W of V in $H_1(F,\mathbb{Q})$ has dimension at least $1 + g_F - g_\Sigma$.

Define $H \subset H_1(F,\mathbb{Z})$ to be

$$H = \{x \in H_1(F,\mathbb{Z}) \mid \exists n \in \mathbb{Z}, n \neq 0, nx \in W\}.$$

It is a well-known fact that the Seifert form of F is identically zero on H. As all of the maps considered are equivariant, we get that also H is invariant under the action of ρ_* on $H_1(F,\mathbb{Z})$.

By the considerations above, the rank of H satisfies the inequalities stated in the proposition.

Finally, let $\alpha \in H$ and let $\Delta \subset M$ be a 2-chain such that $\partial\Delta = n\alpha$ for some integer $n \neq 0$. As M is disjoint from the disk D of fixed points, it follows that $\mathrm{lk}(n\alpha, \partial D) = \#(\Delta \cap D) = 0$, hence $\mathrm{lk}(\alpha, \partial D) = 0$. □

We use now the result given Proposition 4.3 to define a notion of equivariant algebraic concordance for directed strongly invertible knots.

Definition 4.4. Let R be a commutative and unital ring. An *equivariant Seifert system* is a tuple $(\theta, \rho, h, \widetilde{\mathrm{lk}})$, where

- $\theta : M \times M \longrightarrow \mathbb{Z}$ is a bilinear form on a free $\mathbb{Z}$-module M of even rank,
- $\rho : M \longrightarrow M$ is a linear involution,
- $\theta(\rho(x), \rho(y)) = \theta^t(x,y) := \theta(y,x)$ for every $x, y \in M$,
- $\theta - \theta^t$ is unimodular,
- $h, \widetilde{\mathrm{lk}} \in \mathrm{Hom}(M,\mathbb{Z})$,
- $h \circ \rho = -h$,
- $\widetilde{\mathrm{lk}} \circ \rho = \widetilde{\mathrm{lk}}$.

An equivariant Seifert system $(\theta, \rho, h, \widetilde{\mathrm{lk}})$ on M is said to be *equivariantly metabolic* if there exists a submodule $H \subset M$ such that

- $\mathrm{rank}\, M = 2 \cdot \mathrm{rank}\, H$
- $\rho(H) = H$, that is, H is *ρ-invariant*,
- θ is identically zero on $H \times H$,
- $H \subset \ker(h) \cap \ker(\widetilde{\mathrm{lk}})$.

Let (K, ρ, h) be a directed strongly invertible knot and let F be an invariant Seifert surface for K of type n for some n. Fix an auxiliary orientation on F.

We see now how F determines an equivariant Seifert system. As ρ reverses the orientation on F, it is immediate to check that $\theta_F(\rho_*(x), \rho_*(y)) = \theta_F(y,x)$ for every $x, y \in H_1(F,\mathbb{Z})$, where θ_F is the Seifert form of F. As $h \subset F$, we have that h represents a class in $H_1(F, \partial F, \mathbb{Z})$. By duality and universal coefficients, we can consider h as a homomorphism $h : H_1(F,\mathbb{Z}) \longrightarrow \mathbb{Z}$, which maps an oriented curve c in F to the algebraic intersection $\#(c \cap h)$. Finally, let A be the oriented fixed axis of ρ. Then we have a homomorphism

$$\widetilde{\mathrm{lk}} : H_1(F,\mathbb{Z}) \longrightarrow \mathbb{Z}$$

$$c \longmapsto \mathrm{lk}(c^+, A) + \mathrm{lk}(c^-, A),$$

where the $c^\pm$ is a nearby copy of c outside F in the positive/negative direction. It is immediate to check that the tuple $(\theta_F, \rho_*, h, \widetilde{\mathrm{lk}})$ is an equivariant Seifert system. We will denote by $\mathcal{S}(F)$ the equivariant Seifert system determined by F.

Definition 4.5. Let $(\theta_i, \rho_i, h_i, \widetilde{\mathrm{lk}}_i)$ for $i = 1, 2$ be two equivariant Seifert systems defined over M and N, respectively. Their *orthogonal sum* $(\theta_1, \rho_1, h_1, \widetilde{\mathrm{lk}}_1) \oplus (\theta_2, \rho_2, h_2, \widetilde{\mathrm{lk}}_2)$ is the tuple $(\theta, \rho, h, \widetilde{\mathrm{lk}})$ defined by

$$\theta : (M \oplus N) \times (M \oplus N) \longrightarrow \mathbb{Z}$$

$$((x_1, x_2), (y_1, y_2)) \longmapsto \theta_1(x_1, y_1) + \theta_2(x_2, y_2)$$

$$\rho : M \oplus N \longrightarrow M \oplus N$$

$$\rho(x, y) = (\rho_1(x), \rho_2(y))$$

$$h, \widetilde{\mathrm{lk}} : M \oplus N \longrightarrow \mathbb{Z}$$

$$h(x, y) = h_1(x) + h_2(y)$$

$$\widetilde{\mathrm{lk}}(x, y) = \widetilde{\mathrm{lk}}_1(x) + \widetilde{\mathrm{lk}}_2(y).$$

We say that $(\theta_i, \rho_i, h_i, \widetilde{\mathrm{lk}}_i)$, $i = 1, 2$ are *equivariantly concordant* if the orthogonal sum between $(\theta_1, \rho_1, h_1, \widetilde{\mathrm{lk}}_1)$ and $(-\theta_2^t, \rho_2, h_2, \widetilde{\mathrm{lk}}_2)$ is equivariantly metabolic.

Definition 4.6. We define the *equivariant algebraic concordance group* $\widetilde{\mathcal{G}}^{\mathbb{Z}}$ to be the set of equivalence classes of equivariant Seifert systems up to equivariant concordance. It is not difficult to prove that the operation of orthogonal sum defines a group structure on $\widetilde{\mathcal{G}}^{\mathbb{Z}}$, by adapting the proof of Levine [17, 18] in the case of the classical algebraic concordance.

Theorem 4.7. *Let $F \subset S^3$ be an invariant Seifert surface of type 0 for a directed strongly invertible knot (K, ρ, h) and choose an orientation on F. The class of the equivariant Seifert system $\mathcal{S}(F)$ in $\widetilde{\mathcal{G}}^{\mathbb{Z}}$ depends only on the equivariant concordance class of K. In particular, we have a well-defined group homomorphism*

$$\Phi : \widetilde{\mathcal{C}} \longrightarrow \widetilde{\mathcal{G}}^{\mathbb{Z}}$$

$$[K, \rho, h] \longmapsto [\mathcal{S}(F)].$$

Proof. Let G be an invariant surface of type 0 for another directed strongly invertible knot J. We can equivariantly perform the connected sum of F and G along their boundary so that $F \natural G$ is an invariant Seifert surface of type 0 for the $K \widetilde{\#} J$. It is immediate to see that $\mathcal{S}(F \natural G) = \mathcal{S}(F) \oplus \mathcal{S}(G)$.

Therefore, to prove that the homomorphism Φ is well-defined it is sufficient to show that $\mathcal{S}(F)$ is equivariantly metabolic whenever the knot $K = \partial F$ is equivariantly slice.

Let $\widehat{F}$ be the equivariant Seifert surface for $\widehat{L}_b^0(K)$ obtained by cutting F. By Proposition 4.3, there exists a ρ_*-invariant submodule H of $H_1(\widehat{F}, \mathbb{Z})$, such that $2 \operatorname{rank} H = \operatorname{rank} H_1(\widehat{F}, \mathbb{Z}) + 1$ and the Seifert form of $\widehat{F}$ vanishes on it.

Observe that we can regard $H_1(\widehat{F}, \mathbb{Z})$ as a ρ_*-invariant codimension 1 submodule in $H_1(F, \mathbb{Z})$ through the map induced by the inclusion. Moreover, $H_1(\widehat{F}, \mathbb{Z})$ is easily identified with the kernel of $h : H_1(F, \mathbb{Z}) \longrightarrow \mathbb{Z}$. The restriction of the Seifert form of F on $H_1(\widehat{F}, \mathbb{Z})$ clearly coincides with the Seifert form of $\widehat{F}$. Again by Proposition 4.3, the linking number homomorphism $\widetilde{\operatorname{lk}} : H_1(F, \mathbb{Z}) \longrightarrow \mathbb{Z}$ vanishes on H. Therefore, H is an equivariant metabolizer for the equivariant Seifert system $\mathcal{S}(F)$. □

Remark 4.8. Let G be the equivariant Seifert surface of type -1 for the unknot described in Figure 5. Let F be an invariant Seifert surface of type n for a directed strongly invertible knot (K, ρ, h). Then by the proof of Theorem 4.7 and Proposition 4.2 follows easily that we can compute the equivariant algebraic concordance class of K by

$$\Phi(K) = [\mathcal{S}(F) + n\mathcal{S}(G)] \in \widetilde{\mathcal{G}}^{\mathbb{Z}}.$$

Similarly, observe that in order to compute $\Phi(K)$ it is not relevant in Definition 4.1 that $F \setminus h$ is connected. In fact, suppose that $F \setminus h$ is not connected. Then $F \natural G \natural \overline{G}$ is an equivariant Seifert surface of type 0 for K, where $\overline{G}$ is the mirror image of the surface G in Figure 5. Hence, by definition $\Phi(K) = [\mathcal{S}(F \natural G \natural \overline{G})] = [\mathcal{S}(F)] + [\mathcal{S}(G)] + [\mathcal{S}(\overline{G})]$. As $[\mathcal{S}(\overline{G})] = -[\mathcal{S}(G)]$, it follows that $\Phi(K) = [\mathcal{S}(F)]$.

Proposition 4.9. *Let $\mathcal{A}$ be the concordance group of algebraically slice knots, that is, the kernel of $\varphi : \mathcal{C} \longrightarrow \mathcal{G}^{\mathbb{Z}}$. Then the kernel of $\Phi : \widetilde{\mathcal{C}} \longrightarrow \widetilde{\mathcal{G}}^{\mathbb{Z}}$ contains a copy of $\mathcal{A}$, namely, $\mathfrak{r}(\mathcal{A}) \subset \ker(\Phi)$.*

Proof. Let K be an oriented knot representing a class in $\mathcal{A}$ and let F be a Seifert surface for K. Then we can compute $\Phi(\mathfrak{r}(K))$ using as invariant surface $\mathfrak{r}(F) = F \natural r(F)$, where analogously to Definition 2.15, the involution of $\mathfrak{r}(K)$ exchange F and $r(F)$.

Identifying $H_1(\mathfrak{r}(F), \mathbb{Z}) \cong H_1(F, \mathbb{Z}) \oplus H_1(F, \mathbb{Z})$, it is not difficult to see that the equivariant Seifert system of $\mathfrak{r}(F)$ is of type $\mathcal{S}(\mathfrak{r}(F)) = (\theta, \rho, 0, 0)$, where

$$\theta = \begin{pmatrix} \theta_F & 0 \\ 0 & \theta_F^t \end{pmatrix}$$

$$\rho = \begin{pmatrix} 0 & \mathrm{id} \\ \mathrm{id} & 0 \end{pmatrix}.$$

Therefore, if $H \subset H_1(F, \mathbb{Z})$ is a metabolizer of θ_F then $H \oplus H \subset H_1(\mathfrak{r}(F), \mathbb{Z})$ is an equivariant metabolizer for $\mathcal{S}(\mathfrak{r}(F))$. As $\mathfrak{r}$ is injective (see Corollary 2.16), we have that $\ker(\Phi)$ contains a copy of $\mathcal{A}$. □

Remark 4.10. As a consequence, it follows from [15] that $\ker(\Phi)$ contains a subgroup isomorphic to $\mathbb{Z}^\infty$, and from [19] that it contains a subgroup isomorphic to $\mathbb{Z}_2^\infty$.

4.2 | Equivariant Blanchfield pairing

In this section, we show that $\Phi : \widetilde{\mathcal{C}} \longrightarrow \widetilde{\mathcal{G}}^{\mathbb{Z}}$ lifts the homomorphism $\Psi : \widetilde{\mathcal{C}} \longrightarrow \widetilde{\mathcal{AC}}$ defined in [23]. In particular, Ψ factors through a *reduced* version $\widetilde{\mathcal{G}}^{\mathbb{Z}}_r$ of the equivariant algebraic concordance group.

Definition 4.11. An *equivariant Seifert form* over a ring R is a couple (θ, ρ), where

- $\theta : M \times M \longrightarrow R$ is a bilinear form on a free R-module M of even rank,
- $\rho : M \longrightarrow M$ is a R-linear involution,
- $\theta(\rho(x), \rho(y)) = \theta^t(x, y) := \theta(y, x)$ for every $x, y \in M$,
- $\theta - \theta^t$ is unimodular, that is, induces an isomorphism between M and $\mathrm{Hom}(M, R)$.

We say that an equivariant Seifert form (θ, ρ) on M is *equivariantly metabolic* if there exists a submodule $H \subset M$ such that

- $\operatorname{rank} M = 2 \cdot \operatorname{rank} H$
- $\rho(H) = H$, that is, H is *ρ-invariant*,
- θ is identically zero on $H \times H$.

Similarly to Definitions 4.5 and 4.6, we can define a notion of orthogonal sum between equivariant Seifert forms and construct the *reduced equivariant algebraic concordance group* $\widetilde{\mathcal{G}}^{R}_r$ as the set of equivalence classes of equivariant Seifert forms over R up to equivariant concordance.

In the following, we mainly focus on equivariant Seifert forms over $\mathbb{Z}$ and we will often omit to specify the ring R, implying $R = \mathbb{Z}$. Only Section 6 will be mostly devoted to studying equivariant Seifert forms over $\mathbb{Q}$.

Clearly, there exists a forgetful homomorphism

$$r : \widetilde{\mathcal{G}}^{\mathbb{Z}} \longrightarrow \widetilde{\mathcal{G}}^{\mathbb{Z}}_r,$$

which is surjective, as it admits a natural section

$$s : \widetilde{\mathcal{G}}^{\mathbb{Z}}_r \longrightarrow \widetilde{\mathcal{G}}^{\mathbb{Z}}$$

given by mapping an equivariant Seifert form (θ, ρ) to the equivariant Seifert system $(\theta, \rho, 0, 0)$. In particular, $\widetilde{\mathcal{G}}^{\mathbb{Z}}$ splits as

$$\widetilde{\mathcal{G}}^{\mathbb{Z}} \cong \widetilde{\mathcal{G}}^{\mathbb{Z}}_r \oplus \ker(r).$$

We will denote by Φ_r the map given by the composition

$$\Phi_r : \widetilde{\mathcal{C}} \xrightarrow{\Phi} \widetilde{\mathcal{G}}^{\mathbb{Z}} \xrightarrow{r} \widetilde{\mathcal{G}}^{\mathbb{Z}}_r.$$

Levine [17, 18] showed that the algebraic concordance group is isomorphic to $\mathbb{Z}^\infty \oplus \mathbb{Z}_2^\infty \oplus \mathbb{Z}_4^\infty$ and that the knot concordance group surjects onto it. In Section 6, we provide a partial result, similar to Levine's one, on the structure of $\widetilde{\mathcal{G}}^{\mathbb{Z}}_r$.

The results in Section 6 are obtained by adapting the arguments used in [17, 18] to the strongly invertible setting. However, these ideas do not generalize easily to study $\widetilde{\mathcal{G}}^{\mathbb{Z}}$: while the restriction on the equivariant metabolizers given by the homomorphism h and $\widetilde{\mathrm{lk}}$ provide valuable information (as shown in Example 4.13), it is not clear how to adapt these arguments to manage these additional restrictions.

In [23], Miller and Powell study the action of the strong inversion ρ of a strongly invertible knot K on its Alexander module $\mathcal{A}^{\mathbb{Z}}(K)$ and on the Blanchfield pairing on $\mathcal{A}^{\mathbb{Z}}(K)$. In particular, they show that the action induced by ρ on $\mathcal{A}^{\mathbb{Z}}(K)$ is an anti-isometry of the Blanchfield pairing (Proposition 2.8). Moreover, they define an *equivariant algebraic concordance group* $\widetilde{\mathcal{AC}}$ as the Witt group of *abstract equivariant Blanchfield pairings* (Definition 4.3) and they prove that taking the Blanchfield form of a strongly invertible knot (K, ρ) together with the involution on $\mathcal{A}^{\mathbb{Z}}(K)$ induced by ρ defines a homomorphism (Proposition 4.6)

$$\Psi : \widetilde{\mathcal{C}} \longrightarrow \widetilde{\mathcal{AC}}.$$

In [9], the authors prove that the Alexander module and the Blanchfield pairing of K can be expressed in terms of the Seifert form of a Seifert surface F. If A is a matrix representing the Seifert form of F, with respect to a basis $\mathcal{B}$, and genus$(F) = g$ then $\mathcal{A}^{\mathbb{Z}}(K) \cong \mathbb{Z}[t^{\pm 1}]^{2g}/(tA - A^t)\mathbb{Z}[t^{\pm 1}]^{2g}$ and under this identification the Blanchfield pairing is equivalent to

$$\mathcal{BL} : \mathcal{A}^{\mathbb{Z}}(K) \times \mathcal{A}^{\mathbb{Z}}(K) \longrightarrow \mathbb{Q}(t)/\mathbb{Z}[t^{\pm 1}]$$

$$(x, y) \longmapsto x(t-1)(A - tA^t)^{-1}\overline{y},$$

where $\bar{\cdot}$ is the $\mathbb{Z}$-linear involution given by $t \longmapsto t^{-1}$. It is not difficult to see, as pointed out in the examples in [23], that if F is ρ-invariant and P is the matrix representing the action of ρ on $H_1(F, \mathbb{Z})$ with respect to the basis $\mathcal{B}$, we have that the action of ρ on $\mathcal{A}^{\mathbb{Z}}(K)$ can be read as

$$\rho_* : \mathcal{A}^{\mathbb{Z}}(K) \longrightarrow \mathcal{A}^{\mathbb{Z}}(K)$$

$$x \longmapsto P\overline{x}.$$

The same construction carried out for abstract equivariant Seifert forms and abstract equivariant Blanchfield pairings proves the following theorem.

Theorem 4.12. *There exists a natural group homomorphism*

$$\widetilde{\mathcal{G}}^{\mathbb{Z}}_r \longrightarrow \widetilde{\mathcal{AC}}$$

that makes the following diagram commutative

$$\begin{array}{ccccc} \widetilde{\mathcal{C}} & \xrightarrow{\Phi} & \widetilde{\mathcal{G}}^{\mathbb{Z}} & \xrightarrow{r} & \widetilde{\mathcal{G}}^{\mathbb{Z}}_r \\ & \searrow^{\Psi} & & & \downarrow \\ & & & & \widetilde{\mathcal{AC}}. \end{array}$$

It follows from its definition that $\widetilde{\mathcal{AC}}$ does not distinguish a directed strongly invertible knot from its antipode. On the other hand, in Section 5 we prove that the equivariant signature [1] can be retrieved from $\widetilde{\mathcal{G}}^{\mathbb{Z}}_r$. As the equivariant signature depends on the choice of half-axis for a strongly invertible knot, so is $\widetilde{\mathcal{G}}^{\mathbb{Z}}_r$ (see Remark 5.15).

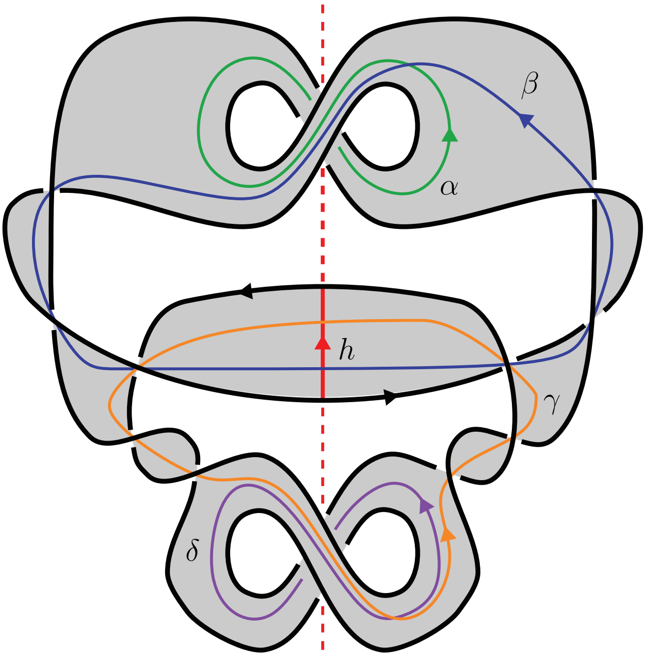


FIGURE 6 The invariant surface F with boundary $K13n1496$. The chosen half-axis h is the solid one. The curves $\alpha, \beta, \gamma, \delta$ form a basis of $H_1(F, \mathbb{Z})$.

We conclude with the following example, which shows that the equivariant algebraic concordance class of a knot is able to obstruct smooth as well as topological equivariant sliceness for knots with trivial Alexander polynomial, contrary to $\widetilde{\mathcal{AC}}$.

Example 4.13. Consider the knot $K13n1496$ as the directed strongly invertible knot (K, ρ, h) that bounds the surface F in Figure 6, where the strong inversion is given by the π-rotation around the vertical axis and the chosen oriented half-axis is the red one in the figure.

One can easily check that K has trivial Alexander polynomial and hence that its image is trivial in the equivariant algebraic concordance group $\widetilde{\mathcal{AC}}$ defined in [23]. However, Boyle and Issa [2] prove that K is not equivariantly slice. We show that the same result can be obtained by using $\widetilde{\mathcal{G}}^{\mathbb{Z}}$.

The surface F is an invariant Seifert surface for K, so we can use it to compute the class of K in $\widetilde{\mathcal{G}}^{\mathbb{Z}}$. With respect to the basis $\{\alpha, \beta, \gamma, \delta\}$ of $H_1(F, \mathbb{Z})$ the Seifert form and the involution ρ_* are represented by the matrices A and P, respectively:

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & -1 & -1 & -1 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$P = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

The homomorphisms h and $\widetilde{\text{lk}}$ are represented by the covectors:

$$h = \begin{pmatrix} 0 & -1 & 1 & 0 \end{pmatrix}$$

$$\widetilde{\text{lk}} = \begin{pmatrix} -2 & -1 & 1 & 2 \end{pmatrix}.$$

One can easily check that $H = \langle \alpha + \delta, 2(\beta - \gamma) - \alpha + \delta \rangle$ is a ρ_*-invariant submodule of rank 2 on which the Seifert form of F vanishes.

Therefore, the class of K represents the identity also in the reduced equivariant algebraic concordance group $\widetilde{\mathcal{G}}^{\mathbb{Z}}_r$.

However, $H = \ker(h) \cap \ker(\widetilde{\text{lk}}) = \langle \beta + \gamma, \alpha + \delta \rangle$ has rank 2 but the Seifert form does not vanish on H, therefore the class of K is nontrivial in $\widetilde{\mathcal{G}}^{\mathbb{Z}}$.

4.3 | Lower bound on the equivariant slice genus

In [23], the authors obtain a lower bound on the equivariant slice genus of a strongly invertible knot using the Blanchfield form. As Miller and Powell's invariant factors through $\widetilde{\mathcal{G}}^{\mathbb{Z}}$, we can get the same lower bound indirectly. However, we prove in this section that it is possible to obtain a different lower bound on the equivariant slice genus using the additional information contained in $\widetilde{\mathcal{G}}^{\mathbb{Z}}$.

Definition 4.14. Let $\Sigma \subset B^4$ be a properly embedded orientable surface, with boundary a strongly invertible knot (K, ρ). Suppose that Σ is invariant under an involution of B^4 which extends ρ, and denote by $D \cong D^2$ the fixed point set of ρ in B^4. Then intersection $\Sigma \cap D$ consists of an arc joining the two fixed points on K and finite set Γ of fixed S^1. We define the *complexity* of Σ as

$$c(\Sigma) = \text{genus}(\Sigma) + |\Gamma|.$$

Then, we define the *slice complexity* of a strongly invertible knot (K, ρ) as

$$sc(K, \rho) = \min_{\Sigma} c(\Sigma),$$

where Σ ranges among the orientable surfaces in B^4 with boundary K, invariant under an involution of B^4 extending ρ.

Remark 4.15. By Smith theory (see [4]), we have that $|\Gamma| \leqslant \text{genus}(\Sigma)$. Therefore, for every strongly invertible knot (K, ρ) the following inequalities hold

$$\widetilde{g}_4(K) \leqslant sc(K) \leqslant 2 \cdot \widetilde{g}_4(K).$$

Let (K, ρ, h) be a directed strongly invertible knot, and let $\Sigma \subset B^4$ an invariant surface for K as in Definition 4.14. Denote by D the fixed point set in B^4, oriented compatible with the half-axis h. Observe that $D \setminus \Sigma$ can be subdivided into two subsurfaces in a checkerboard fashion, as described in Figure 7.

Let S be the subsurface containing the chosen half-axis h, and orient every $\gamma \in \Gamma$ and the fixed arc α as the boundary of S.

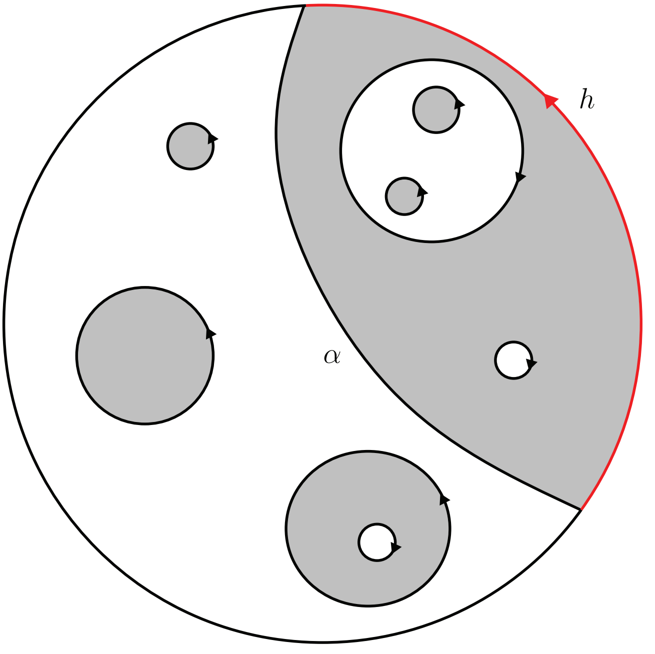


FIGURE 7 An example of how to get the orientation of the fixed point set D.

Let $D \times D^2$ be an equivariant tubular neighborhood of D in B^4. Pick an auxiliary orientation on Σ and observe that Σ induces on every γ a nowhere vanishing section s_γ of $D \times D^2$, which we can regard as a map $s_\gamma : \gamma \cong S^1 \longrightarrow S^1$. We call the degree of s_γ the *framing* $f(\gamma) \in \mathbb{Z}$ of γ. It is easy to see that it does not depend on the auxiliary orientation on Σ.

Similarly, Σ induces on α a nowhere zero section of $D \times D^2$, which we complete to a section on $\alpha \cup h$ using the section induced on h by a band $B \subset S^3$ which gives the 0-butterfly link of K (see Definition 2.8). Call the degree of the associated map $S^1 \longrightarrow S^1$ the *framing* $f(\alpha)$ of α.

Finally, we say that Σ is an *invariant surface type* n for (K, ρ, h), where

$$n = f(\alpha) + \sum_{\gamma \in \Gamma} f(\gamma).$$

Proposition 4.16. *Let $\Sigma \subset B^4$ be an invariant surface of type n for a directed strongly invertible knot (K, ρ, h). Then, there exists an invariant oriented surface $\widehat{\Sigma} \subset B^4$ with boundary $\widehat{L}_b^n(K)$, with no fixed points and such that*

$$\operatorname{genus}(\widehat{\Sigma}) \leqslant \begin{cases} c(\Sigma) - 1 \text{ if } \widehat{\Sigma} \text{ is connected,} \\ c(\Sigma) \text{ if } \widehat{\Sigma} \text{ is not connected.} \end{cases}$$

Proof. On the set of fixed circles $X = \Gamma \cup \{\alpha \cup h\}$ consider the partial order given by the nesting of circles, seen as circles in the fixed disk D. First of all, we want to remove all of the fixed circles. We do so by applying two moves.

Move 1: Suppose there exists a minimal element $\gamma \in \Gamma \subset X$ with framing zero. Let $D_\gamma \subset D$ be the disk bounded by γ. As $f(\gamma) = 0$ the section induced by Σ on γ of the equivariant tubular

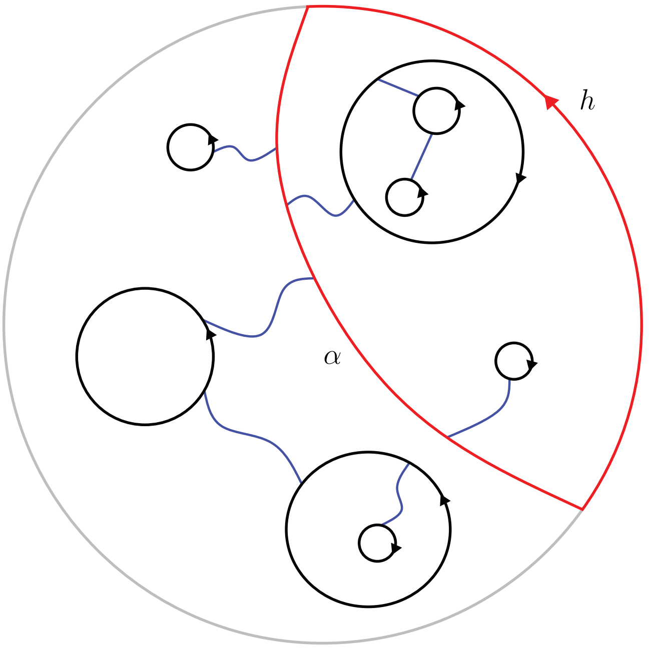


FIGURE 8 An example of choice of the arcs β, in blue.

neighborhood of D extends over D_γ to a nowhere vanishing section, which we can take to be equivariant. Therefore, we can perform an equivariant surgery of Σ along D_γ, obtaining a surface Σ' of the same type, with less genus and fixed circles. Replace Σ by Σ'.

Move 2: Let $\gamma \in \Gamma$ be a minimal element with $f(\gamma) \neq 0$ and let $\xi \in X$ be a circle such that there exists an arc $\beta \subset D \setminus \Sigma$ joining γ and ξ (see Figure 8). Then, we can find an equivariant $D^1 \times D^2$ inside an equivariant tubular neighborhood $D^1 \times D^3$ of β such that $(\partial D^1) \times D^2 \subset \Sigma$. We perform an equivariant surgery along $D^1 \times D^2$, obtaining a surface Σ' with $\mathrm{genus}(\Sigma') = \mathrm{genus}(\Sigma) + 1$. One can check that the circles γ and ξ were joined during the surgery into a new fixed circle with framing $f(\gamma) + f(\xi)$. Therefore, Σ' has the same type of Σ. Replace Σ by Σ'.

Applying Move 1 whenever possible and Move 2 in the other cases, we get an invariant surface Σ' of type n with fixed-point set consisting of only one arc. Observe that we have to apply Move 2 at most $\#\Gamma$ times. As in Remark 2.13, we can consider the equivariant band move on K along h that gives $\widehat{L}^n_b(K)$ as an equivariant cobordism C between K and $\widehat{L}^n_b(K)$. Now glue together C and Σ' along K, obtaining a surface Σ'', with $\mathrm{genus}(\Sigma'') \leqslant c(\Sigma)$. By construction, the fixed point set of the involution on Σ'' consists of a single circle, with framing induced by Σ'' equal to zero. Finally, apply Move 1, obtaining an invariant surface $\widehat{\Sigma}$ with boundary $\widehat{L}^n_b(K)$ and without fixed points. Observe that if the final Move 1 does not disconnect the surface, then $\mathrm{genus}(\widehat{\Sigma}) = \mathrm{genus}(\Sigma'') - 1 \leqslant c(\Sigma) - 1$. Otherwise $\mathrm{genus}(\widehat{\Sigma}) = \mathrm{genus}(\Sigma'') \leqslant c(\Sigma)$. □

Definition 4.17. Let $\mathcal{S} = (\theta, \rho, h, \widetilde{\mathrm{lk}})$ be an equivariant Seifert system on $M \cong \mathbb{Z}^{2m}$. A *partial metabolizer* for $\mathcal{S}$ is a ρ-invariant submodule $H \subset \ker(h) \cap \ker(\widetilde{\mathrm{lk}})$ such that $\theta_{H\times H} \equiv 0$. We define the *algebraic complexity* of $\mathcal{S}$ as $ac(\mathcal{S}) = m - k$, where

$$k = \max\{\mathrm{rank}(H) \mid H \text{ is a partial metabolizer of } \mathcal{S}\}.$$

Proposition 4.18. *The algebraic complexity is constant on the equivariant algebraic concordance classes. Therefore, it induces a well-defined map*

$$ac : \widetilde{\mathcal{G}}^{\mathbb{Z}} \longrightarrow \mathbb{N}.$$

Proof. Let $\mathcal{S}_i = (A_i, P_i, h_i, \widetilde{\mathrm{lk}}_i)$, $i = 0, 1$ be two equivariant Seifert systems. To prove that the algebraic complexity does not depend on the representative of a class in $\widetilde{\mathcal{G}}^{\mathbb{Z}}$ it is sufficient to show that if $\mathcal{S}_0$ is equivariantly metabolic then $ac(\mathcal{S}_0 \oplus \mathcal{S}_1) = ac(\mathcal{S}_1)$. Let H_0 is a metabolizer for $\mathcal{S}_0$ and H_1 is a maximal rank partial metabolizer for $\mathcal{S}_1$ then $H_0 \oplus H_1$ is a partial metabolizer for $\mathcal{S}_0 \oplus \mathcal{S}_1$, showing that

$$ac(\mathcal{S}_0 \oplus \mathcal{S}_1) \leqslant ac(\mathcal{S}_1).$$

For simplicity, consider the equivariant Seifert systems to be defined over $\mathbb{Q}$ coefficients. It can be seen, as in the proof of Lemma 6.1, that passing from $\mathbb{Z}$ to $\mathbb{Q}$ coefficients does not change the maximal rank of a partial metabolizer. Denote by V_i the m_i-dimensional vector space underlying $\mathcal{S}_i$. Let $H \subset V_0 \oplus V_1$ be a maximal partial metabolizer for $\mathcal{S}_0 \oplus \mathcal{S}_1$, H_0 be a metabolizer for $\mathcal{S}_0$ and W a complement of H_0, so that $V_0 = H_0 \oplus W$. Denote by k the dimension of H and let $\{\alpha_i = (x_i, y_i, z_i)\}_{i=1,\dots,k}$ be a basis of H, where $x_i \in H_0$, $y_i \in W$ and $z_i \in V_1$. Up to base change, we can suppose that $y_1, \dots, y_r$ are linearly independent and that $y_{r+1} = \cdots = y_k = 0$. Denote by Y the span of $y_1, \dots, y_r$. Therefore, for $r + 1 \leqslant i \leqslant k$ we have $\alpha_i = (x_i, 0, z_i)$. In particular, observe that the subspace spanned by $\{\alpha_i\}_{r+1 \leqslant i \leqslant k}$ is still invariant under the action of $(P_0 \oplus P_1)$. Repeating the process, we can assume after a change of basis that $z_{r+1}, \dots, z_{r+s}$ are linearly independent and that $z_{r+s+1} =, \dots, = z_k = 0$. Denote by Z the span of $z_{r+1}, \dots, z_{r+s}$. Observe that Z is the projection onto the V_1 summand of $\langle \alpha_i \mid r + 1 \leqslant i \leqslant k \rangle$, hence $P_1(Z) = Z$. Now for $r + 1 \leqslant i, j \leqslant r + s$ we have that

$$0 = (A_0 \oplus A_1)(\alpha_i, \alpha_j) = A_0(x_i, x_j) + A_1(z_i, z_j) = A_1(z_i, z_j),$$

$$0 = h_0(x_i) + h_1(z_i) = h_1(z_i),$$

$$0 = \widetilde{\mathrm{lk}}_0(x_i) + \widetilde{\mathrm{lk}}_1(z_i) = \widetilde{\mathrm{lk}}_1(z_i).$$

In other words, Z is a partial equivariant metabolizer of dimension s for $\mathcal{S}_1$. Hence, it is now sufficient to show that $s \geqslant k - m_0$.

As for $r + s + 1 \leqslant i \leqslant k$ we have that $\alpha_i = (x_i, 0, 0)$ are linearly independent, hence $X = \langle x_i \mid r + s + 1 \leqslant i \leqslant k \rangle$ is a $(k - s - r)$-dimensional subspace of H_0. Moreover, observe that $X \perp Y$, where $\perp$ means orthogonal with respect to the skew-symmetric form $A_0 - A_0^t$. On the other hand, $X \perp H_0$ and $H_0 \cap Y = 0$. As $A_0 - A_0^t$ is nondegenerate we have that

$$2m_0 - (k - r - s) = \dim X^{\perp} \geqslant \dim H_0 + \dim Y = m_0 + r,$$

and therefore $s \geqslant k - m_0$, that is, the opposite inequality $ac(\mathcal{S}_0 \oplus \mathcal{S}_1) \geqslant ac(\mathcal{S}_1)$ holds. □

Theorem 4.19. *Let (K, ρ, h) be a directed strongly invertible knot and let $\Phi(K) \in \widetilde{\mathcal{G}}^{\mathbb{Z}}$ be its equivariant algebraic concordance class. Then the following inequality holds*

$$2\widetilde{g}_4(K) \geqslant sc(K) \geqslant \min_{n \in \mathbb{Z}} ac(\Phi(K) + n\mathcal{S}(G)),$$

where G *is the invariant Seifert surface of type* -1 *for the unknot in Figure 5.*

Proof. Let $\Sigma \subset B^4$ be any invariant orientable surface with boundary K and let n be the type of Σ. Let $\widehat{\Sigma} \subset B^4$ be the invariant orientable surface with boundary $\widehat{L}_b^n(K)$ and no fixed points obtained from Σ by Proposition 4.16. Take now an invariant Seifert surface $F \subset S^3$ of type n for K. By Proposition 4.3 there exists a partial metabolizer $H \subset H_1(F, \mathbb{Z})$, with $\operatorname{rank} H \geqslant g_F - c(\Sigma)$. Therefore, $c(\Sigma) \geqslant g_F - \operatorname{rank} H \geqslant ac(S(F)) \geqslant \min_{n\in\mathbb{Z}} ac(\Phi(K) + nS(G))$. Taking the minimum over Σ, we get

$$sc(K) \geqslant \min_{n\in\mathbb{Z}} ac(\Phi(K) + nS(G)).$$

□

Remark 4.20. As the equivariant slice genus and the slice complexity do not depend on the choice of the direction, one can replace (K, ρ, h) by its antipode (K, ρ, h') in Theorem 4.19 to obtain a (potentially) better lower bound (Figure 8).

5 | EQUIVARIANT GORDON–LITHERLAND FORM

In this section, we define a homomorphism from $\widetilde{\mathcal{G}}_r^{\mathbb{Z}}$ to a simpler group $\widetilde{W}(\mathbb{Q})$ of algebraic concordance, namely an equivariant version of the Witt group of $\mathbb{Q}$. Then we characterize the image of a directed strongly invertible knot in $\widetilde{W}(\mathbb{Q})$ in terms of classical Witt invariants. Finally, we prove that the equivariant signature defined in [1] factors through $\widetilde{W}(\mathbb{Q})$.

Definition 5.1. Let $\mathbb{F}$ be a field. An *equivariant symmetric form* is a pair (Q, ρ) where Q is a symmetric, bilinear, and nondegenerate form on a finite-dimensional $\mathbb{F}$-vector space V and ρ is a Q-isometric involution of V. We say that (Q, ρ) is *equivariantly metabolic* if $\dim V$ is even and there exists a half-dimensional ρ-invariant subspace $W \subset V$ such that $Q_{|W\times W} \equiv 0$.

Again, analogously to Definitions 4.5 and 4.6 we can define a notion of orthogonal sum and concordance between equivariant symmetric forms and define the *equivariant Witt group* $\widetilde{W}(\mathbb{F})$ of $\mathbb{F}$ to be the set of equivalence classes of equivariant symmetric forms up to equivariant concordance.

Given an equivariant Seifert form (θ, ρ) defined over a $\mathbb{Z}$-module M, we can define an equivariant symmetric form on $M \otimes_{\mathbb{Z}} \mathbb{Q}$ by $(\theta + \theta^t, \rho)$.

It is immediate to see that this association induces a group homomorphism

$$\widetilde{\mathcal{G}}_r^{\mathbb{Z}} \longrightarrow \widetilde{W}(\mathbb{Q}).$$

Denote by $\Phi_W : \widetilde{\mathcal{C}} \longrightarrow \widetilde{W}(\mathbb{Q})$ the map given by the composition

$$\widetilde{\mathcal{C}} \longrightarrow \widetilde{\mathcal{G}}_r^{\mathbb{Z}} \longrightarrow \widetilde{W}(\mathbb{Q}).$$

Notice that the map Φ_W maps the equivariant concordance class of a directed strongly invertible knot (K, ρ, h), to the Witt class of the couple $(\mathcal{G}_F, \rho_*)$, where F is an invariant Seifert surface of type 0 for K and $(\mathcal{G}_F, \rho_*)$ is the couple given by the Gordon–Litherland form on $H_1(F, \mathbb{Q})$ and the action induced by ρ.

5.1 A characterization of the equivariant Witt class

Now we show that given a directed strongly invertible knot K, the equivariant Witt class $\Phi_W(K)$ depends only on the (classical) Witt class of K and $\mathfrak{qb}(K)$ (see Definition 2.14).

Remark 5.2. Let (Q, ρ) be an equivariant form over $\mathbb{Q}$ and let E_λ be the λ-eigenspace of ρ, for $\lambda = \pm 1$. Given $v \in E_1$, $w \in E_{-1}$ clearly

$$Q(v, w) = Q(v, -w) = -Q(v, w) \Longrightarrow Q(v, w) = 0.$$

Hence, E_1 and E_{-1} are orthogonal and we can decompose the form as

$$(Q, \rho) = (Q_{|E_1}, \mathrm{id}) \oplus (Q_{|E_{-1}}, -\mathrm{id}).$$

This gives us an isomophism

$$(\pi_+, \pi_-) : \widetilde{W}(\mathbb{Q}) \longrightarrow W(\mathbb{Q}) \oplus W(\mathbb{Q})$$

$$[Q, \rho] \longrightarrow \left(\left[Q_{|E_1}\right], \left[Q_{|E_{-1}}\right]\right).$$

We will denote by $\Phi_W^\pm = \pi_\pm \circ \Phi_W : \widetilde{C} \longrightarrow W(\mathbb{Q})$ the induced homomorphisms.

Using the description above of $\widetilde{W}(\mathbb{Q})$ we can give a new definition of the equivariant signature.

Definition 5.3. Denote by $\sigma : W(\mathbb{Q}) \longrightarrow \mathbb{Z}$ the signature homomorphism. Define the *equivariant signature* as $\widetilde{\sigma} = (\sigma \circ \Phi_W^- - \sigma \circ \Phi_W^+) : \widetilde{C} \longrightarrow \mathbb{Z}$.

We show now how the invariants we just defined are related to some of the invariants defined in [2] and [1].

Definition 5.4. Let A be a nondegenerate symmetric $n \times n$ matrix and let k be a nonzero integer. Define $M_k(A) = k \cdot A$. Clearly, if A is metabolic, $M_k(A)$ is so. Moreover $M_k(A) \oplus M_k(B) = M_k(A \oplus B)$. Therefore, this induces a well-defined homomorphism

$$M_k : W(\mathbb{Q}) \longrightarrow W(\mathbb{Q})$$

$$[A] \longmapsto [M_k(A)].$$

It is immediate to see that $M_k \circ M_k$ is the identity, hence M_k is an isomorphism.

Lemma 5.5. *Let F be an invariant Seifert surface of type 0 for a directed strongly invertible knot (K, ρ, h) and let $\widetilde{F}$ be the corresponding Seifert surface for $\widehat{L}_b^0(K)$. Then the quotient surface $\overline{F} = \widetilde{F}/\rho \subset \overline{S^3} = S^3/\rho$ is a spanning surface for $\mathfrak{qb}(K)$ with zero relative Euler number $e(\overline{F})$ (see Definition 2.17).*

Proof. Pick a representative of the semi-orientation on $\widehat{L}_b^0(K)$ and denote the components of the link by H and J. Let $H^{\widetilde{F}}$ be a nearby longitude of H missing $\widetilde{F}$. Then the projection $\pi(H^{\widetilde{F}})$ is a longitude of $\mathfrak{qb}(K)$ missing $\overline{F}$. To show that $e(\overline{F}) = \mathrm{lk}(\mathfrak{qb}(K), \pi(H^{\widetilde{F}})) = 0$ it is sufficient to prove that $[H^{\widetilde{F}}] = 0 \in H_1(S^3 \setminus H, \mathbb{Z})$. As $H^{\widetilde{F}}$ is disjoint from $\widetilde{F}$, we have that $\mathrm{lk}(H^{\widetilde{F}}, H) + \mathrm{lk}(H^{\widetilde{F}}, J) = 0$. By definition of 0-butterfly link we have that $\mathrm{lk}(H^{\widetilde{F}}, J) = \mathrm{lk}(H, J) = 0$, therefore $\mathrm{lk}(H^{\widetilde{F}}, H) = 0$. In other words, $[H^{\widetilde{F}}] = 0 \in H_1(S^3 \setminus H, \mathbb{Z})$. □

Proposition 5.6. *Let (K, ρ, h) be a directed strongly invertible knot. Then*

$$\Phi^+_W(K) = M_2(\varphi_W(\mathfrak{qb}(K)),$$

where $\varphi_W(\mathfrak{qb}(K))$ is the Witt class of $\mathfrak{qb}(K)$.

Proof. Let F be an invariant Seifert surface of type 0 for K and let $\widetilde{F}$ be the corresponding Seifert surface for $\widehat{L}^0_b(K)$. First of all, observe that F can be obtained from $\widetilde{F}$ by attaching an equivariant band B. As ρ reverses the orientation of the core of B, it is not difficult to see that the dimension of the (-1)-eigenspace of ρ_* increases by one going from $H_1(\widetilde{F}, \mathbb{Q})$ to $H_1(F, \mathbb{Q})$. Hence, the 1-eigenspace of ρ_* is fully contained in $H_1(\widetilde{F})$. Let $\pi : (S^3, \widetilde{F}) \longrightarrow (\overline{S^3}, \overline{F})$ be the quotient projection, given by the action of ρ. The quotient surface $\overline{F}$ is a spanning surface for $\mathfrak{qb}(K)$. Observe that the quotient projection π is a twofold covering $\widetilde{F} \longrightarrow \overline{F}$. Take now an oriented curve c in $\overline{F}$, representing a class in $H_1(\overline{F})$, and lift it to a class $\mathrm{tr}(c) = \pi^{-1}(c) \in H_1(\widetilde{F})$. This defines a transfer homomorphism (see [4] for details) $\mathrm{tr} : H_1(\overline{F}) \longrightarrow H_1(\widetilde{F})$. By construction $\rho_*(\mathrm{tr}(c)) = \mathrm{tr}(c)$, that is, the image of the transfer map is contained in the 1-eigenspace E_1 of ρ_*. The composition $\pi_* \circ \mathrm{tr}$ is given by

$$\pi_* \circ \mathrm{tr} : H_1(\overline{F}) \longrightarrow H_1(\overline{F})$$

$$c \longmapsto 2c,$$

hence tr is injective. Moreover, tr is clearly surjective on E_1: given a ρ-invariant class $d \in E_1$, we can project it by π_* and lift it again, showing that $2d \in \mathrm{im}(\mathrm{tr})$. Finally, we show that the transfer map behaves well with respect to the Gordon–Litherland form. Given $c, d \in H_1(\overline{F})$ let S be an oriented surface in $\overline{S^3}$ with $\partial S = \widetilde{d}$, where $\widetilde{d}$ is d pushed out of $\overline{F}$ "in both directions simultaneously" (as in the definition of the Gordon–Litherlan form). In this way, we have that $\mathcal{G}_{\overline{F}}(c, d) = \mathrm{lk}(c, \widetilde{d}) = \#S \cap c$. Up to a small isotopy, we can suppose S transverse to the branching locus. The lift $\pi^{-1}(S)$ is an oriented surface in S^3 with boundary $\widetilde{\mathrm{tr}(d)}$ and we can use it to calculate

$$\mathrm{lk}(\mathrm{tr}(c), \widetilde{\mathrm{tr}(d)}) = \#(\pi^{-1}(S) \cap \mathrm{tr}(c)) = 2(\#S \cap c).$$

It follows that the Gordon–Litherland forms are related by

$$\mathcal{G}_{\widetilde{F}}(\mathrm{tr}(c), \mathrm{tr}(d)) = 2 \cdot \mathcal{G}_{\overline{F}}(c, d).$$

Therefore, the transfer map gives an isometry

$$\mathrm{tr} : (H_1(\overline{F}), 2 \cdot \mathcal{G}_{\overline{F}}) \longrightarrow (E_1, \mathcal{G}_{\widetilde{F}}).$$

Finally, by Lemma 5.5 we have that $e(\overline{F}) = 0$ and hence by Proposition 2.19 that the Gordon–Litherland form on $\overline{F}$ represents the Witt class of $\mathfrak{qb}(K)$. □

As an immediate consequence, we get the following corollary.

Corollary 5.7. *Let K be a directed strongly invertible knot and let A and B be symmetric matrices representing the (nonequivariant) Witt classes of $\mathfrak{qb}(K)$ and K, respectively. Then the equivariant Witt class of K is represented by the couple*

$$\Phi_W(K) = \left[\begin{pmatrix} 2A & 0 & 0 \\ 0 & -2A & 0 \\ 0 & 0 & B \end{pmatrix}, \begin{pmatrix} \mathrm{id} & 0 & 0 \\ 0 & -\mathrm{id} & 0 \\ 0 & 0 & -\mathrm{id} \end{pmatrix}\right].$$

5.2 | The equivariant signature

We recall now the definition of equivariant signature introduced by Alfieri and Boyle [1] and we prove that it is equivalent to the one in Definition 5.3.

Given a knot $K \subset S^3$ we denote by $\Sigma(K)$ the twofold cover of S^3 branched over K. Given a properly embedded and connected surface $F \subset B^4$, we denote by $\Sigma(F)$ the twofold cover of B^4 branched over F, and by τ the covering transformation of $\Sigma(F)$.

Lemma 5.8 [2, Proposition 12]. *Let ρ be an orientation preserving involution of B^4 such that* $\mathrm{Fix}(\rho)$ *is a 2-disk D. Let $F \subset B^4$ be a properly embedded and connected ρ-invariant surface on which ρ acts nontrivially. Then, there exists a lift $\widetilde{\rho}$ of ρ, that is, the following diagram commutes*

$$\begin{array}{ccc} \Sigma(F) & \xrightarrow{\widetilde{\rho}} & \Sigma(F) \\ \downarrow \pi & & \downarrow \pi \\ B^4 & \xrightarrow{\rho} & B^4. \end{array}$$

In fact, there exist exactly two such lifts, namely $\widetilde{\rho}$ and $\tau\widetilde{\rho}$.

Now let (K, ρ, h) be a directed, strongly invertible knot and let $F \subset B^4$ be a properly embedded connected surface with $\partial F = K$ (not necessarily orientable), invariant with respect to some extension of ρ to B^4 (which we still denote by ρ).

Before introducing the equivariant signature it is useful to better describe the fixed point set of the lifts of ρ given by Lemma 5.8. We do so in the following remark

Remark 5.9. Let D be the fixed point disk of B^4. The intersection $D \cap F = \mathrm{Fix}(\rho_{|F})$ is the disjoint union of an arc joining the fixed point of K and a finite number of S^1 and isolated points. Take $x \in \mathrm{Fix}(\widetilde{\rho})$ and observe that $\rho \circ \pi(x) = \pi \circ \widetilde{\rho}(x) = \pi(x)$, therefore $\mathrm{Fix}(\widetilde{\rho}) \subseteq \pi^{-1}(D)$. Moreover, note that $\widetilde{\rho} \circ \tau(x) = \tau \circ \widetilde{\rho}(x) = \tau(x)$, that is, $\mathrm{Fix}(\widetilde{\rho})$ is τ-invariant. Take now $x \in \pi^{-1}(F \cap D)$. Then $\widetilde{\rho}(x) \in \pi^{-1}(\rho \circ \pi(x)) = \{x, \tau(x)\}$ and as $\tau(x) = x$ we have that $x \in \mathrm{Fix}(\widetilde{\rho})$, that is, $\pi^{-1}(F \cap D)$ is fixed pointwise by $\widetilde{\rho}$. Let $C_1, \dots, C_n$ the connected components of $D \setminus F$. As $\rho_{|C_i}$ is the identity, $\widetilde{\rho}$ and $\tau \circ \widetilde{\rho}$ act either as the identity or as τ on $\pi^{-1}(C_i)$. Therefore, exactly one of the lifts fixes pointwise the preimage of C_i, while the other one has no fixed point in $\pi^{-1}(C_i)$. Let C_i and C_j be *adjacent* components, that is, separated by a circle or an arc in $D \cap F$. Then, $\pi^{-1}(C_i)$ and $\pi^{-1}(C_j)$ cannot be both fixed pointwise by $\widetilde{\rho}$. Otherwise, $\overline{\pi^{-1}(C_i)}$ and $\overline{\pi^{-1}(C_j)}$ would be two fixed surfaces, both contained in $\mathrm{Fix}(\widetilde{\rho})$ and intersecting in a nontrivial way in their interior and this would imply that $\mathrm{Fix}(\widetilde{\rho})$ has a component which is not a manifold. Therefore, if we decompose $D \setminus F = A \sqcup B$, where A and B are union of nonadjacent components, we have that the fixed point sets of the two lifts of ρ are, respectively, $\pi^{-1}(\overline{A})$ and $\pi^{-1}(\overline{B})$.

Observe that by Remark 5.9 exactly one lift $\widetilde{\rho}$ of ρ to $\Sigma(F)$ fixes pointwise $\widetilde{h} = \pi^{-1}(h)$. The fixed point set of $\widetilde{\rho}$ is the disjoint union of a (eventually disconnected) surface Δ, with $\partial\Delta = \widetilde{h}$, and a finite set of points. Recall now that the 0-butterfly link $L_b^0(K)$ is obtained by performing a band move on K along a band parallel to h, in such a way that the linking number between the components of $L_b^0(K)$ is zero. Let γ be one of the arcs of this band parallel to h. As the end-points of γ meet the branching set, its preimage $\widetilde{\gamma}$ in $\Sigma(F)$ is a closed curve. Given a perturbation Δ' of Δ with $\partial\Delta' = \widetilde{\gamma}$ we define the *relative Euler number* $e(\Delta, \widetilde{\gamma})$ as the algebraic intersection $\#(\Delta \cap \Delta')$.

Definition 5.10 [1]. The *equivariant signature* of (K, ρ, h) is defined as

$$\widetilde{\sigma}(K) = \sigma(\Sigma(F), \widetilde{\rho}) - e(\Delta, \widetilde{\gamma}),$$

where $\sigma(\Sigma(F), \widetilde{\rho})$ is the g-signature (see [1] or [11]) of the pair $(\Sigma(F), \widetilde{\rho})$.

Using the G-signature theorem [11], Alfieri and Boyle prove that the equivariant signature is a well-defined invariant for equivariant concordance and in particular defines a homomorphism

$$\widetilde{\sigma} : \widetilde{\mathcal{C}} \longrightarrow \mathbb{Z}.$$

Remark 5.11. Actually, Alfieri and Boyle [1] define the equivariant signature slightly differently, exchanging the role of the two half-axes h and h'. It is immediate to check that our definition of equivariant signature for the directed strongly invertible knot (K, ρ, h) coincides with their definition for the antipode $(K, \rho, h') = a(K, \rho, h)$. Hence, the two invariants are essentially the same. However, it is easier to relate Definition 5.10 to the equivariant algebraic concordance group (see Theorem 5.13).

In [1, section 6], the authors explain how to easily compute the relative Euler number for the equivariant pushoff of a spanning surface in B^4. Using the following proposition it is possible to easily compute the equivariant signature from an equivariant spanning surface.

Proposition 5.12 [2, Proposition 13]. *Let (K, ρ, h) be a directed strongly invertible knot in S^3. Let F be a connected spanning surface for K, with $\rho(F) = F$. We still denote by ρ the radial extension of the involution to B^4. Let $\widehat{F}$ be the surface obtained by equivariantly pushing the interior of F in B^4 and denote by $\widetilde{\rho}$ the preferred lift of ρ to $\Sigma(\widehat{F})$. Then under the identification $(H_1(F), \mathcal{G}_F) \cong (H_2(\Sigma(\widehat{F})), Q)$ the map of lattices $\widetilde{\rho}_* : (H_2(\Sigma(\widehat{F})), Q) \longrightarrow (H_2(\Sigma(\widehat{F})), Q)$ is equivalent to:*

- *$\rho_* : (H_1(F), \mathcal{G}_F) \longrightarrow (H_1(F), \mathcal{G}_F)$ if $h \not\subset F$,*
- *$-\rho_* : (H_1(F), \mathcal{G}_F) \longrightarrow (H_1(F), \mathcal{G}_F)$ if $h \subset F$.*

Theorem 5.13. *The equivariant signature introduced in Definition 5.3 coincides with the one given in Definition 5.10.*

Proof. Let F be an invariant Seifert surface of type 0 for a directed strongly invertible knot (K, ρ, h). According to Definition 5.3, $\widetilde{\sigma}(K)$ is the equivariant signature of $(H_1(F), \mathcal{G}_F, -\rho_*)$. By Lemma 5.8 and Proposition 5.12, this quantity coincides with the g-signature of the pair $(\Sigma(\widehat{F}), \widetilde{\rho})$, where $\Sigma(\widehat{F})$ is the twofold cyclic cover branched over a copy $\widehat{F}$ of F radially pushed into B^4 and $\widetilde{\rho}$ is the preferred lift of the radial extension of ρ to B^4. Hence, it is sufficient to prove that the relative Euler number vanishes. Let γ be a parallel copy of h on F. As cutting F along h produces an equivariant Seifert surface for $\widehat{L}_b^0(K)$, the lift $\widetilde{\gamma}$ of γ in $\Sigma(\widehat{F})$ is the canonical longitude of $\widetilde{h}$. Let D, D' be the traces of h and γ, respectively, along the radial isotopy that pushes the interior of F in B^4. As $\widehat{F}$ is obtained from $F \subset S^3$, one can see that $\mathrm{Fix}(\rho_{\widehat{F}})$ consists solely on an arc joining the fixed points of K.

Then by Remark 5.9 the fixed point set of $\widetilde{\rho}$ consists of the lift Δ of D. The lift Δ' of D' is a perturbation of Δ such that $\partial\Delta' = \widetilde{\gamma}$, and as they are disjoint we have

$$e(\Delta, \widetilde{\gamma}) = \#(\Delta \cap \Delta') = 0$$

that is, the relative Euler number vanishes. □

Remark 5.14. As a consequence of Theorem 5.13 and Corollary 5.7, we obtain the following formula

$$\widetilde{\sigma}(K) = \sigma(K) - 2\sigma(\mathfrak{q}\mathfrak{b}(K))$$

for the equivariant signature of a directed strongly invertible knot K in terms of classical signatures.

Remark 5.15. As shown by Alfieri and Boyle [1, Proposition 7.3], the equivariant signature depends on the choice of the half-axis for a strongly invertible knot. For example, they show in the proof of Proposition 7.2 that $\widetilde{\sigma}(7_4b^+ \widetilde{\#} m7_4b^-) \neq 0$.

As the equivariant algebraic concordance homomorphism $\Psi : \widetilde{\mathcal{C}} \longrightarrow \widetilde{\mathcal{AC}}$ defined in [23] does not distinguish the choice of half-axis, we get that $7_4b^+ \widetilde{\#} m7_4b^-$ has trivial image in $\widetilde{\mathcal{AC}}$.

On the other hand, as $\widetilde{\sigma}$ factors through $\widetilde{\mathcal{G}}_r^{\mathbb{Z}}$, we get that the image of $7_4b^+ \widetilde{\#} m7_4b^-$ is nontrivial in $\widetilde{\mathcal{G}}_r^{\mathbb{Z}}$.

6 | THE STRUCTURE OF THE EQUIVARIANT ALGEBRAIC CONCORDANCE GROUP

In this section, we explore the structure of the reduced equivariant algebraic concordance group $\widetilde{\mathcal{G}}_r^{\mathbb{Z}}$. To do so, we introduce the notion of *symmetric structure* and we describe the equivalence between equivariant Seifert forms over $\mathbb{Q}$ and symmetric structures.

Lemma 6.1. *The natural homomorphism $\widetilde{\mathcal{G}}_r^{\mathbb{Z}} \longrightarrow \widetilde{\mathcal{G}}_r^{\mathbb{Q}}$ given by the extension of coefficients is injective.*

Proof. Let (θ, ρ) be an equivariant Seifert form over a free $\mathbb{Z}$-module M, and denote by $(\theta_{\mathbb{Q}}, \rho_{\mathbb{Q}})$ its extension over $M \otimes \mathbb{Q}$. Suppose $H \subset M \otimes \mathbb{Q}$ is a $\rho_{\mathbb{Q}}$-invariant metabolizer for $\theta_{\mathbb{Q}}$ and let $H_{\mathbb{Z}} = H \cap M$. Then clearly $H_{\mathbb{Z}}$ is a ρ-invariant metabolizer for θ. □

The lemma above ensures that no information is lost by considering rational coefficients instead of integral ones. We focus now on determining the group structure of $\widetilde{\mathcal{G}}_r^{\mathbb{Q}}$. In the following, we will implicitly consider the equivariant Seifert forms to be over $\mathbb{Q}$.

Definition 6.2. A *symmetric structure* is a triple (V, β, S), where

- V is a $\mathbb{Q}$-vector space,
- β is a bilinear, symmetric and nondegenerate form on V,
- $S : V \longrightarrow V$ is a linear isomorphism which is self-adjoint with respect to β.

We say that (V, β, S) is *metabolic* if $\dim V$ is even and there exists a S-invariant half-dimensional subspace $W \subset V$ on which β is identically zero.

We define the *orthogonal sum* of two symmetric structures $\mathcal{S}_i = (V_i, \beta_i, S_i)$, $i = 1, 2$ as

$$\mathcal{S}_1 \oplus \mathcal{S}_2 = (V_1 \oplus V_2, \beta_1 \oplus \beta_2, S_1 \oplus S_2).$$

We say that $\mathcal{S}_1$ and $\mathcal{S}_2$ are *concordant* if $-\mathcal{S}_1 \oplus \mathcal{S}_2$ is metabolic, where $-\mathcal{S}_1 = (V_1, -\beta_1, S_1)$.

Definition 6.3. We define the group $\mathcal{G}_{sym}$ of symmetric structures as the quotient of the set of symmetric structures up to concordance, endowed with the operation of orthogonal sum.

Let (V, θ, ρ) be an equivariant Seifert form. Let T be the endomorphism of V given by the composition

$$V \xrightarrow{\theta - \theta^t} V^* \xrightarrow{(\theta+\theta^t)^{-1}} V,$$

and let β be the bilinear form $\theta + \theta^t$. It is easy to check that the following facts hold:

- T is anti-self-adjoint with respect to β,
- $\rho \circ T + T \circ \rho = 0$,
- $\theta(x, y) = \frac{1}{2}(\beta(x, y) + \beta(Tx, y))$.

In particular, the last property implies that any subspace $H \subset V$ is a ρ-invariant metabolizer for θ if and only if is a $\langle T, \rho \rangle$-invariant metabolizer for β.

Denote now by $V_\pm$ the eigenspace of ρ relative to ± 1. As $\rho \circ T + T \circ \rho = 0$, we have that $T(V_\pm) = V_\mp$, and that $V_\pm$ is T^2-invariant.

Suppose now $H \subset V$ is a $\langle T, \rho \rangle$-invariant metabolizer for β, and let $H_\pm = H \cap V_\pm$. As ρ is a β-isometry, V_+ and V_- are β-orthogonal and hence $H_\pm$ is a metabolizer for $\beta_{|V_\pm}$. As H is T-invariant and T and ρ anticommute, we easily deduce that $T(H_\pm) = H_\mp$ and hence that $H_\pm$ is T^2-invariant. Vice versa, let $H_+ \subset V_+$ be a T^2-invariant metabolizer for $\beta_{|V_+}$. As T is anti-self-adjoint with respect to β, it is immediate to see that $H = H_+ \oplus T(H_+)$ is a $\langle T, \rho \rangle$-invariant metabolizer for β.

Theorem 6.4. *Using the notation above, we have that the following map from equivariant Seifert forms to symmetric structures*

$$(V, \theta, \rho) \longmapsto (V_+, \beta_{|V_+}, T^2_{|V_+})$$

induces an isomorphism between the equivariant algebraic concordance group $\widetilde{\mathcal{G}}_r^{\mathbb{Q}}$ *and the group of symmetric structures* $\mathcal{G}_{sym}$.

Proof. It is immediate to see that the map above is compatible with the orthogonal sum. From the discussion above, we have that (V, θ, ρ) is metabolic if and only if $(V_+, \beta_{|V_+}, T^2_{|V_+})$ is metabolic. Therefore, this map induces a well-defined and injective homomorphism between the two groups. Vice versa, given a symmetric structure (V, β, S), we can construct an equivariant Seifert form on $V \oplus V$, as follows:

$$\theta = \frac{1}{2}\begin{pmatrix} \beta & \beta \cdot S \\ -\beta \cdot S & -\beta \cdot S \end{pmatrix},$$

$$\rho = \begin{pmatrix} \mathrm{id} & 0 \\ 0 & -\mathrm{id} \end{pmatrix}.$$

It is easy to check that $(V \oplus V, \theta, \rho)$ has (V, β, S) as associated symmetric structure. □

Let θ be a Seifert form on a space V. Recall that the Alexander polynomial of θ is defined as $\Delta_\theta(t) = \det(A - tA^t)$, where A is a matrix representing θ with respect to some basis of V (which is well-defined up to squares in $\mathbb{Q}$).

Definition 6.5. Let $p(t) \in \mathbb{Q}[t, t^{-1}]$ be a width $2d$ *symmetric* polynomial (i.e., such that $p(t) = p(-t)$). We denote by $\delta(p)(s) \in \mathbb{Q}[s]$ the polynomial obtained by the following substitution

$$\delta(p)(s) = \left(\left(1 - \lambda^2\right)^d p\left(\frac{1+\lambda}{1-\lambda}\right) \right)_{|s=\lambda^2}.$$

Observe that δ is multiplicative, meaning that $\delta(p \cdot q) = \delta(p) \cdot \delta(q)$. Moreover, δ admits an inverse: given $q(s) \in \mathbb{Q}[s]$ a polynomial of degree d we can define

$$\delta^{-1}(q)(t) = \frac{(t+1)^{2d}}{(4t)^d} q\left(\left(\frac{t-1}{t+1}\right)^2 \right),$$

and it is not difficult to see that δ^{-1} is the inverse of δ.

Remark 6.6. Let (V, θ, ρ) be an equivariant Seifert form, with Alexander polynomial $\Delta_\theta(t)$. Let (V_+, β, S) be the associated symmetric structure. Then, from the discussion above, we have that up to units in $\mathbb{Q}$ and factors $(s - 1)$, the characteristic polynomial of S is given by $\delta(\Delta_\theta)(s)$. In the following, given a directed strongly invertible knot, we will denote by $\delta_K(s)$ the polynomial obtained by the formula in Definition 6.5 applied to the Alexander polynomial $\Delta_K(t)$ of K (which is well-defined if we require that $\Delta_K(t) = \Delta_K(t^{-1})$).

Let (V, β, S) be a symmetric structure and $p(s) \in \mathbb{Q}[s]$ be an irreducible polynomial. We define the p-component of V as

$$V_p = \bigcup_{N>0} \ker(p(S)^N).$$

Clearly, V_p is an S-invariant subspace of V.

Lemma 6.7. *Let p and q be distinct irreducible polynomials. Then V_p and V_q are orthogonal with respect to β.*

Proof. Let $N > 0$ be big enough so that $V_q = \ker(q(S)^N)$. As p is irreducible and $p \neq q$, we have that the restriction of $q(S)^N$ is an isomorphism from V_p to itself. Given now $v \in V_p$ and $w \in V_q$, there exists $v' \in V_p$ such that $v = q(S)^N v'$. We compute now

$$\beta(v, w) = \beta(q(S)^N v', w) = \beta(v', q(S)^N w) = \beta(v', 0) = 0,$$

therefore V_p and V_q are orthogonal. □

In particular, the restriction of β on V_p is nondegenerate and hence $(V_p, \beta_{|V_p}, S_{|V_p})$ is a symmetric structure. The following theorem is an immediate consequence of the lemma above.

Theorem 6.8. *Let $\mathcal{F} \subset \mathbb{Q}[s]$ be the set of irreducible and monic polynomials different from $q(s) = s$. Then the group $\mathcal{G}_{sym}$ splits as*

$$\mathcal{G}_{sym} = \bigoplus_{p \in \mathcal{F}} \mathcal{G}^p_{sym},$$

where $\mathcal{G}^p_{sym}$ is the subgroup of $\mathcal{G}_{sym}$ determined by symmetric structures with characteristic polynomial given by a power of $p(s)$. In particular, the projection of a class $(V, \beta, S) \in \mathcal{G}_{sym}$ onto $\mathcal{G}^p_{sym}$ is given by $(V_p, \beta_{|V_p}, S_{|V_p})$.

Therefore, it is sufficient to study the summands $\mathcal{G}^p_{sym}$ separately.

Proposition 6.9. *Let $(V, \beta, S) \in \mathcal{G}^p_{sym}$. Then (V, β, S) is concordant to $(\overline{V}, \overline{\beta}, \overline{S})$ so that $\overline{V} =$* $\ker p(\overline{S})$.

Proof. Let $N \geqslant 0$ be the least integer so that $V = \ker(p(S)^N)$ and suppose $N \geqslant 2$. Let $W = \operatorname{im} p(S)^{N-1}$ and let $W^\perp$ be its orthogonal. Clearly, W is invariant under S, and so is $W^\perp$. Observe that for $v, w \in V$

$$\beta(p(S)^{N-1} v, p(S)^{N-1} w) = \beta(v, p(S)^{2N-2} w) = \beta(v, 0) = 0,$$

as $2N - 2 \geqslant N$ for $N \geqslant 2$. Therefore, $W \subset W^\perp$, and by definition W is the radical of $\beta_{|W^\perp}$. Hence, β induces a symmetric nondegenerate form $\overline{\beta}$ on the quotient $\overline{V} = W^\perp / W$. Similarly, as W and $W^\perp$ are S-invariant, we have an induced map $\overline{S}$ on the quotient.

It is easy to check that $(\overline{V}, \overline{\beta}, \overline{S})$ is again a symmetric structure, and that by construction $\overline{V} = \ker(p(\overline{S})^{N-1})$

Consider $H = \{(w, [w]) \in V \oplus \overline{V} \mid w \in W^\perp\}$. It is easy now to check that H is a metabolizer for $(V, -\beta, S) \oplus (\overline{V}, \overline{\beta}, \overline{S})$, showing that (V, β, S) and $(\overline{S}, \overline{\beta}, \overline{S})$ are concordant. □

Theorem 6.10. *Let $p(s) \in \mathbb{Q}[s]$ be an irreducible polynomial and consider the field $\mathbb{F} = \mathbb{Q}[s]/(p(s))$. Then, we have a natural isomorphism*

$$\mathcal{G}^p_{sym} \longrightarrow W(\mathbb{F}).$$

Proof. Take $(V, \beta, S) \in \mathcal{G}^p_{sym}$. By Proposition 6.9, we can suppose that $V = \ker(p(S))$. Therefore, V is naturally a $\mathbb{F}$-vector space. Denote by $\mathrm{tr} : \mathbb{F} \longrightarrow \mathbb{Q}$ the trace map. As the symmetric $\mathbb{Q}$-bilinear form

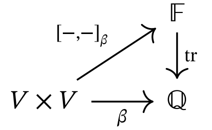
$$\mathbb{F} \times \mathbb{F} \longrightarrow \mathbb{Q}$$

$$(x, y) \longmapsto \mathrm{tr}(x \cdot y)$$

is nondegenerate, for every $u, v \in V$ there exists a unique $e \in \mathbb{F}$ such that for all $f \in \mathbb{F}$

$$\beta(f \cdot u, v) = \mathrm{tr}(e \cdot f).$$

This correspondence defines a symmetric $\mathbb{F}$-bilinear form $[-,-]_\beta$ on V such that



is a commutative diagram.

Now let H be a $\mathbb{Q}$-subspace of V. Then H is S-invariant if and only if is a $\mathbb{F}$-subspace. Moreover, it is easy to see that β vanishes identically on a $\mathbb{F}$-subspace H if and only if $[-,-]_\beta$ does. Therefore, the homomorphism induced by this correspondence is an isomorphism. □

We can now summarize Theorems 6.8 and 6.10 in the following theorem on the structure of $\widetilde{\mathcal{G}}^{\mathbb{Q}}_r$.

Theorem 6.11. *The reduced equivariant algebraic concordance group $\widetilde{\mathcal{G}}^{\mathbb{Q}}_r$ is isomorphic to*

$$\widetilde{\mathcal{G}}^{\mathbb{Q}}_r \cong \bigoplus_{\substack{p(s) \neq s \\ p \text{ irreducible}}} W(\mathbb{Q}[s]/(p(s))).$$

Using classical results on the Witt groups of finite extension of $\mathbb{Q}$ (see [22] for details), one can check that Theorem 6.11 implies that

$$\widetilde{\mathcal{G}}^{\mathbb{Q}}_r \cong \mathbb{Z}^\infty \oplus \mathbb{Z}/2\mathbb{Z}^\infty \oplus \mathbb{Z}/4\mathbb{Z}^\infty \oplus \mathbb{Z}/8\mathbb{Z}^\infty,$$

where G^∞ stands for the direct sum of countable many copies of G.

Remark 6.12. It would be very interesting to know whether there exists any directed strongly invertible knot K whose image has order 8 in $\widetilde{\mathcal{G}}^{\mathbb{Q}}_r$. However, we would like to point out that several of the summands in Theorem 6.11 have trivial intersection with $\widetilde{\mathcal{G}}^{\mathbb{Z}}_r$: it is not difficult to check that many polynomials do not appear as factors of $\delta_K(s)$ for any knot K.

For example, suppose that $p(s) = s - 10$ appears as a factor of $\delta_K(s) = p(s)q(s)$ for some knot K and some other polynomial q. Following Remark 6.6, we compute the Alexander polynomial of K (up to units) as $\delta^{-1}(\delta_K(s))(t) = \delta^{-1}(p)(t) \cdot \delta^{-1}(q)(t) = \frac{1}{4}(-9t - 22 - 9t^{-1}) \cdot \delta^{-1}(q)(t)$. Therefore, $\Delta_K(t)$ would be divisible by the primitive polynomial $\Delta(t) = 9t + 22 + 9t^{-1}$. As $\Delta(1) = 40$, we get a contradiction with the fact that $\Delta_K(1) = \pm 1$.

We conclude this section with a consequence of Theorem 6.11, which can be interpreted as an *equivariant Fox–Milnor condition*.

Theorem 6.13. *Let K be a strongly invertible knot and let $\Delta_K(t)$ be its Alexander polynomial, normalized so that $\Delta_K(t) = \Delta_K(t^{-1})$ and $\Delta_K(1) = 1$. If K is equivariantly slice then $\Delta_K(t)$ is a square.*

Proof. Suppose $\Delta_K(t)$ does not satisfy the condition above. It follows that there exists an irreducible polynomial $g(t) \neq t$ appearing with an odd power in the prime decomposition of $\Delta_K(t)$. If $g(t^{-1}) = \pm t^n g(t)$ for some n, then one can easily see that $\Delta_K(t)$ does not satisfy the (nonequivariant) Fox–Milnor condition, therefore K is not even slice. On the other hand, if $g(t) \neq \pm t^n g(t)$, we know that $g(-t)$ appears in the factorization of $\Delta_K(t)$ with the same exponent $2k+1$, as Δ_K is symmetric. Let $p(s) = \delta(g(t)g(t^{-1}))(s)$. We prove that $p(s)$ is irreducible. Suppose that $p(s) = p_1(s)p_2(s)$ is a nontrivial decomposition. Then we would have a factorization of $g(t)g(t^{-1}) = \delta^{-1}(p_1)(t)\delta^{-1}(p_2)(t)$, where $\delta^{-1}(p_i)(t)$ are nontrivial symmetric polynomials. But this is not possible as $g(t)$ and $g(t^{-1})$ are nonsymmetric and irreducible. It follows that $p(s)$ appears in the irreducible factorization of $\delta_K(s)$ with exponent $2k+1$. Hence, the class of K in $\widetilde{\mathcal{G}}_r^{\mathbb{Q}}$ is nontrivial (with respect to any choice of direction), as its projection on the summand $W(\mathbb{Q}[s]/(p(s)))$ is represented by a form of odd rank. Therefore, K is not equivariantly slice. □

Observe that Theorem 6.13 highlights an important difference between equivariant and nonequivariant algebraic concordance. In fact, given a knot K, its nonequivariant algebraic concordance class splits into components depending on the irreducible and symmetric factors of $\Delta_K(t)$, while nonsymmetric factors of $\Delta_K(t)$ do not contribute. On the other hand, if K is a strongly invertible knot, its class in $\widetilde{\mathcal{G}}_r^{\mathbb{Q}}$ splits into components depending on all irreducible factors of $\delta_K(s)$, and these factors can correspond to irreducible nonsymmetric factors of $\Delta_K(t)$.

7 | NEW EQUIVARIANT SIGNATURES

In this section, we introduce new equivariant signatures, dependent on a parameter $\lambda \in \mathbb{R}$. Then we clarify the relation between $\widetilde{\sigma}_\lambda$, the Levine–Tristram signatures and the equivariant signature defined in [1]. Furthermore, we give an analysis of the discontinuities of these equivariant signatures similar to the one in [21], and we use the signature jumps to obtain lower bounds on the equivariant slice genus.

From now on, we always assume to work with coefficients in $\mathbb{R}$ if not mentioned otherwise. In particular, when we refer to equivariant Seifert form or symmetric structure, we always implicitly consider their natural extensions given by tensoring with $\mathbb{R}$.

Definition 7.1. Let (A, P) be an equivariant Seifert form. Given $\lambda \in \mathbb{R}$ consider the hermitian form

$$A_\lambda = \frac{(A + A^t)}{2}((1-\lambda)I - (1+\lambda)P) + i(A - A^t).$$

We define the *equivariant signature* in λ as

$$\widetilde{\sigma}_\lambda(A, P) = \lim_{\varepsilon \to 0^+} \frac{\sigma(A_{\lambda+\varepsilon}) + \sigma(A_{\lambda-\varepsilon})}{2},$$

and the *equivariant signature jump* in λ as

$$\widetilde{J}_\lambda(A,P) = \lim_{\varepsilon\to 0^+} \frac{\sigma(A_{\lambda+\epsilon}) - \sigma(A_{\lambda-\epsilon})}{2}.$$

It is immediate to see that $\widetilde{\sigma}_\lambda$ and $\widetilde{J}_\lambda$ define homomorphisms

$$\widetilde{\mathcal{G}}_r \longrightarrow \mathbb{Z}.$$

In the following, given a directed strongly invertible knot K, we will denote by $\widetilde{\sigma}_\lambda(K)$ and $\widetilde{J}_\lambda(K)$ the compositions of the homomorphisms above with the map $\widetilde{C} \longrightarrow \widetilde{\mathcal{G}}_r^{\mathbb{Z}}$. In other words, $\widetilde{\sigma}_\lambda(K) = \widetilde{\sigma}_\lambda(A,P)$ and $\widetilde{J}_\lambda(A,P)$, where (A,P) is the equivariant Seifert form given by an equivariant Seifert surface of type 0 for K.

Remark 7.2. Up to a base change, we can always suppose that

$$P = \begin{pmatrix} \mathrm{id} & 0 \\ 0 & -\mathrm{id} \end{pmatrix}.$$

In such a base, we have that

$$A = \begin{pmatrix} B & C \\ -C^t & D \end{pmatrix}.$$

and hence

$$A_\lambda = 2\begin{pmatrix} -\lambda B & iC \\ -iC^t & D \end{pmatrix}$$

which is easily seen to be congruent to

$$2\begin{pmatrix} -\lambda B - CD^{-1}C^t & 0 \\ 0 & D \end{pmatrix}.$$

If we denote by (V,β,S) the symmetric structure associated with (A,P), we have that $2B$ represents β, while S is given by $-B^{-1}C^tD^{-1}C$. Therefore, it is easy to see that $-2\lambda B - 2C^tD^{-1}C$ represents the symmetric bilinear form over V given by

$$\beta_{S-\lambda} : V \times V \longrightarrow \mathbb{R}$$

$$(x,y) \longmapsto \beta((S-\lambda)x,y).$$

It follows that A_λ is nondegenerate for every $\lambda \in \mathbb{R}$ except when λ is a root of the characteristic polynomial of S. When A_λ is nondegenerate then $\widetilde{J}_\lambda(A,P) = 0$ and $\widetilde{\sigma}_\lambda(A,P)$ is equal to the signature of A_λ. Moreover, for every $\lambda \in \mathbb{R}$ we have that

$$\widetilde{J}_\lambda(A,P) = \lim_{\varepsilon\to 0^+} \frac{\sigma(\beta_{S-\lambda-\epsilon}) - \sigma(\beta_{S-\lambda+\epsilon})}{2}.$$

Finally, observe that

$$\lim_{\lambda\to-\infty}\widetilde{\sigma}_\lambda(A,P)=\sigma(B)+\sigma(D)\qquad\text{and}\qquad\lim_{\lambda\to+\infty}\widetilde{\sigma}_\lambda(A,P)=\sigma(D)-\sigma(B).$$

Therefore, given a directed strongly invertible knot K, we get from Theorem 5.13 that

$$\lim_{\lambda\to-\infty}\widetilde{\sigma}_\lambda(K)=\sigma(K)\qquad\text{and}\qquad\lim_{\lambda\to+\infty}\widetilde{\sigma}_\lambda(K)=\widetilde{\sigma}(K).$$

Proposition 7.3. *Let K be a directed strongly invertible knot. Then for any $\lambda<0$ and $\omega\in S^1$ such that $\lambda(\omega-1)^2=(\omega+1)^2$ we have*

$$\widetilde{\sigma}_\lambda(K)=\sigma_\omega(K),$$

Proof. Let (A,P) be an equivariant Seifert form associated with K. As in Remark 7.2, we can suppose that

$$A=\begin{pmatrix}B & C\\ -C^t & D\end{pmatrix}\qquad P=\begin{pmatrix}\mathrm{id} & 0\\ 0 & -\mathrm{id}\end{pmatrix}\qquad\qquad A_\lambda=2\begin{pmatrix}-\lambda B & iC\\ -iC^t & D\end{pmatrix}.$$

As $\lambda<0$, we have that $\sqrt{-\lambda}\in\mathbb{R}$ and hence A_λ is congruent to

$$2\begin{pmatrix}B & i\sqrt{-\lambda}C\\ -i\sqrt{-\lambda}C^t & D\end{pmatrix}=(A+A^t)+i\sqrt{-\lambda}(A-A^t).$$

Finally, observe that $(A+A^t)+i\sqrt{-\lambda}(A-A^t)$ is a positive multiple of $A_\omega=(1-\omega)A+(1-\overline{\omega})A^t$ for $\omega=-\frac{1\pm i\sqrt{-\lambda}}{1\mp i\sqrt{-\lambda}}$. Therefore, $\widetilde{\sigma}_\lambda(K)=\sigma(A_\omega)=\sigma_\omega(K)$. Finally, observe that for a fixed $\lambda<0$, the solution to the equation $\lambda(\omega-1)^2=(\omega+1)^2$ are exactly $\omega=-\frac{1\pm i\sqrt{-\lambda}}{1\mp i\sqrt{-\lambda}}$. □

Let (V,β,S) be a symmetric structure. Given $\lambda\in\mathbb{R}$, we denote by $V^\lambda=\bigcup_{N>0}\ker((S-\lambda)^N)$. Similarly to Lemma 6.7, it is easy to check that if $\lambda\neq\mu$ the V^λ and V^μ are β-orthogonal.

Lemma 7.4. *Let (V,β,S) be a symmetric structure. Suppose that $\lambda\in\mathbb{R}$ is not a root of the characteristic polynomial of S. Then for every $\alpha\in\mathbb{R}$ we have*

$$\sigma(V^\alpha,\beta_{S-\lambda})=\mathrm{sign}(\alpha-\lambda)\cdot\sigma(V^\lambda,\beta).$$

Proof. By a slight abuse of notation, we denote the restriction of S and β to V^α again by S and β. For $t\in[0,1]$ let $S_t=(1-t)S+t\alpha\,\mathrm{id}$ and $\beta_{S_t-\lambda}(x,y)=\beta((S_t-\lambda)x,y)$. Observe that the only eigenvalue of S_t is α for every $t\in[0,1]$, therefore we have that $S_t-\lambda\,\mathrm{id}$ is nonsingular and $\beta_{S_t-\lambda}$ is nondegenerate for every t. It follows that the signature is constant in t, and hence

$$\sigma(V^\alpha,\beta_{S-\lambda})=\sigma(V^\alpha,(\alpha-\lambda)\beta)=\mathrm{sign}(\alpha-\lambda)\sigma(V^\alpha,\beta)$$

by evaluating at $t=0$ and $t=1$. □

Proposition 7.5. *Let (A,P) be an equivariant Seifert form, and let (V,β,S) be the associated symmetric structure. Then for every $\lambda\in\mathbb{R}$ we have that $\widetilde{J}_\lambda(A,P)$ is equal to $-\sigma(V^\lambda,\beta)$.*

Proof. It follows from Remark 7.2 that is sufficient to study how the signature of $\beta_{S-\lambda}$ varies in λ. Let $W \subset V$ the β-orthogonal to $\bigoplus_{\lambda\in\mathbb{R}} V^{\lambda}$. Observe that W is S-invariant and that $S_{|W}$ has no real eigenvalues. Therefore, the signature of the restriction of $\beta_{S-\lambda}$ to W is constant in λ.

As if $\mu_1 \neq \mu_2$ then V^{μ_1} and V^{μ_2} are orthogonal with respect to β and hence with respect to $\beta_{S-\lambda}$, it is sufficient to consider the variation of the signature separately on every V^{α}.

Let now $\epsilon > 0$ be small enough so that λ is the only eigenvalue of S in $[\lambda-\epsilon, \lambda+\epsilon]$.

Then by Lemma 7.4 we have that $\sigma(V^{\mu}, \beta_{S-\lambda+\epsilon}) = \sigma(V^{\mu}, \beta_{S-\lambda-\epsilon})$ for every $\mu \neq \lambda$, while

$$\sigma(V^{\lambda}, \beta_{S-\lambda+\epsilon}) = -\sigma(V^{\lambda}, \beta_{S-\lambda-\epsilon}) = -\sigma(V^{\lambda}, \beta).$$

As the signature of $\beta_{S-\lambda}$ varies only on the summand V^{λ}, we get

$$\widetilde{J}_{\lambda}(A,P) = -\sigma(V^{\lambda}, \beta_{|V^{\lambda}}).$$

□

Corollary 7.6. *Let K be a directed strongly invertible knot, and let (A,P) be an equivariant Seifert form obtained from an equivariant Seifert surface for K of any type. Then for every $\lambda \in \mathbb{R}$, $\lambda \neq 1$ we have that $\widetilde{J}_{\lambda}(K) = \widetilde{J}_{\lambda}(A,P)$, that is, for $\lambda \neq 1$, $\widetilde{J}_{\lambda}$ does not depend on the type of the equivariant Seifert surface.*

Proof. Let (A,P) be the equivariant Seifert form given by any equivariant Seifert surface F for K, and let (V,β,S) be the associated symmetric structure. Recall that we can obtain a surface of type 0 for K by performing a boundary connected sum of F with the type -1 surface G (or its mirror image) as Lemma 4.2. Observe that the equivariant Seifert form given by G is

$$A_G = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad P_G = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

and hence the associated symmetric structure is

$$\beta_G = (1) \qquad S_G = (1).$$

It follows that adding some copies of (A_G, P_G) to (A,P) does not modify the λ-eigenspace of S, for $\lambda \neq 1$. Therefore, by Proposition 7.5 we can compute $\widetilde{J}_{\lambda}(K)$ with any equivariant Seifert surface, for $\lambda \neq 1$. □

Theorem 7.7. *Given a directed strongly invertible knot K, for every $\lambda \in \mathbb{R}$, $\lambda \neq 1$ we have that*

$$\widetilde{g}_4(K) \geqslant \frac{|\widetilde{J}_{\lambda}(K)|}{4}.$$

Proof. Let $\mathcal{S} = (A,P,h,\widetilde{\mathrm{lk}})$ be the equivariant Seifert system associated with an equivariant Seifert surface F for K of any type. It follows easily from Definitions 4.17 and 7.1 that for every $\lambda \in \mathbb{R}$

$$ac(\mathcal{S}) \geqslant |\widetilde{\sigma}_{\lambda}(A,P)|/2,$$

and therefore

$$ac(\mathcal{S}) \geqslant |\widetilde{J}_{\lambda}(A,P)|/2.$$

From Corollary 7.6, we know that for $\lambda \neq 1$ the left-hand side in the inequality above does not depend on the type of the surface F. Therefore, using Theorem 4.19 we get

$$\widetilde{g}_4(K) \geqslant \frac{sc(K)}{2} \geqslant \frac{|\widetilde{J}_\lambda(K)|}{4}. \qquad \square$$

Remark 7.8. Observe that the roots of $\delta_K(s)$ correspond to pairs of roots of $\Delta_K(t)$. First of all, notice that $-1, 0, 1$ cannot be roots of $\Delta_K(t)$. It follows from Definition 6.5 that if $z \in \mathbb{C} \setminus \{-1, 0, 1\}$ is a root of $\Delta_K(t)$ then $\mu(z)$ is a root of $\delta_K(s)$, where

$$\mu : \mathbb{C} \setminus \{-1, 0, 1\} \longrightarrow \mathbb{C} \setminus \{0, 1\}$$

$$z \longmapsto \left(\frac{z-1}{z+1}\right)^2.$$

It is not difficult to see that $\mu(z_1) = \mu(z_2)$ if and only if $z_1 = (z_2)^{-1}$ and that $\mu(z) \in \mathbb{R}$ if and only if $z \in \mathbb{R}$ or $z \in S^1$. Therefore, if $\{\lambda_1, \lambda_1^{-1}, \dots, \lambda_d, \lambda_d^{-1}\}$ is the set of roots of $\Delta_K(t)$ (which come in pairs because the polynomial is symmetric), we get that $\{\mu(\lambda_1), \dots, \mu(\lambda_d)\}$ is the set of roots of $\delta_K(s)$, and this correspondence preserves the multiplicity. In particular, real negative roots of $\delta_K(s)$ correspond to unitary roots of $\Delta_K(t)$, while real positive roots of $\delta_K(s)$ correspond to real roots of $\Delta_K(t)$.

Using Theorem 7.7, we obtain the following result, which can be seen as a generalization of [23, Theorem 5.7].

Theorem 7.9. *Let K and J be directed strongly invertible knots. Suppose that there exists $\Lambda \in \mathbb{R} \cup S^1$ such that*

- *Λ is a root of $\Delta_K(t)$ with odd multiplicity,*
- *Λ is not a root of $\Delta_J(t)$.*

Then for every $n, m \in \mathbb{Z}$, we have that $\widetilde{g}_4(K^n \widetilde{\#} J^m) \geqslant |n|/4$, where $K^n \widetilde{\#} J^m$ is the connected sum of n copies of K and m copies of J taken in any order.

Proof. Let (V_K, β_K, S_K) and (V_J, β_J, S_J) be any symmetric structures associated with K and J, respectively, and let $\lambda = \mu(\Lambda)$. From Remark 7.8, we know that λ is a real root of $\delta_K(s)$ with odd multiplicity. Therefore, $\dim(V_K^\lambda)$ is odd and $|\widetilde{J}_\lambda(K)| \geqslant 1$ by Proposition 7.5. Vice versa, eas λ is not a root of $\delta_J(s)$, we have that $\dim(V_J^\lambda) = 0$ and hence $\widetilde{J}_\lambda(J) = 0$. It follows that $|\widetilde{J}_\lambda(K^n \widetilde{\#} J^m)| \geqslant |n|$, and hence $\widetilde{g}_4(K^n \widetilde{\#} J^m) \geqslant |n|/4$ by Theorem 7.7. $\square$

We conclude this section with an example of application of Theorem 7.9.

Example 7.10. Let $\{K_i\}_{i \in I}$ be a family of (algebraically) slice strongly invertible knots and $\{\lambda_i\}_{i \in I} \subset \mathbb{R}$ such that:

(1) λ_i is a root of $\Delta_{K_i}(t)$ with odd multiplicity,
(2) $\Delta_{K_j}(\lambda_i) \neq 0$ for $i \neq j$.

Endow each K_i with any choice of direction and consider the subgroup $G_I \subset \widetilde{\mathcal{C}}$ generated by $\{K_i\}_{i \in I}$. It follows immediately from Theorem 7.9 that the image of G_I in $\widetilde{\mathcal{G}}_r^{\mathbb{Q}}$ has rank $|I|$.

For example, take $I \subset \mathbb{Z}[t]$ to be an infinite family of irreducible, nonsymmetric polynomials of odd degree, such that

- $f(1) = 1$ for all $f \in I$,
- $f(t) \neq t^{\deg g} g(t^{-1})$ for all $f, g \in I$.

From [24], we know that for all $f \in I$ there exists a strongly invertible knot K_f with Alexander polynomial $\Delta_{K_f}(t) = f(t)f(t^{-1})$. In particular, each K_f is algebraically slice and the family $\{K_f\}_{f\in I}$ satisfies the conditions above (as each f has at least one real root with multiplicity one), therefore it generates a subgroup of $\widetilde{C}$ of infinite rank.

Remark 7.11. Let $G_I \subset \widetilde{C}$ be a subgroup defined as in Example 7.10, with I a countable infinite set. Observe that the equivariant signature defined by Alfieri and Boyle takes values in $\mathbb{Z}$, hence it vanishes on a subgroup $H_I \subset G_I$ such that G_I/H_I is either 0 or $\mathbb{Z}$. Moreover, all Levine–Tristram signatures vanish on H_I, as by hypothesis is spanned by algebraically slice knots. However, using the equivariant signature jumps arguing as above, we have that H_I surjects onto $\mathbb{Z}^\infty$. In particular, this shows that the $\{\widetilde{J}_\lambda\}_{\lambda>0}$ are actually new invariants, independent from Levine–Tristram and Alfieri–Boyle signatures. For other concrete examples regarding the independence of $\widetilde{J}_\lambda$ from Levine–Tristram signatures, see the Appendix.

APPENDIX A: EQUIVARIANT SLICE GENUS OF 2-BRIDGE KNOTS

Recall that every 2-bridge knot is strongly invertible, see [25]. We already know from [7] that any given 2-bridge knot is not equivariantly slice and that has infinite order in $\widetilde{C}$[†], independently of the choice of strong inversion and direction.

In this appendix,[‡] we apply the results of Section 7 to study the equivariant slice genus of 2-bridge knots.

Definition A.1. Let K be a directed strongly invertible knot. We denote the *maximal signature jump* of K by

$$\widetilde{J}(K) = \sup_{\lambda\neq 1} |\widetilde{J}_\lambda(K)|.$$

Let $\mathcal{F}$ be the family of 2-bridge knots with crossing number less or equal to 12 and with (averaged) Levine–Tristram signature function σ_ω identically zero. For every knot K in $\mathcal{F}$, we report in Table A1 its

- name,
- 2-bridge notation p/q,
- order in C, denoted by Ord,
- maximal signature jump $\widetilde{J}$[§].

[†] This result is stated in the smooth category in [7]. However, all the arguments can be adapted to work in the topological category.

[‡] We used [20] to gather the Alexander polynomial, Levine–Tristram signature function and topological concordance order of all knots considered in the Appendix.

[§] See Lemma A.2.

TABLE A1 Concordance orders and maximal signature jumps of some 2-bridge knots.

Name	p/q	Ord	$\widetilde{J}$	Name	p/q	Ord	$\widetilde{J}$
4_1	5/2	2	1	$12a_{204}$	173/76	∞	1
6_1	9/7	1	1	$12a_{221}$	169/66	1	1
6_3	13/5	2	0	$12a_{243}$	133/60	∞	1
7_7	21/8	∞	0	$12a_{303}$	153/64	∞	1
8_1	13/11	∞	1	$12a_{307}$	157/69	∞	1
8_3	17/4	2	1	$12a_{385}$	161/66	∞	1
8_8	25/9	1	0	$12a_{425}$	81/37	1	0
8_9	25/7	1	1	$12a_{437}$	149/65	∞	1
8_{12}	29/12	2	1	$12a_{447}$	121/43	1	1
8_{13}	29/11	∞	0	$12a_{471}$	85/38	2	1
9_{14}	37/14	∞	0	$12a_{477}$	169/70	1	1
9_{19}	41/16	∞	0	$12a_{482}$	93/22	∞	1
9_{27}	49/19	1	1	$12a_{497}$	209/81	∞	0
10_1	17/15	∞	1	$12a_{499}$	233/89	2	0
10_3	25/6	1	1	$12a_{506}$	185/68	2	1
10_{10}	45/17	∞	0	$12a_{510}$	193/81	2	1
10_{13}	53/22	∞	1	$12a_{518}$	157/34	∞	1
10_{17}	41/9	2	0	$12a_{550}$	149/34	?	1
10_{22}	49/36	1	1	$12a_{583}$	161/45	∞	1
10_{26}	61/44	∞	1	$12a_{585}$	181/50	∞	1
10_{28}	53/19	∞	0	$12a_{596}$	81/14	∞	0
10_{31}	57/25	∞	0	$12a_{644}$	113/30	?	0
10_{33}	65/18	2	0	$12a_{650}$	165/46	∞	1
10_{34}	37/13	∞	0	$12a_{690}$	89/20	?	1
10_{35}	49/20	1	1	$12a_{691}$	77/12	∞	1
10_{37}	53/23	2	0	$12a_{715}$	169/50	1	1
10_{42}	81/31	1	1	$12a_{744}$	61/8	∞	0
10_{43}	73/27	2	1	$12a_{774}$	89/16	?	0
10_{45}	89/34	2	1	$12a_{792}$	85/24	∞	0
$11a_{13}$	61/28	∞	0	$12a_{803}$	21/2	∞	1
$11a_{84}$	101/57	∞	1	$12a_{1029}$	81/19	1	0
$11a_{91}$	129/50	∞	1	$12a_{1034}$	121/32	1	0
$11a_{96}$	121/50	1	1	$12a_{1039}$	137/37	2	1
$11a_{98}$	77/18	∞	0	$12a_{1127}$	97/22	2	1
$11a_{110}$	97/35	∞	1	$12a_{1129}$	105/23	∞	0
$11a_{119}$	77/34	∞	0	$12a_{1139}$	101/18	?	0
$11a_{180}$	89/64	∞	0	$12a_{1140}$	97/18	?	1
$11a_{185}$	109/30	∞	1	$12a_{1166}$	33/4	∞	1
$11a_{190}$	85/67	∞	1	$12a_{1273}$	61/11	2	1
$11a_{195}$	53/8	∞	0	$12a_{1275}$	149/44	2	1
$11a_{210}$	73/16	∞	0	$12a_{1277}$	121/37	1	1

(Continues)

TABLE A1 (Continued)

Name	p/q	Ord	$\widetilde{J}$	Name	p/q	Ord	$\widetilde{J}$
$11a_{333}$	65/14	∞	0	$12a_{1281}$	109/33	2	1
$12a_{197}$	69/32	∞	1	$12a_{1287}$	37/6	2	1

The purpose of Table A1 is to show that given a directed strongly invertible knot K, the equivariant signature jumps can be used to prove that the (topological) equivariant slice genus of K^n grows linearly in $|n|$ even when the Levine–Tristram signature function vanishes.

Given a knot K in the family $\mathcal{F}$, it is not possible to obtain a lower bound on the smooth or topologically slice genus of $n \cdot K$, $n \in \mathbb{Z}$ using the Levine–Tristram signatures, as $\sigma_\omega(K) \equiv 0$.

On the other hand, observe from Table A1 that for most of the knots in $\mathcal{F}$, we have $\widetilde{J}(K) = 1$. Therefore, we easily get the following lower bound on the topological equivariant slice genus of K^n

$$\widetilde{g}_4(K^n) \geqslant |n|/4,$$

by applying Theorem 7.7.

We rely on the following lemma in order to compute $\widetilde{J}(K)$ for K in $\mathcal{F}$.

Lemma A.2. *For any knot K in $\mathcal{F}$, we have*

$$\widetilde{J}(K) = \begin{cases} 0 & \text{if} \quad \Delta_K(t) \quad \text{has no real root,} \\ 1 & \text{if} \quad \Delta_K(t) \quad \text{has at least one real root,} \end{cases}$$

and in particular $\widetilde{J}(K)$ does not depend on the choice of strong inversion nor of direction on K.

Proof. First of all we check that each root of $\Delta_K(t)$ has multiplicity one. It follows that $\Delta_K(t)$ has no root $z \in S^1$, otherwise we would have a discontinuity of the signature function in z and hence $\sigma_\omega(K) \neq 0$ for some $\omega \in S^1$. Using Remark 7.8 and arguing as in the proof of Theorem 7.7, one can see that in this case $\widetilde{J}(K)$ does not depend on the choice of strong inversion nor of direction on K, and that

$$\widetilde{J}(K) = \begin{cases} 0 & \text{if} \quad \Delta_K(t) \quad \text{has no real root,} \\ 1 & \text{if} \quad \Delta_K(t) \quad \text{has at least one real root.} \end{cases} \qquad \square$$

Remark A.3. Observe that as each root of $\Delta_K(t)$ has multiplicity one for K in $\mathcal{F}$, $\Delta_K(t)$ cannot be a square. Hence, using Theorem 6.13 we recover the result in [7]: none of the knots in $\mathcal{F}$ is equivariantly slice.

Remark A.4. In [5], the authors provide the first examples of strongly invertible knots with smooth equivariant slice genus arbitrarily larger than their smooth slice genus. Miller and Powell in [23] find examples of knots on which the quantity $\widetilde{g}_4(K) - g_4(K)$ assume arbitrarily large values.

In the family $\mathcal{F}$, we can find knots K with smooth and topological order $\leqslant 2$ and such that $\widetilde{J}(K) = 1$ (8_9). As the smooth and topological slice genus of K^n is bounded, applying Theorem 7.7 we get that both $\widetilde{g}_4(K^n) - g_4(K^n)$ grows linearly in $|n|$ and hence they provide new examples of the phenomenon described above. The knot 8_9 is the simplest example of such knots in $\mathcal{F}$ that does not fall into the examples in [5, 23].

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