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Stochastic Problems for Turbulent Fluids in Domains with Boundaries

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Abstract

This dissertation explores various related aspects on how stochasticity may influence the analysis of turbulent 2D fluids in domains with boundaries. Boundaries can exert a direct influence, affecting the boundary conditions within the vorticity and velocity equations, or they can induce more indirect effects by introducing the instabilities generated by boundaries in the system and subsequently considering the properties of this modified model. These two lines of research are both considered in this work.

Regarding the first aspect, we initially delve into the global well-posedness and interior regularity of the Navier-Stokes equations with stochastic boundary conditions. The model employed provides an idealization of fluid velocity in oceanography, where our domain represents a vertical slice of the ocean. The noise, white in time and colored in space, captures the physical laws governing the atmosphere-ocean interface, representing the balance of shear stress from the ocean and horizontal wind forces. This result is the first example of global well-posedness for a fluid dynamical system subjected to boundary noise that is white in time. Secondly, we analyze the convergence of stochastic point vortex dynamics toward fluid flux in a bounded domain with no-slip boundary conditions. It is well-known that no-slip boundary conditions at the velocity level correspond to non-homogeneous Neumann-type boundary conditions at the vorticity level. The generation of vorticity at the boundary exhibits nonlinear and nonlocal dependence on the vorticity itself. Although the correct analysis of vorticity equations remains an open problem, we present a sequence of interacting reflecting diffusions converging to the solution of the vorticity equation subject to a given Neumann boundary conditions in a smooth, bounded convex domain. Despite differing from the boundary conditions encountered in formulating the Navier-Stokes equations with no-slip boundaries, the introduced Neumann boundary condition serves as a valuable approximation, particularly locally in time. This analysis is ultimately related to providing a Lagrangian framework for understanding properties of the boundary layer and the generation of vorticity at boundaries.

In the second line of research, we address the often-overlooked phenomenon of boundary roughness in the mathematical description of fluid dynamical models. This leads, in a suitable scaling limit, to the introduction of a Gaussian source of randomness diffused inside the fluid domain in fluid dynamic equations. Consequently, a natural question arises regarding the analysis of such modified systems. While their well-posedness is a standard fact under quite general assumptions on the regularity of the noise, we provide several results on the analysis of the inviscid limit, considering different models for turbulent fluids (i.e., Navier-Stokes equations and Second-Grade fluid equations) with noise white in time, colored in space, scaling according to the viscosity. After proving the validity of the inviscid limit under natural assumptions compared to deterministic cases (utilizing a Kato-type criterion for the Navier-Stokes equations and a specific scaling between the elasticity and viscosity of the model for the second-grade fluid equations), we analyze the probability of significant fluctuations in the zero noise-zero viscosity limit. While the result concerning the Navier-Stokes equations depends on a Kato-type condition stronger than that assumed for simple convergence, we provide an explicit example where this condition is satisfied. This analysis is ultimately related to providing an Eulerian framework for understanding some properties of the boundary layer and the generation of vorticity at boundaries.

Summary

Chapter 1

Introduction

This dissertation is devoted to explore several aspects of incompressible turbulent fluid with constant density in 2D bounded domains and how stochastic analysis may help in two relevant issues related to such topic: model description and new insights in understanding of turbulence. Before starting to analyze how this plan is followed along this thesis in section 1.3, section 1.4, section 1.5, it is relevant to describe the main keywords of the previous sentence. We refer to [17], [126], [84], [171] for more details on the physics of fluids, just sketched here.

We start describing the equation of motion of an incompressible fluid. Denoting by u the velocity of the fluid evolving in $\mathcal{O} \subseteq \mathbb{R}^2$, then it evolves according the second Newton law: consider a very small portion of fluid, identified by a point $x(t)$. The acceleration $x''(t)$ is equal to the sum of the forces. But the velocity $x'(t)$ is equal to $u(t, x(t))$, by definition of u. Hence

$$\frac{d}{dt} u(t, x(t)) = \text{forces}.$$

This reads

$$\partial_t u + u \cdot \nabla u = \text{forces}.$$

along the trajectory $x(t)$. The forces acting on the fluid are of two kinds, internal and external. Internal forces are described by the Cauchy Stress tensor and are due to pressure and the characteristics of the fluid (e.g. viscosity, elasticity). In particular Navier-Stokes equations correspond to the choice

$$f_{int} = \operatorname{div} \sigma_{NS}, \quad \sigma_{NS} = -PI + \nu A_1, \quad A_1 = \frac{\nabla u + \nabla u^T}{2}. \tag{1.1}$$

Conservation of mass, incompressibility and constant density lead u to satisfies also to $\operatorname{div} u = 0$. Therefore, thanks to the assumption (1.1), we derived the celebrated Navier-Stokes equations to which the great part of this work is devoted

$$\begin{gathered}\partial_t u + u\cdot\nabla u + \nabla P = \nu\Delta u + f_{est} \quad x\in\mathcal{O}, \\ \operatorname{div} u = 0 \quad x\in\mathcal{O},\end{gathered} \tag{1.2}$$

$u = u\,(t,x)$ being a vector field and the pressure $P = P\,(t,x)$ being a scalar field. Since we are working in a bounded domain $\mathcal{O}$, equations (1.2) must be supplemented by boundary and initial conditions. The fluid is assumed to be viscous, namely we assume $\nu > 0$.

It is generally accepted that this fact has, as a consequence, the no-slip boundary condition $u|_{\partial\mathcal{O}} = 0$, because viscous fluids must be at rest on solid boundaries. However, while the condition $u\cdot\hat{n}|_{\partial\mathcal{O}} = 0$, corresponding to the assumption that an incompressible fluid cannot penetrate an impermeable solid surface, is unanimously accepted, the condition $u\cdot\hat{\tau}|_{\partial\mathcal{O}} = 0$ is not satisfied in particular regimes, see [172, Chapter 19] for a review. More in general, the component of the fluid velocity tangent to the surface, $u\cdot\hat{\tau}|_{\partial\mathcal{O}}$, is proportional to the rate of strain, (or shear rate) at the surface

$$u\cdot\hat{\tau}|_{\partial\mathcal{O}} = 2\lambda\hat{n}\cdot A_1(I-\hat{n}\otimes\hat{n})|_{\partial\mathcal{O}}. \tag{1.3}$$

$\lambda\in[0,+\infty]$ has the unit of a length, and is referred to as the slip length. In case $\mathcal{O} = \mathbb{T}\times I$, where $\mathbb{T}$ is the one dimensional torus and $I\subset\mathbb{R}$ is an open interval, λ has a clear interpretation. Indeed, for a pure shear flow, it can be seen as the fictitious distance below the surface where the no-slip boundary condition would be satisfied. According to [91], [148], [149], additional forcing terms of the form

$$f_{bdd} = \lambda g_{st} + g_m,$$

can be added to (1.3), the first describing an external shear stress acting on the surface, the latter relative velocity between the two plates of $\mathcal{O}$. However, we point out that, except for chapter 2, we only consider along this thesis the case $u|_{\partial\mathcal{O}} = 0$. We stress that the no-slip condition $u|_{\partial\mathcal{O}} = 0$ provokes large stress near the boundary, if u is large nearby and this stress, when the viscosity is small enough, may lead to instabilities and generate vortices. This is the so–called phenomenon of the emergence of a boundary layer: close to the boundary the fluid presents a turbulent behavior for $\nu\to 0$. The thickness of the boundary layer and some control on the behavior of the fluid in this region are very challenging and mostly open questions, see [14] for a review on the topic.

1.1 The Role of Stochasticity in Modeling Turbulence

We ended the previous section without a discussion on the concept of turbulent fluid. The goal of this section is, therefore, trying to give some insights, at least heuristically, in how stochasticity may help in modeling turbulent fluids, in particular in the case of domains with boundaries.

Turbulence is not a unique and well–defined concept. The interested reader may refers to [84], [17], [126], for reference works in this field. However, it is easy to describe examples of turbulent fluids and some universal characteristics of such motions. For instance, think of a fast fluid along a solid boundary, developing a turbulent boundary layer, or a shear flow developing a turbulent region by instability. Turbulent fluids present large–scale motion and structures, specific to the geometry of the flow, which coexist with small–scale ones, maybe more universal. For instance, in the case of the turbulent boundary layer, we observe a mean flow and possibly other large–scale elements like large scale–vortex structures, superimposed on an extremely complex small–scale motion made of small hairpin vortices arising at the boundary, others apparently detaching from the boundary and traveling in the interior, others arising from the previous ones by further instabilities and so on. We refer for instance to [100] for a review on the complexity of the so–called turbulence coherent structures.

From previous examples, we can understand that boundaries plays a crucial role in the generation of small scales and developing turbulence. Indeed, vortices are produced by instability even on a flat boundary. This fact, however, is already incorporated in a mathematical model based on deterministic Navier–Stokes equations in a domain with smooth boundary; thus it does not require the artificial introduction of a noise. Different is the case of vortices produced by irregularities of the boundary or by several small or complicated obstacles in the middle of the fluid domain. In principle, if we describe precisely such irregularities in the mathematical model, then the deterministic model should be sufficient. But this is never done in practice, the irregularities being too detailed for a mathematical description. However, some attempts in this direction can be found in [15], [35]. It is here that it is meaningful to introduce noise: as a phenomenological replacement of a realistic element which is discarded by the deterministic part of the mathematical model. In the case of irregularities of a boundary this is important, since the consequences in the fluid motion of such irregularities are relevant, visible, macroscopic.

The precise physical description of the generation of vortices is a difficult topic in itself. Here we take a phenomenological viewpoint: emergence of vortices near obstacles is commonly observed and we content ourselves with an ad hoc inclusion of this fact into the equations.

Assume that the velocity field at time t is $u\,(t,x)$. Assume that, as a consequence of an obstacle in the domain (Figure 1.1[1]) or at the boundary (Figure 1.2[2] and Figure 1.3[3]), a modification occurs and in a very short time we have a field $u\,(t+\Delta t,x)$ which is not just equal to the smooth evolution of $u\,(t,x)$. We may idealize and think that at time t we had a jump:

$$u\left(t^{+},x\right)=u\left(t^{-},x\right)+\sigma\,(x),$$

where $\sigma\,(x)$ is presumably localized in space and corresponds to a vortex structure. Continuum mechanics does not make jumps; we idealize a fast change due to an instability as a jump, for

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[1] https://visibleearth.nasa.gov/images/117121/cloud-vortices-off-madeira-and-canary-islands

[2] https://eol.jsc.nasa.gov/SearchPhotos/photo.pl?mission=ISS030&roll=E&frame=162344

[3] https://visibleearth.nasa.gov/images/148350/lake-erie-astir/148350f

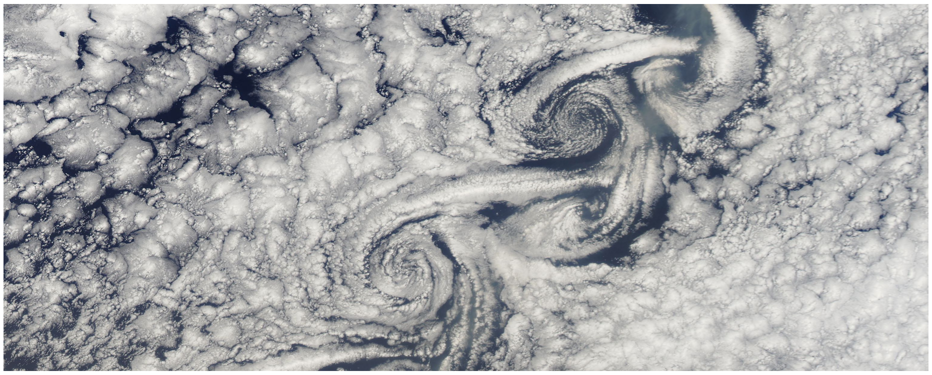

Figure 1.1: Cloud vortices off Madeira and Canary Islands. Images by the MODIS Rapid Response team, NASA.

Figure 1.2: Ice floes, Kamchatka Coast, Russia. Image courtesy of the Earth Science and Remote Sensing Unit, NASA Johnson Space Center, eol.jsc.nasa.gov. NASA photo ID: ISS030-E-162344.

Figure 1.3: Lake Erie Astir. NASA Earth Observatory images by Joshua Stevens.

a cleaner mathematical description.

We emphasize that vortices produced by irregularities (as well as by instabilities) appear as discrete events. Figure 1.1, Figure 1.2 and Figure 1.3 have the merit of showing very isolated and clearly visible vortices. In general, the complexity of a rough boundary profile produces a more disordered pattern of vortices, as schematically represented in Figure 1.4. A wonderful example in nature is shown in Figure 1.5:[4] thanks to the different colouring due to phytoplankton, we may appreciate the complexity of vortical structures close to a rough boundary.

Assume that, due to several obstacles in the boundary at certain locations x_k, $k \in \mathcal{K}$, we may observe jumps of the form

$$u\left(t^{+}, x\right) = u\left(t^{-}, x\right) + \sigma_k\left(x\right), \tag{1.4}$$

where $\sigma_k\left(x\right)$ is a perturbation around x_k. We assume that $\mathcal{K}$ is finite, but one can generalize, see [70].

The way to incorporate these jumps into the Navier–Stokes equations (1.2) is by means of an impulsive force:

$$\partial_t u + u \cdot \nabla u + \nabla P = \nu \Delta u + \sum_{k \in \mathcal{K}} \sum_{i} \delta\left(t - t_i^k\right) \sigma_k.$$

Here, for each $k \in \mathcal{K}$, we denote by $t_1^k < t_2^k < \ldots$ the sequence of jump times of class k. This way the fluid moves according to the free Navier–Stokes equations between two consecutive jumps times (reorder the full family $\left\{t_i^k; k \in \mathcal{K}, i \in \mathbb{N}\right\}$ and consider two consecutive elements) and

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[4] https://visibleearth.nasa.gov/images/65000/phytoplankton-bloom-off-argentina

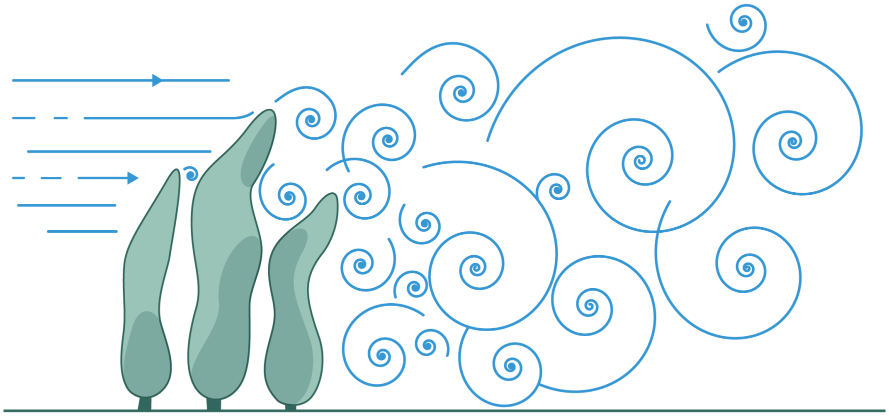

Figure 1.4: Schematic representation of several vortices produced by a complex family of boundary obstacles. Picture by Claudia Flandoli.

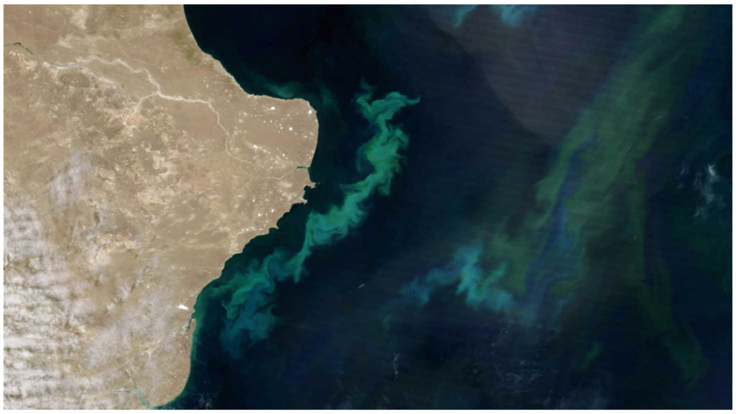

Figure 1.5: Phytoplankton bloom off Argentina. Jacques Descloitres, MODIS Rapid Response Team, NASA/GSFC.

fulfils (1.4) at the jump times, with the correct $k \in \mathcal{K}$. The previous one enters the framework of fluid mechanics SPDEs with kick force, see for instance [29], [45], [124] [123].

We may assume that the jump times are random or deterministic (for the latter case, think of Karman vortices past an obstacle, as in one of the pictures above). For some purposes it is the same, for others it is mathematically more convenient to assume them random, thus we do so. We assume that $t_{i+1}^k - t_i^k$ has exponential distribution with mean time τ^k, $\mathbb{P}\left(t_{i+1}^k - t_i^k > s\right) = e^{-s/\tau^k}$, and that all these random time intervals are independent. We may equivalently describe this by means of a family $\left\{\left(N_t^k\right)_{t\geq 0}; k \in \mathcal{K}\right\}$ of independent standard (rate 1) Poisson processes, rescale their times as N_{t/τ^k}^k and define $t_1^k < t_2^k < ...$ as the random times when the Poisson process N_{t/τ^k}^k jumps (at time t_1^k it jumps from 0 to 1, at time t_2^k from 1 to 2 and so on). We have

$$\sum_{k\in\mathcal{K}}\sum_i \delta\left(t - t_i^k\right)\sigma_k = \sum_{k\in\mathcal{K}}\sigma_k \frac{dN_{t/\tau^k}^k}{dt},$$

where the time derivative of the jump process N_{t/τ^k}^k is understood in the sense of distributions.

It is then clear that we introduce the function

$$W(t,x) = \sum_{k\in\mathcal{K}}\sigma_k(x)\, N_{t/\tau^k}^k = \sum_{k\in\mathcal{K}}\sum_{i\in\mathbb{N}: t_i^k\leq t}\sigma_k(x)$$

and write the equation in the form

$$\partial_t u + u\cdot\nabla u + \nabla P = \nu\Delta u + \partial_t W. \tag{1.5}$$

We refer to [124] for a review on some issues related to equation (1.5), including uniqueness of invariant measures and their inviscid limit in the 2D framework without boundaries.

In many examples, however, the vortices appear in opposite pairs $\pm\sigma(x)$ as in the wake after an obstacle of Figure 1.1 above. At a boundary, usually the primary vortices always have the same sign but secondary vortices are often in pairs. With a large degree of idealization (this issue certainly requires more investigation) let us assume that each vortex σ_k appears in pairs by means of two independent Poisson processes $N_{t/\tau^k}^{k,1}, N_{t/\tau^k}^{k,2}$ with the same rate:

$$\frac{1}{\sqrt{2}}\left(\sigma_k(x)\frac{dN_{t/\tau^k}^{k,1}}{dt} - \sigma_k(x)\frac{dN_{t/\tau^k}^{k,2}}{dt}\right).$$

The factor $\frac{1}{\sqrt{2}}$ is just to normalize and maintain the notation τ^k for the mean time between consecutive generations, now understanding the generations of $\pm\sigma_k$ as a single process. The full process $W(t,x)$ thus has the form

$$W(t,x) = \sum_{k\in\mathcal{K}}\frac{1}{\sqrt{2}}\sigma_k(x)\left(N_{t/\tau^k}^{k,1} - N_{t/\tau^k}^{k,2}\right). \tag{1.6}$$

Let us parametrize by n the jump times and the vortex intensities, as:

$$W_n(t,x) = \sum_{k\in\mathcal{K}} \frac{1}{n}\sigma_k(x)\frac{N^{k,1}_{n^2t/\tau^k} - N^{k,2}_{n^2t/\tau^k}}{\sqrt{2}}.$$

The heuristics is that we make many more jumps but of smaller size. The precise rescaling has been chosen in order to have a non-zero finite limit. Indeed, the average of $W_n(t,x)$ is zero and the variance is equal to

$$\mathbb{E}\left[|W_n(t,x)|^2\right] = t\sum_{k\in\mathcal{K}} \frac{|\sigma_k(x)|^2}{\tau^k},$$

which is finite and non–zero in the limit when $n\to\infty$. Let us check the previous result: since $\mathbb{E}\left[N^{k,j}_{n^2t/\tau^k}\right] = \frac{n^2t}{\tau^k}$, $Var\left[N^{k,j}_{n^2t/\tau^k}\right] = \frac{n^2t}{\tau^k}$, and $N^{k,1}_{n^2t/\tau^k}, N^{k,2}_{n^2t/\tau^k}$ are independent,

$$\begin{aligned}
&\mathbb{E}\left[\left|\frac{1}{n}\sigma_k(x)\frac{N^{k,1}_{n^2t/\tau^k} - N^{k,2}_{n^2t/\tau^k}}{\sqrt{2}}\right|^2\right]\\
&= \frac{1}{2n^2}|\sigma_k(x)|^2\mathbb{E}\left[\left|N^{k,1}_{n^2t/\tau^k} - \frac{n^2t}{\tau^k} - N^{k,2}_{n^2t/\tau^k} + \frac{n^2t}{\tau^k}\right|^2\right]\\
&= \frac{1}{2n^2}|\sigma_k(x)|^2 2Var\left[N^{k,j}_{n^2t/\tau^k}\right] = t\frac{|\sigma_k(x)|^2}{\tau^k}
\end{aligned}$$

and then a similar argument applies to the sum in k.

The Donsker invariance principle (see [27]) claims that, as $n\to\infty$,

$$\frac{1}{n}\left(N_{n^2t} - n^2t\right) \to W_t \text{ (Brownian motion)}$$

the convergence being in law and uniform on compact sets. A multidimensional version of the Donsker theorem similarly gives us that the stochastic process $W_n(t,x)$ converges in law to

$$W(t,x) := \sum_{k\in\mathcal{K}} \frac{1}{\sqrt{\tau^k}}\sigma_k(x)W^k_t,$$

where $\left(W^k_t\right)_{t\geq 0}$ are independent Brownian motions. The corresponding Navier–Stokes equations, in the usual language of stochastic differential equations, have the form

$$du + (u\cdot\nabla u + \nabla P)\,dt = \nu\Delta u dt + \sum_{k\in\mathcal{K}} \frac{1}{\sqrt{\tau^k}}\sigma_k dW^k_t. \tag{1.7}$$

The analysis described in this section motivates the interest in studying Navier–Stokes equations with Gaussian additive noise, in particular in domains with boundaries. While classical issues on (1.7) like global well-posedness, existence and uniqueness of invariant measures are nowadays well established results under quite general assumptions on the coefficients σ_k, see for example [68], [78], [99], the last part of this thesis, namely chapter 4 and chapter 5, would be devoted to treating some issues related to the so-called zero-noise zero-viscosity limit in domains with boundaries and no-slip boundary conditions. We will come back in section 1.5 to the content of chapter 4 and chapter 5.

1.2 Notation

To facilitate comprehension of the upcoming content, we have collected here frequently used notation in the subsequent sections and chapters. We suggest initially skipping this section during the first read-through and returning to it only if certain notations remain unclear. This section is divided into several parts, allowing readers to quickly locate the necessary notation.

Since chapter 2 relies on a different geometry than chapter 3, chapter 4, chapter 5, sometimes we need to specialize the notation of this section to the different geometries. In order to stress this thing, let $\mathcal{O}_1 = \mathbb{T} \times (0, a)$, where a is a positive constant and we denote by

$$\Gamma_b = \mathbb{T} \times \{0\} \quad \text{and} \quad \Gamma_u = \mathbb{T} \times \{a\},$$

while $\mathcal{O}_2 \subseteq \mathbb{R}^2$ is a bounded, smooth, convex. If we need to specialize our notation to one of the particular geometries we will use $\mathcal{O}_1$ or $\mathcal{O}_2$ in place of $\mathcal{O}$.

General Notation

Consider a complete filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$, a separable Hilbert space $\mathcal{U}$ and a cylindrical $\mathcal{F}-$Brownian motion $(W_{\mathcal{U}}(t))_{t\geq 0}$ on $\mathcal{U}$. If $\mathcal{Q}$ is a trace-class operator on $\mathcal{U}$, we can endow the space $\mathcal{H}_0 := \mathcal{Q}^{1/2}\mathcal{U}$ with the metric induced by $\mathcal{Q}$, that is

$$\langle g, h\rangle_{\mathcal{H}_0} = \langle \mathcal{Q}^{-1/2} g, \mathcal{Q}^{-1/2} h\rangle_{\mathcal{U}}$$

which makes $\mathcal{H}_0$ a Hilbert space. The norm induced by this inner product will be denoted $\|\cdot\|_{\mathcal{H}_0}$. When $\mathcal{Q}$ is the covariance operator of a Wiener process $\{W(t)\}_{t\in[0,T]}$ on $\mathcal{U}$, we call $\mathcal{H}_0$ the reproducing kernel Hilbert space of $W(t)$, or simply RKHS. We also define the space

$$S^N := S^N(\mathcal{H}_0) := \Big\{ v \in L^2(0, T; \mathcal{H}_0) : \quad \int_0^T \|v(s)\|^2_{\mathcal{H}_0} ds \leq N \Big\},$$

which makes a Polish space when endowed with the weak topology. We denote by $\mathcal{P}_2 := \mathcal{P}_2(\mathcal{H}_0)$ the space of $\mathcal{H}_0$-valued, $\mathcal{F}_t$-predictable and $\mathbb{P}$-a.s. square integrable processes. Next we define

$$\mathcal{P}_2^N := \{\varphi \in \mathcal{P}_2 : \ \varphi(\omega) \in S^N \quad \mathbb{P} - a.s.\}$$

For square integrable semimartingales taking value in separable Hilbert spaces $\mathcal{U}_1$, $\mathcal{U}_2$ we denote by $[M,N]_t$ the quadratic covariation process. If M,N take values in the same separable Hilbert space $\mathcal{U}$ with orthonormal basis u_i, we denote by

$$\langle\langle M,N\rangle\rangle_t = \sum_{i\in\mathbb{N}}[\langle M,u_i\rangle_{\mathcal{U}}, \langle N,u_i\rangle_{\mathcal{U}}]_t.$$

We will consider in the following $\{W_t^k\}_{k\in\mathbb{N}}$, a family of mutually independent, real valued Brownian motions adapted to $\mathcal{F}_t$, and $\{B_t^k\}_{k\in\mathbb{N}}$, a family of mutually independent, $\mathbb{R}^2$-valued Brownian motions adapted to $\mathcal{F}_t$. We denote by $L^p(\mathcal{F}_{t_0},\mathcal{U})$ the space of p integrable random variables with values in $\mathcal{U}$, measurable with respect to $\mathcal{F}_{t_0}$. We denote by $L^p(0,T;\mathcal{U})$ the space of measurable functions from $[0,T]$ to $\mathcal{U}$ such that

$$\|u\|_{L^p(0,T;\mathcal{U})} := \left(\int_0^T \|u_t\|_{\mathcal{U}}^p \, dt\right)^{1/p} < +\infty, \quad 1\le p<\infty$$

and obvious generalization for $p=\infty$. For any $r,\ p\ge 1$, we denote by

$$L^p(\Omega,\mathcal{F},\mathbb{P};L^r(0,T;\mathcal{U}))$$

the space of processes with values in $\mathcal{U}$ such that

1. $u(\cdot,t)$ is progressively measurable.
2. $u(\omega,t)\in\mathcal{U}$ for almost all (ω,t) and

$$\mathbb{E}\left[\|u(\omega,\cdot)\|_{L^r(0,T;\mathcal{U})}^p\right] < +\infty.$$

Obvious generalizations for $p=\infty$ or $r=\infty$. In case $p=q=2$ we simply write $L^2_{\mathcal{F}}(0,T;\mathcal{U})$ in place of $L^2(\Omega,\mathcal{F},\mathbb{P};L^2(0,T;\mathcal{U}))$. Lastly we denote by $C_{\mathcal{F}}([0,T];\mathcal{U})$ the space of continuous adapted processes with values in $\mathcal{U}$ such that

$$\mathbb{E}\left[\sup_{t\in[0,T]}\|u_t\|_{\mathcal{U}}^2\right] < +\infty.$$

By C we will denote several constants, perhaps changing value line by line. If we want to keep track of the dependence of C from some parameter ξ we will use the symbol $C(\xi)$. Sometimes we will use the notation $a\lesssim b$ (resp. $a\lesssim_\xi b$), if it exists a constant such that $a\le Cb$ (resp. $a\le C(\xi)b$). If $p\in[1,\infty]$ we denote by p' its conjugate exponent, $1/p+1/p'=1$.

In what follows, dx denotes the Lebesgue measure restricted to the domain $\mathcal{O}$, and $d\sigma(x)$ the (1-dimensional) volume measure on the smooth boundary $\partial\mathcal{O}$. We denote by $\mathcal{B}(\mathcal{T})$, $\mathcal{B}_b(\mathcal{T})$ respectively the spaces of real-valued Borel and bounded Borel functions on a topological space $\mathcal{T}$. When considering the norm (resp. the inner product) in L^2 spaces we omit the subscript of the norm (resp. the inner product). Now on $q\in(1,\infty)$. For an integer $k\ge 1$, $W^{k,q}(\mathcal{O})$ denotes the usual Sobolev spaces. In the non-positive and non-integer case $s \in (-\infty,\infty)\setminus\mathbb{N}$, we let $W^{s,q}(\mathcal{O}) := B^s_{q,q}(\mathcal{O})$ where $B^s_{q,q}(\mathcal{O})$ is the Besov space with smoothness s, and integrability q and microscopic integrability q (in particular $W^{0,q}(\mathcal{O}) \neq L^q(\mathcal{O})$). Moreover, $H^{s,q}(\mathcal{O})$ denotes the Bessel potential spaces. Both Besov and Bessel potential spaces can be defined by means of Littlewood-Paley decompositions and restrictions (see e.g. [157], [155, Section 6]) or using the interpolation methods starting with the standard Sobolev spaces $W^{k,q}(\mathcal{O})$ (see e.g. [20, Chapter 6]). Finally, we set $\Lambda^{s,q}(\mathcal{O};\mathbb{R}^d) := (\Lambda^{s,q}(\mathcal{O}))^d$ for an integer $d \geq 1$, a domain $\mathcal{O}$ and $\Lambda \in \{W, H\}$.

Let $\mathcal{K}$ and Y be a Hilbert and a Banach space, respectively. We denote by $\gamma(\mathcal{K}, Y)$ the set of γ-radonifying operators, see e.g. [102, Chapter 9] for basic definitions and properties. In particular, we need the following Fubini-type result:

$$\Lambda^{s,q}(\mathcal{O};\mathcal{K}) = \gamma(\mathcal{K}, \Lambda^{s,q}(\mathcal{O})) \quad \text{for all } s \in \mathbb{R},\ q \in (1,\infty),\ \Lambda \in \{W, H\}.$$

The above follows from [102, Theorem 9.3.6] and interpolation.

Stokes Semigroups

Let $f \in L^q(\mathcal{O};\mathbb{R}^2)$ and let $\psi_f \in W^{1,q}(\mathcal{O})$ be the unique solution to the following elliptic problem

$$\begin{cases} \Delta\psi_f = \operatorname{div} f & \text{on } \mathcal{O}, \\ \nabla\psi_f \cdot \hat{n} = f \cdot \hat{n} & \text{on } \partial\mathcal{O}. \end{cases} \tag{1.8}$$

Of course, the above elliptic problem is interpret in its natural weak formulation:

$$\int_{\mathcal{O}} \nabla\psi_f \cdot \nabla\varphi \, dx = \int_{\mathcal{O}} f \cdot \nabla\varphi \, dx \quad \text{for all } \varphi \in C^\infty(\mathcal{O}). \tag{1.9}$$

By [150, Corollary 7.4.4], we have $\psi_f \in W^{1,q}(\mathcal{O})$ and $\|\nabla\psi_f\|_{L^q(\mathcal{O};\mathbb{R}^2)} \lesssim \|f\|_{L^q(\mathcal{O};\mathbb{R}^2)}$. Then the Helmholtz projection is given by $\mathbf{P}_q$ and is defined as

$$\mathbf{P}_q f = f - \nabla\psi_f, \quad f \in L^q(\mathcal{O};\mathbb{R}^2).$$

Let $\mathbb{L}^q := \mathbf{P}_q(L^q(\mathcal{O};\mathbb{R}^2))$. In particular, it holds

$$\mathbf{P}_q(L^q(\mathcal{O}_1;\mathbb{R}^2)) = \{f \in L^q(\mathcal{O}_2;\mathbb{R}^2) : \operatorname{div} f = 0, \quad f_2|_{\Gamma_b\cup\Gamma_u} = 0\},$$
$$\mathbf{P}_q(L^q(\mathcal{O}_2;\mathbb{R}^2)) = \{f \in L^q(\mathcal{O}_2;\mathbb{R}^2) : \operatorname{div} f = 0, \quad \hat{n}\cdot f|_{\partial\mathcal{O}_2} = 0\}.$$

Next we define the Stokes operator on $L^q(\mathcal{O};\mathbb{R}^2)$. We define the operator $A_q : \mathsf{D}(A_q) \subseteq \mathbb{L}^q \to \mathbb{L}^q$ (resp. $\mathcal{A}_q : \mathsf{D}(\mathcal{A}_q) \subseteq \mathbb{L}^q \to \mathbb{L}^q$) where

$$\begin{aligned} \mathsf{D}(A_q) = \big\{f = (f_1, f_2) \in W^{2,q}(\mathcal{O}_1;\mathbb{R}^2) \cap \mathbb{L}^q \ :\ & f|_{\Gamma_b} = 0, \\ & f_2|_{\Gamma_u} = \partial_2 f_1|_{\Gamma_u} = 0\big\} \end{aligned}$$
$$(\text{resp. } \mathsf{D}(\mathcal{A}_q) = \big\{f = (f_1, f_2) \in W^{2,q}(\mathcal{O}_2;\mathbb{R}^2) \cap \mathbb{L}^q \ :\ f|_{\partial\mathcal{O}_2} = 0\big\}),$$

and $A_q u = -\mathbf{P}_q \Delta u$ for $u \in \mathsf{D}(A_q)$ (resp. $\mathcal{A}_q u = -\mathbf{P}_q \Delta u$ for $u \in \mathsf{D}(\mathcal{A}_q)$). We show in chapter 2 (resp. it is well known, see for example [150]) that $-A_q$ (resp. $-\mathcal{A}_q$) generates an analytic semigroup on $\mathbb{L}^q$ which we denote by $S_q(t)$ (resp. $\mathcal{S}_q(t)$). For $s \in (0,1]$ we write

$$\mathbb{H}^{2s,q} = \mathsf{D}(A_q^s).$$

and simply $\mathbf{P}$, A, $S(t)$ and $\mathbb{H}^{2s}$ in case of $q = 2$. The dual spaces of the $\mathbb{H}^{2s}$, $s \in [0,1]$, will be denoted by negative indexes, i.e. $\mathbb{H}^{-2s} = \left(\mathbb{H}^{2s}\right)^*$, $s \in [0,1]$.

In the second case we are interested only to the case $q = 2$ and we write

$$H = \mathbb{L}^2, \quad V = \mathsf{D}(\mathcal{A}_2^{1/2})$$

and simply $\mathcal{A}$, $\mathcal{S}(t)$ in case of $q = 2$. It is well known that $V = H \cap H_0^1(\mathcal{O}_2; \mathbb{R}^2)$.
We introduce the furthers spaces

$$\mathcal{E}_0^{NS} = H \cap H^3(\mathcal{O}_2; \mathbb{R}^2), \quad \mathcal{E}_0^{SG} = V \cap H^3(\mathcal{O}_2; \mathbb{R}^2), \quad \mathcal{D}(\mathcal{O}) := \{f \in C_c^\infty(\mathcal{O}; \mathbb{R}^2) \, : \, \operatorname{div} f = 0\}.$$

Every time X is a reflexive Banach space such that the embedding $X \hookrightarrow H$ is continuous and dense, denoting by X^* the dual of X, the scalar product $\langle \cdot, \cdot \rangle$ in H extends to the dual pairing between X and X^*. We will simplify the notation accordingly.

Heat Semigroup with Neumann Boundary Conditions

We define the operator $\mathcal{B}_q : \mathsf{D}(\mathcal{B}_q) \subseteq L^q \to L^q$ where

$$\mathsf{D}(\mathcal{B}_q) = \left\{ f \in W^{2,q}(\mathcal{O}_2), : \; \nabla f \cdot \hat{n}|_{\mathcal{O}_2} = 0 \right\},$$

and $\mathcal{B}_q u = -\Delta u$ for $u \in \mathsf{D}(\mathcal{B}_q)$. It is well known, see for example [145, Section 7.3], [98] that $-\mathcal{B}_q$ generates an analytic semigroup on L^q which we denote by $P_q(t)$. For $s \in (0,1]$ we write

$$\mathcal{H}^{2s,q} = \mathsf{D}((I + \mathcal{B}_q)^s).$$

and simply $\mathcal{B}$, $\mathcal{H}^{2s}$ in case of $q = 2$. With some abuse of notation, we always forget the subscript in the heat semigroup with Neumann boundary conditions, write $P(t)$ in place of $P_q(t)$ and denote by $p_t(x,y)$ the heat kernel, such that the action of $P(t)$ on $f \in L^1$ is given by

$$P(t)f(x) = \int_{\mathcal{O}_2} p_t(x,y) f(y) \, dy, \quad t > 0.$$

Boundary Value Problems

Given $g \in W^{s,q}(\mathbb{T})$ we denote by $\mathcal{N}[g]$ the unique divergence free vector field solving, in a weak sense, the non-homogeneous boundary value problem

$$\begin{cases} -\Delta u + \nabla \pi = 0, & \text{on } \mathcal{O}_1, \\ \operatorname{div} u = 0, & \text{on } \mathcal{O}_1, \\ u(\cdot, 0) = 0, & \text{on } \Gamma_b, \\ \partial_2 u_1(\cdot, a) = g, & \text{on } \Gamma_u, \\ u_2 = 0, & \text{on } \Gamma_u. \end{cases} \tag{1.10}$$

We refer to chapter 2 for the properties of the map $\mathcal{N}$.

Given $g \in W^{s,q}(\partial\mathcal{O}_2)$ we denote by $\mathsf{N}[g]$ (resp. $\mathsf{D}[g]$) the unique scalar solving, in a weak sense, the non-homogeneous boundary value problem

$$\begin{cases} (I-\Delta)u+ = 0, & \text{on } \mathcal{O}_2, \\ \hat{n} \cdot \nabla u(\cdot) = g, & \text{on } \partial\mathcal{O}_2. \end{cases} \quad \left(\text{resp.} \begin{cases} \Delta u = 0, & \text{on } \mathcal{O}_2, \\ u(\cdot) = g, & \text{on } \partial\mathcal{O}_2. \end{cases} \right) \tag{1.11}$$

We refer to [170, Chapter 4] for the properties of the map $\mathsf{N}[g]$ (resp. $\mathsf{D}[g]$).

1.3 Boundaries as a Forcing Term at the Velocity Level: Stochastic Wind Driven Boundary Conditions

The first topic we discuss in this thesis is related to forcing terms that act on the behavior of a fluid through the boundary at the level of the velocity equation. According to previous discussion, an external shear stress acting on the surface, or a term related to the velocity between the two plates of $\mathcal{O}$ can be added to (1.3). In particular, given a positive constant a we focus on the case $\mathcal{O} = \mathbb{T} \times (0, a)$, where Γ_b and Γ_u are the bottom and the upper part of the boundary of $\mathcal{O}$ introduced in section 1.2, respectively and consider the following problem

$$\begin{cases} \partial_t u + u \cdot \nabla u + \nabla P = \Delta u, & \text{on } (0,T) \times \mathcal{O}, \\ \operatorname{div} u = 0, & \text{on } (0,T) \times \mathcal{O}, \\ u = 0, & \text{on } (0,T) \times \Gamma_b, \\ \partial_2 u_1 = g_{st}, & \text{on } (0,T) \times \Gamma_u, \\ u_2 = 0, & \text{on } (0,T) \times \Gamma_u, \\ u(0) = u_0, & \text{on } \mathcal{O}. \end{cases} \tag{1.12}$$

Before stating some assumption on g_{st}, let us discuss the meaning of problem (1.12). Following the books by Pedlosky [148, 149] and Gill [91], (1.12) is a good idealization of the velocity of the ocean. Indeed even if, in principle, one should consider a free surface, instead of

Γ_u, depending of the time, the approximation of such surface as independent of the time, although highly unrealistic, is justified by the fact that the behavior of the fluid around the surface is in general very turbulent. Hence, as emphasized in [62], only a modelization is tractable and meaningful. The model considered in (1.12) corresponds to a strip of the ocean. Therefore, according to the discussion below (1.3), we consider $\lambda = 0$ (i.e. no-slip boundary conditions) on the bottom of the ocean, Γ_b, instead we are interested to $\lambda = +\infty$ and some inhomogeneous boundary conditions on the contact between the ocean and the atmosphere, Γ_u. The inhomogeneous boundary condition appearing in (1.12) is therefore interpreted as the physical law describing the driving mechanism on the atmosphere-ocean interface as a balance of the shear stress of the ocean and the horizontal wind force, see [129] for details. We take the inhomogeneous boundary conditions of the form

$$g_{st} = h_b \dot{W}_{\mathcal{U}}. \tag{1.13}$$

The reason to consider a forcing term white in time and coloured in space is twofold. From the physical view point, since our inhomogeneous boundary condition describes the action of the wind which varies on time scales much smaller than the one present in the oceanic dynamics is therefore reasonable, from the prospective of the ocean, that wind acts as a noise. From the mathematical view point, partial differential equations with white noise boundary conditions are an intriguing problem. Indeed, from the seminal paper [53], it is well-known that also in the one dimensional case, the solutions of the heat equation with white noise Dirichlet or Neumann Boundary conditions have low regularity compared to the case of noise diffused inside the domain. In particular, in the case of Dirichlet boundary conditions the solution is only a distribution. Some improvements in the analysis of the interior regularity of the solutions of these problems and some nonlinear variants have been obtained exploiting specific properties of the heat kernel and of suitable nonlinearities. For some results in this direction we refer to [6, 59, 64, 93]. All these issues make the problem of treating non-linear partial differential equations with boundary noise coming from fluid dynamical models an, almost untouched, field of open problems. In [3] we studied the global well-posedness and the interior regularity of (1.12) subjected to the boundary conditions (1.13) under quite general regularity assumptions on h_b. Following the idea of [52] we split the analysis of (1.12) in two parts. First we consider the stochastic linear problem with non-homogeneous boundary conditions

$$\begin{cases} \partial_t w + \nabla \rho = \Delta w, & \text{on } (0,T) \times \mathcal{O}, \\ \operatorname{div} w = 0, & \text{on } (0,T) \times \mathcal{O}, \\ w = 0, & \text{on } (0,T) \times \Gamma_b, \\ \partial_2 w_1 = h_b \dot{W}_{\mathcal{U}}, & \text{on } (0,T) \times \Gamma_u, \\ w_2 = 0, & \text{on } (0,T) \times \Gamma_u, \\ w(0) = 0, & \text{on } \mathcal{O}, \end{cases} \tag{1.14}$$

The solution to the above linear equation (1.14) can be treated in mild form as in [53, 54], namely writing formally

$$w(t) = A_q \int_0^t S_q(t-s)\mathcal{N}[h_b(s)]\, dW_{\mathcal{U}}(s). \tag{1.15}$$

and giving a meaning to (1.15) in regular enough space of functions. Secondly, denoting by $v = u - w$, we consider the Navier-Stokes equations with random coefficients

$$\begin{cases} \partial_t v + (v+w)\cdot\nabla(v+w) + \nabla(P-\rho) = \Delta v, & \text{on } (0,T)\times\mathcal{O}, \\ \operatorname{div} v = 0, & \text{on } (0,T)\times\mathcal{O}, \\ v = 0, & \text{on } (0,T)\times\Gamma_b, \\ \partial_2 v_1 = 0, & \text{on } (0,T)\times\Gamma_u, \\ v_2 = 0, & \text{on } (0,T)\times\Gamma_u, \\ v(0) = u_0, & \text{on } \mathcal{O}. \end{cases} \tag{1.16}$$

As discussed in [54, Chapter 13], if h_b, u_0, $W_{\mathcal{U}}(t)$ would be regular enough, then $u = v + w$ will be a classical solution of the Navier-Stokes equations with inhomogeneous boundary conditions (1.12). In our rougher framework, we say that a process u with paths $\mathbb{P} - a.s.$ in $C([0,T];\mathbb{L}^2)\cap L^4(0,T;\mathbb{L}^4)$ and progressively measurable with respect to these topologies, is a pathwise weak solution of (1.12) if $u = v+w$, where w has paths in $C([0,T];\mathbb{L}^2)\cap L^4(0,T;\mathbb{L}^4)$, it is progressively measurable with respect to these topologies and is a mild solution of (1.14) while v has paths in $C([0,T];\mathbb{L}^2)\cap L^2(0,T;\mathbb{H}^1)$, it is progressively measurable with respect to these topologies and is a Leray solution of (1.16). The first main result of [3] is the following:

Theorem 1.1. *Let $q > 2,\ p > 2q,\ \alpha \in [0, \frac{1}{q} - \frac{2}{p})$ be such that:*

$$-\frac{1}{q} - \alpha + \frac{1}{2} > 0.$$

Assuming that $h_b : (0,T) \in L^p(0,T;W^{-\alpha,q}(\Gamma_u;\mathcal{U}))$ and $u_0 \in \mathbb{L}^2$, there exists a unique weak solution u to (1.12).

The proof of Theorem 1.1 relies strongly on the fact that $w \in L^4(0,T;\mathbb{L}^4)$. This is the minimal requirement such that $\mathbf{P}(w\cdot\nabla w) \in L^2(0,T;\mathbb{H}^{-1})$ and follows by the analysis of the mild formula (1.15). However, we note that the case $q = 2$ is not allowed in Theorem 1.1. Indeed in such a case, it seems not possible to prove by stochastical maximal regularity techniques that w has paths in $L^4(0,T;\mathbb{L}^4)$. We, therefore, relies on the stochastical maximal L^p regularity techniques introduced in [173]. In order to prove the needed regularity of (1.15) we need some technical results on the semigroup associated to (1.14) in a non Hilbertian framework. Indeed, we show that for all $q \in (1,\infty)$, the operator A_q is invertible and has a bounded H^∞-calculus of angle $< \frac{\pi}{2}$. This implies that $-A_q$ generates an analytic semigroup on $\mathbb{L}^q$. This results allows us to apply the maximal L^p regularity techniques introduced of [173]. Moreover, we can characterize the domains of the fractional powers of the Stokes operator, $\mathsf{D}(A_q^\alpha)$, $\alpha \in [0,1]$, $\alpha \neq \frac{1}{2q}$, $\frac{1}{2} + \frac{1}{2q}$. Therefore we can show that, for sufficiently regular inputs, the Neumann map $\mathcal{N}$ takes value in $\mathsf{D}(A_q^\alpha)$ for $\alpha < \frac{1}{2} + \frac{1}{2q}$. Having already proved the regularity of the stochastic convolution w, the existence and uniqueness of Leray solutions of (1.16) follows by standard arguments.

A second mathematical issue, classical when studying Navier-Stokes equations or boundary noise problems, is showing that the solution, even if may presents some singularities close to the boundary, is smooth in the interior of $\mathcal{O}$. This is exactly the case. Indeed, the second main result of [3] reads as

Theorem 1.2. *Under the same assumptions of Theorem 1.1, if u is the unique weak solution of (1.12) provided by Theorem 1.1 then for all $t_0 \in (0,T)$ and $\mathcal{O}_0 \subset \mathcal{O}$ such that* $\operatorname{dist}(\mathcal{O}_0, \partial\mathcal{O}) > 0$,

$$u \in C([t_0,T]; C^\infty(\mathcal{O}_0;\mathbb{R}^2)) \quad \mathbb{P}-a.s.$$

According to [161] (see also [127, Section 13.1]), it seems not possible to gain high-order interior time-regularity for the Navier-Stokes problem. This fact is in contrast to the case of the heat equation with white noise boundary conditions, see [34]. The reason behind this is the presence of the unknown pressure P which, due to its non-local nature, provides a connection between the interior and the boundary regularity. Similarly to the proof of Theorem 1.1, we analyze the interior regularity of u combining the interior regularity of w and the interior regularity of v. The interior regularity of w is obtained introducing a proper weak formulation which we prove being equivalent to the mild formulation (1.15). The weak formulation satisfied by w is more convenient in order to study its interior regularity since, considering suitable compactly supported test functions, we reduce to study the interior regularity of a distributional solution of the heat equation. Instead, the regularity of v is analyzed via a Serrin's argument exploiting the aforementioned regularity of w.

Trying to understand the well-posedness of 2D Navier-Stokes equation with Dirichlet type boundary noise, corresponding to the case $\lambda = 0$ and $g_m = h_m\dot{W}_{\mathcal{U}}$ in (1.3) is an open problem. We end this section mentioning [4], where this problem is considered replacing the cylindrical Brownian motion by a cylindrical fractional Brownian motion, $W_{\mathcal{U}}^h$, with Hurst index $h \in (\frac{3}{4}, 1)$. Exploiting a more refined splitting compared to the one of [3] described in this section, indeed, we are able to prove the validity of Theorem 1.1, Theorem 1.2 in this more singular framework.

1.4 Boundaries as a Forcing Term at the Vorticity Level: A Lagrangian Description of the Generation of Vorticity

Let us now consider a smooth convex bounded domain $\mathcal{O}$ and $\lambda = 0$ in (1.3), i.e. no-slip boundary conditions. As discussed before, the condition $u|_{\partial\mathcal{O}} = 0$ provokes large stress near the boundary and this lead to instabilities and generate vortices. This can be seen, at least heuristically, considering the equation satisfied by $\omega = \operatorname{curl} u = \partial_1 u_2 - \partial_2 u_1$. Indeed, taking formally the curl of equation (1.2) for $f_{est} = 0$ leads to

$$\partial_t \omega + u \cdot \nabla\omega = \nu\Delta\omega,$$

an equation that can be put in closed form expressing $u = K[\omega]$ in terms of ω by means of Biot-Savart law,

$$K[\omega](x) = \int_{\mathcal{O}} K(x,y)\omega(y)\,dy, \quad K(x,y) = -\nabla_x^{\perp} G(x,y), \quad G = (-\Delta_{Dir})^{-1},$$

where the choice of Dirichlet conditions for the Green function $(-\Delta_{Dir})^{-1}$ makes it so that the vorticity dynamics encodes the impermeability condition $u \cdot \hat{n}|_{\partial\mathcal{O}} = 0$ at the boundary. The tangential part $u \cdot \hat{\tau}|_{\partial\mathcal{O}} = 0$ of the no-slip boundary condition, on the other hand, is harder to express in terms of ω only, giving rise to a non-local condition on vorticity which we derive here heuristically assuming $u_0 \cdot \hat{\tau}|_{\partial\mathcal{O}} = 0$. The no-slip boundary condition implies

$$\begin{aligned}
0 &= \partial_t u \cdot \hat{\tau}|_{\partial\mathcal{O}} \\
&= -\nabla^{\perp}(-\Delta_{Dir})^{-1}\partial_t\omega \cdot \hat{\tau}|_{\partial\mathcal{O}} \\
&= -\partial_{\hat{n}}(-\Delta_{Dir})^{-1}\left(\nu\Delta\omega - u\cdot\nabla\omega\right)|_{\partial\mathcal{O}} \\
&= -\partial_{\hat{n}}(-\Delta_{Dir})^{-1}\left(\nu\Delta(\omega - \mathsf{D}[\omega]) - u\cdot\nabla\omega\right)|_{\partial\mathcal{O}} \\
&= \nu(\partial_{\hat{n}}\omega - \partial_{\hat{n}}\mathsf{D}[\omega]) + \partial_{\hat{n}}(-\Delta_{Dir})^{-1} u\cdot\nabla\omega|_{\partial\mathcal{O}}.
\end{aligned}$$

In conclusion, we showed that two dimensional Navier-Stokes equation with no-slip boundary conditions reads in vorticity formulation as

$$\begin{cases}
\partial_t\omega + K[\omega]\cdot\nabla\omega = \nu\Delta\omega \quad x \in \mathcal{O} \\
\nabla\omega\cdot\hat{n} = \partial_{\hat{n}}\mathsf{D}[\omega] - \frac{1}{\nu}\partial_{\hat{n}}(-\Delta_{Dir})^{-1}u\cdot\nabla\omega \quad x \in \partial\mathcal{O}, \\
\omega(0) = \omega_0 \quad x \in \mathcal{O}.
\end{cases} \tag{1.17}$$

The heuristic computations above can be made rigorous in some functional framework, in particular showing that there exists a unique solution of (1.17) which correspond to the curl of the unique Leray solution of (1.2) with no-slip boundary conditions, we refer to [128] and the references therein for details. In particular, the right hand side of the boundary conditions in (1.17) can be seen as a forcing term modeling the so-called boundary shear stress.

Physically speaking, the condition $u\cdot\hat{\tau}|_{\partial\mathcal{O}}=0$ is expected to force production of vorticity, especially in the proximity of the boundary, so that the tangential velocity at $\partial\mathcal{O}$ induced by the bulk of the fluid be compensated [18]. This fact becomes an issue in the study of approximation methods for Navier-Stokes equations in vorticity form, which is a natural approach in the 2-dimensional setting, see [139]. The derivation of (1.2) with no-slip boundary conditions by means of an approximation method explicitly describing the effect of the boundary as a singular source of vorticity is nowadays a mostly open problem, we refer to [44], [51] for some attempts and reviews on the topic. In [95] we construct a system of N particles evolving according to the stochastic differential equations with reflecting boundary

$$dx_i = F\left(\sum_{j=1}^{N}\omega_j K_n(x_i,x_j)\right)dt+\sqrt{2\nu}\,dB^i-dk_i, \tag{1.18}$$

and show that its regularized empirical measure converges to the unique Leray solution of

$$\begin{cases}\partial_t\omega+K[\omega]\cdot\nabla\omega=\nu\Delta\omega & x\in\mathcal{O},\\ \nabla\omega\cdot\hat{n}=g & x\in\partial\mathcal{O},\\ \omega(0)=\omega_0 & x\in\mathcal{O}.\end{cases} \tag{1.19}$$

System (1.19) acts as a natural proxy for the (physical) no-slip condition if one seeks to mathematically describe vorticity production in the boundary layer by means of the source term of Neumann boundary condition in (1.17).

Each particle in (1.18) starts its evolution at time $t_i\in[0,T]$ at $x_i(t_i)$, either located in the interior of $\mathcal{O}$ if $t_i=0$ or on $\partial\mathcal{O}$ if $t_i>0$. The system is closely related to the one considered in [105], the main difference being the generation mechanism. Let us briefly and informally describe the various parts of the dynamics, precise definitions are all deferred to chapter 3.

Initialization of each of the SDEs (1.18), happens either in the interior $\mathcal{O}$ at time 0 (so to model the initial datum of the limiting PDE) or at the boundary $\partial\mathcal{O}$ at later times (so to model the source term g of the Neumann boundary condition). Concretely, we will consider a grid spanning $(\{0\}\times\mathcal{O})\cup((0,T]\times\partial\mathcal{O})$ indexed by i, and generate the particle x_i at grid point i with an intensity ω_i. The mesh of the grid (both in time and space) will be $1/n$, thus $n\in\mathbb{N}$, $n\to\infty$ is a convenient parameter to rule the macroscopic limit: for instance the global number of particles will have order $N=N(n)\simeq n^2$. Weights (or intensities) ω_i of particles will correspond to local averages of the initial datum ω_0 or the boundary datum g around the starting point of the trajectory of x_i.

Each particle interacts with the others by a regularized version of the singular kernel K, obtained by applying the Neumann heat semigroup,

$$K_n(x,y)=\int_{\mathcal{O}}K(x,z)p_\varepsilon(z,y)\,dz,\quad \varepsilon=\varepsilon(n),\quad x,y\in\mathcal{O}, \tag{1.20}$$

with $\varepsilon\to 0$ with a slow enough rate in terms of n. Kernel smoothing by means of the heat semigroup $P(t)$, compared for instance to smoothing by convolution with bump functions, has many advantages in our setting: it is well suited for regularizing functions keeping their support in $\mathcal{O}$, $P(t)$ preserves the L^1 norm of non-negative functions, and precise estimates on p_t and its derivatives are available.

The interaction term of (1.18) also includes a cutoff,

$$F:\mathbb{R}^2\to\mathbb{R}^2,\quad F(v)=\frac{v}{|v|}(|v|\wedge M),\quad M>0, \tag{1.21}$$

which is Lipschitz continuous, and its Lipschitz constant is uniformly bounded in $M>0$. In principle, the presence of the cutoff makes it so that the macroscopic behavior of the particle system is described by a different PDE, that is

$$\begin{cases}\partial_t\omega+\operatorname{div}(F(K[\omega])\omega)=\nu\Delta\omega & x\in\mathcal{O},\\ \nabla\omega\cdot\hat{n}=g & x\in\partial\mathcal{O},\\ \omega(0)=\omega_0 & x\in\mathcal{O}\end{cases} \tag{1.22}$$

coinciding with (1.19) only when the velocity field in the latter is bounded. As revealed by suitable *a priori* estimates, this is in fact the case, so the cutoff F is just a technical device that will naturally disappear in the macroscopic limit taking $M>0$ large enough. Regularization of interactions is important to ease uniform convergence of empirical measures and well-posedness of the SDE system.

The Brownian noise acts independently on each particle models viscosity, the boundary terms $-dk_i$ are defined so that $\partial\mathcal{O}$ acts on particles as a reflecting boundary, in the standard setting of [163, 131]. In terms of boundary conditions for the macroscopic limit, this would produce the null Neumann condition $\hat{n}\cdot\nabla\omega=0$, so an additional effect must be included to model the source g.

Our main result concerns the uniform convergence (in space) of kernel-smoothed empirical measures of the particle system (1.18) under consideration,

$$\mathsf{S}^n(t)=\sum_{i\in\mathsf{A}^n_t}\omega_i\delta_{x_i(t)},\quad \omega^n(t,x)=P(\varepsilon)\mathsf{S}^n(t), \tag{1.23}$$

where, $P(\varepsilon)$ stands for the action of Neumann heat semigroup on measures, and A^n_t collects the indices of particles generated up to time t. The generation of new particles at the boundary produces discontinuities in time at the level of $\omega_n(t,x)$, therefore our limit theorem is going to be set in the space of $C(\bar{\mathcal{O}})$-valued *càdlàg* functions, $\mathbb{D}([0,T];C(\bar{\mathcal{O}}))$, endowed with the usual Skorohod topology. Informally, [95, Theorem 3.4] reads as follows:

Theorem 1.3. *If $\varepsilon\gtrsim n^{-1/2}$, the regularized empirical measures $\omega_n(t,x)$ converge as $n\to\infty$, probability in the topology of $\mathbb{D}([0,T];C(\bar{\mathcal{O}}))$, to the unique weak solution of (1.19) with Lipschitz continuous initial datum ω_0 and bounded Neumann boundary datum g.*

As detailed in chapter 3, we actually obtain a stronger convergence in Sobolev norms. The proof or Theorem 1.3 consists in a compactness argument, therefore the technical core of this result resides in uniform estimates on the regularized empirical measures. In order to ease the informal description of the steps needed in order to prove Theorem 1.3, let us assume that $\omega_0, g \geq 0$. Relying on the dynamics on the particles (1.18), by Itô formula it follows that $\omega^n(t,x)$ satisfies

$$\begin{aligned}\omega^n(t,x) &= \omega^n(0,x) + \sum_{i\in \mathsf{A}^n_t\setminus \mathsf{A}^n_0} \omega_i p_\varepsilon(x, x_i(t_i)) + \nu \int_0^t \Delta\omega^n(s,x)ds \\ &\quad + \int_0^t \int_{\mathcal{O}} F\left(\int_{\mathcal{O}} K_n(y,z)\mathsf{S}^n(s,dz)\right)\nabla_y p_\varepsilon(x,y)\mathsf{S}^n(s,dy) \\ &\quad + \sqrt{2\nu}\int_0^t \sum_{i\in\mathsf{A}^n_s} \omega_i \nabla_y p_\varepsilon(x,x_i(s))dB^i_s, \end{aligned} \tag{1.24}$$

which can be interpreted in mild form as

$$\begin{aligned}\omega^n(t) &= P(t)\omega^n(0) + \sum_{i\in \mathsf{A}^n_t\setminus \mathsf{A}^n_0} \omega_i P(t-t_i)p_\varepsilon(\cdot, x_i(t_i)) \\ &\quad + \sqrt{2\nu}\int_0^t \sum_{i\in\mathsf{A}^n_s} \omega_i P(t-s)\nabla_y p_\varepsilon(\cdot,x_i(s))dB^i_s \\ &\quad + \int_0^t P(t-s)\int_{\mathcal{O}} F\left(\int_{\mathcal{O}} K_n(y,z)\mathsf{S}^n(s,dz)\right)\nabla_y p_\varepsilon(\cdot,y)\mathsf{S}^n(s,dy). \end{aligned} \tag{1.25}$$

The strong formulation (1.24) allows us to show that for each stopping time τ_n, $p,q \in (2,+\infty)$ the increments of $\omega^n(t,x)$ can be bounded uniformly in some negative Sobolev norms, i.e. it holds

$$\mathbb{E}\left[\left\|(I+\mathcal{B}_p)^{-1}\left(\omega^n((\tau^n+r)\wedge T) - \omega^n(\tau^n\wedge T)\right)\right\|^q_{L^p}\right] \lesssim_{T,p,q} r^{q/2} + \frac{1}{n^q}. \tag{1.26}$$

On the contrary, the mild formulation (1.25) allows us to show that for each $p,q\in(2,+\infty)$, $\alpha \in (2/p,1)$ $\omega^n(t,x)$ can be bounded uniformly in time in some positive Sobolev norms, i.e. it holds

$$\mathbb{E}\left[\sup_{t\in[0,T]}\left\|(I+\mathcal{B}_p)^{\frac{\alpha}{2}}\omega^n(t)\right\|^q_{L^p}\right] \lesssim_{T,\alpha,p,q} 1. \tag{1.27}$$

Combining (1.26) and (1.27), by interpolation and Aldous criterion, we can show the tightness of the laws of ω^n in $\mathbb{D}([0,T];C(\bar{\mathcal{O}}))$. By classical Prokhorov's theorem and Skorokhod's representation theorem we can introduce an auxiliary probability space and a limit object $\bar{\omega}^M \in \mathbb{D}([0,T];C(\bar{\mathcal{O}}))$ such that $\omega^n \to \bar{\omega}^M$ in $\mathbb{D}([0,T];C(\bar{\mathcal{O}}))$ almost surely. Passing to the limit the equation satisfied by ω^n one can see that $\bar{\omega}^M$ is a weak solution of (1.22). Uniqueness of the weak solutions of (1.22) and (1.19) in $\mathbb{D}([0,T];C(\bar{\mathcal{O}}))$ allows us to show the convergence of ω^n to $\bar{\omega}^M$ in probability in the original probability space. Lastly as explained before, if M is large enough, depending only on ω_0 and g, the velocity field associated to the unique weak solution of (1.19) is bounded, therefore $\bar{\omega}^M \equiv \omega$ and this gives us Theorem 1.3.

The main additional difficulty compared to the works [71, 79] inspiring our technique is of course the presence of the boundary, which complicates many analytic operations. For instance, the absence of translation invariance in the models under consideration prevents derivatives and heat semigroups from commuting, so one has to resort instead to suitable gradient estimates on the heat kernel in order to exploit the regularizing properties of heat semigroup. Uniform convergence at time 0 and at the boundary also requires a careful control due to particle generation, but we are able to deal with rather natural assumptions on ω_0, g (compared to the obtained convergence), and to avoid further technical hypothesis on initial data such as [71, Assumption 1.1.3].

The particle system introduced can be thought of as a (regularized) stochastic version of the well-known point vortex system. However, there are some relevant differences concerning in particular the effect of the boundary on the vortex dynamics, see Remark 3.16 below. Let us only briefly discuss here some qualitative aspects for the sake of a comparison with the related literature. Due to the regularization of interaction at the microscopic level our approximants share some features with the systems of *moderately interacting* particles (see [69, 71] for a comparison), but our focus is on the approximation procedure rather than on the scaling of $\varepsilon = \varepsilon(n)$. If we were to omit the regularization of the interaction kernel and the cutoff F (and neglecting for a moment the complications of boundary effects), the scaling of weights ω_i would make our particle system (1.18) a mean-field rescaled version of the stochastic point vortex model.

1.5 Stochastic Models and Turbulence in Bounded Domains: The Problem of the Inviscid Limit

An important role in the understanding of the behavior of turbulent fluids is given by the analysis of the so-called inviscid limit. In a naive way, given u^ν and $\bar{u}$ solutions, in a suitable sense, of (1.2) with viscosity $\nu > 0$ and of (1.2) in the ideal case of a non-viscous fluid, i.e. $\nu = 0$, the problem of the inviscid limit consists in showing that u^ν, the solution of the Navier-Stokes equations, converges to $\bar{u}$, the solution of the corresponding Euler equations, as $\nu \to 0$ in the topology $L^\infty(0,T;L^2(\mathcal{O}))$. In order to simplify the notation we assume here on that $f_{est} \equiv 0$ in (1.2).

The difficulty of answering to this problem changes drastically considering different boundary conditions. Indeed, the problem is well-understood in case of periodic boundary conditions or if λ depends on $x \in \partial\mathcal{O}$ and it holds $\lambda(x) = \frac{1}{4\kappa(x)}$, $\kappa(x)$ being the curvature of the domain at

$x \in \partial\mathcal{O}$, see [56], [130] for details. In the case of no-slip boundary conditions, i.e. $\lambda = 0$, the convergence of u^ν to $\bar{u}$ in the topology $L^\infty(0,T;L^2(\mathcal{O}))$ is an open problem with few results available:

1. Unconditioned results. They are based on strong assumptions about the symmetry of the domain and of the data [133], or real analytic data [154], [13].
2. Conditioned results. They are based on stating some criteria about the behavior of the solutions of the Navier-Stokes equations in the boundary layer in order to prove that the inviscid limit holds. This line of research started with the seminal work by Kato [111], see also [48], [169], [177], [114] and the references therein for other results in this direction.

We employ the following notation to denote the boundary layer of the fluid: given $c > 0$, we write Γ_c in place of $\{x \in \mathcal{O} : \ d(x, \partial\mathcal{O}) \leq c\}$. Having this notation in mind, the main result of [111], valid in general dimension, reads as follows:

Theorem 1.4. *If u^ν is a Leray's solution of (1.2) with no-slip boundary conditions and $\bar{u}$ is a classical solution of the Euler equations such that $\|u_0^\nu - \bar{u}_0\| \to 0$ as $\nu \to 0$, the following are equivalent:*

1.

$$\sup_{t\in[0,T]} \|u^\nu(t) - \bar{u}(t)\| \to 0 \quad \text{as } \nu \to 0; \tag{1.28}$$

2.

$$u^\nu(t) \rightharpoonup \bar{u}(t) \quad \text{in} \quad L^2(\mathcal{O}) \quad \text{for each} \quad t \in [0,T]\,; \tag{1.29}$$

3.

$$\nu \int_0^T \|\nabla u^\nu(t)\|^2 dt \to 0 \quad \text{as } \nu \to 0; \tag{1.30}$$

4.

$$\nu \int_0^T \|\nabla u^\nu(t)\|^2_{L^2(\Gamma_{c\nu})} dt \to 0 \quad \text{as } \nu \to 0. \tag{1.31}$$

Concerning the possible links between inviscid limit in bounded domains and turbulence, a not completely exhaustive view, is that, under suitable assumptions, for which we refer for example to [154], the solution of the Navier–Stokes equations with no-slip boundary conditions can be split in two parts: a regular part far from the boundary of the domain which is the solution of the Euler equations and a rougher part in the boundary layer of the domain which is the solution of the so–called Prandtl equations. Without entering into the details of the assumptions about the validity of previous result and the meaning of the Prandtl equations, let us simply point out that, so far, the validity of this decomposition under the natural assumptions of existence of Leray's solutions is an open problem. Indeed, the validity of such decomposition, which implies (1.28), due to Theorem 1.4 is in contrast to the so-called *Kolmogorov's zeroth law of turbulence*, see [115], [116], [117]. The latter describing the physical evidence that the anomalous dissipation of the kinetic energy holds for three dimensional fluids at high Reynolds number. While, nowadays, the *Kolmogorov's zeroth law of turbulence* is a well-accepted assumption for three dimensional fluids where counterexamples to (1.30) has been shown in the case of deterministic forcing and domains without boundaries, see for example [32] for an explicit counterexample and [108], [147] for some numerical discussions, the situation is less clear in the two dimensional case. This is due the fact that either Navier-Stokes and Euler's flows preserve smooth solutions. We refer to [49] and the references therein for further discussions on this topic. As a consequence of Theorem 1.4, however, it is enough to control that no anomalous dissipation occurs in a boundary layer of width $c\nu$ for some $c > 0$ in order to prove the validity of the inviscid limit.

1.5.1 The Inviscid Limit in the Stochastic Framework

Due to the discussion above, from the physical view point it seems that there is no hope to carry on this analysis in the 3D and we continue to assume that $\mathcal{O} \subset \mathbb{R}^2$, smooth, convex and bounded. However, according to section 1.1, we model the roughness of the boundaries and the presence of obstacles by additive noise and we are interested to study the inviscid limit for (1.7) subjected to no-slip boundary conditions. In particular, we consider (1.7) with a particular scaling between the stochastic forcing and the viscosity, this is coherent to [124, Chapter 10], and we study the problem of the inviscid limit for the following system

$$\begin{cases} du^\nu = (\nu\Delta u^\nu - u^\nu \cdot \nabla u^\nu + \nabla P^\nu)dt + \sqrt{\nu}\sum_{k\in\mathcal{K}} \sigma_k dW^k_t, \\ \operatorname{div} u^\nu = 0, \\ u^\nu|_{\partial\mathcal{O}} = 0, \\ u^\nu(0) = u^\nu_0. \end{cases} \tag{1.32}$$

The existence and uniqueness of solutions of Leray's solutions of (1.32), strong from the probabilistic view point, is a well-known result under general assumptions on the coefficients σ_k, e.g. $\mathcal{K}$ is finite and $\sigma_k \in \mathsf{D}(\mathcal{A})$, and $u^\nu_0 \in H$, e.g. $\mathbb{E}\left[\|u^\nu_0\|^2\right] < +\infty$. The first results we discuss in chapter 4, introduced in [136], is the validity of the inviscid limit in this stochastic framework (1.32).

Theorem 1.5. *If $\bar{u}_0 \in C^{1,\varepsilon}(\bar{\mathcal{O}})$ and under previous assumptions on u^ν_0 and σ_k, if*

$$\lim_{\nu\to 0} \mathbb{E}\left[\|u^\nu_0 - \bar{u}_0\|^2\right] = 0,$$

then the following conditions are equivalent:

1. $\lim_{\nu\to 0}\mathbb{E}\left[\sup_{t\in[0,T]}\|u^\nu(t)-\bar{u}(t)\|^2\right]=0.$
2. $u^\nu(t)\rightharpoonup\bar{u}(t)$ *in* $L^2(\Omega\times\mathcal{O})$ *for each* $t\in[0,T]$.
3. $\lim_{\nu\to 0}\nu\int_0^T\mathbb{E}\left[\|\nabla u^\nu(t)\|^2\right]\ dt=0.$
4. $\lim_{\nu\to 0}\nu\int_0^T\mathbb{E}\left[\|\nabla u^\nu(t)\|^2_{L^2(\Gamma_{c\nu})}\right]\ dt=0.$

We refer to item 4 in Theorem 1.5 as *weak Kato hypothesis* in the following. The weak Kato hypothesis implies the validity of the inviscid limit under more general assumptions on the initial condition of the Euler's equations than $\bar{u}_0\in C^{1,\varepsilon}(\bar{\mathcal{O}})$, see Theorem 4.20 below. However, in order to ease the presentation, we postpone the introduction of this result to chapter 4. We only point out that the solutions of the Euler equations considered in Theorem 4.20 are not classical, therefore this provide also an improvement of Theorem 1.4, see [136, Section 5] for further remarks in the deterministic frameworks which will be not considered in chapter 4.

Concerning the proof of Theorem 1.5, item 1 $\Longrightarrow$ item 2, item 2 $\Longrightarrow$ item 3 and item 3 $\Longrightarrow$ item 4 are simple implications. The difficult point is showing that item 4 $\Longrightarrow$ item 1. We need to estimate

$$\|u^\nu(t)-\bar{u}(t)\|^2=\|u^\nu(t)\|^2+\|\bar{u}(t)\|^2+2\langle u^\nu(t),\bar{u}(t)\rangle.$$

The first two terms can be treated by Itô formula for (1.32) and the conservation of the energy for classical solutions of the Euler equations, therefore we need to obtain some cancellations from the last term. However in order to treat $\langle u^\nu(t),\bar{u}(t)\rangle$ we cannot work directly and we use the following classical idea in the study of the inviscid limit with no-slip boundary conditions. According to [111], if $\bar{u}$ is smooth, for each $\delta>0$ we can introduce a function

$$v:[0,T]\times\overline{\mathcal{O}}\to\mathbb{R}^2$$

which is called the *boundary layer corrector* of $\bar{u}$. v has support in Γ_δ, it has the same regularity of $\bar{u}$, $\operatorname{div}(\bar{u}-v)=0$ and $\bar{u}-v|_{\partial\mathcal{O}}=0$. In order to complete the proof the correct choice is $\delta=c\nu$, c being fixed in the weak Kato hypothesis. In particular, the support of v goes to 0 as $\nu\to 0$. This allows to consider $\bar{u}-v$ as a test function in the weak formulation satisfied by u^ν and obtain a proper bound on $\langle u^\nu(t),\bar{u}(t)-v(t)\rangle$ which provides also the required cancellations to control the positive terms coming from $\|u^\nu(t)\|^2$ and $\|\bar{u}(t)\|^2$. The term $\langle u^\nu(t),v(t)\rangle$ is negligible in the limit $\nu\to 0$ due to the properties of the support of v. Obviously, due to the presence of stochastic integrals in the Itô formula for $\|u^\nu(t)\|^2$, our control on $\|u^\nu(t)-\bar{u}(t)\|^2$ can be only in mean. The weak Kato hypothesis plays a crucial role in order to study $\langle u^\nu(t),\bar{u}(t)-v(t)\rangle$. Indeed, during the computations we need to show that $\int_0^t|\langle u^\nu(s)\cdot\nabla v(s),u^\nu(s)\rangle|ds$ is negligible in mean. This is possible thanks to the properties of v and its support. Indeed we show in chapter 4 that

$$\int_0^t|\langle u^\nu(s)\cdot\nabla v(s),u^\nu(s)\rangle|ds\lesssim\nu\int_0^t\|\nabla u^\nu(s)\|^2_{L^2(\Gamma_\delta)}ds,\tag{1.33}$$

which is negligible in mean in the limit due to the weak Kato hypothesis.

Following similar ideas, Theorem 1.5 has been generalized in [178], [94] to different scaling between the viscosity and the noise than the one considered in (1.32).

Due to Theorem 1.5, a natural question is understanding the asymptotic behavior for the probability of large fluctuations away from the zero-noise and zero-viscosity limit. In [38], we examined this issue utilizing Large Deviations Principle techniques. In order to state the main results of [38] we need to introduce further notation. Denoting by H_0 the reproducing kernel Hilbert space associated to the covariance of our noise,

$$W(t)=\sum_{k\in\mathcal{K}}\sigma_k W^k_t,$$

$u_0\in\mathcal{E}_0^{Ns}$ and $f(s)\in\mathcal{P}_2^N$ for some $N\in\mathbb{N}$ we denote by $\mathcal{G}^{NS,\nu}\left(u_0,\sqrt{\nu}W(\cdot)+\int_0^{\cdot}f(s)ds\right)$ the unique Leray solution, strong from the probabilistic view point, of

$$\begin{cases}du+(u\cdot\nabla u+\nabla P)\,dt=\nu\Delta u dt+fdt+\sqrt{\nu}dW(t),\\ \operatorname{div}u=0,\\ u|_{\partial\mathcal{O}}=0,\\ u(0)=u_0\end{cases}\tag{1.34}$$

and by $\mathcal{G}^{NS,0}\left(u_0,\int_0^{\cdot}f(s)ds\right)$ the unique *smooth* solution, of

$$\begin{cases}\partial_t u+u\cdot\nabla u+\nabla P=f,\\ \operatorname{div}u=0,\\ u\cdot\hat{n}|_{\partial\mathcal{O}}=0,\\ u(0)=u_0.\end{cases}\tag{1.35}$$

The maps $\mathcal{G}^{NS,\nu}$ and $\mathcal{G}^{NS,0}$ will be rigorously defined in section 5.2 to which we refer for details. Having this notation in mind we can introduce what we call *strong Kato hypothesis.*

Hypothesis 1.6. For each $N\in\mathbb{N}$, $u_0^\nu,\ u_0\in\mathcal{E}_0^{NS}$ and $f^\nu,\ f\in\mathcal{P}_2^N$ such that $u_0^\nu\to u_0$ in $\mathcal{E}_0^{NS}$ and $f^\nu\to_{\mathcal{L}} f$ in S^N, if $(\Omega,\mathcal{F},\mathcal{F}_t,\mathbb{P})$ is a filtered probability space where all f^ν, f are defined together and $f^\nu\to f$ $\mathbb{P}-a.s.$ in S^N, then, it exists $c>0$ such that for every $\delta>0$

$$\mathbb{P}\left(\nu\int_0^T\left\|\nabla\mathcal{G}^{NS,\nu}\left(u_0^\nu,\sqrt{\nu}W(\cdot)+\int_0^{\cdot}f^\nu(s)ds\right)\right\|^2_{L^2(\Gamma_{c\nu})}ds>\delta\right)\to 0.$$

Loosely speaking, this condition requires a control on the behaviour of the anomalous dissipation in the boundary layer of the solutions of the stochastic Navier-Stokes system with respect to all kind of forcings in the reproducing kernel Hilbert space of our noise. This is much more than what we ask to ensure the validity of the inviscid limit. However with such assumption we are able to control the probability of the fluctuactions in the zero-noise, zero-viscosity limit. Indeed, the following holds:

Theorem 1.7. *Assuming Hypothesis 1.6, the family* $\{u^{NS,\nu} = \mathcal{G}^{NS,\nu}(u_0, \sqrt{\nu}W(\cdot))\}_{u_0\in\mathcal{E}_0^{NS}}$ *satisfies the uniform Laplace principle with the rate function*

$$\begin{aligned} I_{u_0}^{NS}(w) &= \inf_{\{f\in L^2(0,T;H_0):\ w=\mathcal{G}^{NS,0}(u_0,\int_0^\cdot f(s)ds)\}} \frac{1}{2}\int_0^T \|f(s)\|_{H_0}^2 ds \\ &= \frac{1}{2}\int_0^T \|\partial_t w(t) + \mathbf{P}(w(t)\cdot\nabla w(t))\|_{H_0}^2 dt. \end{aligned}$$

where $u_0 \in \mathcal{E}_0^{NS}$, $w \in C([0,T];H)$, *with the convention that* $I_{u_0}^{NS}(w) = +\infty$ *anytime* w *is not in the range of* $\mathcal{G}^{NS,0}(u_0,\cdot)$.

The proof of Theorem 1.7 relies on the so-called *Weak Convergence Approach* developed in [36]. Even if, in general, large deviations principles for models with additive noise are treated via the techniques introduced by Freidlin and Wentzell based on a discretization of the equation and the application of the so-called *contraction principle*, see [83], we cannot deal with it since the vanishing of the viscosity together with the noise constitutes a technical issue that cannot be addressed via a classical contraction argument. Analogous issues has been addressed via the Weak Convergence Approach in [25] for showing the validity of a Large Deviation Principle for the inviscid limit of the Navier-Stokes equations with periodic or free boundary conditions. Similarly to other results with these kind of boundary conditions [124], [47], the result of [25] is based on the validity of the enstrophy equality, which allows to obtain stable estimates in the limit $\nu \to 0$ for two dimensional fluids stronger than the one guaranteed by the energy equality. These relations are not available in the case of no-slip boundary conditions, due to the generation of vorticity close the boundary. Therefore, the introduction of some Kato-type hypothesis is required in order to show the validity of the Large Deviation Principle, similarly to the validity of the inviscid limit.

The main issue in the proof of Theorem 1.7 relies in showing that if $u_0^\nu \to u_0$ in $\mathcal{E}_0^{NS}$ and $f_\cdot^\nu, f_\cdot \in \mathcal{P}_N^2$ such that in law

$$f_\cdot^\nu \rightharpoonup f_\cdot \quad \text{in } L^2(0,T;H_0),$$

then in law

$$\mathcal{G}^{NS,\nu}\left(u_0^\nu, \sqrt{\nu}W(\cdot) + \int_0^\cdot f^\nu(s)ds\right) \to \mathcal{G}^{NS,0}\left(u_0^\nu, \int_0^\cdot f(s)ds\right) \quad \text{in } C([0,T];H). \tag{1.36}$$

In order to show (1.36), we move by Prokhorov Theorem and Skorohod Theorem to a common probability space where $f_\cdot^\nu \rightharpoonup f_\cdot$ in $L^2(0,T;H_0)$ *a.s.* Then we prove that (1.36) holds in probability in this auxiliary probability space. The proof of this claim follows by introducing a splitting of $\mathcal{G}^{NS,\nu}\left(u_0^\nu, \sqrt{\nu}W(\cdot) + \int_0^\cdot f^\nu(s)ds\right)$ and $\mathcal{G}^{NS,0}\left(u_0^\nu, \int_0^\cdot f(s)ds\right)$ of the form

$$\begin{aligned} \mathcal{G}^{NS,\nu}\left(u_0^\nu, \sqrt{\nu}W(\cdot) + \int_0^\cdot f^\nu(s)ds\right) &= z^\nu + v^\nu, \\ \mathcal{G}^{NS,0}\left(u_0^\nu, \int_0^\cdot f(s)ds\right) &= z^0 + v^E, \end{aligned}$$

where $z(t) = \int_0^t f(s)ds$ and $z^\nu(t)$ solves the Stokes equations with random forcing f^ν. Relying on the mild formula satisfied by $z^\nu(t)$, the convergence of $f^\nu_.$ to $f.$ allows as to show that

$$z^\nu(t) \to z(t) \quad \text{a.s. uniformly in time in some positive Sobolev norms.}$$

The aforementioned convergence allows us to prove that

$$v^\nu(t) \to v^E(t) \quad \text{in probability in } C([0,T];H). \tag{1.37}$$

The proof of (1.37), as one can expect, is the more demanding part of the argument. It relies in introducing a random corrector of the boundary layer for v^E and exploiting in a more careful way some ideas of Theorem 1.5. The strong Kato hypothesis introduced above plays a role in order to study terms of the form (1.33) in this framework.

Even if the strong Kato hypothesis may looks not physical and impossible to check according to the Kolmogorov's zeroth law of turbulence, one may also thinks that it is an empty statement, we introduce in [38] a particular stochastic framework, based on some symmetries described in [133], where Hypothesis 1.6 holds. We postpone the discussion of this framework to chapter 5.

One way of improving Theorem 1.7 could be to show that the strong Kato hypothesis can be deduced from the weak Kato hypothesis. In some sense this would requires to be able to pass information between systems with different forcings. We shall notice that the forcing that we are working with all live in the reproducing kernel of $W(t)$, therefore we might switch from one system to another just by a Girsanov transformation; however this correction explodes exponentially fast in the limit $\nu \to 0$. A posteriori, if one is able to prove that the large deviation principle holds for some forcing, one expects that the explosion of the Girsanov correction gets compensated by the exponential decay of the law of the solutions. However, this is for us an open problem at the moment we are writing these notes.

1.5.2 The Case of the Second Grade Fluid Equations

The second-grade fluid equations are a model for viscoelastic fluids, with two parameters: $\alpha > 0$, corresponding to the elastic response, and $\nu > 0$, corresponding to viscosity. For such fluids, assuming their density being constant, the stress tensor is given by

$$\sigma_{SG} = -\mathsf{P}I + \nu A_1 + \alpha A_2 - \alpha A_1^2,$$

where

$$A_2 = \partial_t A_1 + A_1 \nabla \mathsf{u} + (\nabla \mathsf{u})^T A_1.$$

Given this stress tensor, the equations of motion for an incompressible homogeneous fluid of grade 2 are given by

$$\begin{cases} \partial_t \mathsf{v} = \nu \Delta \mathsf{u} - \operatorname{curl}(\mathsf{v}) \times \mathsf{u} + \nabla \mathsf{P} + \mathsf{f} \\ \operatorname{div} \mathsf{u} = 0 \\ \mathsf{v} = \mathsf{u} - \alpha \Delta \mathsf{u} \\ \mathsf{u}|_{\partial \mathcal{O}} = 0 \\ \mathsf{u}(0) = \mathsf{u}_0, \end{cases} \tag{1.38}$$

where f describes some external forces acting on the fluid. We refer to [63], [153] for further details on the physics behind this system. The analysis of the second-grade fluid equations started with [46], where some results, not restricted to the two dimensional case, for global existence and uniqueness of solutions of the problem above have been shown. Setting, formally, $\alpha = 0$ the system above reduces to the Navier-Stokes system. Thus, it can be seen as a generalization of the Navier-Stokes equations. Moreover, the convergence of the solution of the second-grade fluid equations to the solution of the Navier-Stokes equations has been shown rigorously in [103]. Due to these good properties of the system, it is a legitimate question trying to understand if the second-grade fluid equations behave better than the Navier-Stokes ones in problems related to turbulence, like the inviscid limit for domains with boundaries and no-slip boundary conditions. This is, indeed, the case: in the deterministic framework the problem of the inviscid limit in bounded domains and no-slip boundary conditions has been solved for the second grade fluid equations, at least in a suitable regime of the parameters, see [134], [132]. In [135], [38] we discuss if the issues raised in subsection 1.5.1 can be solved for (1.38) in case of stochastic forcings without any Kato-type hypothesis. Here we just state at the informal level the main results of [135], [38] related to this topic, postponing to chapter 4 and chapter 5 the rigorous statements of the results and their proofs. For the matter of notation, in the following we denote by $\mathcal{G}^{SG,0}(\cdot,\cdot)$ the unique *smooth* solution of (1.35) with initial condition in $\mathcal{E}_0^{SG}$ and forcing in the reproducing kernel Hilbert space of the noise.

Theorem 1.8 (Informal Statement, see Theorem 4.22, Theorem 4.37)**.** *Let u^α be the unique solution weak solution of (1.38) with initial condition u_0^α and forcing, f, of the form of transport or additive noise scaling as $\sqrt{\alpha}$, i.e.*

$$\mathsf{f}(t) = \sqrt{\alpha}\dot{W}(t), \quad \text{or} \quad \mathsf{f}(t) = \sqrt{\alpha} \circ \dot{W}(t) \cdot \nabla \mathsf{u}^\alpha(t),$$

and $\bar{u}$ be the unique smooth solution of the Euler equations with initial condition $\bar{u}_0$ and no forcing. If $\nu = O(\alpha)$, u_0^α, $\bar{u}_0 \in \mathcal{E}_0^{SG}$ are such that

$$\mathbb{E}\left[\|\mathsf{u}_0^\alpha - \bar{u}_0\|^2\right] \to 0, \quad \alpha \mathbb{E}\left[\|\nabla \mathsf{u}_0^\alpha\|^2\right] = o(1), \quad \alpha^3 \mathbb{E}\left[\|\mathsf{u}_0^\alpha\|_{H^3}^2\right] = O(1),$$

the inviscid limit holds, i.e.

$$\lim_{\alpha \to 0} \mathbb{E}\left[\sup_{t \in [0,T]} \|\mathsf{u}^\alpha(t) - \bar{u}(t)\|^2\right] = 0.$$

Theorem 1.9 (Informal Statement, see Theorem 5.19)**.** *Assuming $\nu = O(\alpha)$, the family made by the solutions of* (1.38) *with $u_0 \in \mathcal{E}_0^{SG}$ and forcing, f, of the form of additive noise scaling as $\sqrt{\alpha}$, satisfies the uniform Laplace principle with the rate function*

$$\begin{aligned} I_{u_0}^{SG}(w) &= \frac{1}{2} \inf_{f \in L^2(0,T;H_0):\ w = \mathcal{G}^{SG,0}(\mathsf{u}_0, \int_0^{\cdot} f(s)ds)} \int_0^T \|f(s)\|_{H_0}^2 ds \\ &= \frac{1}{2} \int_0^T \|\partial_t w(t) + \mathbf{P}(w(t) \cdot \nabla w(t))\|_{H_0}^2 ds \end{aligned}$$

where $w \in C([0,T];H)$, with the convention that $I_{\mathsf{u}_0}^{SG}(w) = +\infty$ anytime w is not in the range of $\mathcal{G}^{SG,0}(\mathsf{u}_0, \cdot)$.

The proofs of Theorem 1.8, Theorem 1.9 rely on ideas similar to Theorem 1.5 and Theorem 1.7, namely introductions of boundary layer correctors and weak convergence approach. The main difference is that, considering for example no forcing terms, we have another estimate available. Indeed, denoting by $\mathsf{q}^\alpha = \operatorname{curl} \mathsf{v}^\alpha$, it follows

$$\partial_t \|\mathsf{q}^\alpha\|^2 = -\frac{2\nu}{\alpha} \langle \mathsf{q}^\alpha - \operatorname{curl} \mathsf{u}^\alpha, \mathsf{q}^\alpha \rangle.$$

The estimate above is stable in the inviscid limit under the assumptions of Theorem 1.8, i.e. a particular scaling between ν and α and on the initial conditions u_0^α, and allows us to have strong enough controls on u^α in order to treat terms of the form (1.33).

1.6 What is not Included Here

In the past four years, we worked with some collaborators on different projects which cannot be linked directly to the results of the previous sections. Some of them were an opportunity to learn new techniques, some others may be seen as the starting point of lines of research never properly explored. Lastly, a significant portion of our works with Professor Franco Flandoli, aimed to understand the properties of the so-called Itô-Stratonovich diffusion limit. In this section, we collect briefly the results of the second and third group.

1.6.1 Gibbs Equilibrium Fluctuations of Point Vortex Dynamics

The existence and uniqueness of solutions for the 2D Euler equations is a settled problem, at least within the Yudovich well-posedness class [180], unlike the 3D case. For more general initial conditions, non-uniqueness phenomena are known to occur, as demonstrated by convex integration techniques, see for example [31], [43].

In [96], our focus is on the existence of 2D Euler dynamics with a special class of initial distributions involving much rougher initial conditions referred to as Energy-Enstrophy measures. These measures hold paramount physical importance as both kinetic energy and

enstrophy are the main conserved quantities, and thus these measures are formally left invariant by the Euler dynamics.
We consider the point vortex system on a bounded domain $\mathcal{O} \subset \mathbb{R}^2$ with smooth boundary. If x_i are the vortex positions and ω_i their intensities, the equations of motions are

$$\dot{x}_i(t) = \frac{1}{\sqrt{N}} \sum_{j \neq i}^{N} \omega_j K(x_i(t), x_j(t)) + \frac{1}{\sqrt{N}} \omega_i \nabla^{\perp} g(x_i(t), x_i(t)), \quad i = 1, \dots, N, \tag{1.39}$$

where g takes into account the effect of the boundary and prefactors $1/\sqrt{N}$ fix intensities to the fluctuation scale. A series of works initiated with [140, 138, 141, 139] provided a derivation of (1.39) as a limit of *vortex patch* solutions of 2-dimensional Euler equations in the vorticity formulation,

$$\partial_t \omega + K[\omega] \cdot \nabla \omega = 0. \tag{1.40}$$

In [96] we consider vortex intensities having the same magnitude $\xi_i = \pm 1$ and subjected to the *neutrality condition*

$$\sum_{j=1}^{N} \omega_j = 0.$$

Under this assumption one-particle marginals of the (Mean-Field rescaled) Canonical Gibbs ensemble,

$$d\nu_\beta^N(x_1, \dots, x_N) = \frac{1}{Z_\beta^N} \exp\left(-\frac{\beta}{N} H(x_1, \dots, x_N) \right) dx_1 \cdots dx_N,$$

with Hamiltonian function

$$H(x_1, \dots, x_n) = \sum_{j=1}^{N} \sum_{i<j}^{N} \omega_i \omega_j G(x_i, x_j) + \frac{1}{2} \sum_{i=1}^{N} \omega_i^2 g(x_i, x_i),$$

converge to the uniform distribution on $\mathcal{O}$ [39, 40], while particle correlations decay to zero. Therefore, in this scaling regime, the empirical measure $\frac{1}{N} \sum_{i=1}^{N} \omega_i \delta_{x_i}$ converges to 0 under the Canonical Ensemble.

Building up on the study initiated in [97], we consider in [96]

$$\omega^N(t) = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \omega_i \delta_{x_i(t)},$$

describing fluctuations at equilibrium around the null Mean-Field limit of the empirical measure and establish convergence of the full space-time fluctuation field $(\omega^N(t))_{t \in [0,T]}$ of point vortex dynamics preserving ν_β^N at finite temperature $\beta > 0$ to weak vorticity solutions of Euler equations preserving the Energy-Enstrophy measure. Our main result can be summarized as follows:

The flow $t \mapsto \omega^N(t)$ *converges to a stochastic process* $t \mapsto \omega(t)$ *whose trajectories satisfy* (1.40) *in a weak sense.*
No external forcing is involved in the dynamics, and stochasticity is only due to the invariant measure from which the initial datum is sampled. The limiting dynamics can be described as a measure-preserving weak solution of (1.40).

It is worth noting that in this context, vortex positions are not independent, contrary to [65]. Furthermore, we tackled the more challenging problem of motion within a bounded domain, where the influence of boundary effects must be taken into account in the point vortex dynamics. The limit fluctuation dynamics $t \to \omega(t)$ possesses some interesting and distinguished characteristics. The evolution is still non-linear in the limit, and it satisfies a generalized version of the equations of which point vortices are a discretization. As a consequence, while single-time marginals of $\omega(t)$ have Gaussian distribution corresponding to the Energy-Enstrophy measure of (1.40) multi-time distributions are not jointly Gaussian.

1.6.2 Stochastic Models and Turbulence: The Itô-Stratonovich Diffusion Limit

Starting from [88], it has been shown in several settings that in a suitable scaling limit, active and passive scalars subjected to transport noise and evolving in a periodic domain converges to the solutions of a deterministic equation with an additional diffusive operator. The introduction of transport noise in fluid dynamical models has some physical intuitions, see [120], [121], [122], and can be rigorously justified basing on the idea of separation of scales; we refer to [101], [80], [61], [144] for some results in this direction. The paradigmatic example to have in mind in dimension two is the following: let $\nu, \nu_T > 0$ and consider the unique weak solution of the following stochastic partial differential equation with transport noise

$$d\omega^N = (\nu\Delta\omega^N - K[\omega]\cdot\nabla\omega)dt + \sqrt{2\nu_T}\sum_{k\in\mathbb{Z}_0^2}\sigma_k^N\cdot\nabla\omega^N \circ dW_t^k, \tag{1.41}$$

where $\theta^N = (\theta_k^N)_k \in \ell^2(\mathbb{Z}_0^2)$, and $\sigma_k^N,\ \theta_k^N$ are chosen satisfying

$$\sum_{k\in\mathbb{Z}_0^2}(\theta_k^N)^2 = 1, \quad \theta_k^N = \theta_l \quad \text{if } |k| = |l|, \quad \left\|\theta^N\right\|_{\ell^\infty} \to 0$$

and

$$\sigma_k(x) = \frac{k^\perp}{|k|}e_k(x), \quad e_k(x) = \sqrt{2}\begin{cases}\cos(k\cdot x) & \text{if } k\in\mathbb{Z}_+^2,\\ \sin(k\cdot x) & \text{if } k\in\mathbb{Z}_-^2.\end{cases}$$

Above we denoted by $k^\perp = (k_2, -k_1)$, $\mathbb{Z}_+^2 := \{k \in \mathbb{Z}_0^2 : (k_1 > 0) \text{ or } (k_1 = 0,\ k_2 > 0)\}$ and $\mathbb{Z}_-^2 := -\mathbb{Z}_+^2$. Then, as $N \to +\infty$, ω^N converges in probability to $\overline{\omega}$ in $C([0,T]; H^{-\delta}(\mathbb{T}^2))$ for each $\delta > 0$, $\overline{\omega}$ being the unique solution of the deterministic Navier-Stokes equations in vorticity formulation with viscosity $\nu+\nu_T$. This stochastic approach is related to explain Boussinesq hypothesis that "turbulent fluctuations are dissipative on large scales" [30]. Therefore, we talk of ν_T as a *turbulent viscosity*.

We classify such kind of results as *Itô-Stratonovich stochastic diffusion limit*, in analogy with the deterministic and stochastic theory of diffusion limits based on homogenization techniques (see for instance [137], [118]). The approach however is different, strongly based on the Itô-Stratonovich corrector, from which we give this name. In our approach, the Itô-Stratonovich corrector appearing when we rewrite the term $\sqrt{2\nu_T}\sum_{k\in\mathbb{Z}_0^2}\sigma_k^N\cdot\nabla\omega^N\circ dW_t^k$ from the Stratonovich form to the Itô one, corresponds to the mean field term in the physical literature under the most natural closure formulae. But in our rigorous derivation there is still the Itô term, which should disappear in the limit when $N\to\infty$. There is no obvious reason why it should disappear, and in general it contains fluctuations which may be very important and alter the limit result. In scalar and 2D models, where stretching of vector quantities does not appear, it is possible to prove that the Itô term is negligible in the limit (see for instance [88], [76], [66], [67], [81]). In 3D, the problem is extremely difficult. Indeed, when considering passive vector fields, e.g. the magnetic field, or active vector fields, e.g. the vorticity of a 3D fluids, previous discussions about the separation of scales lead to add a further stochastic stretching term in (1.41). Having a good control on this term, keeping a nontrivial Itô-Stratonovich corrector is an almost open problem, the only result available being [72], [37].

Besides from the reviews [76], [75], our contributions to this topic include the following results:

1. In [74] we consider the heat diffusion and transport in an infinite 2D channel. The velocity field is white noise in time, modelling phenomenologically a turbulent fluid. We prove that, in the long run, the solution is close, in a weak sense, to the stationary deterministic solution of the heat equation with augmented diffusion coefficients.

2. In [41] we extend the results of [67] and [74] to the two-layer quasi-geostrophic model. This popular intermediate complexity model describes large scale atmosphere and ocean dynamics at the mid-latitudes and does not fall into the general framework of [67]. We show, in a quantitative way, that the solutions converge to the deterministic two-layer quasi-geostrophic model with enhanced dissipation. Moreover, these solutions converge to the deterministic stationary with enhanced dissipation on the long time horizon.

3. The works quoted so far on the Itô-Stratonovich diffusion limit are limited to the case of *linear* limit dissipation term, namely turbulent viscosity independent of the solution. Smagorinsky type models, see [21], [50], are excluded from the previous analysis and it was not clear for some time how to incorporate them into this new theory. In [73] we solved this problem, proving that a version of Smagorinsky Large Eddy model for a 2D fluid in vorticity form is the scaling limit of a suitable stochastic models for large scales, where the influence of small turbulent eddies is modeled by a nonlinear transport type noise.

4. In [37] we introduce a stochastic model for a passive magnetic field in a three dimensional thin domain. The turbulent fluid, white in time, acts on the magnetic field as a transport-stretching noise. We prove, in a quantitative way, that, in the simultaneous scaling limit of the thickness of the thin layer and the separation of scales, the mean on the thin direction of the magnetic field is close to the solution of the equation of the magnetic field with additional dissipation. In certain choice of noises with correlation between their components, without mirror symmetry and with a non zero mean helicity, we identify an alpha-term, in addition to the extra dissipation term. However, it does not produce dynamo; consequently, we extend a no-dynamo theorem to thin layers. This is coherent with conclusions of the Plasma Physics literature which state that dynamo does not appear when the magnetic field depends only on two variables, see for instance [90, Section 3.5]. This happens only in the limit of the thin domain, not for the approximating equation, but the final lack of dynamo remains true. The closest previous result to [37] is [72], a 2D-3C (two-dimensional, three component) model. However, the thin domain treated in [37] is a truly 3D-3C model, which reduces to 2D-3C only in the limit. The treatment of the stretching term in full generality, namely not in the particular case of thin domains and the setting of assumptions described in [37, Section 2.3], is an open problem, perhaps related to the role of intermittency in 3D turbulent fluids.

Chapter 2

2D Navier-Stokes Equations with Stochastic Wind Driven Boundary Conditions

As discussed in section 1.3, this chapter covers the result of [3], namely the global well posedness and the interior regularity of

$$\begin{cases} \partial_t u + u \cdot \nabla u + \nabla P = \Delta u, & \text{on } (0,T) \times \mathcal{O}, \\ \operatorname{div} u = 0, & \text{on } (0,T) \times \mathcal{O}, \\ u = 0, & \text{on } (0,T) \times \Gamma_b, \\ \partial_2 u_1 = h_b \dot{W}_{\mathcal{U}}, & \text{on } (0,T) \times \Gamma_u, \\ u_2 = 0, & \text{on } (0,T) \times \Gamma_u, \\ u(0) = u_0, & \text{on } \mathcal{O}. \end{cases} \tag{2.1}$$

We recall for the convenience of the reader that $T,\ a > 0$, $\mathcal{O} = \mathbb{T} \times (0,a)$ with $\mathbb{T}$ the one dimensional torus and

$$\Gamma_b = \mathbb{T} \times \{0\} \quad \text{and} \quad \Gamma_u = \mathbb{T} \times \{a\}, \tag{2.2}$$

the bottom and the upper part of the boundary of $\mathcal{O}$, respectively. We are interested on proving the following results:

Theorem 2.1. *Let* $q > 2,\ p > 2q,\ \alpha \in [0, \frac{1}{q} - \frac{2}{p})$ *be such that:*

$$-\frac{1}{q} - \alpha + \frac{1}{2} > 0.$$

Assuming that $h_b : (0,T) \in L^p(0,T; W^{-\alpha,q}(\Gamma_u; \mathcal{U}))$ *and* $u_0 \in \mathbb{L}^2$, *there exists a unique weak solution* u *to* (2.1).

Theorem 2.2. *Under the same assumptions of Theorem 2.1, if* u *is the unique weak solution of* (2.1) *provided by Theorem 2.1 then for all* $t_0 \in (0, T)$ *and* $\mathcal{O}_0 \subset \mathcal{O}$ *such that* $\mathrm{dist}(\mathcal{O}_0, \partial\mathcal{O}) > 0$*,*

$$u \in C([t_0, T]; C^\infty(\mathcal{O}_0; \mathbb{R}^2)) \quad \mathbb{P} - a.s.$$

As discussed in section 1.3, the proof of Theorem 2.1 relies strongly on the mild formula (1.15) and the properties of the map $\mathcal{N}$ and the Stokes operator A_q to which a part of this chapter is devoted. In particular, we devote section 2.1 to the functional analytic setup of our problem and study the properties of $\mathcal{N}$ and A_q. The proof of Theorem 2.1 is the object of section 2.2. In particular, in subsection 2.2.1 we consider the linear problem (1.14), while in subsection 2.2.2 we consider the nonlinear problem (1.16). The proof of Theorem 2.2 is the object of section 2.3. Similarly we study the interior regularity of the solution of the linear problem (1.14) and the nonlinear one (1.16) in subsection 2.3.1 and subsection 2.3.2 respectively.

Before starting with the technical part of this chapter, let us recall some results which are related to our model.

Theorem 2.1 shares strong similarities with [22, Theorem 1.2], which addresses the well-posedness of certain 2D deterministic Navier-Stokes equations with non-homogeneous non-smooth Navier-type boundary conditions. However, it is important to note that our model focuses on a different phenomenon than the one studied in [22]. For this reason, contrary to us, they stress the regularity of the boundary condition of the normal trace of the velocity. From a mathematical viewpoint, the white noise appearing in equation (2.1) is rougher both in time and in space compared to the boundary conditions discussed in [22]. However, as discussed in [53], Neumann boundary conditions are more regular than Dirichlet boundary conditions and allow us to treat rougher inputs. Due to these differences, the two results have different ranges of applicability and do not cover each other. Moreover, the tools introduced in [3] differ significantly from the techniques involved in [22].
Concerning boundary noise and fluid dynamical models, the local well-posedness of 3D primitive equations with boundary noise modeling wind forces has been studied in [28]. Both their strategy and ours are based on the splitting technique introduced in [52]. After showing suitable regularity properties of the stochastic convolution via stochastic maximal L^p-regularity techniques (cf. Proposition 2.10 and [28, Proposition 4.3]), a thorough analysis of certain nonlinear models is required. In contrast, we conduct this analysis within a suitable Hilbertian framework, enabling us to derive energy estimates essential for establishing the global well-posedness of (2.1) (cf. Theorem 2.12 and [28, Section 5.3]). The difference between the global well-posedness result which we are able to obtain and [28, Theorem 5.1] can be seen as consequence of the fact that the 2D Navier-Stokes equations are globally well-posed in the weak setting, while the same cannot be asserted for the primitive equations (cf. [106]). Therefore, in order to prove their local well-posedness result, the authors in [28] need to work with a notion of solution which mixes strong and weak regularity in the space variables. As a byproduct of this fact we are able to consider a noise rougher in space compared to them.

Additionally, a minor distinction lies in the boundary conditions applied to the bottom part of the domain Γ_b. We introduce no-slip boundary conditions to accurately model the bottom of the ocean, a choice with theoretical underpinning in works such [57, 58, 91, 148, 149]. In contrast, [28] considered some form of homogeneous Neumann boundary conditions, a choice related to the functional analytic setup of the primitive equations (cf. [28, Remark 3.3]). Beyond the distinct justifications from a modeling perspective, our choice leads to differences in the analysis of the corresponding linear elliptic systems (cf. subsection 2.1.2 and [28, Section 3.5]).

2.1 Preliminaries

2.1.1 The Stokes operator and its spectral properties

In this subsection we introduce the functional analytic setup in order to define all the object necessarily in the following. We refer to section 1.2 for a description of the notation used here without further explanations.

In the main arguments we need stochastic maximal L^q-regularity estimates for stochastic convolutions. By [173], it is enough to show the boundedness of the H^∞-calculus for A_q. For the main notation and basic results on the H^∞-calculus we refer to [150, Chapters 3 and 4] and [102, Chapter 10]. This assumption on the operator A_q is guaranteed by the following theorem:

Theorem 2.3. *For all $q \in (1,\infty)$, the operator A_q is invertible and has a bounded H^∞-calculus of angle $< \frac{\pi}{2}$. Moreover the domain of the fractional powers of A_q is characterized as follows:*

1. $\mathsf{D}(A_q^s) = H^{s,q}(\mathcal{O};\mathbb{R}^2) \cap \mathbb{L}^q$ *if* $0 \le s < \frac{1}{2q}$.
2. $\mathsf{D}(A_q^s) = \big\{ f \in H^{s,q}(\mathcal{O};\mathbb{R}^2) \cap \mathbb{L}^q \,|\, f|_{\Gamma_b} = 0,\ f_2|_{\Gamma_u} = 0 \big\}$ *if* $\frac{1}{2q} < s < \frac{1}{2} + \frac{1}{2q}$.
3. $\mathsf{D}(A_q^s) = \big\{ f \in H^{s,q}(\mathcal{O};\mathbb{R}^2) \cap \mathbb{L}^q \,|\, f|_{\Gamma_b} = 0,\ f_2|_{\Gamma_u} = \partial_2 f_1|_{\Gamma_u} = 0 \big\}$ *if* $\frac{1}{2} + \frac{1}{2q} < s < 1$.

The above implies that $-A_q$ generates an analytic semigroup on $\mathbb{L}^q$. Theorem 2.3 could be known to experts. For the reader's convenience, we prefer to postpone to section 2.4 a complete and relatively short proof based on the recent strategy used in [125] for the H^∞-calculus for the Stokes operator on Lipschitz domains [125, Theorem 16].

2.1.2 The Neumann map

Now we are interested in L^q-estimates for the Neumann map, i.e. we are interested in studying the weak solutions of the elliptic problem

$$\begin{cases} -\Delta u + \nabla \pi = 0, & \text{on } \mathcal{O}, \\ \operatorname{div} u = 0, & \text{on } \mathcal{O}, \\ u(\cdot, 0) = 0, & \text{on } \Gamma_b, \\ \partial_2 u_1(\cdot, a) = g, & \text{on } \Gamma_u, \\ u_2 = 0, & \text{on } \Gamma_u. \end{cases} \tag{2.3}$$

To state the main result of this subsection, we need to formulate (2.3) in the weak setting. To this end, we argue formally. Take $\varphi = (\varphi_1, \varphi_2) \in C^\infty(\mathcal{O}; \mathbb{R}^2)$ such that $\operatorname{div} \varphi = 0$,

$$\varphi(\cdot, 0) = 0, \quad \text{and} \quad \varphi_2(\cdot, a) = 0.$$

A formal integration by parts shows that (2.3) implies

$$\int_{\mathcal{O}} \nabla u : \nabla \varphi \, dx = -\int_{\mathbb{T}} g(x) \varphi_1(x, a) \, d\sigma(x). \tag{2.4}$$

In particular, the RHS of (2.4) makes sense even in case g is a distribution if we interpret $\int_{\mathbb{T}} g(x)\varphi_1(x,a)\,d\sigma(x) = \langle \varphi_1(\cdot, a), g\rangle$.

Theorem 2.4. *Let $q \in (1, \infty)$, for all $g \in W^{-1/q,q}(\Gamma_u)$ there exists a unique $(u, \pi) \in W^{1,q}(\mathcal{O}; \mathbb{R}^2) \times L^q(\mathcal{O})/\mathbb{R}$ weak solution of (2.3). Moreover (u, π) satisfy*

$$\|u\|_{W^{1,q}(\mathcal{O};\mathbb{R}^2)} + \|\pi\|_{L^q(\mathcal{O})/\mathbb{R}} \leq C \|g\|_{W^{-1/q,q}(\Gamma_u)}. \tag{2.5}$$

Finally, if $g \in W^{1-1/q,q}(\Gamma_u)$, then $(u, \pi) \in W^{2,q}(\mathcal{O}; \mathbb{R}^2) \times W^{1,q}(\mathcal{O})/\mathbb{R}$ and

$$\|u\|_{W^{2,q}(\mathcal{O};\mathbb{R}^2)} + \|\pi\|_{W^{1,q}(\mathcal{O})/\mathbb{R}} \leq C \|g\|_{W^{1-1/q,q}(\Gamma_u)}. \tag{2.6}$$

Proof. We divide the proof into three steps.

Step 1: Proof of (2.5). Let A_q be as in section 2.1. We prove (2.5) by applying the Lax-Milgram theorem of [119, Theorem 1.1] to the form $a : Y_1 \times Y_2 \to \mathbb{R}$ where

$$a(u, \varphi) = \int_{\mathcal{O}} \nabla u : \nabla \varphi \, dx, \quad Y_1 = \mathsf{D}(A_q^{1/2}), \quad Y_2 = \mathsf{D}(A_{q'}^{1/2}).$$

Recall that, by Theorem 2.3,

$$\mathsf{D}(A_q^{1/2}) = \{v = (v_1, v_2) \in \mathbb{H}^{1,q}(\mathcal{O}) \,:\, v|_{\Gamma_b} = 0,\ v_2|_{\Gamma_u} = 0\}.$$

Since $W^{1,q'}(\mathcal{O}) \ni \varphi \mapsto \varphi_1|_{\Gamma_u} \in W^{1-1/q',q'}(\Gamma_u) = W^{1/q,q'}(\Gamma_u)$, we have

$$|\langle \varphi_1(\cdot,a), g\rangle| \leq \|g\|_{W^{-1/q,q}(\Gamma_u)} \|\varphi\|_{W^{1/q,q'}(\Gamma_u)} \lesssim \|g\|_{W^{-1/q,q}(\Gamma_u)} \|\varphi\|_{W^{1,q'}(\mathcal{O})}. \tag{2.7}$$

Hence the Lax-Milgram theorem of of [119, Theorem 1.1] implies the existence of u as in (2.5) provided, for all $v \in \mathsf{D}(A_q^{1/2})$,

$$\|\nabla v\|_{L^q(\mathcal{O};\mathbb{R}^2)} \eqsim \sup\Big\{ \int_{\mathcal{O}} \nabla v : \nabla f \, dx \,\Big|\, f \in \mathsf{D}(A_{q'}^{1/2}) \text{ and } \|f\|_{\mathsf{D}(A_{q'}^{1/2})} \leq 1 \Big\}. \tag{2.8}$$

The case $\gtrsim$ of (2.8) follows from the Hölder's inequality. To prove the opposite inequality, we argue by duality. We start by discussing some known facts about the "Sobolev tower" of spaces associated the operator A_q:

$$\begin{aligned} X_{\alpha,A_q} &= \mathsf{D}(A_q^\alpha) && \text{for } \alpha \geq 0, \\ X_{\alpha,A_q} &= (\mathbb{L}^q, \|A_q^\alpha \cdot\|_{\mathbb{L}^q})^\sim && \text{for } \alpha < 0. \end{aligned}$$

Here $\sim$ denotes the completion (since $0 \in \rho(A_q)$ by Theorem 2.3, we have that $f \mapsto \|A_q^\alpha f\|_{\mathbb{L}^q}$ is a norm for all $\alpha < 0$). Since $(A_q)^* = A_{q'}$, it follows that (see e.g. [7, Chapter 5, Theorem 1.4.9])

$$(X_{\alpha,A_q})^* = X_{-\alpha,A_{q'}}. \tag{2.9}$$

Now we can proceed in the proof of $\lesssim$ in (2.8). Firstly, as $\mathsf{D}(A_q) \hookrightarrow \mathsf{D}(A_q^{1/2})$ is dense for all $q \in (1,\infty)$, we can prove such inequality assuming $v \in \mathsf{D}(A_q)$. In the latter case, the duality (2.9) and the Hahn-Banach theorem imply the existence of $g \in X_{-\alpha,A_{q'}}$ of unit norm such that

$$\begin{aligned} \|A_q^{1/2} v\|_{L^q(\mathcal{O};\mathbb{R}^2)} &= \int_{\mathcal{O}} A_q^{1/2} v \cdot A_{q'}^{-1/2} g \, dx \\ &\overset{(i)}{=} \int_{\mathcal{O}} A_q v \cdot A_{q'}^{-1} g \, dx \\ &\overset{(ii)}{=} -\int_{\mathcal{O}} \Delta v \cdot A_{q'}^{-1} g \, dx \\ &\overset{(iii)}{=} -\int_{\mathcal{O}} \nabla v : \nabla (A_{q'}^{-1} g) \, dx \end{aligned}$$

where in (i) we used that $A_q^{1/2} v = A_q^{-1/2}(A_q v)$ and $(A_q^{-1/2})^* = A_{q'}^{-1/2}$, in (ii) that $A_q = -\mathbf{P}_q \Delta_q$ and therefore $\mathbf{P}_{q'} A_{q'}^{-1} g = A_{q'}^{-1} g$ as $A_{q'}^{-1} g \in \mathsf{D}(A_{q'}^{1/2}) \subseteq \mathbb{L}^{q'}(\mathcal{O})$. Finally, in (iii) we used that no boundary terms appear due to the boundary conditions and $\partial_2 v_1(\cdot,a) = 0$ as $v \in \mathsf{D}(A_q)$.

Hence the case $\lesssim$ of (2.8) follows from the above chain of equality, the fact that $\mathsf{D}(A_q^{1/2}) \hookrightarrow W^{1,q}(\mathcal{O};\mathbb{R}^2)$ and $A_{q'}^{-1} : X_{-1/2,A_{q'}} \to X_{1/2,A_{q'}}$ is an isomorphism.

Now, the existence of the pressure π satisfying the estimate (2.5) is standard and follows from the De Rham theorem, see e.g. [86, Corollary III.5.1, Lemma IV.1.1].

Step 2: Proof of (2.6). By Step 1, it suffices to prove the existence of a solution $(u,\pi)\in W^{2,q}(\mathcal{O})\times W^{1,q}(\mathcal{O})/\mathbb{R}$ for which (2.3) holds. In case of $g\in C^\infty(\Gamma_u)$, the conclusion follows from standard L^2-theory and we will present the argument in this case at the end of the proof. In the remaining case we argue by density. Note that, arguing as in the proof of Proposition 2.24, a localization argument and [150, Theorem 7.2.1] (applied with time as a dummy variable) yield the following a-priori estimates for solutions $(u,\pi)\in W^{2,q}(\mathcal{O};\mathbb{R}^2)\times W^{1,q}(\mathcal{O})/\mathbb{R}$ to (2.3):

$$\begin{aligned}\|u\|_{W^{2,q}(\mathcal{O};\mathbb{R}^2)}+\|\nabla\pi\|_{W^{1,q}(\mathcal{O};\mathbb{R}^2)}&\leq C\|u\|_{W^{2-2/q,q}(\mathcal{O};\mathbb{R}^2)}+C\|g\|_{W^{1-1/q,q}(\Gamma_u)}\\&\leq\varepsilon\|u\|_{W^{2,q}(\mathcal{O};\mathbb{R}^2)}+C_\varepsilon\|u\|_{W^{1,q}(\mathcal{O};\mathbb{R}^2)}+C\|g\|_{W^{1-1/q,q}(\Gamma_u)}\\&\leq\varepsilon\|u\|_{W^{2,q}(\mathcal{O};\mathbb{R}^2)}+C_\varepsilon\|g\|_{W^{1-1/q,q}(\Gamma_u)},\end{aligned}$$

where $\varepsilon>0$ is arbitrary and in the last step we applied Step 1.

The above shows $\|u\|_{W^{2,q}(\mathcal{O};\mathbb{R}^2)}+\|\nabla\pi\|_{W^{1,q}(\mathcal{O})}\lesssim\|g\|_{W^{1-1/q,q}(\mathbb{T})}$ for all solutions $(u,\pi)\in W^{2,q}(\mathcal{O};\mathbb{R}^2)\times W^{1,q}(\mathcal{O})/\mathbb{R}$ to (2.3). Combining this, the density of $C^\infty(\Gamma_u)$ in $W^{1-1/q,q}(\Gamma_u)$, and the above mentioned solvability for $g\in C^\infty(\Gamma_u)$; one readily obtains the existence of solutions to (2.3) in the class $W^{2,q}(\mathcal{O})\times W^{1,q}(\mathcal{O})/\mathbb{R}$.

Step 3: Proof of the regularity of (u,π) *in case of* $g\in C^\infty(\mathbb{T})$. The proof of this fact follows the lines of Proposition 2.22. First, by Lax-Milgram Lemma and [168, Proposition 1.1, Proposition 1.2], there exists a unique couple, $(u,\pi)\in\mathbb{H}^1\times L^2(\mathcal{O})$ such that

$$\int_{\mathcal{O}}\nabla u:\nabla\varphi\,dx=-\int_{\mathbb{T}}g(x)\varphi_1(x,a)\,d\sigma(x)\quad\forall\varphi\in\mathbb{H}^1\tag{2.10}$$

$$\langle-\Delta u+\nabla\pi,\varphi\rangle=0\quad\forall\varphi\in C_c^\infty(\mathcal{O};\mathbb{R}^2)\tag{2.11}$$

$$\|u\|_{\mathbb{H}^1}+\|\pi\|_{L^2/\mathbb{R}}\lesssim\|g\|_{H^{-1/2}(\Gamma_u)}.\tag{2.12}$$

Now, let us fix $h>0$, extend periodically either u and g in the x direction and consider $\varphi=\tau_h\tau_{-h}u$ as a test function in (2.10), where $\tau_hv=\frac{v(x+h,z)-v(x,z)}{h}$. Then by change of variables, it follows that

$$\begin{aligned}\|\tau_h\nabla u\|^2_{L^2(\mathcal{O})}&\leq C\|\tau_hg\|_{L^2(\Gamma_u)}\|\tau_hu\|_{L^2(\Gamma_u)}\\&\leq C\|\tau_hg\|_{L^2(\Gamma_u)}\|\tau_hu\|_{L^2(\mathcal{O})}+C\|\tau_hg\|_{L^2(\Gamma_u)}\|\tau_h\nabla u\|_{L^2(\mathcal{O})}\\&\leq C\|g\|_{C^1(\Gamma_u)}\|\tau_hu\|_{L^2(\mathcal{O})}+\frac{\|\tau_h\nabla u\|^2_{L^2(\mathcal{O})}}{2}+C\|g\|^2_{C^1(\Gamma_u)}.\end{aligned}$$

Therefore

$$\|\tau_h\nabla u\|^2_{L^2(\mathcal{O})}\leq C\|g\|_{C^1(\Gamma_u)}\|\tau_hu\|_{L^2(\mathcal{O})}+C\|g\|^2_{C^1(\Gamma_u)}.\tag{2.13}$$

Since $u\in\mathbb{H}^1$ and (2.12) holds the right hand side of inequality (2.13) is uniformly bounded in $h\to0$ and this implies

$$\|\partial_1\nabla u\|^2_{L^2(O)}\leq C\|g\|^2_{C^1(\Gamma_u)}.\tag{2.14}$$

Let us now consider $\varphi = \partial_1\psi$, $\psi \in \mathcal{D}(\mathcal{O})$ as test function in (2.10). Thanks to [168, Proposition 1.1, Proposition 1.2], $\partial_1\pi \in L^2(\mathcal{O})$ and $\|\partial_1\pi\| \lesssim \|g\|_{C^1(\Gamma_u)}$. Since u is divergence free and (2.11) holds, then

$$\partial_2\pi = \partial_{1,1}u_2 - \partial_{1,2}u_1 \in L^2(\mathcal{O}).$$

Therefore $\|\nabla\pi\| \lesssim \|g\|_{C^1(\Gamma_u)}$. Lastly, again by relation (2.11)

$$\partial_{2,2}u_1 = \partial_1\pi - \partial_{1,2}u_1 \in L^2(\mathcal{O}).$$

Combining all the information obtained we get

$$\|u\|_{H^2(\mathcal{O};\mathbb{R}^2)} + \|\pi\|_{H^1(\mathcal{O})/\mathbb{R}} \leq C\|g\|^2_{C^1(\Gamma_u)}$$

Iterating the argument one gets that $(u,\pi) \in H^{k+1}(\mathcal{O};\mathbb{R}^2) \times H^k(\mathcal{O})$ provided $g \in C^k(\Gamma_u)$ for all $k \geq 1$. Now the claim of Step 3 follows from Sobolev embeddings. ☐

Next we denote by $\mathcal{N}$ the solution map defined by Theorem 2.4 which associate to a boundary datum g the velocity u solution of (2.3), i.e. $\mathcal{N}[g] := u$. From the above result we obtain

Corollary 2.5. *Let $\mathcal{N}$ and $\mathcal{U}$ be the Neumann map and a Hilbert space, respectively. Then, for all $q \geq 2$ and $\varepsilon > 0$,*

1. $\mathcal{N} \in \mathscr{L}(W^{-\alpha,q}(\Gamma_u;\mathcal{U}); \gamma(\mathcal{U}, \mathsf{D}(A_q^{\frac{1-\alpha}{2}+\frac{1}{2q}-\varepsilon})))$ *for* $\alpha \in [0, \frac{1}{q}]$.
2. $\mathcal{N} \in \mathscr{L}(L^q(\Gamma_u;\mathcal{U}); \gamma(\mathcal{U}, \mathsf{D}(A_q^{\frac{1}{2}+\frac{1}{2q}-\varepsilon})))$.

Proof. To begin, recall that $W^{s,q}(\Gamma_u;\mathcal{U}) = \gamma(\mathcal{U}, W^{s,q}(\Gamma_u))$ for all $s \in \mathbb{R}$ and $q \in (1,\infty)$, see section 1.2. Hence, due to the ideal property of γ-radonifying operators [102, Theorems 9.1.10 and 9.1.20], it is enough to consider the scalar case $\mathcal{U} = \mathbb{R}$.

(1): By interpolating with the real method $(\cdot,\cdot)_{\theta,q}$ where $\theta \in (0,1)$ (see e.g. [20, Theorem 6.4.5]), the estimates in Theorem 2.4 yield

$$\mathcal{N} : W^{\theta-1/q,q}(\Gamma_u) \to W^{\theta+1,q}(\mathcal{O}) \quad \text{for all } \theta \in (0,1).$$

Moreover, by construction $\mathcal{N}[u]$ satisfies

$$\mathcal{N}[u]|_{\Gamma_b} = 0, \qquad \text{and} \qquad (\mathcal{N}[u])_2|_{\Gamma_u} = 0,$$

where $(\cdot)_2$ denotes the second component. Hence (1) follows from the description of the fractional power of A_q in Theorem 2.3 and that $B^{1+\theta}_{q,q}(\mathcal{O};\mathbb{R}^2) \hookrightarrow H^{\theta+1-\varepsilon,q}(\mathcal{O};\mathbb{R}^2)$.

(2): Follows from (1) and $L^q(\Gamma_u) \hookrightarrow B^0_{q,q}(\Gamma_u)$ as $q \geq 2$. ☐

2.1.3 Deterministic Navier-Stokes equations

Let us consider the deterministic Navier-Stokes equations with homogeneous boundary conditions

$$\begin{cases} \partial_t \overline{u} + \overline{u} \cdot \nabla \overline{u} + \nabla \overline{\pi} = \Delta \overline{u} + \overline{f}, & \text{on } (0,T) \times \mathcal{O}, \\ \operatorname{div} \overline{u} = 0, & \text{on } (0,T) \times \mathcal{O}, \\ \overline{u} = 0, & \text{on } (0,T) \times \Gamma_b, \\ \partial_2 \overline{u}_1 = 0, & \text{on } (0,T) \times \Gamma_u, \\ \overline{u}_2 = 0, & \text{on } (0,T) \times \Gamma_u, \\ \overline{u}(0) = \overline{u}_0, & \text{on } \mathcal{O}. \end{cases} \tag{2.15}$$

Define the trilinear form $b : \mathbb{L}^4 \times \mathbb{H}^1 \times \mathbb{L}^4 \to \mathbb{R}$ as

$$b\left(u,v,w\right) = \sum_{i,j=1}^{2} \int_{\mathcal{O}} u_i \partial_i v_j w_j \, dx = \int_{\mathcal{O}} \left(u \cdot \nabla v\right) \cdot w \, dx \tag{2.16}$$

which is well–defined and continuous on $\mathbb{L}^4 \times \mathbb{H}^1 \times \mathbb{L}^4$ by the Hölder's inequality. Since by Sobolev embedding theorem $\mathbb{H}^1 \subset \mathbb{L}^4$, b is also defined and continuous on $\mathbb{H}^1 \times \mathbb{H}^1 \times \mathbb{H}^1$. Moreover, by standard interpolation inequalities,

$$\|f\|^2_{L^4(\mathcal{O})} \leq C \|f\|_{L^2(\mathcal{O})} \|f\|_{H^1(\mathcal{O})} \quad \text{for all } f \in H^1\left(\mathcal{O}\right). \tag{2.17}$$

Integrating by parts, the standard oddity relation below holds

$$b\left(u,v,w\right) = -b\left(u,w,v\right)$$

if $u \in \mathbb{L}^4$, $v, w \in \mathbb{H}^1$.

Lastly we introduce the operator

$$B : \mathbb{L}^4 \times \mathbb{L}^4 \to \mathbb{H}^{-1}$$

defined by the identity

$$\left\langle B\left(u,v\right), \varphi \right\rangle = -b\left(u,\varphi,v\right) = -\int_{\mathcal{O}} \left(u \cdot \nabla \varphi\right) \cdot v \, dx$$

for all $\varphi \in \mathbb{H}^1$. When $v \in \mathbb{H}^1$, we may also write

$$\left\langle B\left(u,v\right), \varphi \right\rangle = b(u,v,\varphi).$$

Moreover, when $u \cdot \nabla v \in L^2\left(\mathcal{O}; \mathbb{R}^2\right)$, it is explicitly given by

$$B\left(u,v\right) = \mathbf{P}(u \cdot \nabla v).$$

We have to define our notion of weak solution for problem (2.15).

Definition 2.6. Given $\overline{u}_0 \in H$ and $\overline{f} \in L^2\left(0, T; \mathbb{H}^{-1}\right)$, we say that

$$\overline{u} \in C\left([0, T]; H\right) \cap L^2\left(0, T; \mathbb{H}^1\right)$$

is a weak solution of equation (2.15) if for all $\varphi \in \mathsf{D}(A)$ and $t \in [0, T]$,

$$\begin{aligned} &\langle \overline{u}(t), \varphi \rangle - \int_0^t b\left(\overline{u}(s), \varphi, \overline{u}(s)\right) ds \\ &= \langle \overline{u}_0, \varphi \rangle - \int_0^t \langle \overline{u}(s), A\varphi \rangle \, ds + \int_0^t \left\langle \overline{f}(s), \varphi \right\rangle_{\mathbb{H}^{-1}, \mathbb{H}^1} ds. \end{aligned}$$

The following results are simple adaptations of classical results, see for instance [77, 130, 167, 168].

Lemma 2.7. *If* $u, v \in L^4\left(0, T; \mathbb{L}^4\right)$ *then*

$$B(u, v) \in L^2\left(0, T; \mathbb{H}^{-1}\right). \tag{2.18}$$

Moreover,

$$|b(u, v, w)| \leq \varepsilon \|v\|_{\mathbb{H}^1}^2 + \varepsilon' \|u\|_{\mathbb{H}^1}^2 + \frac{C}{\varepsilon^2 \varepsilon'} \|u\|^2 \|w\|_{\mathbb{L}^4}^4 \tag{2.19}$$

$$|b(u, v, w)| \leq \varepsilon \|v\|_{\mathbb{H}^1}^2 + \varepsilon' \|w\|_{\mathbb{H}^1}^2 + \frac{C}{\varepsilon^2 \varepsilon'} \|w\|^2 \|u\|_{\mathbb{L}^4}^4, \tag{2.20}$$

where C *is a constant independent of* ε *and* ε'.

Theorem 2.8. *For every* $\overline{u}_0 \in H$ *and* $\overline{f} \in L^2\left(0, T; \mathbb{H}^{-1}\right)$ *there exists a unique weak solution of equation* (2.15). *It satisfies*

$$\|\overline{u}(t)\|^2 + 2\nu \int_0^t \|\nabla \overline{u}(s)\|^2 ds = \|\overline{u}_0\|^2 + 2 \int_0^t \left\langle \overline{f}(s), \overline{u}(s) \right\rangle_{\mathbb{H}^{-1}, \mathbb{H}^1} ds.$$

If $\left(\overline{u}_0^n\right)_{n \in \mathbb{N}}$ *is a sequence in* H *converging to* $\overline{u}_0 \in H$ *and* $\left(\overline{f}^n\right)_{n \in \mathbb{N}}$ *is a sequence in* $L^2\left(0, T; \mathbb{H}^{-1}\right)$ *converging to* $\overline{f} \in L^2\left(0, T; \mathbb{H}^{-1}\right)$, *then the corresponding unique solutions* $\left(\overline{u}^n\right)_{n \in \mathbb{N}}$ *converge to the corresponding solution* $\overline{u}$ *in* $C\left([0, T]; H\right)$ *and in* $L^2\left(0, T; \mathbb{H}^1\right)$.

2.1.4 Stochastic Maximal L^p-regularity

Let $\mathcal{U}$ and $(W_{\mathcal{U}}(t))_{t \geq 0}$ be a Hilbert space and a cylindrical $\mathcal{F}-$Brownian motion on $\mathcal{U}$, respectively. The following result was proven in [173], see also [174, Section 7] and [5, Section 3] for additional references. Below, for a Banach space Y, $H^{s,q}(\mathbb{R}_+; Y)$ denotes the Y-valued Bessel potential space on $\mathbb{R}_+$ with smoothness $s \in \mathbb{R}$ and integrability q; such space can be defined either by complex interpolation (see e.g. [150, Chapter 3, Section 4.5]) or by restriction from $\mathbb{R}$ (see e.g. [2, Subsection 3.1]). For the notion of H^∞-calculus and γ-radonifying operators $\gamma(\mathcal{U}, Y)$ we refer to [102, Chapter 9 and 10].

Theorem 2.9. *Let X be a Banach space isomorphic to a closed subspace of $L^q(D,\mu)$ where $q \in [2,+\infty)$ and $(D,\mathscr{A},\mu)$ is a σ-finite measure space. Let $\mathcal{A}$ be an invertible operator and assume that it admits a bounded H^∞ calculus of angle $< \pi/2$ on X and let $(\mathcal{S}(t))_{t\geq 0}$ the bounded analytic semigroup generated by $-\mathcal{A}$. For all $\mathcal{F}-$adapted $G \in L^p(\mathbb{R}_+ \times \Omega; \gamma(\mathcal{U};X))$ the stochastic convolution process*

$$U(t) = \int_0^t \mathcal{S}(t-s)G(s)\, dW_{\mathcal{U}}(s) \quad t \geq 0,$$

is well defined in X, takes values in the fractional domain $\mathsf{D}(\mathcal{A}^{1/2})$ almost surely and for all $2 < p < +\infty$ the following space-time regularity estimate holds: $\forall \theta \in [0,\frac{1}{2})$

$$\mathbb{E}\left[\|U(t)\|^p_{H^{\theta,p}(\mathbb{R}_+;\mathsf{D}(\mathcal{A}^{1/2-\theta}))}\right] \leq C_\theta^p \mathbb{E}\left[\|G\|^p_{L^p(\mathbb{R}_+;L^q(O;\mathcal{U}))}\right] \tag{2.21}$$

with a constant C_θ independent of G.

2.2 Well-Posedness

2.2.1 Stokes Equations

As discussed in section 1.3, we start by considering the linear problem

$$\begin{cases} \partial_t w + \nabla\rho = \Delta w, & \text{on } (0,T)\times\mathcal{O}, \\ \operatorname{div} w = 0, & \text{on } (0,T)\times\mathcal{O}, \\ w = 0, & \text{on } (0,T)\times\Gamma_b, \\ \partial_2 w_1 = h_b \dot{W}_{\mathcal{U}}, & \text{on } (0,T)\times\Gamma_u, \\ w_2 = 0, & \text{on } (0,T)\times\Gamma_u, \\ w(0) = 0, & \text{on } \mathcal{O}. \end{cases}$$

According to [53, 54], the mild solution w of the former problem is formally given by

$$w(t) = A_q \int_0^t S_q(t-s)\mathcal{N}[h_b(s)]\, dW_{\mathcal{U}}(s). \tag{2.22}$$

Here A_q is (minus) the Stokes operator with homogeneous boundary conditions, and $(S_q(t))_{t\geq 0}$ its corresponding semigroup (cf. Theorem 2.3).

Next step is to prove that $w(t)$ is well defined in some functional spaces and has some regularities useful to treat the nonlinearity of the Navier-Stokes equations.

Proposition 2.10. *Let $\alpha \in [0,\frac{1}{q}]$ and assume that $h_b \in L^p(0,T;W^{-\alpha,q}(\Gamma_u))$. Then the process w defined in (2.22) is a well defined process with $\mathbb{P}-a.s.$ paths in*

$$H^{\theta,p}(0,T;\mathsf{D}(A_q^{\frac{1}{2q}-\frac{\alpha}{2}-\theta-\varepsilon})) \quad \textit{for all} \quad \theta\in[0,\tfrac{1}{2}),\ \varepsilon > 0.$$

In particular, if h_b *satisfies the assumptions of* Theorem 2.1*, then* w *has* $\mathbb{P}-a.s.$ *trajectories in* $C([0,T];H)\cap L^4(0,T;\mathbb{L}^4)$*.*

Proof. Let $\varepsilon>0$ be fixed later. From Corollary 2.5 and Theorem 2.9 we have that $\mathbb{P}-a.s.$ and for each $\theta\in[0,\frac{1}{2})$

$$\tilde{w}=\int_0^{\cdot} S_q(\cdot-s)A_q^{\frac{1-\alpha}{2}+\frac{1}{2q}-\varepsilon}\mathcal{N}[h_b(s)]\,dW_{\mathcal{U}}(s)\in H^{\theta,p}(0,T;\mathsf{D}(A_q^{1/2-\theta}))$$

Therefore, $\mathbb{P}-a.s.$,

$$w=A_q^{\frac{1+\alpha}{2}-\frac{1}{2q}+\varepsilon}\tilde{w}\in H^{\theta,p}(0,T;\mathsf{D}(A_q^{\frac{1}{2q}-\frac{\alpha}{2}-\theta-\varepsilon})).$$

Finally, note that under the assumptions of Theorem 2.1, Theorem 2.3 and the Sobolev embeddings (see e.g. [5, Proposition 2.7]) we can find $\theta_1,\theta_2\in[0,\frac{1}{2})$ and $\varepsilon>0$ such that

$$H^{\theta_1,p}(0,T;\mathsf{D}(A_q^{\frac{1}{2q}-\frac{\alpha}{2}-\theta_1-\varepsilon}))\hookrightarrow C([0,T];H),$$
$$H^{\theta_2,p}(0,T;\mathsf{D}(A_q^{\frac{1}{2q}-\frac{\alpha}{2}-\theta_2-\varepsilon}))\hookrightarrow L^4(0,T;\mathbb{L}^4).$$

Hence the proof is complete. □

2.2.2 Auxiliary Navier–Stokes Type Equations

Having solved the Stokes problem we introduce the auxiliary variable

$$v\,(t)=u\,(t)-w(t),$$

which satisfies (1.16), i.e.

$$\begin{cases}\partial_t v+(v+w)\cdot\nabla\,(v+w)+\nabla\,(P-\rho)=\Delta v, & \text{on } (0,T)\times\mathcal{O},\\ \operatorname{div} v=0, & \text{on } (0,T)\times\mathcal{O},\\ v=0, & \text{on } (0,T)\times\Gamma_b,\\ \partial_2 v_1=0, & \text{on } (0,T)\times\Gamma_u,\\ v_2=0, & \text{on } (0,T)\times\Gamma_u,\\ v(0)=u_0, & \text{on } \mathcal{O}.\end{cases}$$

This first equation in the above system has the form

$$\partial_t v+v\cdot\nabla v+\nabla\pi=\nu\Delta v-L\,(v,w)$$

with the affine function

$$L\,(v,w)=v\cdot\nabla w+w\cdot\nabla v+w\cdot\nabla w.$$

For each $\omega \in \Omega$ fixed, the Navier–Stokes structure is preserved and the auxiliary equation for v with homogeneous boundary conditions is solvable similarly to the classical Navier–Stokes equations. Therefore, let us introduce the notion of weak solution of the deterministic problem (1.16) with random coefficients. Recall that A and b are (minus) the Stokes operator on $\mathbb{L}^2$ and defined in (2.16), respectively.

Definition 2.11. Given $u_0 \in H$ and $w \in L^4\left(0,T;\mathbb{L}^4\right)$, we say that

$$v \in C\left([0,T];H\right) \cap L^2\left(0,T;\mathbb{H}^1\right)$$

is a weak solution of equation (1.16) if

$$\begin{aligned}&\langle v(t),\varphi\rangle - \int_0^t b\left(v(s)+w(s),\varphi,v(s)+w(s)\right)ds\\ &= \langle u_0,\varphi\rangle - \int_0^t \langle v(s),A\varphi\rangle\, ds\end{aligned}$$

for every $\varphi \in \mathsf{D}(A)$ and $t \in [0,T]$.

Theorem 2.12. *For every $u_0 \in H$ and $w \in L^4\left(0,T;\mathbb{L}^4\right)$, there exists a unique weak solution v of equation (1.16). Moreover, v satisfies for all $t \in [0,T]$*

$$\|v(t)\|^2 + 2\int_0^t \|\nabla v(s)\|_{L^2}^2\, ds = \|u_0\|^2 + 2\int_0^t \left(b(v,v,w)+b(w,v,w)\right)(s)\, ds. \tag{2.23}$$

If $\left(u_0^n\right)_{n\in\mathbb{N}}$ is a sequence in H converging to $u_0 \in H$ and $\left(w^n\right)_{n\in\mathbb{N}}$ is a sequence in $L^4\left(0,T;\mathbb{L}^4\right)$ converging to $w \in L^4\left(0,T;\mathbb{L}^4\right)$, then the corresponding unique solutions $\left(v^n\right)_{n\in\mathbb{N}}$ converge to the corresponding solution v in $C\left([0,T];H\right)$ and in $L^2\left(0,T;\mathbb{H}^1\right)$.

Proof. We split the proof into several steps.

Step 1: Uniqueness. Let $v^{(i)}$ be two solutions. The function $z = v^{(1)} - v^{(2)}$ satisfies

$$\begin{aligned}&\langle z(t),\varphi\rangle - \int_0^t \left(b\left(v^{(1)}+w,\varphi,v^{(1)}+w\right) - b\left(v^{(2)}+w,\varphi,v^{(2)}+w\right)\right)ds\\ &= -\int_0^t \langle z(s),A\varphi\rangle\, ds\end{aligned}$$

for every $\varphi \in \mathsf{D}(A)$. A simple manipulation gives us

$$\begin{aligned}&b\left(v^{(1)}+w,\varphi,v^{(1)}+w\right) - b\left(v^{(2)}+w,\varphi,v^{(2)}+w\right) - b(z,\varphi,z)\\ &= b\left(v^{(2)}+w,\varphi,z\right) + b\left(z,\varphi,v^{(2)}+w\right)\end{aligned}$$

hence

$$\langle z(t), \varphi\rangle - \int_0^t b(z(s), \varphi, z(s))\, ds = -\int_0^t \langle z(s), A\varphi\rangle\, ds + \int_0^t \left\langle \tilde{f}(s), \varphi\right\rangle ds$$

where

$$\tilde{f} = -B\left(v^{(2)} + w, z\right) - B\left(z, v^{(2)} + w\right).$$

By Lemma 2.7, $\tilde{f} \in L^2\left(0, T; \mathbb{H}^{-1}\right)$. Then, by Theorem 2.8,

$$\|z(t)\|^2 + 2\int_0^t \|\nabla z(s)\|_{L^2}^2\, ds = 2\int_0^t b\left(z, z, v^{(2)} + w\right)(s)\, ds.$$

Again by Lemma 2.7, we have

$$\begin{aligned}\left|b\left(z, z, v^{(2)} + w\right)\right| &\le \left|b\left(z, z, v^{(2)}\right)\right| + |b(z, z, w)| \\ &\le \varepsilon\|z\|_{\mathbb{H}^1}^2 + \varepsilon\|z\|_{\mathbb{H}^1}^2 + \frac{C}{\varepsilon^3}\|z\|^2\|v^{(2)}\|_{\mathbb{L}^4}^4 \\ &\quad + \varepsilon\|z\|_{\mathbb{H}^1}^2 + \varepsilon\|z\|_{\mathbb{H}^1}^2 + \frac{C}{\varepsilon^3}\|z\|^2\|w\|_{\mathbb{L}^4}^4 \\ &= 4\varepsilon\|z\|_{\mathbb{H}^1}^2 + \frac{C}{\varepsilon^3}\|z\|^2\left(\|v^{(2)}\|_{\mathbb{L}^4}^4 + \|w\|_{\mathbb{L}^4}^4\right).\end{aligned}$$

Summarizing, with $4\varepsilon = 1$, using the fact that $\|z\|_{\mathbb{H}^1}^2 = \|\nabla z\|_{L^2}^2$, renaming the constant C,

$$\|z(t)\|^2 + \int_0^t \|\nabla z(s)\|_{L^2}^2\, ds \le C\int_0^t \|z(s)\|^2\left(1 + \|v^{(2)}(s)\|_{\mathbb{L}^4}^4 + \|w(s)\|_{\mathbb{L}^4}^4\right) ds.$$

We conclude $z = 0$ by the Grönwall's lemma, using the assumption on w and the integrability properties of $v^{(2)}$.

Step 2: Existence. Define the sequence (v^n) by setting $v^0 = 0$ and for every $n \ge 0$, given $v^n \in C([0, T]; H) \cap L^2\left(0, T; \mathbb{H}^1\right)$, let v^{n+1} be the solution of equation (2.15) with initial condition u_0 and with

$$-B(v^n, w) - B(w, v^n) - B(w, w)$$

in place of f. In particular

$$\begin{aligned}&\left\langle v^{n+1}(t), \varphi\right\rangle - \int_0^t b\left(v^{n+1}(s), \varphi, v^{n+1}(s)\right) ds \\ &= \langle u_0, \varphi\rangle - \int_0^t \left\langle v^{n+1}(s), A\varphi\right\rangle ds \\ &- \int_0^t \langle (B(v^n, w) + B(w, v^n) + B(w, w))(s), \varphi\rangle\, ds\end{aligned}$$

for every $\varphi \in \mathsf{D}(A)$. In order to claim that this definition is well done, we notice that

$$B(v^n, w), B(w, v^n), B(w, w) \in L^2\left(0, T; \mathbb{H}^{-1}\right)$$

by Lemma 2.7.

Then let us investigate the convergence of (v^n). First, let us prove a bound. From the previous identity and Theorem 2.8 we get

$$\begin{aligned} &\|v^{n+1}(t)\|^2 + 2\int_0^t \|\nabla v^{n+1}(s)\|_{L^2}^2\, ds \\ &= \|u_0\|^2 + 2\int_0^t \left(b\left(v^n, v^{n+1}, w\right) + b\left(w, v^{n+1}, v^n\right) + b\left(w, v^{n+1}, w\right)\right)(s)\ ds. \end{aligned}$$

It gives us (using Lemma 2.7)

$$\begin{aligned} &\|v^{n+1}(t)\|^2 + \int_0^t \|\nabla v^{n+1}(s)\|_{L^2}^2\, ds \\ &= \|u_0\|^2 + \varepsilon\int_0^t \|v^n(s)\|_{\mathbb{H}^1}^2\, ds \\ &+ C_\varepsilon \int_0^t \|v^n(s)\|^2\left(1 + \|w(s)\|_{\mathbb{L}^4}^4\right)\ ds + C_\varepsilon\int_0^t \|w(s)\|_{\mathbb{L}^4}^4\, ds. \end{aligned}$$

Choosing a small constant ε, one can find $R > \|u_0\|^2$ and $\overline{T}$ small enough, depending only from $\|u_0\|$ and $\|w\|_{L^4(0,T;\mathbb{L}^4)}$, such that if

$$\sup_{t\in\left[0,\overline{T}\right]} \|v^n(t)\|^2 \le R, \qquad \int_0^{\overline{T}} \|v^n(s)\|_{\mathbb{H}^1}^2\, ds \le R \tag{2.24}$$

then the same inequalities hold for v^{n+1}.

Set $w_n = v^n - v^{n-1}$, for $n \ge 1$. From the identity above,

$$\begin{aligned} &\langle w_{n+1}(t), \varphi\rangle - \int_0^t \left(b\left(v^{n+1}, \varphi, v^{n+1}\right) - b\left(v^n, \varphi, v^n\right)\right)(s)\ ds \\ &= -\int_0^t \langle w_{n+1}(s), A\varphi\rangle\ ds - \int_0^t \left\langle\left(B(v^n, w) - B\left(v^{n-1}, w\right)\right)(s), \varphi\right\rangle\ ds \\ &- \int_0^t \left\langle\left(B(w, v^n) - B\left(w, v^{n-1}\right)\right)(s), \varphi\right\rangle\ ds. \end{aligned}$$

Again as above, since

$$\begin{aligned} &b\left(v^{n+1}, \varphi, v^{n+1}\right) - b\left(v^n, \varphi, v^n\right) - b\left(w_{n+1}, \varphi, w_{n+1}\right) \\ &= b\left(v^n, \varphi, w_{n+1}\right) + b\left(w_{n+1}, \varphi, v^n\right) \end{aligned}$$

we may rewrite it as

$$\begin{aligned}&\left\langle w_{n+1}\left(t\right),\varphi\right\rangle - \int_0^t b\left(w_{n+1}\left(s\right),\varphi,w_{n+1}\left(s\right)\right)\,ds\\&= -\int_0^t \left\langle w_{n+1}\left(s\right),A\varphi\right\rangle\,ds - \int_0^t \left\langle \left(B\left(w_n,w\right)+B\left(w,w_n\right)\right)\left(s\right),\varphi\right\rangle\,ds\\&+\int_0^t \left(b\left(v^n,\varphi,w_{n+1}\right)+b\left(w_{n+1},\varphi,v^n\right)\right)\left(s\right)\,ds.\end{aligned}$$

One can check as above the applicability of Theorem 2.8 and get

$$\begin{aligned}&\left\|w_{n+1}\left(t\right)\right\|^2 + 2\int_0^t \left\|\nabla w_{n+1}\left(s\right)\right\|_{L^2}^2\,ds\\&= 2\int_0^t \left(b\left(w_n,w_{n+1},w\right)+b\left(w,w_{n+1},w_n\right)\right)\left(s\right)\,ds\\&+2\int_0^t b\left(w_{n+1},w_{n+1},v^n\right)\left(s\right)\,ds.\end{aligned}$$

As above we deduce

$$\left|b\left(w_{n+1},w_{n+1},v^n\right)\right| \le \frac{1}{2}\|w_{n+1}\|_{\mathbb{H}^1}^2 + C\|w_{n+1}\|^2\|v^n\|_{\mathbb{L}^4}^4.$$

But

$$\left|b\left(w_n,w_{n+1},w\right)+b\left(w,w_{n+1},w_n\right)\right| \le \frac{1}{2}\|w_{n+1}\|_{\mathbb{H}^1}^2 + \frac{1}{4}\|w_n\|_{\mathbb{H}^1}^2 + C\|w_n\|^2\|w\|_{\mathbb{L}^4}^4.$$

Hence

$$\begin{aligned}&\left\|w_{n+1}\left(t\right)\right\|^2 + \int_0^t \left\|\nabla w_{n+1}\left(s\right)\right\|_{L^2}^2\,ds\\&\le C\int_0^t \left\|w_{n+1}\left(s\right)\right\|^2\left(1+\left\|v^n\left(s\right)\right\|_{\mathbb{L}^4}^4\right)\,ds\\&+\frac{1}{4}\int_0^t \left\|w_n\left(s\right)\right\|_{\mathbb{H}^1}^2\,ds + C\int_0^t \left\|w_n\left(s\right)\right\|^2\left\|w\left(s\right)\right\|_{\mathbb{L}^4}^4\,ds.\end{aligned}$$

Now we work under the bounds (2.24) and deduce, using the Grönwall's Lemma, for $\overline{T}$, depending only from $\|u_0\|$ and $\|w\|_{L^4(0,T;\mathbb{L}^4)}$, possibly smaller than the previous one,

$$\begin{aligned}&\sup_{t\in\left[0,\overline{T}\right]}\left\|w_{n+1}\left(t\right)\right\|^2 + \int_0^{\overline{T}} \left\|w_{n+1}\left(s\right)\right\|_{\mathbb{H}^1}^2\,ds\\&\le \frac{1}{2}\left(\sup_{t\in\left[0,\overline{T}\right]}\left\|w_n\left(t\right)\right\|^2 + \int_0^{\overline{T}} \left\|w_n\left(s\right)\right\|_{\mathbb{H}^1}^2\,ds\right).\end{aligned}$$

It implies that the sequence (v^n) is Cauchy in $C\left(\left[0,\overline{T}\right];H\right)\cap L^2\left(0,\overline{T};\mathbb{H}^1\right)$. The limit v has the right regularity to be a weak solution and satisfies the weak formulation; in the identity above for v^{n+1} and v^n we may prove that

$$\begin{aligned}\int_0^t b\left(v^{n+1}(s),\varphi,v^{n+1}(s)\right)ds &\to \int_0^t b\left(v(s),\varphi,v(s)\right)ds\\ \int_0^t b\left(v^n(s),\varphi,w(s)\right)ds &\to \int_0^t b\left(v(s),\varphi,w(s)\right)ds\\ \int_0^t b\left(w(s),\varphi,v^n(s)\right)ds &\to \int_0^t b\left(w(s),\varphi,v(s)\right)ds.\end{aligned}$$

All these convergences can be proved easily by recalling the definition of b. Similarly, we can pass to the limit in the energy identity. After proving existence and uniqueness in $[0,\overline{T}]$ we can reiterate the existence procedure and in a finite number of steps cover the interval $[0,T]$.

Step 3: Continuous dependence on the data. Let v^n (resp. v) the unique solution of (1.16) with data u_0^n, w^n (resp. u_0, w). Since $u_0^n\to u_0$ in H (resp. $w^n\to w$ in $L^4(0,T;\mathbb{L}^4)$) the family $(u_0^n)_{n\in\mathbb{N}}$ is bounded in H (resp. the family $(w^n)_{n\in\mathbb{N}}$ is bounded in $L^4(0,T;\mathbb{L}^4)$), by (2.23) one can show easily that the family $(v^n)_{n\in\mathbb{N}}$ is bounded in $C([0,T];H)\cap L^2(0,T;\mathbb{H}^1)\hookrightarrow L^4(0,T;\mathbb{L}^4)$. Moreover for each $t\in[0,T]$, $z^n:=v^n-v$ satisfies the energy relation

$$\begin{aligned}\frac{1}{2}\|z^n(t)\|^2+\int_0^t\|\nabla z^n(s)\|^2\,ds=\frac{1}{2}\|u_0^n-u_0\|^2&\\ +\int_0^t b(v^n(s)+w^n(s),z^n(s),w^n(s)-w(s))\,ds&\\ +\int_0^t b(z^n(s),z^n(s),v(s)+w(s))\,ds&\\ +\int_0^t b(w^n(s)-w(s),z^n(s),v(s)+w(s))\,ds.&\end{aligned}\tag{2.25}$$

We can easily bound the right hand side of relation (2.25) by Young's inequality and Hölder's inequality obtaining

$$\begin{aligned}\frac{1}{2}\|z^n(t)\|^2+\frac{1}{2}\int_0^t\|\nabla z^n(s)\|^2\,ds\le\frac{1}{2}\|u_0^n-u_0\|^2&\\ +C\int_0^t\|z^n(s)\|^2\left(\|v(s)\|_{\mathbb{L}^4}^4+\|w(s)\|_{\mathbb{L}^4}^4\right)ds&\\ +C\|w^n-w\|_{L^4(0,T;\mathbb{L}^4)}^2\left(\|w^n\|_{L^4(0,T;\mathbb{L}^4)}^2+\|w\|_{L^4(0,T;\mathbb{L}^4)}^2\right)&\\ +C\|w^n-w\|_{L^4(0,T;\mathbb{L}^4)}^2\left(\|v^n\|_{L^4(0,T;\mathbb{L}^4)}^2+\|v\|_{L^4(0,T;\mathbb{L}^4)}^2\right).&\end{aligned}\tag{2.26}$$

Applying Grönwall's inequality to relation (2.26) the claim follows immediately. □

Remark 2.13. Freezing the variable $\omega \in \Omega$ and solving (1.16) for each ω does not allow to obtain information about the measurability properties of v. However, measurability of v with respect of the progressive σ-algebra follows from the continuity of the solution map with respect to u_0 and w. Therefore we have the required measurability properties for v with w being the mild solution of (1.14). In particular v has $\mathbb{P}$-a.s. paths in $C([0,T];H)\cap L^2(0,T;\mathbb{H}^1)$, it is progressively measurable with respect to these topologies and

$$\begin{aligned} &\langle v\left(t\right),\varphi\rangle - \int_0^t b\left(v\left(s\right)+w\left(s\right),\varphi, v\left(s\right)+w\left(s\right)\right)\, ds \\ &= \langle u_0,\varphi\rangle - \int_0^t \langle v\left(s\right), A\varphi\rangle\, ds \quad \mathbb{P}-a.s. \end{aligned} \tag{2.27}$$

for every $\varphi \in \mathsf{D}\left(A\right)$ and $t\in[0,T]$.

Proof of Theorem 2.1. It follows immediately combining Proposition 2.10, Theorem 2.12 and Remark 2.13. □

Remark 2.14. Without additional difficulties we could also consider in equation (2.1) an additive noise diffused inside the domain of the form $h_d(t)\, d\tilde{W}_{\mathcal{U}}(t)$, where $\tilde{W}_{\mathcal{U}}$ is a cylindrical Brownian motion on $\mathcal{U}$ independent of $W_{\mathcal{U}}$ and $h_d \in L^p\left(0,T;\gamma\left(\mathcal{U},X_{-\lambda,A_q}\right)\right)$, with $p>2,\ q\geq 2,\ \lambda\in[0,\frac{1}{2})$ such that $1-\frac{2}{p}-2\lambda>0$ and there exists $\theta\in[0,\frac{1}{2})$ satisfying

$$\theta \leq \frac{3}{4}-\lambda-\frac{1}{q}, \qquad \theta\geq \frac{1}{p}-\frac{1}{4}.$$

The case $q=p=2$ and $\lambda=0$ is also allowed, see [55, Chapter 5]. To see this, note that, under these assumptions, arguing as in Proposition 2.10 the solution $\tilde{w}$ to

$$\begin{cases} \partial_t \tilde{w} + \nabla\tilde{\rho} = \Delta\tilde{w} + h_d\dot{\tilde{W}}_{\mathcal{U}}, & \text{on } (0,T)\times\mathcal{O}, \\ \operatorname{div}\tilde{w} = 0, & \text{on } (0,T)\times\mathcal{O}, \\ \tilde{w} = 0, & \text{on } (0,T)\times\Gamma_b, \\ \partial_2\tilde{w}_1 = 0, & \text{on } (0,T)\times\Gamma_u, \\ \tilde{w}_2 = 0, & \text{on } (0,T)\times\Gamma_u, \\ \tilde{w}(0)=0, & \text{on } \mathcal{O}, \end{cases} \tag{2.28}$$

can be obtained as a stochastic convolution. In particular, the above assumptions on h_d imply that $\tilde{w}$ is a progressively measurable process with values in $C([0,T];H)\cap L^4(0,T;\mathbb{L}^4)$. Therefore this term adds no difficulties in order to analyze the well-posedness of equation (1.16). For this reason we prefer to not consider this classical source of randomness.

2.3 Interior Regularity

2.3.1 Stokes Equations

We start showing a lemma, concerning the relation between the mild and the weak formulation of (1.14) defined below.

Definition 2.15. Under the same assumptions of Theorem 2.1, we say that a stochastic process w is a weak solution of (1.14) if it is $\mathcal{F}$-progressively measurable with $\mathbb{P}-a.s.$ paths in

$$w \in C([0,T];H) \cap L^4(0,T;\mathbb{L}^4)$$

and for each $\varphi \in \mathsf{D}(A)$

$$\langle w(t), \varphi \rangle = -\int_0^t \langle w(s), A\varphi \rangle \ ds + \int_0^t \langle h_b(s), \varphi \rangle_{W^{-\alpha,q}(\Gamma_u), W^{\alpha,q'}(\Gamma_u)} \, dW_{\mathcal{U}}(s) \tag{2.29}$$

for each $t \in [0,T]$, $\mathbb{P}-a.s.$

Note that the last term in (2.29) is well-defined as $\alpha < 1/2$ and $q' < 2$.

Remark 2.16. In the definition above, the term $\langle h_b(s), \varphi \rangle_{W^{-\alpha,q}(\Gamma_u), W^{\alpha,q'}(\Gamma_u)}$ is given by the following linear operator on $\mathcal{U}$:

$$\mathcal{U} \ni h' \mapsto \langle h_b(s)h', \varphi \rangle_{W^{-\alpha,q}(\Gamma_u), W^{\alpha,q'}(\Gamma_u)} = L_\varphi(h_b(s)h')$$

where $L_\varphi := \langle \cdot, \varphi \rangle_{W^{-\alpha,q}(\Gamma_u), W^{\alpha,q'}(\Gamma_u)}$. By the ideal property of γ-spaces and $\gamma(\mathcal{U}, W^{-\alpha,q}(\Gamma_u)) = W^{-\alpha,q}(\Gamma_u;\mathcal{U})$ we have

$$\|\langle h_b(s), \varphi \rangle_{W^{-\alpha,q}(\Gamma_u), W^{\alpha,q'}(\Gamma_u)}\|_{\mathcal{U}^*} \lesssim \|h_b(s)\|_{W^{-\alpha,q}(\Gamma_u;\mathcal{U})} \|\varphi\|_{W^{\alpha,q'}(\Gamma_u)},$$

a.e. on $(0,T)$. Whence, the stochastic integral in (2.29) is well-defined with scalar value as

$$\begin{aligned}
&\mathbb{E}\left[\left| \int_0^T \langle h_b(s), \varphi \rangle_{W^{-\alpha,q}(\Gamma_u), W^{\alpha,q'}(\Gamma_u)} \, dW_{\mathcal{U}}(s) \right|^2\right] \\
&\quad = \int_0^T \|\langle h_b(s), \varphi \rangle_{W^{-\alpha,q}(\Gamma_u), W^{\alpha,q'}(\Gamma_u)}\|^2_{\mathcal{U}^*} \, ds \\
&\quad \lesssim \int_0^T \|h_b(s)\|^2_{W^{-\alpha,q}(\Gamma_u;\mathcal{U})} \|\varphi\|^2_{W^{-\alpha,q'}(\Gamma_u)} \, ds < \infty,
\end{aligned}$$

where the last estimate follows from the assumptions of Theorem 2.1.

Lemma 2.17. *Under the same assumptions of Theorem 2.1, there exists a unique weak solution of (1.14) in the sense of Definition 2.15 and it is given by the formula (2.22).*

Proof. We split the proof into two steps.

Step 1: There exists a unique weak solution of (1.14) *and it is necessarily given by the mild formula* (2.22). Let $\psi \in C^1([0,T];\mathsf{D}(A))$. Arguing as in the first step of the proof of [77, Theorem 1.7], see also [74, Proposition 17], one can readily check that w satisfies

$$\begin{aligned}\langle w(t),\psi(t)\rangle = \int_0^t \langle w(s),\partial_s\psi(s)\rangle\,ds - \int_0^t \langle w(s),A\psi(s)\rangle\,ds \\ + \int_0^t \langle h_b(s),\psi(s)\rangle_{W^{-\alpha,q}(\Gamma_u),W^{\alpha,q'}(\Gamma_u)}\,dW_{\mathcal{U}}(s)\end{aligned} \tag{2.30}$$

for each $t\in[0,T]$, $\mathbb{P}-a.s.$ The stochastic integral in the relation above is well-defined arguing as in Remark 2.16. Now consider $\varphi\in\mathsf{D}(A^2)$ and use $\psi_t(s)=S_{q'}(t-s)\varphi,\ s\in[0,t]$ as test function in (2.30) obtaining, since $S_{q'}(t)|_H=S(t)$,

$$\langle w(t),\varphi\rangle = \int_0^t \langle h_b(s),S_{q'}(t-s)\varphi\rangle_{W^{-\alpha,q}(\Gamma_u),W^{\alpha,q'}(\Gamma_u)}\,dW_{\mathcal{U}}(s). \tag{2.31}$$

Recalling the definition of the Neumann map $\mathcal{N}$, (2.31) can be rewritten as

$$\langle w(t),\varphi\rangle = \int_0^t \langle \mathcal{N}[h_b(s)],A_{q'}S_{q'}(t-s)\varphi\rangle\,dW_{\mathcal{U}}(s). \tag{2.32}$$

Then, exploiting the self-adjointness property of S_q and A_q we have that weak solutions of (1.14) satisfy the mild formulation. Therefore they are unique.

Step 2: The mild formula (2.22) *is a weak solution of* (1.14) *in the sense of Definition 2.15.* We begin by noticing that w has the required regularity due to Proposition 2.10. Let us test our mild formulation (2.22) against functions $\varphi\in\mathsf{D}(A^2)\subseteq\mathsf{D}(A_{q'}^2)$. It holds, since $S_{q'}(t)|_H=S(t),\ A_{q'}|_{\mathsf{D}(A)}=A$ and exploiting self-adjointness property of S_q and A_q

$$\begin{aligned}\langle w(t),\varphi\rangle &= \int_0^t \langle \mathcal{N}[h_b(s)],AS(t-s)\varphi\rangle\,dW_{\mathcal{U}}(s) \\ &= \int_0^t \langle h_b(s),S(t-s)\varphi\rangle_{W^{-\alpha,q}(\Gamma_u),W^{\alpha,q'}(\Gamma_u)}\,dW_{\mathcal{U}}(s) \quad \mathbb{P}-a.s.,\end{aligned}$$

where in the last step we used the definition of Neumann map. In order to complete the proof of this step it is enough to show that

$$\begin{aligned}\int_0^t \langle h_b(s),S(t-s)\varphi\rangle_{W^{-\alpha,q}(\Gamma_u),W^{\alpha,q'}(\Gamma_u)}\,dW_{\mathcal{U}}(s) = -\int_0^t \langle w(s),A\varphi\rangle\,ds \\ + \int_0^t \langle h_b(s),\varphi\rangle_{W^{-\alpha,q}(\Gamma_u),W^{\alpha,q'}(\Gamma_u)}\,dW_{\mathcal{U}}(s) \quad \mathbb{P}-a.s.\end{aligned} \tag{2.33}$$

Relation (2.33) is true. Indeed,

$$\int_0^t \langle w(s), A\varphi\rangle \, ds = \int_0^t ds \int_0^s \langle \mathcal{N}[h_b(r)], S(s-r)A^2\varphi\rangle \, dW_{\mathcal{U}}(r) \quad \mathbb{P}-a.s. \tag{2.34}$$

The double integrals in (2.34) can be exchanged via stochastic Fubini's Theorem, see [55], since

$$\begin{aligned}&\int_0^t ds \left(\int_0^s dr \|\langle \mathcal{N}[h_b(r)], S(s-r)A^2\varphi\rangle\|_{\mathcal{U}}^2\right)^{1/2} \\ &\le C(T,q)\|A^2\varphi\|_{\mathbb{L}^2}\|h_b\|_{L^2(0,T;W^{-\alpha,q}(\Gamma_u;\mathcal{U}))} < +\infty\end{aligned}$$

Therefore the double integral in the right hand side of (2.34) can be rewritten as

$$\begin{aligned}&\int_0^t ds \int_0^s \langle \mathcal{N}[h_b(r)], S(s-r)A^2\varphi\rangle \, dW_{\mathcal{U}}(r) \\ &= \int_0^t dW_{\mathcal{U}}(r) \int_r^t \langle \mathcal{N}[h_b(r)], S(s-r)A^2\varphi\rangle \, ds \\ &= \int_0^t \langle \mathcal{N}[h_b(r)], A\varphi\rangle \, dW_{\mathcal{U}}(r) \\ &- \int_0^t \langle \mathcal{N}[h_b(r)], AS(t-r)\varphi\rangle \, dW_{\mathcal{U}}(r) \\ &= \int_0^t \langle h_b(r), \varphi\rangle_{W^{-\alpha,q}(\Gamma_u),W^{\alpha,q'}(\Gamma_u)} \, dW_{\mathcal{U}}(r) \\ &- \int_0^t \langle h_b(r), S(t-r)\varphi\rangle_{W^{-\alpha,q}(\Gamma_u),W^{\alpha,q'}(\Gamma_u)} \, dW_{\mathcal{U}}(r) \quad \mathbb{P}-a.s.\end{aligned}$$

Inserting this expression in (2.34), (2.33) holds and the proof is complete. □

Thanks to the weak formulation guaranteed by Lemma 2.17 we can easily obtain the interior regularity of the linear stochastic problem (1.14). Let N_0 be the $\mathbb{P}$ null measure set where at least one between $w \notin C([0,T];H) \cap L^4(0,T;\mathbb{L}^4)$, $v \notin C([0,T];H) \cap L^2(0,T;V)$, (2.29) and (2.27) is not satisfied. In the following we will work pathwise in $\Omega \setminus N_0$ even if not specified.

Corollary 2.18. *Let w be the unique weak solution of (1.14) in the sense of Definition 2.15 under the assumptions of Theorem 2.1. Then, for all $0 < t_1 \le t_2 < T$, $x_0 \in \mathcal{O}$, $r > 0$ such that* $\operatorname{dist}(B(x_0,r), \partial\mathcal{O}) > 0$,

$$w \in C([t_1,t_2], C^\infty(B(x_0,r);\mathbb{R}^2)) \quad \mathbb{P}-a.s.$$

Proof. Denote $\omega_w = \operatorname{curl} w \in C([0,T]; H^{-1}(\mathcal{O}))$ $\mathbb{P}-a.s.$ Since $\operatorname{dist}(B(x_0,r),\partial\mathcal{O}) > 0$, $0 < t_1 \le t_2 < T$ we can find ε small enough such that $0 < t_1 - 2\varepsilon < t_2 + 2\varepsilon < T$, $\operatorname{dist}(B(x_0, r+2\varepsilon), \partial\mathcal{O}) > 0$. Let us consider $\psi \in C_c^\infty(\mathcal{O})$ and use $\nabla^\perp\psi$ as test function in (2.29). This implies that ω_w is a distributional solution of the heat equation

$$\partial_t\omega_w = \Delta\omega_w.$$

Since ω_w solves the heat equation in distributions, a standard localization argument and regularity results for the heat equation (see e.g. [165, Chapter 6, Section 1]) imply that

$$\omega_w \in C([t_1-\varepsilon, t_2+\varepsilon]; C^\infty(B(x_0, r+\varepsilon))) \quad \mathbb{P}-a.s.$$

Let us now consider a test function $\varphi \in C_c^\infty(B(x_0, r+\varepsilon))$ identically equal to one on $B(x_0, r+\varepsilon/2)$. Since $\operatorname{div} w = 0$, we have that $\hat{w} = \varphi w$ solves the elliptic problem

$$\Delta\hat{w} = \nabla^\perp\omega_w\varphi + \Delta\varphi w + 2\nabla\varphi\cdot\nabla w, \quad \hat{w}|_{\partial B(x_0,r+\varepsilon)} = 0.$$

Since $w \in C([t_1-\varepsilon, t_2+\varepsilon]; L^2(B(x_0, r+\varepsilon)))$ $\mathbb{P}-a.s.$ by Proposition 2.10, it follows that

$$\nabla^\perp\omega_w\varphi + \Delta\varphi w + 2\nabla\varphi\cdot\nabla w \in C([t_0-\varepsilon, T]; H^{-1}(B(x_0, r+\varepsilon))) \quad \mathbb{P}-a.s.$$

Therefore, by standard elliptic regularity theory

$$\hat{w} \in C([t_1-\varepsilon, t_2+\varepsilon]; H^1(B(x_0, r+\varepsilon))) \quad \mathbb{P}-a.s.$$

From the fact that $\varphi \equiv 1$ on $B(x_0, r+\varepsilon/2)$ it follows that

$$w \in C([t_1-\varepsilon/4, t_2+\varepsilon/4]; H^1(B(x_0, r+\varepsilon/4))) \quad \mathbb{P}-a.s.$$

Therefore, the required regularity of w is established by inductively reiterating this argument and by considering test functions $\varphi \in C_c^\infty(B(x_0, r+\frac{\varepsilon}{2^{2n}}))$ identically equal to one on $B(x_0, r+\frac{\varepsilon}{2^{2n+1}})$. □

2.3.2 Auxiliary Navier–Stokes Type Equations

In order to deal with the interior regularity of (1.16) we perform a Serrin type argument, see [127, 161]. The regularity of w guaranteed by Corollary 2.18 will play a crucial role to treat the linear terms appearing in (1.16). We start with the following lemma.

Lemma 2.19. *Let v be the unique solution of (1.16) in the sense of Definition 2.11 under the assumptions of Theorem 2.1, where w is as in Corollary 2.18. Then, for all $0 < t_1 \le t_2 < T$, $x_0 \in \mathcal{O}$, $r > 0$ such that $\operatorname{dist}(B(x_0,r),\partial\mathcal{O}) > 0$,*

$$v \in C([t_1,t_2], H^{3/2}(B(x_0,r);\mathbb{R}^2)) \quad \mathbb{P}-a.s.$$

Proof. As described in Lemma 2.17, arguing as in the proof of [77, Theorem 1.7], we can extend the weak formulation satisfied by v to time dependent test functions $\varphi \in C^1([0,T];H)\cap C([0,T];\mathsf{D}(A))$ obtaining that for each $t\in[0,T]$

$$\begin{aligned}\langle v(t),\varphi(t)\rangle - \langle u_0,\varphi(0)\rangle &= \int_0^t \langle v(s),\partial_s\varphi(s)\rangle\, ds - \int_0^t \langle v\left(s\right),A\varphi(s)\rangle\, ds\\ &\quad + \int_0^t b\left(v\left(s\right)+w\left(s\right),\varphi(s),v\left(s\right)+w\left(s\right)\right) ds \quad \mathbb{P}-a.s.\end{aligned}$$

Choosing $\varphi=-\nabla^{\perp}\chi,\ \chi\in C_c^{\infty}((0,T)\times\mathcal{O})$ in the weak formulation above and denoting by

$$\begin{aligned}\omega &= \operatorname{curl} v \in C([0,T];H^{-1})\cap L^2((0,T)\times\mathcal{O}) \quad \mathbb{P}-a.s.,\\ \omega_w &= \operatorname{curl} w \in C([t_1,t_2];C^{\infty}(B(x_0,r))) \quad \mathbb{P}-a.s.\end{aligned}$$

it follows that

$$\begin{aligned}-\int_0^T \langle\omega(s),\partial_s\chi(s)\rangle + \langle\omega(s),\Delta\chi(s)\rangle\, ds &= \int_0^T \langle \operatorname{curl}(w(s)\otimes w(s)),\nabla\chi(s)\rangle\, ds\\ &\quad+\int_0^T \langle \operatorname{curl}(w(s)\otimes v(s)),\nabla\chi(s)\rangle\, ds\\ &\quad+\int_0^T \langle \operatorname{curl}(v(s)\otimes w(s)),\nabla\chi(s)\rangle\, ds\\ &\quad+\int_0^T \langle \omega(s), v(s)\cdot\nabla\chi(s)\rangle\, ds.\end{aligned}$$

This means that ω is a distributional solution in $(0,T)\times\mathcal{O}$ of the partial differential equation

$$\begin{aligned}\partial_t\omega + v\cdot\nabla\omega = \Delta\omega - \operatorname{div}\Bigg(&\operatorname{curl}(w(s)\otimes w(s))\\ &+\operatorname{curl}(w(s)\otimes v(s)) + \operatorname{curl}(v(s)\otimes w(s))\Bigg).\end{aligned}$$

Since $\operatorname{dist}(B(x_0,r),\partial\mathcal{O})>0,\ 0<t_1\le t_2<T$ we can find ε small enough such that $0<t_1-2\varepsilon<t_1\le t_2<t_2+2\varepsilon<T,\ \operatorname{dist}(B(x_0,r+2\varepsilon),\partial\mathcal{O})>0$. Let us consider $\psi\in C_c^{\infty}((0,T)\times\mathcal{O})$ supported in $[t_1-\varepsilon,t_2+\varepsilon]\times B(x_0,r+\varepsilon)$ such that it is equal to one in $[t_1-\varepsilon/2,t_2+\varepsilon/2]\times B(x_0,r+\varepsilon/2)$. Let us denote by $\tilde{\omega}=\omega\psi\in L^2((0,T)\times\mathbb{R}^2)$ supported in $[t_1-\varepsilon,t_2+\varepsilon]\times B(x_0,r+\varepsilon)$, then $\tilde{\omega}$ is a distributional solution in $(0,T)\times\mathbb{R}^2$ of

$$\partial_t\tilde{\omega} = \Delta\tilde{\omega} - v\cdot\nabla\tilde{\omega} - w\cdot\nabla\tilde{\omega} + g \tag{2.35}$$

with

$$g = \partial_t\psi\omega - 2\nabla\psi\cdot\nabla\omega - \Delta\psi\omega + v\cdot\nabla\psi\omega - \psi w\cdot\nabla\omega_w - \psi v\cdot\nabla\omega_w + w\cdot\nabla\psi\omega.$$

Due to Corollary 2.18 the term

$$-\psi w\cdot\nabla\omega_w-\psi v\cdot\nabla\omega_w+w\cdot\nabla\psi\omega\in L^2((0,T)\times\mathbb{R}^2)\quad\mathbb{P}-a.s.$$

Therefore $g\in L^2(0,T;H^{-1}(\mathbb{R}^2))+L^1(0,T;L^2(\mathbb{R}^2))$ $\mathbb{P}-a.s.$ Then, arguing as in the first step of the proof of [127, Theorem 13.2], the fact that $\tilde\omega$ is a distributional solution of (2.35) implies that $\tilde\omega\in C([0,T];L^2(\mathbb{R}^2))\cap L^2(0,T;H^1(\mathbb{R}^2))$. Therefore

$$\begin{aligned}\omega\in&C([t_1-\varepsilon/4,t_2+\varepsilon/4];L^2(B(x_0,r+\varepsilon/4)))\\&\cap L^2(t_1-\varepsilon/4,t_2+\varepsilon/4,H^1(B(x_0,r+\varepsilon/4)))\quad\mathbb{P}-a.s.\end{aligned}$$

Introducing $\varphi\in C_c^\infty(B(x_0,r+\varepsilon/4))$ equal to one in $B(x_0,r+\varepsilon/8)$, since $\omega=\operatorname{curl}v$, then φv satisfies

$$\Delta(\varphi v)=\nabla^\perp\omega\varphi+\Delta\varphi v+2\nabla\varphi\cdot\nabla v,\quad(\varphi v)|_{\partial B(x_0,r+\varepsilon/4)}=0.\tag{2.36}$$

From the regularity of ω, by standard elliptic regularity theory (see for example [8]), it follows that $\varphi v\in C([t_1-\varepsilon/4,t_2+\varepsilon/4];H^1(B(x_0,r+\varepsilon/4);\mathbb{R}^2))\cap L^2(t_1-\varepsilon/4,t_2+\varepsilon/4;H^2(B(x_0,r+\varepsilon/4);\mathbb{R}^2))$ $\mathbb{P}-a.s.$ Therefore, since $\varphi\equiv1$ on $B(x_0,r+\varepsilon/8)$

$$\begin{aligned}v\in&C([t_1-\frac{\varepsilon}{16},t_2+\frac{\varepsilon}{16}];H^1(B(x_0,r+\frac{\varepsilon}{16});\mathbb{R}^2))\\&\cap L^2(t_1-\frac{\varepsilon}{16},t_2+\frac{\varepsilon}{16};H^2(B(x_0,r+\frac{\varepsilon}{16});\mathbb{R}^2))\quad\mathbb{P}-a.s.\end{aligned}\tag{2.37}$$

Let us now consider $\hat\psi\in C_c^\infty((t_1-\frac{\varepsilon}{16},t_2+\frac{\varepsilon}{16})\times B(x_0,r+\frac{\varepsilon}{16}))$ such that it is equal to one in $[t_1-\frac{\varepsilon}{32},t_2+\frac{\varepsilon}{32}]\times B(x_0,r+\frac{\varepsilon}{32})$. Let us denote by $\hat\omega=\omega\hat\psi\in C([0,T];L^2(\mathbb{R}^2))\cap L^2(0,T;H^1(\mathbb{R}^2))$ supported in $(t_1-\frac{\varepsilon}{16},t_2+\frac{\varepsilon}{16})\times B(x_0,r+\frac{\varepsilon}{16})$, then $\hat\omega$ is a distributional solution in $(0,T)\times\mathbb{R}^2$ of

$$\partial_t\hat\omega=\Delta\hat\omega+\hat g\tag{2.38}$$

with

$$\begin{aligned}\hat g=&-v\cdot\nabla\hat\omega-w\cdot\nabla\hat\omega+\partial_t\hat\psi\omega-2\nabla\hat\psi\cdot\nabla\omega-\Delta\hat\psi\omega+v\cdot\nabla\hat\psi\omega\\&-\hat\psi w\cdot\nabla\omega_w-\hat\psi v\cdot\nabla\omega_w+w\cdot\nabla\hat\psi\omega.\end{aligned}$$

By Corollary 2.18 and relation (2.37) it follows that $\hat g\in L^2(0,T;H^{-1/2}(\mathbb{R}^2))$ $\mathbb{P}-a.s.$ Therefore $\hat\omega\in C([0,T];H^{1/2}(\mathbb{R}^2))\cap L^2(0,T;H^{3/2}(\mathbb{R}^2))$ $\mathbb{P}-a.s.$ and arguing as above

$$\begin{aligned}v\in&C([t_1-\frac{\varepsilon}{64},t_2+\frac{\varepsilon}{64}];H^{3/2}(B(x_0,r+\frac{\varepsilon}{64});\mathbb{R}^2))\\&\cap L^2(t_1-\frac{\varepsilon}{64},t_2+\frac{\varepsilon}{64};H^{5/2}(B(x_0,r+\frac{\varepsilon}{64});\mathbb{R}^2))\quad\mathbb{P}-a.s.\end{aligned}$$

This concludes the proof of Lemma 2.19. □

Corollary 2.20. *Let v be the unique weak solution of* (1.16) *in the sense of Definition 2.11 under the assumptions of Theorem 2.1, where w is as in Corollary 2.18. Then, for all $0 < t_1 \le t_2 < T$, $x_0 \in \mathcal{O}$, $r > 0$ such that* $\operatorname{dist}(B(x_0,r),\partial\mathcal{O}) > 0$,

$$v \in C([t_1,t_2];C^\infty(B(x_0,r);\mathbb{R}^2)) \quad \mathbb{P}-a.s.$$

Proof. Since $\operatorname{dist}(B(x_0,r),\partial\mathcal{O}) > 0,\ 0 < t_1 \le t_2 < T$ we can find ε small enough such that $0 < t_1 - 2\varepsilon < t_1 \le t_2 < t_2 + 2\varepsilon < T,\ \operatorname{dist}(B(x_0,r+2\varepsilon),\partial\mathcal{O}) > 0$ and $\psi \in C_c^\infty((0,T)\times\mathcal{O})$ supported in $[t_1-\varepsilon,t_2+\varepsilon]\times B(x_0,r+\varepsilon)$ such that it is equal to one in $[t_1-\varepsilon/2,t_2+\varepsilon/2]\times B(x_0,r+\varepsilon/2)$. From Lemma 2.19 and Sobolev embedding theorem we know that $v \in C([t_1-\varepsilon,t_2+\varepsilon];L^\infty(B(x_0,r+\varepsilon);\mathbb{R}^2))$ $\mathbb{P}-a.s.$ Denoting by

$$\begin{aligned}&\omega = \operatorname{curl} v \in C([0,T];H^{-1}) \cap L^2((0,T)\times\mathcal{O}),\\ &\omega_w = \operatorname{curl} w \in C([t_1-2\varepsilon,t_2+\varepsilon],C^\infty(B(x_0,r+2\varepsilon))) \quad \mathbb{P}-a.s.\end{aligned}$$

and $\tilde\omega = \omega\psi \in L^2((0,T)\times\mathbb{R}^2)$ supported in $[t_1-\varepsilon,t_2+\varepsilon]\times B(x_0,r+\varepsilon)$, then, arguing as in the proof of Lemma 2.19, it follows that $\tilde\omega$ is a distributional solution in $(0,T)\times B(x_0,r+\varepsilon)$ of

$$\partial_t\tilde\omega = \Delta\tilde\omega + \tilde g \tag{2.39}$$

with

$$\begin{aligned}\tilde g = &-v\cdot\nabla\tilde\omega - w\cdot\nabla\tilde\omega + \partial_t\psi\omega - 2\nabla\psi\cdot\nabla\omega - \Delta\psi\omega + v\cdot\nabla\psi\omega\\ &-\psi w\cdot\nabla\omega_w - \psi v\cdot\nabla\omega_w + w\cdot\nabla\psi\omega.\end{aligned}$$

From the regularity of $\omega,\ v,\ \tilde\omega,\ \omega_w,\ w$, then $\tilde g \in L^2(t_1-\varepsilon,t_2+\varepsilon;H^{-1}(B(x_0,r+\varepsilon)))$ $\mathbb{P}-a.s.$ By standard regularity theory for the heat equation, see for example Step 2 in [127, Theorem 13.1], a solution of (2.39) with $\tilde g \in L^2(t_1-\varepsilon,t_2+\varepsilon;H^{k-1}(B(x_0,r+\varepsilon)))$, $k\in\mathbb{N}$, belongs to $C([t_1-\varepsilon/2,t_2+\varepsilon/2];H^k(B(x_0,r+\varepsilon/2)))\cap L^2(t_1-\varepsilon/2,t_2+\varepsilon/2;H^{k+1}(B(x_0,r+\varepsilon/2)))$. Therefore

$$\begin{aligned}\tilde\omega \in\ & C([t_1-\varepsilon/2,t_2+\varepsilon/2];L^2(B(x_0,r+\varepsilon/2)))\\ &\cap L^2(t_1-\varepsilon/2,t_2+\varepsilon/2;H^1(B(x_0,r+\varepsilon/2))) \quad \mathbb{P}-a.s.\end{aligned}$$

which implies

$$\begin{aligned}\omega \in\ & C([t_1-\varepsilon/4,t_2+\varepsilon/4];L^2(B(x_0,r+\varepsilon/4)))\\ &\cap L^2(t_1-\varepsilon/4,t_2+\varepsilon/4;H^1(B(x_0,r+\varepsilon/4))) \quad \mathbb{P}-a.s.\end{aligned}$$

since $\psi\equiv 1$ on $(t_1-\varepsilon/2,t_2+\varepsilon/2)\times B(x_0,r+\varepsilon/2)$. Considering now $\varphi\in C_c^\infty(\mathcal{O})$ supported on $B(x_0,r+\varepsilon/4)$ such that $\varphi\equiv 1$ on $B(x_0,r+\varepsilon/8)$, since $\operatorname{curl} v = \omega$ then φv satisfies

$$\Delta(\varphi v) = \nabla^\perp\omega\varphi + \Delta\varphi v + 2\nabla\varphi\cdot\nabla v, \quad (\varphi v)|_{\partial B(x_0,r+\varepsilon/4)} = 0. \tag{2.40}$$

Since

$$\begin{aligned}\nabla^\perp \omega\varphi + \Delta\varphi v + 2\nabla\varphi\cdot\nabla v \in C([t_1-\varepsilon/4, t_2+\varepsilon/4]; H^{-1}(B(x_0, r+\varepsilon/4)))\\ \cap L^2(t_1-\varepsilon/4, t_2+\varepsilon/4; L^2(B(x_0, r+\varepsilon/4))) \quad \mathbb{P}-a.s.,\end{aligned}$$

by standard elliptic regularity theory (see for example [8]),

$$\begin{aligned}\varphi v \in C([t_1-\varepsilon/4, t_2+\varepsilon/4]; H^{1}(B(x_0, r+\varepsilon/4)))\\ \cap L^2(t_1-\varepsilon/4, t_2+\varepsilon/4; H^2(B(x_0, r+\varepsilon/4))) \quad \mathbb{P}-a.s.\end{aligned}$$

Since $\varphi\equiv 1$ on $B(x_0, r+\varepsilon/8)$ then

$$\begin{aligned}v \in C([t_1-\frac{\varepsilon}{16}, t_2+\frac{\varepsilon}{16}]; H^{1}(B(x_0, r+\frac{\varepsilon}{16})))\\ \cap L^2(t_1-\frac{\varepsilon}{16}, t_2+\frac{\varepsilon}{16}; H^2(B(x_0, r+\frac{\varepsilon}{16}))) \quad \mathbb{P}-a.s.\end{aligned}$$

Reiterating the argument as in Step 3 in [127, Theorem 13.1] the result follows. □

Proof of Theorem 2.2. The claim follows by Corollary 2.18, Corollary 2.20 and a localization argument. Moreover, to obtain the claimed smoothness up to time $t=T$, let us consider the extension by 0 of h_b on $[0,T+1]$, i.e.

$$\tilde{h}_b(t) = \begin{cases} h_b(t), & \text{if } t\in(0,T),\\ 0, & \text{if } t\in(T,T+1).\end{cases}$$

Let $\tilde{u}$ be the unique weak solution (2.1) provided by Theorem 2.1 with T replaced by $T+1$. Then, by Corollary 2.18, Corollary 2.20 and a standard covering argument, for all $t_0\in(0,T)$, $\mathcal{O}_0\subset\mathcal{O}$ such that $\operatorname{dist}(\mathcal{O}_0,\partial\mathcal{O})>0$,

$$\tilde{u}\in C([t_0,T]; C^\infty(\mathcal{O}_0;\mathbb{R}^2)) \quad \mathbb{P}-a.s. \tag{2.41}$$

Now, let u be the unique weak solution of (2.1) provided by Theorem 2.1. By uniqueness, we have $u=\tilde{u}|_{[0,T]}$ and the conclusion follows from (2.41). □

2.4 H^∞-calculus for the Stokes operator

Now we prove Theorem 2.3. Here we use the transference result proven in [125]. In this section we also need the concept of $\mathcal{R}$-sectoriality, which can be again found in [150, Chapters 3 and 4] and [102, Chapter 10].

To discuss the main idea in the proof of Theorem 2.3, let us define the operator $B_q v := -\Delta v$ on $L^q(\mathcal{O};\mathbb{R}^2)$ with domain

$$\begin{aligned}\mathsf{D}(B_q) := \big\{ f=(f_1,f_2)\in W^{2,q}(\mathcal{O};\mathbb{R}^2) \,:\, f|_{\Gamma_b}=0,\\ f_2|_{\Gamma_u}=\partial_2 f_1|_{\Gamma_u}=0\big\}.\end{aligned}$$

Note that B_q has the same boundary conditions of A_q. However, B_q is considerably more simple than A_q since, to study its spectral properties, it is possible to use reflection arguments which are not available for the Stokes operator, cf. subsection 2.4.2.

We prove Theorem 2.3 by using the transference techniques developed in [125]. By [125, Theorem 9], we divide the proof into the following steps:

1. Boundedness of the H^∞-calculus for A_2 and B_2 (i.e. in the Hilbertian case).
2. Boundedness of the H^∞-calculus for B_q for all $q \in (1, \infty)$.
3. $\mathcal{R}$-sectoriality of A_q.
4. Conclusion via transference and interpolation [125, Theorems 5 and 9].

2.4.1 The Hilbertian case

Here we analyse the L^2-case of Theorem 2.3, i.e. the operators $A := A_2$ and $B := B_2$ acting on $\mathbb{L}^2$ and $L^2(\mathcal{O}; \mathbb{R}^2)$, respectively.

Proposition 2.21. *A and B are positive self-adjoint operators, and*

$$\mathsf{D}(A^\gamma) = \mathsf{D}((A^*)^\gamma) = H^{2\gamma}(\mathcal{O}; \mathbb{R}^2) \cap H, \quad \mathsf{D}(B^\gamma) = \mathsf{D}((B^*)^\gamma) = H^{2\gamma}(\mathcal{O}; \mathbb{R}^2) \tag{2.42}$$

for all $\gamma \in (0, \frac{1}{4})$.

The above result and [102, Proposition 10.2.23] imply that A and B have bounded H^∞-calculus of angle 0. Below we mainly focus on the operator A. The argument to treat B is analogous and simpler.

Consider the elliptic problem associated to A, i.e.

$$\begin{cases} -\Delta u + \nabla \pi = f, & \text{on } \mathcal{O}, \\ \operatorname{div} u = 0, & \text{on } \mathcal{O}, \\ u = 0, & \text{on } \Gamma_b, \\ \partial_2 u_1 = 0, & \text{on } \Gamma_u, \\ u_2 = 0, & \text{on } \Gamma_u. \end{cases} \tag{2.43}$$

If $f \in \mathbb{H}^{-1}$, the definition of weak solutions $u \in \mathbb{H}^1$ is standard and similar to the one of (2.3). The well-posedness of (2.43) is analysed below.

Proposition 2.22. *For each $f \in \mathbb{H}^{-1}$ there exists a unique solution of problem (2.43). Moreover if $f \in H$ then $u \in \mathsf{D}(A)$ and*

$$\|u\|_{\mathsf{D}(A)} + \|\pi\|_{H^1/\mathbb{R}} \le C\|f\|.$$

Proof. Existence of weak solutions follows immediately by Lax-Milgram Lemma, since Poincaré inequality holds in $\mathbb{H}^1$. Therefore we can endow $\mathbb{H}^1$ with the norm $\|\nabla u\|_{\mathbb{H}^1} := \|\nabla u\|$ equivalent to the standard H^1 norm. Let now $f \in H$, therefore the weak formulation satisfied by u reduces to

$$\langle \nabla u, \nabla \varphi \rangle = \langle f, \varphi \rangle \quad \forall \varphi \in V. \tag{2.44}$$

Considering $\varphi \in \mathcal{D}(\mathcal{O})$ it follows that

$$\langle \Delta u + f, \varphi \rangle_{\mathcal{D}(\mathcal{O})', \mathcal{D}(\mathcal{O})} = 0,$$

therefore by [168, Proposition 1.1, Proposition 1.2] it exists $\pi \in L^2(D)$ such that

$$-\Delta u + \nabla \pi = f \tag{2.45}$$

in the sense of distributions and

$$\|\pi\|_{L^2/\mathbb{R}} \leq C\|\Delta u\|_{H^{-1}} + \|f\| \lesssim \|f\|.$$

The higher regularity follows by the standard Niremberg's method of finite difference quotients. Therefore, fix $h > 0$, extending periodically either u and f in the x direction and consider $\varphi = \tau_h \tau_{-h} u$ as a test function in (2.44), where $\tau_h v = \frac{v(x+h,z) - v(x,z)}{h}$. Then by change of variables it follows that

$$\|\tau_{-h} \nabla u\| \leq C\|f\|.$$

The relation above implies that

$$\|\partial_1 \nabla u\| \leq C\|f\|.$$

Let us now consider $\varphi = \partial_1 \psi, \ \psi \in \mathcal{D}(\mathcal{O})$ as test function in (2.44). Therefore arguing as above it follows that $\partial_1 \pi \in L^2(\mathcal{O})$ and $\|\partial_1 \pi\| \lesssim \|f\|$. Since equation (2.45) is satisfied in the sense of distribution and u is divergence free it follows that

$$\partial_2 \pi = f_2 + \partial_{1,1} u_2 - \partial_{1,2} u_1 \in L^2(\mathcal{O}).$$

This implies that $\|\nabla \pi\| \lesssim \|f\|$. Lastly u_1 satisfies

$$-\partial_{2,2} u_1 = -\partial_1 \pi + \partial_{1,1} u_1 + f_1 \in L^2(\mathcal{O}),$$

which completes the proof. □

We are ready to prove Proposition 2.21.

Proof of Proposition 2.21. Step 1: A and B are a positive self-adjoint operators. As above we only discuss the operator A. The positivity of A is clear. Next, note that, integrating by parts

$$\langle Au, v\rangle = \langle u, Av\rangle \quad \forall u, v \in \mathsf{D}(A).$$

This means that A is symmetric. It remains to show that $\mathsf{D}(A^*) = \mathsf{D}(A)$ and $\forall u \in \mathsf{D}(A),\ A^*u = Au$. By definition

$$\mathsf{D}(A^*) = \{u \in H : \quad F : D(A) \subseteq H \to \mathbb{R}, \quad F(v) = \langle u, Av\rangle \\ \text{has a linear bounded extension on } H\}.$$

For each $u \in \mathsf{D}(A^*)$, $F(v) = \langle u, Av\rangle = \langle f_u, v\rangle$ therefore $A^*u = f_u$. In particular, $\forall u \in \mathsf{D}(A^*)$ $\langle u, Av\rangle = \langle A^*u, v\rangle$. Thanks to the fact that A is symmetric we have $\mathsf{D}(A) \subseteq \mathsf{D}(A^*)$. Given now $v \in \mathsf{D}(A^*)$, $f_v = A^*v \in H$, let us consider the boundary value problem (2.43) with forcing term equal to f_v. By Proposition 2.22 it has a unique solution $(w, \pi) \in \mathsf{D}(A) \times H^1(\mathcal{O})$, this implies that $Aw = f_v = A^*v$. For each $z \in H$, let us consider the boundary value problem (2.43) with forcing term equal to z. By Proposition 2.22 it exists a unique $s_z \in \mathsf{D}(A)$ such that $As_z = z$. Therefore $\langle z, w - v\rangle = \langle As_z, w - v\rangle = \langle s_z, Aw - A^*v\rangle = 0$ thanks to the fact that A is symmetric. Since z is arbitrary, then $v = w$ and the claim follows.

Step 2: Proof of (2.42). We begin by proving the first identity in (2.42). Note that $\mathsf{D}(B^\gamma) = \mathsf{D}((B^*)^\gamma)$ for $\gamma < 1/2$ follows from [109, Theorem 1.1] and Step 1. By Step 1 and [102, Proposition 10.2.23], B has bounded H^∞-calculus and in particular B has the bounded imaginary powers property, [150, Subsection 3.4]. By [150, Theorem 3.3.7], $\mathsf{D}(B^\gamma) = [L^2(\mathcal{O}; \mathbb{R}^2), \mathsf{D}(B)]_\gamma$ for all $\gamma < 1$. The latter gives $\mathsf{D}(B^\gamma) = H^{2\gamma}(\mathcal{O}; \mathbb{R}^2)$ in case $\gamma < 1/4$ by [159]. The second identity in (2.42) follows analogously, where one uses the argument in [85] (see also [165, Proposition 5.5, Chapter 17]) to deduce $\mathsf{D}(A^\gamma) = \mathbb{H}^{2\gamma}(\mathcal{O})$ from the first identity in (2.42). □

2.4.2 Bounded H^∞-calculus for Laplace operators

In this subsection we prove the boundedness of the H^∞-calculus for B_q. The basic idea is to use the product structure of the domain $\mathcal{O}$ and to write $B_q u = (L_{q,\mathrm{R}} u_1, L_{q,\mathrm{D}} u_2)$ where

$$\begin{aligned} \mathsf{D}(L_{q,\mathrm{D}}) &:= \big\{f \in W^{2,q}(\mathcal{O}) \,:\, f|_{\Gamma_b} = f|_{\Gamma_u} = 0\big\}, & L_{q,\mathrm{D}} f &:= \Delta f, \\ \mathsf{D}(L_{q,\mathrm{R}}) &:= \big\{f \in W^{2,q}(\mathcal{O}) \,:\, \partial_2 f|_{\Gamma_b} = 0,\ f|_{\Gamma_u} = 0\big\}, & L_{q,\mathrm{R}} f &:= \Delta f. \end{aligned}$$

Proposition 2.23 (Bounded H^∞-calculus for Laplace operators)**.** *Let $q \in (1, \infty)$ and let $\mathcal{O}$ be as above. Then $-L_{q,\mathrm{D}}$ and $-L_{q,\mathrm{R}}$ have a bounded H^∞-calculus of angle* 0*. In particular B_q has a bounded H^∞-calculus of angle* 0*.*

The above statement also holds for the Neumann Laplacian, but it will not be needed below.

Proof. We divide the proof into three steps. In the first step, we exploit the product structure of our domain to reduce the problem to a one dimensional situation.

Step 1: Reduction to the 1d case. Then the Dirichlet and the Robin Laplacian in 1d are given by

$$\begin{aligned} \mathsf{D}(\ell_{q,\mathrm{D}}) &:= \big\{f \in W^{2,q}(0,a) \,:\, f(0)=f(a)=0\big\}, && \ell_{q,\mathrm{R}} f := \partial_{1,1} f,\\ \mathsf{D}(\ell_{q,\mathrm{R}}) &:= \big\{f \in W^{2,q}(0,a) \,:\, \partial_1 f(0)=f(a)=0\big\}, && \ell_{q,\mathrm{R}} f := \partial_{1,1} f. \end{aligned}$$

Let us consider $\ell_{q,\mathrm{D}}$, the other case is analogue. In this step we assume that $-\ell_{q,\mathrm{D}}$ has a bounded H^∞-calculus of angle 0. Let $\ell_{q,\mathrm{P}}$ be the Laplacian on the periodic torus $\mathbb{T}$ with domain $W^{2,q}(\mathbb{T})$. The boundedness of the H^∞-calculus for $-\ell_{q,\mathrm{P}}$ such operator follows from the periodic version of [102, Theorem 10.2.25] and $\omega_{H^\infty}(\ell_{q,\mathrm{P}})=0$.

On $L^2(\mathcal{O})$ considers the operator

$$(\ell^{(x)}_{q,\mathrm{D}} f)(x,z) = (\ell_{q,\mathrm{D}} f(\cdot,z))(x), \qquad (\ell^{(y)}_{q,\mathrm{P}} f)(x,z) = (\ell_{q,\mathrm{D}} f(x,\cdot))(z),$$

with the corresponding natural domains. It is clear that both $-\ell^{(x)}_{q,\mathrm{D}}$ and $-\ell^{(z)}_{q,\mathrm{P}}$ have bounded H^∞-calculus of angle 0. Now by sum of commuting operators [150, Corollary 4.5.8], the sum operator $-A_q := -\ell^{(x)}_{q,\mathrm{D}} - \ell^{(z)}_{q,\mathrm{P}}$ has a bounded H^∞-calculus of angle 0 with domain

$$\mathsf{D}(A_q) = \mathsf{D}(\ell^{(x)}_{q,\mathrm{D}}) \cap \mathsf{D}(\ell^{(z)}_{q,\mathrm{P}}) = \mathsf{D}(L_{q,\mathrm{D}})$$

where the last equality follows from elliptic regularity.

Step 2: $-L_{q,\mathrm{D}}$ has a bounded H^∞-calculus of angle 0. By rescaling and translation we may replace $(0,a)$ by $(-\pi,\pi)$. Let $\ell_{q,\mathrm{P}}$ be the Laplacian on the periodic torus $\mathbb{T}=(-\pi,\pi)$ (as measure space) with domain $W^{2,q}(\mathbb{T})$. Let

$$Y := \Big\{f \in L^2(\mathbb{T}) \,:\, f = \sum_{n\geq 0} f_n \sin(nx) \text{ where } (f_n)_{n\geq 1} \in \ell^2\Big\}.$$

It is clear that $Y \subseteq L^2(\mathbb{T})$ is closed, and

$$(\lambda - \ell_{q,\mathrm{P}})^{-1} : Y \to Y \quad \text{for all} \quad \lambda \in \rho(L_{q,\mathrm{D}}).$$

Now note that $L_{q,\mathrm{D}}$ is the part of $\ell_{q,\mathrm{P}}$ on Y, i.e.

$$\begin{aligned} \mathsf{D}(L_{q,\mathrm{D}}) &= \{f \in \mathsf{D}(\ell_{q,\mathrm{P}}) \cap Y \,:\, \ell_{q,\mathrm{P}} f \in Y\},\\ L_{q,\mathrm{D}} f &= \ell_{q,\mathrm{P}} f \quad \text{for all } f \in \mathsf{D}(L_{q,\mathrm{D}}). \end{aligned}$$

Now the claim of Step 1 follows from [102, Proposition 10.2.18] and the periodic version of [102, Theorem 10.2.25].

Step 2: $-L_{q,\mathrm{R}}$ has a bounded H^∞-calculus of angle 0. As in the above step, by rescaling we replace $(0,a)$ by $(0,\pi)$. Consider the reflection map

$$Rf(z) := \begin{cases} f(z) & z \in (0,\pi), \\ f(-z) & z \in (-\pi,0). \end{cases}$$

Let $L_{q,\mathrm{D}}$ be the Dirichlet Laplacian on $(-\pi,\pi)$. Then one can readily check that $\rho(L_{q,\mathrm{D}}) \subseteq \rho(L_{q,\mathrm{R}})$ and for all $\lambda \in \rho(L_{q,\mathrm{D}})$

$$(\lambda - L_{q,\mathrm{R}})^{-1} f = \big[(\lambda - L_{q,\mathrm{D}})^{-1} Rf\big]\Big|_{(0,\pi)}.$$

Now the claim follows from Step 1 and the definition of H^∞-calculus. □

2.4.3 $\mathcal{R}$-sectoriality for the Stokes operator

For the notion of $\mathcal{R}$-boundedness of a family of linear operators we refer to [102, Chapter 8]. For a family of linear operators $\mathscr{J}$, the $\mathcal{R}$-bound is denoted by $\mathcal{R}(\mathscr{J})$. As in [102, Chapter 10], we said that operator T on a Banach space X is said to be $\mathcal{R}$-sectorial if there exists $\varphi \in (0,\pi)$ such that $\rho(A) \subseteq \{\lambda \in \mathcal{C} \,|\, |\arg\lambda| \geq \pi - \varphi\}$ and

$$\mathcal{R}(\lambda(\lambda - T)^{-1} \,|\, |\arg\lambda| > \pi - \varphi) < \infty.$$

The $\mathcal{R}$-sectoriality angle is the infimum over all $\varphi \in (0,\pi)$ for which the above holds. The main result of this subsection reads as follows.

Proposition 2.24. *For all $q \in (1,\infty)$, the operator A_q is $\mathcal{R}$-sectorial with $\mathcal{R}$-sectoriality angle $< \pi/2$.*

Proof. Fix $q \in (1,\infty)$. For simplicity we first prove the statement for a shifted Stokes operator and in a second step we conclude by a simple translation argument.

Step 1: There exists λ_q such that $\lambda_q + A_q$ is $\mathcal{R}$-sectorial with $\mathcal{R}$-sectoriality angle $< \pi/2$. Due to the the well-known equivalence of maximal L^q-regularity and $\mathcal{R}$-sectoriality proven by L. Weis [179] (see also [150, Subsection 4.2, Chapter 3]), it is enough to show that, for all $f \in L^q(0,1;\mathbb{L}^q)$, the Stokes problem on $\mathcal{O}$,

$$\begin{cases} \partial_t u = \Delta u - \nabla P + f, & \text{on } (0,1) \times \mathcal{O}, \\ \operatorname{div} u = 0, & \text{on } (0,1) \times \mathcal{O}, \\ u = 0, & \text{on } (0,1) \times \Gamma_b, \\ u_2 = \partial_2 u_1 = 0, & \text{on } (0,1) \times \Gamma_u, \\ u(0) = 0, & \text{on } \mathcal{O}, \end{cases} \tag{2.46}$$

admits a unique solution in the class

$$u \in W^{1,q}(0,1;\mathbb{L}^q) \cap L^q(0,1;\mathbb{W}^{2,q}(\mathcal{O})), \quad P \in L^q(0,1;W^{1,q}(\mathcal{O})). \tag{2.47}$$

The proof follows a standard localization argument. Let $(\varphi_j)_{j=1}^N$ be a smooth partition of the unity such that, for all $j \in \{1, \dots, N\}$, $\operatorname{diam}(\operatorname{supp}\varphi_k) < \frac{1}{2}$

$$\text{either} \quad \operatorname{supp}\varphi_j \cap (\mathbb{T} \times \{0\}) = \varnothing \quad \text{or} \quad \operatorname{supp}\varphi_j \cap (\mathbb{T} \times \{a\}) = \varnothing.$$

Fix $k \in \{1, \dots, N\}$. Multiplying (2.46) by φ_k, we obtain for $u_k := \varphi_k u$ and $P_k = \varphi_k P$ either

$$\begin{cases} \partial_t u_k = \Delta u_k - \nabla P_k + \varphi_k f + \mathcal{L}_k(u, P), & \text{on } (0,1) \times \mathbb{R} \times (0, \infty), \\ \operatorname{div} u_k = \nabla \varphi_k \cdot u, & \text{on } (0,1) \times \mathbb{R} \times (0, \infty), \\ u_k = 0, & \text{on } (0,1) \times \mathbb{R} \times \{0\}, \\ u_k(0) = 0, & \text{on } \mathbb{R} \times (0, \infty), \end{cases} \tag{2.48}$$

or

$$\begin{cases} \partial_t u_k = \Delta u_k - \nabla P_k + \varphi_k f + \mathcal{L}_k(u, P), & \text{on } (0,1) \times \mathbb{R} \times (-\infty, a), \\ \operatorname{div} u_k = \nabla \varphi_k \cdot u, & \text{on } (0,1) \times \mathbb{R} \times (-\infty, a), \\ \varphi_k u_2 = 0, & \text{on } (0,1) \times \mathbb{R} \times \{a\}, \\ \partial_2(\varphi_k u_1) = u_1 \partial_2 \varphi_k, & \text{on } (0,1) \times \mathbb{R} \times \{a\}, \\ u_k(0) = 0, & \text{on } \mathbb{R} \times (-\infty, a). \end{cases} \tag{2.49}$$

Here $\mathcal{L}_k$ denotes a lower order operator w.r.t. to the maximal regularity space for (u, P) in (2.47).

Maximal $L^p(L^q)$-regularity estimates for (2.48) and (2.49) are proven in [150, Theorem 7.2.1] in the case of no-slip or pure-slip, respectively (see conditions (7.16) and (7.17) on [150, p. 323]). Now a-priori estimates for solutions as in (2.47) in the maximal $L^p(L^q)$-regularity class $W^{1,q}(0,1;\mathbb{L}^q) \cap L^q(0,1;\mathbb{W}^{2,q}(\mathcal{O}))$ follows by repeating the localization argument of [150, Subsection 3.4 in Chapter 7] to adsorb the lower order terms.

It remains to discuss the existence of solutions as in (2.47). Arguing as in step 3 of Theorem 2.4 and using L^2-theory, one can prove existence of smooth solutions to equation (2.46) in case of smooth data f. Hence the existence follows from a standard density argument and the a-priori estimates obtained above for solutions in the class $W^{1,q}(0,1;\mathbb{L}^q) \cap L^q(0,1;\mathbb{W}^{2,q}(\mathcal{O}))$.

Step 2: Conclusion. By Step 1, it remains to remove the shift λ_q. Arguing as in [1, Proposition 2.2], by holomorphicity of the resolvent and [150, Proposition 4.1.12], it is enough to show that

$$\rho(A_q) \subseteq \{\lambda \in \mathcal{C} \,|\, |\arg z| > \psi\}, \qquad \text{for some } \psi < \pi/2.$$

In the case $q > 2$, noticing that $(\lambda - A_q) = (\lambda - \lambda_q - A_q) + \lambda_q$ and that $\rho(\lambda_q + A_q) \subseteq \{\lambda \in \mathcal{C} \,|\, |\arg z| > \varphi\}$ for some $\varphi < \pi/2$ by Step 1, the conclusion can be obtained by using a standard bootstrap method via Sobolev embeddings.

In the case $q < 2$ one uses $(A_q)^* = A_{q'}$. □

2.4.4 Proof of Theorem 2.3

As the proof of Theorem 2.3 follows the one of [125, Theorem 16], we only provide a sketch.

Proof of Theorem 2.3 - Sketch. Step 1: There exists $\beta > 0$ for which the following estimates hold:

$$\begin{aligned}\sup_{1\leq t,s\leq 2} \mathcal{R}(\varphi(2^{j+\ell}sA_2)\psi(t2^jB_2)) \lesssim 2^{-\beta\ell},\\ \sup_{1\leq t,s\leq 2} \mathcal{R}(\varphi(2^{j+\ell}sA_2)^*\mathbf{P}_p^*\psi(t2^jB_2)^*) \lesssim 2^{-\beta\ell}.\end{aligned} \tag{2.50}$$

Recall that $\mathcal{R}(\mathscr{J})$ stand for the $\mathcal{R}$-bound of the family of operators $\mathscr{J}$, see [102, Chapter 9] for details on $\mathcal{R}$-boundedness.

By elliptic regularity we have $\mathbf{P} : H^1(\mathcal{O};\mathbb{R}^2) \to H^1(\mathcal{O};\mathbb{R}^2) \cap H$. Interpolating we obtain $\mathbf{P} : H^s(\mathcal{O};\mathbb{R}^2) \to H^s(\mathcal{O};\mathbb{R}^2) \cap H$ for all $s \in (0,1)$. Hence $\mathbf{P} : \mathsf{D}(B^\gamma) \to \mathsf{D}(A^\gamma)$ for all $\gamma \in (0,1/4)$. The estimate (2.50) now follows from [125, Proposition 10] and (2.42).

Step 2: Boundedness of the H^∞-calculus. Next we argue as in the proof of [125, Theorem 5]. Let $q \in (1,\infty)$ be as in the statement of Theorem 2.3 and fix $p \in (q,\infty)$. By $\mathcal{R}$-sectoriality of A_p and B_p (i.e. Proposition 2.24) and [102, Proposition 10.3.2],

$$\mathcal{R}(\varphi(tA_p) \,:\, t > 0) \leq c_0 \quad \text{and} \quad \mathcal{R}(\psi(sB_p) \,:\, s > 0) \leq c_0. \tag{2.51}$$

Note that $(A_r)_{r\in(1,\infty)}$, $(B_r)_{r\in(1,\infty)}$ are consistent family of operators. Hence, by complex interpolation and [102, Proposition 8.4.4], we have that (2.50) holds for some $\beta = \beta(r,p) > 0$ and with (A_2, B_2) replaced by (A_q, B_q). Now the boundedness of the H^∞-calculus follows from [125, Theorem 9].

Step 3: Description of the fractional powers. To obtain the description of the fractional powers of A_q and B_q one can argue as in the proof of (2.42) by using the bounded imaginary power property and [85, 159]. □

Chapter 3

Stochastic Particle Approximation of the 2D Navier-Stokes Equations with Vorticity Generation

This chapter covers the content of [95], namely we provide a stochastic interacting particle system approximating the following nonlinear, scalar PDE

$$\begin{cases} \partial_t \omega + K[\omega] \cdot \nabla \omega = \nu \Delta \omega \quad x \in \mathcal{O}, \\ \nabla \omega \cdot \hat{n} = g \quad x \in \partial \mathcal{O}, \\ \omega(0) = \omega_0 \quad x \in \mathcal{O}. \end{cases} \tag{3.1}$$

in the sense discussed in section 1.4. We recall that $T > 0$ and $\mathcal{O} \subseteq \mathbb{R}^2$ is a smooth, convex, bounded domain and we set $\nu = 1$ for the matter of notation. The proof of Theorem 1.3 relies on a compactness argument for the regularized empirical measure of our particle system and some a priori estimates on the solutions of equation (3.1). In particular, section 3.1 is devoted to recall some properties of the Heat semigroup with Neumann boundary conditions and discussing some issues related to equation (3.1), like uniqueness of solutions in certain function spaces. Subsequently, in section 3.2 we introduce rigorously our particle system and state the main result of the chapter. We provide in section 3.3 and section 3.4 the detailed proofs of the estimates (1.26), (1.27) and Theorem 3.17 in the case of $\omega_0, g \geq 0$. Lastly, in section 3.5, we describe what we need to change in order to treat the general case of functions not having a fixed sign.

Beside the improved notion of convergence to the PDE we are able to obtain, the main point of this chapter is the approach to particle approximation for (3.1): our approach is almost completely functional analytic and based on semigroup theory, in contrast with results in the perhaps more classical framework of propagation of chaos such as [105, 82]. The motivation for our approach is that our technique circumvents the need for fine controls on particle dynamics. We regard this as a step forward in attempting to build a similar particle approximation of the physical no-slip Navier-Stokes equations (1.17), since in that case particle creation has to be linked with the velocity at the boundary induced by the bulk of the fluid, making single particle dynamics very hard to treat (as already commented in [105]).

3.1 Functional Analytic Setup

3.1.1 Sobolev Spaces and Neumann Heat Semigroup

Consider the heat equation with Neumann boundary conditions

$$\begin{cases} \partial_t u = \Delta u, & x \in \mathcal{O},\ t > 0, \\ \hat{n} \cdot \nabla u = 0, & x \in \partial\mathcal{O},\ t > 0, \\ u_0(x) = f(x), & x \in \mathcal{O}, \end{cases} \tag{3.2}$$

with f a bounded measurable function on $\mathcal{O}$. It is well-known that the PDE problem is well-posed, we denote its solution by $P(t)f(x) = u(t,x)$, $t \geq 0$, $x \in \mathcal{O}$; moreover $P(t)f \in C^\infty(\mathcal{O})$ for all $t > 0$. Operators $P(t)$, $t > 0$ form a Markov semigroup (in the sense of [10]), associated to the *reflected Brownian motion* in $\mathcal{O}$ (see subsection 3.2.2 below).

We now recall classical facts concerning the action of the semigroup $P(t)$ on $L^p(\mathcal{O})$ and Sobolev spaces: we refer to [145, Section 7.3] for a detailed discussion. For all $p \in [1,\infty)$, $P(t)$ extends to an analytic semigroup of contractions $P(t) : L^p \to L^p$ with infinitesimal generator

$$-\mathcal{B}_p : \mathsf{D}(\mathcal{B}_p) \subset L^p(\mathcal{O}) \to L^p(\mathcal{O}).$$

The spectrum of $\mathcal{B}_p$ is included in $[0,+\infty)$. A first consequence of the fact that $P(t)$ is an analytic semigroup is *ultracontractivity* on L^p spaces: we refer to [9, Section 7.3.2] for the following result.

Proposition 3.1. *If* $1 \leq p < q \leq \infty$*, then it holds*

$$\|P(t)\|_{L^p \to L^q} \leq \frac{C_{T,p,q}}{t^{1/p-1/q}}, \quad t \in [0,T]. \tag{3.3}$$

Another consequence of analiticity of $P(t)$ is that we can introduce fractional powers of $I + \mathcal{B}_p$ as operators on L^p by means of

$$(I+\mathcal{B}_p)^{-\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} e^{-t} P(t)dt, \quad \alpha > 0,$$

where the integral converges in the uniform operator topology and defines injective operators (*cf.* [145, Section 2.6]). We set

$$(I+\mathcal{B}_p)^0 = I, \quad (I+\mathcal{B}_p)^\alpha = ((I+\mathcal{B}_p)^\alpha)^{-1} \text{ for } \alpha > 0,$$

and denote by

$$\mathcal{H}^{\alpha,p} = \mathsf{D}((I+\mathcal{B}_p)^{\alpha/2})$$

the domain of fractional powers for all $\alpha \geq 0$. On the contrary, for $\alpha < 0$ we set by $\mathcal{H}^{\alpha,p}$ the completion of L^p with respect to the norm $\left\|(I+\mathcal{B}_p)^{\alpha/2}\right\|_{L^p}$, namely the extrapolated space or order $\alpha/2$.

The following proposition collects standard facts on the fractional powers of the Laplacian with Neumann boundary conditions.

Proposition 3.2. *Let $p \in (1,\infty)$. It holds:*

- *for $\alpha, \beta \in \mathbb{R}$ and $u \in \mathcal{H}^{2\gamma,p}$, $\gamma = \alpha \vee \beta \vee (\alpha+\beta)$,*
$$(I+\mathcal{B}_p)^{\alpha+\beta} u = (I+\mathcal{B}_p)^{\alpha}(I+\mathcal{B}_p)^{\beta} u;$$
- *$\mathcal{B}_p^* = \mathcal{B}_{p'}$ with $\frac{1}{p}+\frac{1}{p'}=1$;*
- *$\mathcal{H}^{2\alpha,p}$ can be identified with the interpolation space $[L^p, \mathsf{D}(\mathcal{B}_p)]_\alpha$ for all $\alpha \in [0,1]$; moreover it holds*
$$\mathcal{H}^{\alpha,p} = \begin{cases} H^{\alpha,p}(\mathcal{O}) & \text{for} \quad 0 \leq \alpha < 1+\frac{1}{p}; \\ \{u \in H^{\alpha,p}(\mathcal{O}) : \hat{n}\cdot\nabla u|_{\partial\mathcal{O}} = 0\} & \text{for} \quad 1+\frac{1}{p} < \alpha < 3+\frac{1}{p}; \\ \{u \in H^{\alpha,p}(\mathcal{O}) : \hat{n}\cdot\nabla \mathcal{B}_p^l u|_{\partial\mathcal{O}} = 0 \ \forall l \in \mathbb{N},\ l \leq k\} & \\ & \text{for} \quad 1+2k+\frac{1}{p} < \alpha < 3+2k+\frac{1}{p}. \end{cases}$$
- *for $\alpha \in \mathbb{R}$, $P(t)(L^p) \subset \mathcal{H}^{2\alpha,p}$ for all $t > 0$ and operators $P(t)$ and $(I+\mathcal{B}_p)^\alpha$ commute on $\mathcal{H}^{2\alpha,p}$; moreover*
$$\|(I+\mathcal{B}_p)^\alpha\, P(t)\|_{L^p\to L^p} \leq \frac{C_{\alpha,p}}{t^\alpha}, \quad t \in [0,T]. \tag{3.4}$$

The action of $P(t)$ on $f \in L^1$ is given by

$$P(t)f(x) = \int_{\mathcal{O}} p_t(x,y) f(y) dy, \quad t > 0,$$

where the *heat kernel* $p_t(x,y) \geq 0, t > 0, x, y \in \bar{\mathcal{O}}$, is a smooth function, $p_t \in C^\infty(\bar{\mathcal{O}}^2)$, $t > 0$, satisfying (3.2) with the initial condition replaced by $u(t,x) \to \delta_y$ as $t \to 0$. The Neumann heat kernel can be controlled with the free heat kernel (on the whole plane) up to a multiplicative constant at the exponent, that is:

$$p_t(x,y) \leq \frac{C}{t \wedge |\mathcal{O}|} e^{-|x-y|^2/(ct)}, \quad x, y \in \bar{\mathcal{O}}, \tag{3.5}$$

with $C, c > 0$ depending only on the domain $\mathcal{O}$, *cf.* [176, Eq. (3.2)]. The following statement recalls the *gradient estimates* on heat semigroup and kernel, we refer to [176, Theorem 1.2, Corollary 1.3, Lemma 3.1] for a proof.

Proposition 3.3. *There exist a constant* $C > 0$ *depending only on* $\mathcal{O}$ *such that,*

$$|\nabla P(t)f(x)| \leq e^{Ct} P(t) \, |\nabla f| \, (x), \quad t \geq 0, \, x \in \mathcal{O}, \, f \in C^1_b(\mathcal{O}). \tag{3.6}$$

Moreover, there exist constants $C, c, c' > 0$ *depending only on* $\mathcal{O}$ *such that*

$$\nabla p_t(x,y) \leq C(\mathbf{1}_{0<t\leqslant 1} t^{-3/2} + \mathbf{1}_{t>1} e^{-c't}) e^{-|x-y|^2/(ct)}. \tag{3.7}$$

As a consequence, for all $t > 0$ *and* $p \in (1, \infty)$*,*

$$\|\nabla P(t)\|_{L^p \to C(\bar{\mathcal{O}})} \leq \frac{C_p}{(1 \wedge t)^{1/2+1/p}}. \tag{3.8}$$

As reported in [176, Theorem 3.2], the latter implies the following:

Corollary 3.4. *For all* $p > 1$, $\left\|\nabla(I + \mathcal{B}_p)^{-1/2}\right\|_{L^p \to L^p} \leq C_p$.

We will also employ a gradient estimate for the heat semigroup acting on distribution spaces.

Lemma 3.5. *For all* $t > 0$, $\alpha > 0$, $p \in (1, \infty)$*,*

$$\|\nabla P(t)\|_{\mathcal{H}^{-\alpha,p} \to C(\bar{\mathcal{O}})} \leq \frac{C_{\alpha,p}}{t^{(\alpha+1)/2+1/p}}. \tag{3.9}$$

Proof. Observe first that, by definition of $\mathcal{H}^{-\alpha,p}$ and density of $\mathcal{H}^{\alpha,p}$ in L^p, it holds for all $s > 0$

$$\begin{aligned}
\|P(s)\|_{\mathcal{H}^{-\alpha,p} \to L^p} &= \sup_{f \in L^p} \frac{\left\|P(s)(I + \mathcal{B}_p)^{\alpha/2} f\right\|_{L^p}}{\|f\|_{L^p}} \\
&= \sup_{f \in \mathcal{H}^{\alpha,p}} \frac{\left\|P(s)(I + \mathcal{B}_p)^{\alpha/2} f\right\|_{L^p}}{\|f\|_{L^p}} \leq \left\|(I + \mathcal{B}_p)^{\alpha/2} P(s)\right\|_{L^p \to L^p} \leq \frac{C_{\alpha,p}}{s^{\alpha/2}},
\end{aligned}$$

the last step using (3.4). For $0 < s < t$ we can combine (3.8) and the estimate of above in order to obtain

$$\begin{aligned}
\|\nabla P(t)\|_{\mathcal{H}^{-\alpha,p} \to C(\bar{\mathcal{O}})} &= \|\nabla P(t-s) P(s)\|_{\mathcal{H}^{-\alpha,p} \to C(\bar{\mathcal{O}})} \\
&= \|\nabla P(t-s)\|_{L^p \to C(\bar{\mathcal{O}})} \, \|P(s)\|_{\mathcal{H}^{-\alpha,p} \to L^p} \leq \frac{C_{\alpha,p}}{(t-s)^{1/2+1/p} s^{\alpha/2}},
\end{aligned} \tag{3.10}$$

from which the result follows minimizing the right-hand side with respect to the parameter s. □

The following statement collects technical passages involving heat semigroups that will appear often in our computations, since they concern (duality) relations between operator norms of semigroups and uniform estimates of their kernels over the space domain $\mathcal{O}$.

Lemma 3.6. *Let $\alpha \in \mathbb{R}$, $p \in (1,\infty)$, $t \geqslant 0$, $\varepsilon > 0$. For all $y \in \bar{\mathcal{O}}$ and $f \in \mathcal{H}^{\alpha,p'}(\mathcal{O}) \cap L^{p'}(\mathcal{O})$ it holds:*

$$\begin{aligned}&\int_{\mathcal{O}} (I+\mathcal{B}_p)^{\alpha/2} P(t) p_\varepsilon(\cdot,y)(x) f(x) dx \\ &\qquad = P(t+\varepsilon)(I+\mathcal{B}_{p'})^{\alpha/2} f(y) = (I+\mathcal{B}_{p'})^{\alpha/2} P_{t+\varepsilon} f(y),\end{aligned} \tag{3.11}$$

$$\begin{aligned}&\int_{\mathcal{O}} (I+\mathcal{B}_p)^{\alpha/2} P(t) \nabla_y p_\varepsilon(\cdot,y)(x) f(x) dx \\ &\qquad = \nabla P(t+\varepsilon)(I+\mathcal{B}_{p'})^{\alpha/2} f(y) = \nabla (I+\mathcal{B}_{p'})^{\alpha/2} P(t+\varepsilon) f(y).\end{aligned} \tag{3.12}$$

As a consequence

$$\sup_{y\in\bar{\mathcal{O}}} \|P(t)p_\varepsilon(\cdot,y)\|_{\mathcal{H}^{\alpha,p}(\mathcal{O})} = \left\| P(t+\varepsilon)(I+\mathcal{B}_{p'})^{\alpha/2}\right\|_{L^{p'}(\mathcal{O})\to C(\bar{\mathcal{O}})}, \tag{3.13}$$

$$\sup_{y\in\bar{\mathcal{O}}} \|P(t)\nabla_y p_\varepsilon(\cdot,y)\|_{\mathcal{H}^{\alpha,p}(\mathcal{O})} = \left\| \nabla P(t+\varepsilon)(I+\mathcal{B}_{p'})^{\alpha/2}\right\|_{L^{p'}(\mathcal{O})\to C(\bar{\mathcal{O}})}. \tag{3.14}$$

Proof. The first two claims follow from (3.5). For $y \in \bar{\mathcal{O}}$ consider a sequence $y_n \in \mathcal{O}$ such that $y_n \to y$; as for (3.11), the following chain of equalities holds due to the regularity of f and p_ε:

$$\begin{aligned}\int_{\mathcal{O}} (I+\mathcal{B}_p)^{\alpha/2} P(t) p_\varepsilon(\cdot,y)(x) f(x) dx &= \int_{\mathcal{O}} p_\varepsilon(\cdot,y)(x)(I+\mathcal{B}_{p'})^{\alpha/2} P(t) f(x) dx \\ &= \int_{\mathcal{O}} \lim_{n\to+\infty} p_\varepsilon(\cdot,y_n)(x)(I+\mathcal{B}_{p'})^{\alpha/2} P(t) f(x) dx \\ &= \int_{\mathcal{O}} \lim_{n\to+\infty} p_\varepsilon(y_n,\cdot)(x)(I+\mathcal{B}_{p'})^{\alpha/2} P(t) f(x) dx \\ &= \lim_{n\to+\infty} \int_{\mathcal{O}} p_\varepsilon(y_n,\cdot)(x)(I+\mathcal{B}_{p'})^{\alpha/2} P(t) f(x) dx \\ &= \lim_{n\to+\infty} P(\varepsilon)(I+\mathcal{B}_{p'})^{\alpha/2} P(t) f(y_n).\end{aligned}$$

This implies (3.11) thanks to the fact that $(I+\mathcal{B}_{p'})^{\alpha/2}$ and the heat semigroup commute. The exchange between limit and integral is allowed thanks to (3.5). The proof of (3.12) follows from a similar argument. Indeed,

$$\begin{aligned}&\int_{\mathcal{O}} (I+\mathcal{B}_p)^{\alpha/2} P(t) \nabla_y p_\varepsilon(\cdot,y)(x) f(x) dx \\ &\qquad = \int_{\mathcal{O}} \nabla_y p_\varepsilon(\cdot,y)(x)(I+\mathcal{B}_p)^{\alpha/2} P(t) f(x) dx \\ &\qquad = \nabla_y \int_{\mathcal{O}} p_\varepsilon(\cdot,y)(x)(I+\mathcal{B}_p)^{\alpha/2} P(t) f(x) dx\end{aligned}$$

$$\begin{aligned}
&= \nabla_y \int_{\mathcal{O}} \lim_{n\to+\infty} p_\varepsilon(\cdot, y_n)(x)(I+\mathcal{B}_p)^{\alpha/2}P(t)f(x)dx \\
&= \nabla_y \int_{\mathcal{O}} \lim_{n\to+\infty} p_\varepsilon(y_n, \cdot)(x)(I+\mathcal{B}_p)^{\alpha/2}P(t)f(x)dx \\
&= \nabla_y \lim_{n\to+\infty} P(\varepsilon)(I+\mathcal{B}_{p'})^{\alpha/2}P(t)f(y_n).
\end{aligned}$$

Equation 3.12 follows from the regularity of $P(\varepsilon)(I+\mathcal{B}_{p'})^{\alpha/2}P(t)$ and the fact that $(I+\mathcal{B}_{p'})^{\alpha/2}$ and the heat semigroup commute. After these preliminaries, (3.13) and (3.14) are easy to prove by duality. Indeed, for each $y \in \bar{\mathcal{O}}$,

$$\begin{aligned}
\|P(t)p_\varepsilon(\cdot, y)\|_{\mathcal{H}^{\alpha,p}} &= \sup_{f\in\mathcal{H}^{\alpha,p'}, \|f\|_{L^{p'}}=1} \left|\int_{\mathcal{O}} (I+\mathcal{B}_p)^{\alpha/2}P(t)p_\varepsilon(\cdot, y)(x)f(x)dx\right| \\
&= \sup_{f\in\mathcal{H}^{\alpha,p'}, \|f\|_{L^{p'}}=1} \left|(I+\mathcal{B}_{p'})^{\alpha/2}P(t+\varepsilon)f(y)\right| \\
&= \sup_{\|f\|_{L^{p'}}=1} \left|(I+\mathcal{B}_{p'})^{\alpha/2}P(t+\varepsilon)f(y)\right|.
\end{aligned}$$

Considering the supremum of both sides for $y \in \bar{\mathcal{O}}$ (3.13) follows immediately due to the fact that $(I+\mathcal{B}_{p'})^{\alpha/2}P(t+\varepsilon) \in C(\bar{\mathcal{O}})$. (3.14) is similar. Indeed

$$\begin{aligned}
\|P(t)\nabla_y p_\varepsilon(\cdot, y)\|_{\mathcal{H}^{\alpha,p}} &= \sup_{f\in\mathcal{H}^{\alpha,p'}, \|f\|_{L^{p'}}=1} \left|\int_{\mathcal{O}} (I+\mathcal{B}_p)^{\alpha/2}P(t)\nabla_y p_\varepsilon(\cdot, y)(x)f(x)dx\right| \\
&= \sup_{f\in\mathcal{H}^{\alpha,p'}, \|f\|_{L^{p'}}=1} \left|\nabla(I+\mathcal{B}_{p'})^{\alpha/2}P(t+\varepsilon)f(y)\right| \\
&= \sup_{\|f\|_{L^{p'}}=1} \left|\nabla(I+\mathcal{B}_{p'})^{\alpha/2}P(t+\varepsilon)f(y)\right|.
\end{aligned}$$

Equation 3.14 then follows considering the supremum of both sides for $y \in \bar{\mathcal{O}}$. □

We conclude the paragraph recalling the regularizing property of Biot-Savart kernel, following from the one of Green's function for Dirichlet boundaries, for which we refer to [98, Section 3].

Lemma 3.7. *The linear operator $K[\omega] = -\nabla^\perp(-\Delta_{Dir})^{-1}\omega$, defined first for $\omega \in C^\infty(\mathcal{O})$, extends to continuous linear maps (still denoted by K)*

$$\begin{aligned}
K &: L^2(\mathcal{O}) \to L^q(\mathcal{O};\mathbb{R}^2), \quad q \in [1,\infty), \\
K &: L^p(\mathcal{O}) \to C(\bar{\mathcal{O}};\mathbb{R}^2), \quad p \in (2,\infty), \\
K &: H^{\alpha,p}(\mathcal{O}) \to H^{\alpha+1,p}(\mathcal{O};\mathbb{R}^2), \quad p \in (1,\infty), \alpha \geq 0.
\end{aligned}$$

Moreover, for all the above extensions $K[\omega]$ is divergence-less (in the sense of distributions), and its normal trace on $\partial\mathcal{O}$ (when defined) vanishes.

3.1.2 *Càdlàg* Functions and Aldous' Criterion

For $T > 0$, we denote by $\mathbb{D}([0,T],S)$ the space of *càdlàg* functions on $[0,T]$ taking values in a complete metric space (S,d). We will always endow $\mathbb{D}([0,T],S)$ with the Skorohod metric: we refer to [143, Section 2.1] for a definition of the latter, which we will not be using directly since we can rely on the Aldous' criterion for tightness, [143, Theorem 3.2].

Proposition 3.8. *Consider a sequence of filtered probability spaces $(\Omega_n,\mathcal{F}_n,\mathbb{P}_n)_{n\in\mathbb{N}}$ on each of which it is defined a* càdlàg *adapted process $(X^n(t))_{t\in[0,T]}$ taking values in a complete separable metric space (S,d). The laws of processes $(X^n(t))_{t\in[0,T]}$ are tight on $\mathbb{D}([0,T],S)$ if the following two conditions are satisfied:*

1. *for every t in a dense subset of $[0,T]$ the laws of $X^n(t)$ are tight on S;*
2. *for all $\varepsilon,\delta>0$ there exists $r_0>0$ and $n_0\in\mathbb{N}$ such that, for any sequence $(\tau_n)_{n\in\mathbb{N}}$ with τ_n a $\mathcal{F}_n$-stopping time, it holds*

$$\sup_{n\geq n_0}\sup_{r\in[0,r_0]}\mathbb{P}_n\left(d\left(X^n\left((\tau_n+r)\wedge T\right),X^n\left(\tau_n\wedge T\right)\right)>\delta\right)<\varepsilon. \tag{3.15}$$

3.1.3 Well-Posedness of Navier-Stokes Equations with Neumann Boundary

In order to identify the limit dynamics of our interacting particle system with solutions of (3.1), we will need a uniqueness result and *a priori* estimates. We set the discussion of the limit PDE in the space of *càdlàg* functions since convergence of the particle system will take place in that space.

Definition 3.9. Let $T>0$; $\omega\in\mathbb{D}([0,T];L^2(\mathcal{O}))$, is a weak solution of (3.1) if for all $\varphi\in\mathcal{H}^{2,2}(\mathcal{O})$, for $t\in[0,T]$,

$$\begin{aligned}&\int_{\mathcal{O}}\varphi\omega(t)dx-\int_{\mathcal{O}}\varphi\omega_0dx\\&\qquad=\int_0^t\int_{\mathcal{O}}\Delta\varphi\cdot\omega(s)dxds+\int_0^t\int_{\mathcal{O}}\nabla\varphi\cdot K[\omega(s)]\omega(s)dxds+\int_0^t\int_{\partial\mathcal{O}}\varphi g(s)d\sigma(x)ds.\end{aligned}$$

Remark 3.10. The latter is a good definition: thanks to Lemma 3.7, integrability in space of the nonlinear term $\nabla\varphi\cdot K[\omega]\omega$ follows from Hölder's inequality, since both $\nabla\varphi$ and $K[\omega]$ are L^q for all $q>2$.

Proposition 3.11. *Given $\omega_0\in L^2$ and $g\in L^2([0,T],L^2)$, there exists a unique weak solution of (3.1) in the sense of Definition 3.9 with $\omega(0)=\omega_0$. Moreover, the unique solution belongs to $C([0,T];L^2(\mathcal{O}))\cap L^2([0,T];\mathcal{H}^{1,2}(\mathcal{O}))$.*

The same statement holds for the cutoff PDE (1.22) for which in fact, thanks to the bounded nonlinearity, much stronger well-posendenss results can be obtained.

Proof. Consider two weak solutions $\omega, \tilde{\omega} \in \mathbb{D}([0,T]; L^2(\mathcal{O}))$, let $w(t) = \omega(t) - \tilde{\omega}(t)$. With standard passages (we refer for instance to [74, Section 3]) one can extend the weak formulation of the PDE to time-dependent tests

$$\varphi \in C^1([0,T]; L^2) \cap C([0,T]; \mathcal{H}^{2,2})$$

(this makes use of the right continuity hypothesis on ω), then, taking $\varphi_t(s,x) = P(t-s)\psi(x)$, $\psi \in \mathcal{H}^{2,2}$ transforms the PDE into the variation-of-constants form

$$\omega(t) = P(t)\omega_0 - \int_0^t P(t-s)(\operatorname{div}(K[\omega(s)]\omega(s)))ds + \int_0^t (I-\Delta)P(t-s)\mathsf{N}[g(s)]ds.$$

A similar formula holds for $\tilde{\omega}(t)$. Therefore $w(t)$ satisfies

$$w(t) = -\int_0^t P(t-s)(\operatorname{div}(K[w(s)]\tilde{\omega}(s)))ds - \int_0^t P(t-s)(\operatorname{div}(K[\omega(s)]w(s)))ds.$$

By Sobolev embedding, for all $q > 2$,

$$K[w(s)],\ K[\omega(s)] \in \mathbb{D}([0,T]; L^q(\mathcal{O}; \mathbb{R}^2)),$$

hence, by Holder inequality, again for all $q > 2$,

$$K[w(s)]\tilde{\omega}(s),\ K[\omega(s)]w(s) \in \mathbb{D}\left([0,T]; L^{\frac{2q}{q+2}}(\mathcal{O}; \mathbb{R}^2)\right).$$

We can thus control nonlinear terms by means of ultracontractivity (3.3), Hölder's inequality and Lemma 3.7:

$$\begin{aligned}
\|w(t)\| &\lesssim_{T,q} \int_0^t \frac{1}{(t-s)^{\frac{q+2}{2q}-\frac{1}{2}}} \left\| P\left(\frac{t-s}{2}\right)(\operatorname{div}(K[w(s)]\tilde{\omega}(s)))\right\|_{L^{\frac{2q}{q+2}}} ds \\
&\quad + \int_0^t \frac{1}{(t-s)^{\frac{q+2}{2q}-\frac{1}{2}}} \left\| P\left(\frac{t-s}{2}\right)(\operatorname{div}(K[\omega(s)]w(s)))\right\|_{L^{\frac{2q}{q+2}}} ds \\
&\lesssim_q \int_0^t \frac{1}{(t-s)^{\frac{q+2}{2q}}} \left(\|K[w(s)]\tilde{\omega}(s)\|_{L^{\frac{2q}{q+2}}} + \|K[\omega(s)]w(s)\|_{L^{\frac{2q}{q+2}}} \right) ds \\
&\lesssim_q \int_0^t \frac{\|\omega(s)\| + \|\tilde{\omega}(s)\|}{(t-s)^{\frac{q+2}{2q}}} \|w(s)\|\, ds.
\end{aligned}$$

The uniqueness statement now follows from Grönwall's Lemma.

The last statement of the proposition follows from existence of a weak solution belonging to $C([0,T]; L^2(\mathcal{O})) \cap L^2([0,T]; H^{1,2}(\mathcal{O}))$. This can be proved with a standard Galerkin approximation together with the usual energy estimate (see the beginning of the proof of Proposition 3.12 below), we refer to [105, Theorem 2.6] for details. □

Proposition 3.12. *Any weak solution of* (3.1) *in the sense of Definition 3.9 in* $\mathbb{D}([0,T];L^p)$ *with* $\omega_0 \in L^p$*,* $p>2$*, satisfies*

$$\|K[\omega]\|_{L^\infty([0,T];L^\infty)} \lesssim_{T,p} \|\omega_0\|_{L^p} + \|\omega_0\|^2 + \|g\|_{L^2([0,T];L^p)} + \|g\|^2_{L^2([0,T];L^2)}\,. \tag{3.16}$$

As a consequence, there exists $M_0 = M_0(\omega_0, g)$ *such that for all* $M \geq M_0$ *the unique weak solution of* (3.1) *coincides with the one of the cutoff PDE* (1.22) *with the same initial and boundary data.*

In order to establish this *a priori* estimate we first recall a convenient representation for boundary terms appearing in the weak formulation of (3.1).

Lemma 3.13. *Let* $g \in L^p(\partial\mathcal{O})$*,* $p \geq 2$ *and* $\varepsilon > 0$*. There exists a unique weak solution* $u \in H^{1+\frac{1}{p}-\varepsilon,p}(\mathcal{O})$ *of*

$$\begin{cases} (I-\Delta)u(x) = 0, & x \in \mathcal{O}, \\ \hat{n}\cdot\nabla u(x) = g(x), & x \in \partial\mathcal{O}, \end{cases}$$

that is, for all $\varphi \in H^{1,p'}(\mathcal{O})$*,*

$$\int_{\mathcal{O}} (\varphi u + \nabla\varphi\cdot\nabla u)dx = \int_{\partial\mathcal{O}} \varphi g d\sigma(x).$$

Moreover, the solution map $\mathsf{N}[g] = u$ *defines a continuous linear operator* $\mathsf{N} : L^p(\partial\mathcal{O}) \to \mathcal{H}^{1+1/p-\delta,p}$ *for any* $\delta > 0$*.*

We refer to [170, Chapter 4] for the (standard) proof. It follows that

$$\int_{\partial\mathcal{O}} \varphi g d\sigma(x) = \int_{\mathcal{O}} (I+\mathcal{B}_{p'})\varphi \mathsf{N}[g]dx, \quad \varphi \in \mathcal{H}^{2,p'},\ g \in L^p(\partial\mathcal{O}).$$

Proof of Proposition 3.12. For the sake of completeness we sketch the classical argument for the *a priori* estimate in the L^2 setting. By Proposition 3.11 we know that the weak solution actually belongs to $C([0,T];L^2(\mathcal{O})) \cap L^2([0,T];\mathcal{H}^{1,2}(\mathcal{O}))$, so we can evaluate

$$\frac{1}{2}\frac{d}{dt}\|\omega(t)\|^2 = -\|\nabla\omega(t)\|^2 + \int_{\mathcal{O}} \nabla\omega(t)\cdot K[\omega(t)]\omega(t)dx + \int_{\partial\mathcal{O}} \omega(t)g(t)d\sigma(x).$$

The second term on the right-hand side vanishes, and the third one can be controlled thanks to the properties of the trace of functions in $H^1(\mathcal{O})$:

$$\begin{aligned}\int_{\partial\mathcal{O}} \omega(t)g(t)d\sigma(x) &\leq C\left(\|\omega(t)\| + \|\nabla\omega(t)\|\right)\|g(t)\|_{L^2(\partial\mathcal{O})} \\ &\leq \frac{1}{2}\|\omega(t)\|^2 + \frac{1}{2}\|\nabla\omega(t)\|^2 + C\|g(t)\|^2_{L^2(\partial\mathcal{O})}\,.\end{aligned}$$

Therefore, we obtain

$$\frac{1}{2}\frac{d}{dt}\|\omega(t)\|^2 + \frac{1}{2}\|\nabla\omega(t)\|^2 \leq \frac{1}{2}\|\omega(t)\|^2 + C\int_{\partial\mathcal{O}}\omega(t)g(t)d\sigma(x),$$

from which by Grönwall's Lemma we conclude that

$$\|\omega\|^2_{L^\infty([0,T],L^2)} + \|\nabla\omega\|^2_{L^2([0,T],L^2)} \lesssim_T \|\omega_0\|^2 + \|g\|^2_{L^2([0,T],L^2(\partial\mathcal{O}))}\,. \tag{3.17}$$

Arguing as in the proof of Proposition 3.11, extending the weak formulation of the PDE to time-dependent tests

$$\varphi \in C^1([0,T];L^{p'}) \cap C([0,T];\mathcal{H}^{2,p'})$$

and taking $\varphi_t(s,x) = P(t-s)\psi(x)$, $\psi \in \mathcal{H}^{2,p'}$, standard passages allow to rewrite the PDE into the variation-of-constants form

$$\omega(t) = P(t)\omega_0 - \int_0^t P(t-s)(K[\omega(s)]\cdot\nabla\omega(s))ds + \int_0^t (I-\Delta)P(t-s)\mathsf{N}[g(s)]ds, \tag{3.18}$$

from which

$$\|\omega(t)\|_{L^p} \leq \|\omega_0\|_{L^p} + \int_0^t \|P(t-s)(K[\omega(s)]\cdot\nabla\omega(s))\|_{L^p}\,ds + \int_0^t \|(I-\Delta)P(t-s)\mathsf{N}[g(s)]\|_{L^p}\,ds.$$

We control the nonlinear term by means of ultracontractivity (3.3), Hölder's inequality and Lemma 3.7: for $\frac{2p}{p+2} < q < 2 < p$ it holds

$$\begin{aligned}\int_0^t \|P(t-s)(K[\omega(s)]\cdot\nabla\omega(s))\|_{L^p}\,ds &\leq \int_0^t (t-s)^{1/p-1/q}\|(K[\omega(s)]\cdot\nabla\omega(s))\|_{L^q}\,ds\\ &\leq \int_0^t (t-s)^{1/p-1/q}\|K[\omega(s)]\|_{L^{2q/(2-q)}}\|\nabla\omega(s)\|\,ds\\ &\lesssim_q \|\omega\|_{L^\infty([0,T],L^2)}\int_0^t (t-s)^{1/p-1/q}\|\nabla\omega(s)\|\,ds\\ &\lesssim_q \|\omega\|_{L^\infty([0,T],L^2)}\|\omega\|_{L^2([0,T],\mathcal{H}^{1,2}(\mathcal{O}))}\,,\end{aligned}$$

the last step following from Hölder's inequality and the fact that $\int_0^t (t-s)^{2/q-2/p}ds < \infty$. As for the boundary term, by Proposition 3.2, Lemma 3.13 and (3.4), for $\delta > 0$,

$$\begin{aligned}&\int_0^t \|(I-\Delta)P(t-s)\mathsf{N}[g(s)]\|_{L^p}\,ds\\ &\quad= \int_0^t \|(I+\mathcal{B}_p)^{\frac{1}{2}+\delta-\frac{1}{2p}}P(t-s)(I+\mathcal{B}_p)^{\frac{1}{2}-\delta+\frac{1}{2p}}\mathsf{N}\left[g(s)\right]\|_{L^p}ds\\ &\quad\lesssim_{p,\delta} \int_0^t (t-s)^{-1/2-\delta+1/(2p)}\|(I+\mathcal{B}_p)^{\frac{1}{2}-\delta+\frac{1}{2p}}\mathsf{N}\left[g(s)\right]\|_{L^p}ds\\ &\quad\leq \left(\int_0^t (t-s)^{-1-2\delta+1/p}ds\right)^{1/2}\|g\|_{L^2(0,T;L^p)},\end{aligned}$$

where the integral in parentheses is finite if we choose $\frac{1}{2p} > \delta$.

By Lemma 3.7, and combining the estimates above, we deduce that

$$\begin{aligned}\|K[\omega]\|_{L^\infty([0,T],L^\infty)} &\lesssim_p \sup_{t\in[0,T]} \|\omega(t)\|_{L^p} \\ &\lesssim_{T,p} \|\omega_0\|_{L^p} + \|\omega_0\|^2 + \|g\|_{L^2([0,T];L^p)} + \|g\|^2_{L^2([0,T];L^2)},\end{aligned}$$

from which the first statement of the proposition follows.

As for the second statement, it suffices to take

$$M \geq C_{T,p}\left(\|\omega_0\|_{L^p} + \|\omega_0\|^2 + \|g\|_{L^2([0,T];L^p)} + \|g\|^2_{L^2([0,T];L^2)} + 1\right),$$

where $C_{T,p}$ is the constant implied in the previous inequality, making it so that the nonlinear term of solutions of (3.1) and (1.22) coincide by definition of the cutoff F. □

3.2 Definition of the Model and Main Results

Since we are considering a finite, fixed time horizon $T > 0$, for the sake of lightening notation we assume from now, without loss of generality, $T = 1$.

3.2.1 Generation of Particles

Let us introduce a grid of mesh $1/n$ spanning $(\{0\}\times\mathcal{O})\cup((0,1]\times\partial\mathcal{O})$. Let $\gamma : S^1 \simeq (0,1] \to \partial\mathcal{O}$ be a diffeomorphism parametrizing the boundary of $\mathcal{O}$; for $n \geq 1$ we set

$$L^n = \{0\} \times L^n_{in} \cup \left\{\frac{1}{2n}, \frac{3}{2n} \dots \frac{2n-1}{2n}\right\} \times L^n_{bd},$$
$$L^n_{in} = (\mathbb{Z}/n)^2 \cap \mathcal{O}, \quad L^n_{bd} = \{\gamma(h/n) \mid h = 1, \dots n\}.$$

We will write

$$i \mapsto (t^n_i, \zeta^n_i) \in L^n, \quad i = 1, \dots, N(n), \tag{3.19}$$

to enumerate the points of the grid L^n, $N(n)$ being the cardinality of L^n. From now on we simply write $N = N(n)$ implying the dependence on n. We now introduce a partition of $(\{0\} \times \mathcal{O}) \cup ((0,1] \times \partial\mathcal{O})$ whose elements are centered at points ζ_i:

$$Q^n_i = \begin{cases} \left(\zeta^n_i + [-\frac{1}{2n}, \frac{1}{2n}]^2\right) \cap \mathcal{O}, & t^n_i = 0, \\ [t^n_i - \frac{1}{2n}, t^n_i + \frac{1}{2n}] \times \gamma\left(\gamma^{-1}(\zeta_i) + [-\frac{1}{2n}, \frac{1}{2n}]\right), & t^n_i > 0. \end{cases} \tag{3.20}$$

Notice that $N(n) \simeq n^2$ and the area of Q^n_i (both if $t^n_i = 0$ or not) is of order n^{-2}. We denote by

$$\mathsf{A}^n(t) = \{i : 0 \leq t^n_i \leq t\}, \quad A^n_0(t) = \{i : 0 < t^n_i \leq t\}, \tag{3.21}$$

the set of indices relative to particles created before time t. Given $\omega_0 \in \mathcal{B}_b(\mathcal{O})$, $g \in \mathcal{B}_b([0,1] \times \partial\mathcal{O})$ with $g, \omega_0 \geq 0$, we set

$$\omega_i^n = \begin{cases} \int_{Q_i^n} \omega_0(x)dx & t_i^n = 0, \\ \int_{Q_i^n} g(s,x)dsd\sigma(x) & t_i^n > 0. \end{cases} \tag{3.22}$$

From our assumptions on the mesh of the grid, ω_0 and g it follows that

$$\omega_i^n \lesssim \frac{1}{n^2} \quad \forall n \in \mathbb{N},\ i = 1, \dots N.$$

3.2.2 Diffusion Processes and Reflecting Boundaries

Let $(\Omega, \mathcal{F}, \mathbb{P})$ a complete, filtered probability space satisfying the standard hypothesis, on which it is defined a sequence $(B^i)_{i\in\mathbb{N}}$ of independent $\mathcal{F}$-Brownian motions. We have the following (probabilistically strong) well-posedness result:

Proposition 3.14. *For all $n \geq 1$, on $(\Omega, \mathcal{F}, \mathbb{P})$, there exists a unique continuous $\bar{\mathcal{O}}^N$-valued adapted process $x^n(t) = (x_1^n(t), \dots, x_N^n(t))_{t\in[0,T]}$ and a continuous $\mathbb{R}^{2\times N}$-valued adapted process $k^n(t) = (k_1^n(t), \dots, k_N^n(t))_{t\in[0,T]}$ with bounded variation trajectories such that: for $i = 1, \dots, N$, $t \in [t_i^n, 1]$,*

$$x_i^n(t) - \zeta_i^n = \int_{t_i^n}^t F\left(\sum_{j\in \mathsf{A}^n(s)} \omega_j^n K_n(x_i^n(s), x_j^n(s))\right) ds + \sqrt{2}\int_{t_i^n}^t dB_s^i - k_i^n(t),$$

$$k_i^n(t) = \int_{t_i^n}^t \hat{n}(x_i^n(s))d|k_i^n|(s), \quad |k_i^n|(t) = \int_{t_i^n}^t \mathbf{1}_{x_i^n(s)\in\partial\mathcal{O}} d|k_i^n|(s),$$

while for $t < t_i^n$ we impose $k_i^n(t) = 0$ and $x_i^n(t) = \zeta_i^n$.

The proof can be straightforwardly adapted from the one given in [131, 164] for systems of SDEs with regular coefficients and reflecting boundaries: the only difference is generation of new particles at the boundary, which is taken care of applying the well-posedness result on each time interval of length $1/n$ during which particles are not generated. We also refer to [105, Section 4] for details on a SDE system (closely related to ours) including boundary generation at random times. Itô's formula for the process $x^n(t)$ takes the following form (for which we refer again to [131]):

Corollary 3.15. *If $\varphi \in C^2(\bar{\mathcal{O}})$ with $\nabla\varphi \cdot \hat{n} = 0$ on $\partial\mathcal{O}$, for $i = 1, \dots, N$, $t \in [t_i^n, T]$,*

$$d\varphi(x_i^n(t)) = F\left(\sum_{j\in \mathsf{A}^n(t)} \omega_j^n K_n(x_i^n(t), x_j^n(t))\right) \cdot \nabla\varphi(x_i^n(t))dt + \Delta\varphi(x_i^n(t))dt + \sqrt{2}\nabla\varphi(x_i^n(t))dB_t^i.$$

Notice that the hypothesis $\nabla\varphi\cdot\hat{n}=0$ on $\partial\mathcal{O}$ makes it so that reflection terms $-k_i^n$ do not appear in the Itô formula. In our applications, this assumption will be verified thanks to our choice of regularizing the empirical measure with the heat kernel under Neumann bouondary conditions.

Remark 3.16. The classical vortex dynamics in a bounded domain is a system of singular ODEs of the form

$$\dot{x}_i=\sum_{j\neq i}\xi_j K(x_i,x_j)+\xi_i\nabla^\perp\gamma(x_i,x_i),\quad x_i\in\mathcal{O},\,\xi_i\in\mathbb{R}^*,\quad i,j=1,\dots N,$$

$$G=(-\Delta_{Dir})^{-1},\quad \gamma(x,y)=G(x,y)+\frac{1}{2\pi}\log|x-y|\,,$$

including self-interaction terms induced from the boundary effects; this is necessary for the system to satisfy in a weak form (as in [158]) the 2-dimensional Euler equations. Self-interactions diverge logarithmically at $\partial\mathcal{O}$, and this prevents us to include them in our model because the Brownian part of the dynamics (which we must include to model viscosity) might drive particles onto the boundary causing blow-up of the dynamics at finite time. Nevertheless, under the Mean-Field scaling of particle intensities we are considering, self-interactions should be negligible in macroscopic limit. Indeed (heuristically) a self-interaction term $\omega_i\nabla^\perp\gamma(x_i,x_i)$ in our model would scale as n^{-2}, while the nonlinear and noise terms are of order 1 as $n\to\infty$. Hence, self-interactions due to the boundary appear to be irrelevant for the purpose of our discussion.

Having Proposition 3.11 and Proposition 3.14 in mind we can state the main result of the chapter.

Theorem 3.17. *Let* $p>2$, $\frac{2}{p}<\alpha<1$, *and* $\varepsilon=\varepsilon(n)\gtrsim n^{-1/2}$. *Assume that* $\omega_0\in\mathrm{Lip}(\bar{\mathcal{O}})$ *and* $g\in\mathcal{B}_b([0,1]\times\partial\mathcal{O})$, $\omega_0,g\geq 0$, *and let the related notation introduced above prevail. There exists* $M>0$ *(only depending on* ω_0,g*) such that for every* $\eta\in\left(\frac{2}{p},\alpha\right)$, *as* $n\to\infty$, *the kernel-smoothed empirical measure* $\omega^n(t)=\sum_{i\in\mathsf{A}^n(t)}\omega_i^n p_\varepsilon(\cdot,x_i^n(t))$ *converges in probability on* $\mathbb{D}([0,1],\mathcal{H}^{\eta,p})$ *to the unique weak solution (given by Proposition 3.11) of (3.1) with initial datum* ω_0 *and boundary source* g.

By Morrey's inequality, the stated convergence implies that on $\mathbb{D}([0,1],C(\bar{\mathcal{O}}))$.

3.3 Uniform Bounds

The proof of Theorem 3.17 essentially relies on uniformly bounding (in n) the approximating process $\omega^n(t,x)$ in terms of strong norms, allowing the application of Aldous' tightness criterion, Proposition 3.8, and to pass to the limit the dynamics obtaining (3.1). The way we exploit the regularizing effect of the Brownian noise driving particles, that is the parabolic nature of the corresponding PDE dynamics, consists in formulating the evolution problem in Duhamel form (that is the variation-of-constants form, or *mild* formulation), in complete analogy with the *a priori* estimates for the limiting PDE in the previous section.

3.3.1 Duhamel Formulation of Empirical Measure Dynamics

We adopt, for the remainder of the section, the notation of subsection 3.2.2. Let thus $(x_1^n, \dots, x_N^n)$ be the $\bar{\mathcal{O}}^N$-valued stochastic process on $[0,1]$ defined by Proposition 3.14 as the unique strong solution of the approximating particle system, and

$$\mathsf{S}^n(t) = \sum_{i \in \mathsf{A}^n(t)} \omega_i^n \delta_{x_i^n(t)}.$$

A direct application of the Itô formula in Corollary 3.15 shows that, if $\varphi \in C^2(\bar{\mathcal{O}})$ with $\nabla\varphi \cdot \hat{n} = 0$ on $\partial\mathcal{O}$, for all $t \in [0,1]$ it holds

$$\begin{aligned}\int_{\mathcal{O}} \varphi d\mathsf{S}^n(t) - \int_{\mathcal{O}} \varphi d\mathsf{S}_0^n = &\sum_{i \in \mathsf{A}_0^n(t)} \omega_i \varphi(\zeta_i) + \int_0^t \int_{\mathcal{O}} \Delta\varphi d\mathsf{S}^n(s) ds \\ &+ \int_0^t \int_{\mathcal{O}} F\left(\int_{\mathcal{O}} K_n(x,y) d\mathsf{S}^n(s,y)\right) \nabla\varphi(x) d\mathsf{S}^n(s,x) ds \\ &+ \sqrt{2} \int_0^t \sum_{i \in \mathsf{A}^n(s)} \omega_i \nabla\varphi(x_i^n(s)) dB_s^i.\end{aligned} \tag{3.23}$$

Let us now considered the kernel-smoothed empirical measure

$$\omega^n(t,x) = P_\varepsilon \mathsf{S}^n(t) = \sum_{i \in \mathsf{A}^n(t)} \omega_i^n p_\varepsilon(x, x_i^n(t)), \quad x \in \bar{\mathcal{O}},$$

which, in sight of (3.23), must satisfy for all $t \in [0,1]$ and $x \in \bar{\mathcal{O}}$

$$\begin{aligned}\omega^n(t,x) - \omega_0^n(x) = &\sum_{i \in \mathsf{A}_0^n(t)} \omega_i^n p_\varepsilon(x, \zeta_i^n) + \int_0^t \Delta\omega^n(s,x) ds \\ &+ \int_0^t \int_{\mathcal{O}} F\left(\int_{\mathcal{O}} K_n(y,z) d\mathsf{S}^n(s,z)\right) \nabla_y p_\varepsilon(x,y) d\mathsf{S}^n(s,y) ds \\ &+ \sqrt{2} \int_0^t \sum_{i \in \mathsf{A}^n(s)} \omega_i^n \nabla_y p_\varepsilon(x, x_i^n(s)) dB_s^i.\end{aligned} \tag{3.24}$$

As in the proof of Proposition 3.12, we can derive from the latter formulation of the dynamics the variation-of-constants form with standard passages. For all $t \in [0,1]$ and $x \in \bar{\mathcal{O}}$ it holds:

$$\begin{aligned} \omega^n(t,x) = P(t)\omega_0^n(x) &+ \sum_{i \in \mathsf{A}_0^n(t)} \omega_i^n P(t-t_i^n) p_\varepsilon(x,\zeta_i^n) \\ &+ \int_0^t P(t-s) \int_{\mathcal{O}} F\left(\int_{\mathcal{O}} K_n(y,z) d\mathsf{S}^n(s,z) \right) \nabla_y p_\varepsilon(x,y) d\mathsf{S}^n(s,y) ds \\ &+ \sqrt{2} \int_0^t \sum_{i \in \mathsf{A}^n(s)} \omega_i^n P(t-s) \nabla_y p_\varepsilon(x, x_i^n(s)) dB_s^i. \end{aligned} \tag{3.25}$$

3.3.2 Preliminary Estimates on Particle Creation and Diffusion

We begin by estimating separately the terms due to particle generation at $t=0$ and at the boundary at later $t>0$.

Lemma 3.18. *Let $\varepsilon = \varepsilon(n) \gtrsim n^{-2/(3+\alpha)} \gtrsim n^{-1/2}$; for all $p > 2$, $\frac{2}{p} < \alpha < 1$ it holds*

$$\left\| \sum_{i : t_i^n = 0} \omega_n^i p_\varepsilon(\cdot, \zeta_i^n) \right\|_{\mathcal{H}^{\alpha,p}} \leq C_{\alpha,p}.$$

The idea of the proof is that, for n large and ε small,

$$\sum_{i : t_i^n = 0} \omega_i^n p_\varepsilon(x, \zeta_i^n) \approx \int_{\mathcal{O}} p_\varepsilon(x,y) \omega_0(y) dy,$$

and the right-hand side is bounded if, say, ω_0 is Lipschitz continuous.

Proof. We have

$$\begin{aligned} \left\| \sum_{i : t_i^n = 0} \omega_n^i p_\varepsilon(\cdot, \zeta_i^n) \right\|_{\mathcal{H}^{\alpha,p}} &\leqslant \left\| \int_{\mathcal{O}} p_\varepsilon(\cdot, y) \omega_0(y) dy \right\|_{\mathcal{H}^{\alpha,p}} \\ &+ \left\| \int_{\mathcal{O}} p_\varepsilon(\cdot, y) \omega_0(y) dy - \sum_{i : t_i^n = 0} \omega_n^i (I + \mathcal{B}_p)^{\alpha/2} p_\varepsilon(\cdot, \zeta_i^n) \right\|_{\mathcal{H}^{\alpha,p}} \quad =: I_{\omega_0} + R. \end{aligned}$$

We bound the term I_{ω_0}:

$$I_{\omega_0} = \left\| (I + \mathcal{B}_p)^{\alpha/2} P_\varepsilon \omega_0 \right\|_{L^p} = \left\| P_\varepsilon (I + \mathcal{B}_p)^{\alpha/2} \omega_0 \right\|_{L^p} \leqslant \|\omega_0\|_{\mathcal{H}^{\alpha,p}}. \tag{3.26}$$

Concerning R, recall that, for i such that $t_i^n = 0$, $\omega_i^n = \int_{Q_i^n} \omega_0(y)dy$. Hence,

$$\begin{aligned}
&\int_{\mathcal{O}} p_\varepsilon(\cdot,y)(x)\omega_0(y)dy - \sum_{i:t_i^n=0} \omega_n^i p_\varepsilon(\cdot,\zeta_i^n)(x)\\
&= \sum_{i:t_i^n=0} \int_{Q_i^n} p_\varepsilon(\cdot,y)(x) - p_\varepsilon(\cdot,\zeta_i^n)(x)\omega_0(y)dy\\
&= \sum_{i:t_i^n=0} \int_{Q_i^n} (y-\zeta_i^n)\cdot \int_0^1 \nabla_y p_\varepsilon(\cdot,\zeta_i^n+\xi(y-\zeta_i^n))(x)d\xi\, \omega_0(y)dy
\end{aligned}$$

(note that $\zeta_i^n+\xi(y-\zeta_i^n)$ belongs to $\bar{\mathcal{O}}$ since $\bar{\mathcal{O}}$ is convex). Recall that, for every i, $|y-\xi_i^n| \lesssim 1/n$ on Q_i^n. We then have, by (3.14),

$$\begin{aligned}
R &\leqslant \sum_{i:t_i^n=0} \int_{Q_i^n} (y-\zeta_i^n)\cdot \int_0^1 \|\nabla_y p_\varepsilon(\cdot,\zeta_i^n+\xi(y-\zeta_i^n))\|_{\mathcal{H}^{\alpha,p}}\, d\xi\, \omega_0(y)dy\\
&\lesssim \frac{1}{n}\int_{\mathcal{O}} \omega_0(y)dy \sup_{y\in\bar{\mathcal{O}}} \|\nabla_y p_\varepsilon(\cdot,y)\|_{\mathcal{H}^{\alpha,p}}\\
&= \frac{1}{n}\int_{\mathcal{O}} \omega_0(y)dy \left\|\nabla P(\varepsilon)(I+\mathcal{B}_{p'})^{\alpha/2}\right\|_{L^{p'}\to C(\bar{\mathcal{O}})}.
\end{aligned}$$

Thanks to the gradient bound Lemma 3.5, we get

$$R \lesssim_p \frac{1}{n}\varepsilon^{-1/2-\alpha/2-1/p'}\|\omega_0\|_{L^1} \leqslant \frac{1}{n}\varepsilon^{-3/2-\alpha/2}\|\omega_0\|_{L^1}. \tag{3.27}$$

Putting together the bounds (3.26) and (3.27), we conclude that

$$\left\|\sum_{i:t_i^n=0} \omega_n^i p_\varepsilon(\cdot,\zeta_i^n)\right\|_{\mathcal{H}^{\alpha,p}} \lesssim_p \|\omega_0\|_{\mathcal{H}^{\alpha,p}} + \frac{1}{n}\varepsilon^{-3/2-\alpha/2}\|\omega_0\|_{L^1}.$$

By the assumption $\varepsilon=\varepsilon(n)\gtrsim n^{-2/(3+\alpha)}$, we get the desired bound. □

Lemma 3.19. *Let $\varepsilon=\varepsilon(n)\gtrsim n^{-2/(2+\alpha)}\gtrsim n^{-2/3}$; for all $p>2$, $\frac{2}{p}<\alpha<1$ it holds*

$$\sup_{t\in[0,1]} \left\|\sum_{i\in\mathsf{A}_0^n(t)} \omega_i^n P(t-t_i^n)p_\varepsilon(\cdot,\zeta_i^n)\right\|_{\mathcal{H}^{\alpha,p}} \le C_{\alpha,p}.$$

As for the previous lemma, the idea of the proof is that, for n large and ε small,

$$\sum_{i\in\mathsf{A}_0^n(t)} \omega_i^n P(t-t_i^n)p_\varepsilon(\cdot,\zeta_i^n)(x) \approx \int_0^t\int_{\partial\mathcal{O}} P(t-s)p_\varepsilon(x,y)g(s,y)dyds,$$

and the right-hand side is bounded if, say, g is bounded.

Proof. We have

$$\begin{aligned}\Big\|\sum_{i\in \mathsf{A}_0^n(t)} \omega_i^n P(t-t_i^n)p_\varepsilon(\cdot,\zeta_i^n)\Big\|_{\mathcal{H}^{\alpha,p}} &\leqslant \Big\|\int_0^t\int_{\partial\mathcal{O}} P(t-s)p_\varepsilon(\cdot,y)g(s,y)dsd\sigma(y)\Big\|_{\mathcal{H}^{\alpha,p}} \\ &+\Big\|\int_0^t\int_{\partial\mathcal{O}} P(t-s)p_\varepsilon(\cdot,y)g(s,y)dsd\sigma(y)-\sum_{i\in \mathsf{A}_0^n(t)} \omega_i^n P(t-t_i^n)p_\varepsilon(\cdot,\zeta_i^n)\Big\|_{\mathcal{H}^{\alpha,p}} =: I_g+R.\end{aligned}$$

We bound the term I_g first: by (3.11), we get

$$\begin{aligned}&\Big\|\int_0^t\int_{\partial\mathcal{O}} P(t-s)p_\varepsilon(\cdot,y)g(s,y)dsd\sigma(y)\Big\|_{\mathcal{H}^{\alpha,p}} \\ &= \sup_{\|f\|_{L^{p'}}=1}\int_0^t\int_{\partial\mathcal{O}}\langle (I+\mathcal{B}_p)^{\alpha/2}P(t-s)p_\varepsilon(\cdot,y),f\rangle g(s,y)dsd\sigma(y) \\ &= \sup_{\|f\|_{L^{p'}}=1}\int_0^t (I+\mathcal{B}_p')^{\alpha/2}P(t-s+\varepsilon)f(y)g(s,y)dsd\sigma(y) \\ &\le \|g\|_{L^\infty([0,1];L^p(\partial\mathcal{O}))}\int_0^t \Big\|(I+\mathcal{B}_{p'})^{\alpha/2}P(t-s+\varepsilon)\Big\|_{L^{p'}\to L^{p'}(\partial\mathcal{O})}ds.\end{aligned}$$

Using the trace theorem and the contractivity bound (3.4), we get, for some fixed $\delta>0$ such that $\alpha/2+1/(2p')+\delta/2<1$,

$$\begin{aligned}I_g &\leqslant \|g\|_{L^\infty([0,1];L^p(\partial\mathcal{O}))}\int_0^t \Big\|(I+\mathcal{B}_{p'})^{\alpha/2}P(t-s+\varepsilon)\Big\|_{L^{p'}\to H^{1/p'+\delta,p'}(\mathcal{O})}ds \\ &\lesssim \|g\|_{L^\infty([0,1];L^p(\partial\mathcal{O}))}\int_0^t \Big\|(I+\mathcal{B}_{p'})^{\alpha/2+1/(2p')+\delta/2}P(t-s+\varepsilon)\Big\|_{L^{p'}\to L^{p'}}ds \\ &\lesssim_{\alpha,p} \|g\|_{L^\infty([0,1];L^p(\partial\mathcal{O}))}\int_0^t (t-s+\varepsilon)^{-(\alpha/2+1/(2p')+\delta/2)}ds \\ &\lesssim_{\alpha,p} \|g\|_{L^\infty([0,1];L^p(\partial\mathcal{O}))}. \end{aligned}\tag{3.28}$$

Concerning R, recall that $A_0^n(t)=\{i: 0<t_i^n\leqslant t\}$ and, for $i\in A_0^n(t)$, $\omega_i^n=\int_{Q_i^n} g(s,y)dsd\sigma(y)$. In particular, we can split

$$\begin{aligned}\sum_{i\in \mathsf{A}_0^n(t)} \omega_i^n P(t-t_i^n)p_\varepsilon(\cdot,\zeta_i^n) &= \sum_{i:t_i^n+1/(2n)<t}\int_{Q_i^n} P(t-t_i^n)p_\varepsilon(\cdot,\zeta_i^n)g(s,y)dsd\sigma(y) \\ &+\sum_{i:t_i^n\leqslant t\leqslant t_i^n+1/(2n)} \omega_i^n P(t-t_i^n)p_\varepsilon(\cdot,\zeta_i^n)\end{aligned}$$

(the last sum possibly being over an empty set) and similarly

$$\begin{aligned}
&\int_0^t \int_{\partial\mathcal{O}} P(t-s)p_\varepsilon(\cdot,y)g(s,y)dsd\sigma(y)\\
&= \sum_{i:t_i^n+1/(2n)<t} \int_{Q_i^n} P(t-s)p_\varepsilon(\cdot,y)g(s,y)dsd\sigma(y)\\
&\quad + \sum_{i:t_i^n-1/(2n)<t\leqslant t_i^n+1/(2n)} \int_{Q_i^n\cap[0,t]\times\partial\mathcal{O}} P(t-s)p_\varepsilon(\cdot,y)g(s,y)dsd\sigma(y).
\end{aligned}$$

Hence we can split R as

$$\begin{aligned}
R &\leqslant \left\| \sum_{i:t_i^n+1/(2n)<t} \int_{Q_i^n} \left(P(t-s)p_\varepsilon(\cdot,y) - P(t-t_i^n)p_\varepsilon(\cdot,\zeta_i^n)\right) g(s,y)dsd\sigma(y) \right\|_{\mathcal{H}^{\alpha,p}}\\
&\quad + \left\| \sum_{i:t_i^n-1/(2n)<t\leqslant t_i^n+1/(2n)} \int_{Q_i^n\cap[0,t]\times\partial\mathcal{O}} P(t-s)p_\varepsilon(\cdot,y)g(s,y)dsd\sigma(y) \right\|_{\mathcal{H}^{\alpha,p}}\\
&\quad + \left\| \sum_{i:t_i^n\leqslant t\leqslant t_i^n+1/(2n)} \omega_i^n P(t-t_i^n)p_\varepsilon(\cdot,\zeta_i^n) \right\|_{\mathcal{H}^{\alpha,p}}\\
&=: R_1 + R_{21} + R_{22}.
\end{aligned}$$

To bound the term R_1, we start with an observation: setting

$$[t_i^n, s](\xi) := t_i^n + \xi(s-t_i^n), \quad [\zeta_i^n, y](\xi) := \zeta_i^n + \xi(y-\zeta_i^n), \quad \xi\in[0,1]$$

(notice that $[\zeta_i^n, y](\xi)\in\bar{\mathcal{O}}$ for all ξ since $\bar{\mathcal{O}}$ is convex), since by definition of heat semigroup it holds $\partial_t P(t) = -\mathcal{B}_p P(t)$, we can write

$$\begin{aligned}
&P(t-s)p_\varepsilon(\cdot,y)(x) - P(t-t_i^n)p_\varepsilon(\cdot,\zeta_i^n)(x)\\
&= (s-t_i^n)\int_0^1 -\partial_t P(t-[t_i^n,s](\xi))p_\varepsilon(\cdot,[\zeta_i^n,y](\xi))(x)d\xi\\
&\quad + (y-\zeta_i^n)\cdot\int_0^1 \nabla_y P(t-[t_i^n,s](\xi))p_\varepsilon(\cdot,[\zeta_i^n,y](\xi))(x)d\xi\\
&= (s-t_i^n)\int_0^1 \mathcal{B}_p P(t-[t_i^n,s](\xi))p_\varepsilon(\cdot,[\zeta_i^n,y](\xi))(x)d\xi\\
&\quad + (y-\zeta_i^n)\cdot\int_0^1 \nabla_y P(t-[t_i^n,s](\xi))p_\varepsilon(\cdot,[\zeta_i^n,y](\xi))(x)d\xi.
\end{aligned}$$

Applying the latter to R_1 we obtain:

$$\begin{aligned}
R_1 \leqslant{}& \left\| \sum_{i:t_i^n+\frac{1}{2n}<t} \int_{Q_i^n} \int_0^1 \mathcal{B}_p P(t-[t_i^n,s](\xi)) p_\varepsilon(\cdot,[\zeta_i^n,y](\xi)) d\xi (s-t_i^n) g(s,y) ds d\sigma(y) \right\|_{\mathcal{H}^{\alpha,p}} \\
&+ \left\| \sum_{i:t_i^n+\frac{1}{2n}<t} \int_{Q_i^n} \int_0^1 \nabla_y P(t-[t_i^n,s](\xi)) p_\varepsilon(\cdot,[\zeta_i^n,y](\xi)) d\xi (y-\zeta_i^n) g(s,y) ds d\sigma(y) \right\|_{\mathcal{H}^{\alpha,p}} \\
&=: R_{11} + R_{12}.
\end{aligned}$$

Concerning R_{11}, recall that $|s-t_i^n| \leqslant 1/(2n)$ for every s in Q_i^n. Hence by (3.13) we have

$$\begin{aligned}
R_{11} &\leqslant \sum_{i:t_i^n+\frac{1}{2n}<t} \int_{Q_i^n} \int_0^1 \|\mathcal{B}_p P(t-[t_i^n,s](\xi)) p_\varepsilon(\cdot,[\zeta_i^n,y](\xi))\|_{\mathcal{H}^{\alpha,p}} \, d\xi \, |s-t_i^n| g(s,y) ds d\sigma(y) \\
&\leqslant \frac{1}{2n} \int_0^1 \sum_{i:t_i^n+\frac{1}{2n}<t} \int_{Q_i^n} \sup_{x\in\bar{\mathcal{O}}} \|P(t-[t_i^n,s](\xi)) p_\varepsilon(\cdot,x)\|_{\mathcal{H}^{\alpha+2,p}} \, g(s,y) ds d\sigma(y) d\xi, \\
&= \frac{1}{2n} \int_0^1 \sum_{i:t_i^n+\frac{1}{2n}<t} \int_{Q_i^n} \left\| (I+\mathcal{B}_{p'})^{1+\alpha/2} P(t-[t_i^n,s](\xi)+\varepsilon) \right\|_{L^{p'}\to C(\bar{\mathcal{O}})} g(s,y) ds d\sigma(y) d\xi.
\end{aligned}$$

Since $\mathcal{H}^{2,p'}$ is embedded into $C(\bar{\mathcal{O}})$ for $p'>1$, we have

$$R_{11} \lesssim_p \frac{1}{n} \int_0^1 \sum_{i:t_i^n+\frac{1}{2n}<t} \int_{Q_i^n} \left\| (I+\mathcal{B}_{p'})^{2+\alpha/2} P(t-[t_i^n,s](\xi)+\varepsilon) \right\|_{L^{p'}\to L^{p'}} g(s,y) ds d\sigma(y) d\xi.$$

Note that, if $t_i^n + 1/(2n) < t$, then $t-[t_i^n,s](\xi) \geqslant (t-s)/2$ for every s in Q_i^n. Thanks to the contractivity bound (3.4), we get

$$\begin{aligned}
R_{11} &\lesssim_{\alpha,p} \frac{1}{n} \int_0^1 \sum_{i:t_i^n+\frac{1}{2n}<t} \int_{Q_i^n} (t-[t_i^n,s](\xi)+\varepsilon)^{-2-\alpha/2} g(s,y) ds d\sigma(y) d\xi \\
&\lesssim \frac{1}{n} \int_0^t \int_{\partial\mathcal{O}} (t-s+2\varepsilon)^{-2-\alpha/2} g(s,y) d\sigma(y) ds \\
&\lesssim \frac{1}{n} \varepsilon^{-1-\alpha/2} \|g\|_{L^\infty([0,1];L^1(\partial\mathcal{O}))}. \qquad (3.29)
\end{aligned}$$

Concerning R_{12}, recall that $|y-\zeta_i^n| \lesssim 1/n$ for every y in Q_i^n. Hence by (3.14) we have

$$\begin{aligned} R_{12} &\leqslant \sum_{i: t_i^n + \frac{1}{2n} < t} \int_{Q_i^n} \int_0^1 \|\nabla_y P(t-[t_i^n,s](\xi)) p_\varepsilon(\cdot,[\zeta_i^n,y](\xi))\|_{\mathcal{H}^{\alpha,p}}\, d\xi\, |y-\zeta_i^n| g(s,y) ds d\sigma(y) \\ &\lesssim \frac{1}{n} \int_0^1 \sum_{i: t_i^n + \frac{1}{2n} < t} \int_{Q_i^n} \sup_{x\in\bar{\mathcal{O}}} \|\nabla P(t-[t_i^n,s](\xi)) p_\varepsilon(\cdot,x)\|_{\mathcal{H}^{\alpha,p}}\, g(s,y) ds d\sigma(y) d\xi \\ &\lesssim \frac{1}{n} \int_0^1 \sum_{i: t_i^n + \frac{1}{2n} < t} \int_{Q_i^n} \left\| \nabla P(t-[t_i^n,s](\xi)+\varepsilon)(I+\mathcal{B}_{p'})^{\alpha/2} \right\|_{L^{p'}\to C(\bar{\mathcal{O}})} g(s,y) ds d\sigma(y) d\xi. \end{aligned}$$

Thanks to the gradient bound Lemma 3.5, we get

$$\begin{aligned} R_{12} &\lesssim_{\alpha,p} \frac{1}{n} \int_0^1 \sum_{i: t_i^n + \frac{1}{2n} < t} \int_{Q_i^n} (t-[t_i^n,s](\xi)+\varepsilon)^{-1/2-\alpha/2-1/p'} g(s,y) ds d\sigma(y) d\xi \\ &\lesssim \frac{1}{n} \int_0^t \int_{\partial\mathcal{O}} (t-s+2\varepsilon)^{-3/2-\alpha/2} g(s,y) d\sigma(y) ds \\ &\lesssim_{\alpha,p} \frac{1}{n} \varepsilon^{-1/2-\alpha/2} \|g\|_{L^\infty([0,1];L^1(\partial\mathcal{O}))}. \end{aligned} \tag{3.30}$$

We turn now to the bound on R_{21}. By (3.14) we have

$$\begin{aligned} R_{21} &\leqslant \sum_{i: t_i^n - \frac{1}{2n} < t \leqslant t_i^n + \frac{1}{2n}} \int_{Q_i^n \cap [0,t]\times\partial\mathcal{O}} \|P(t-s) p_\varepsilon(\cdot,y)\|_{\mathcal{H}^{\alpha,p}}\; g(s,y) ds d\sigma(y) \\ &\leqslant \sum_{i: t_i^n - \frac{1}{2n} < t \leqslant t_i^n + \frac{1}{2n}} \int_{Q_i^n \cap [0,t]\times\partial\mathcal{O}} \sup_{x\in\bar{\mathcal{O}}} \|P(t-s+\varepsilon) p_\varepsilon(\cdot,x)\|_{\mathcal{H}^{\alpha,p}}\; g(s,y) ds d\sigma(y) \\ &= \sum_{i: t_i^n - \frac{1}{2n} < t \leqslant t_i^n + \frac{1}{2n}} \int_{Q_i^n \cap [0,t]\times\partial\mathcal{O}} \left\| (I+\mathcal{B}_{p'})^{\alpha/2} P(t-s+\varepsilon) \right\|_{L^{p'}\to C(\bar{\mathcal{O}})}\; g(s,y) ds d\sigma(y). \end{aligned}$$

Notice that

$$\bigcup_{i: t_i^n - \frac{1}{2n} < t \leqslant t_i^n + \frac{1}{2n}} Q_i^n \cap ([0,t]\otimes\partial\mathcal{O}) \subseteq [t-1/n,t]\times\partial\mathcal{O}.$$

Therefore, by the embedding $\mathcal{H}^{2,p'} \hookrightarrow C(\bar{\mathcal{O}})$, $p'>1$, and contractivity bound (3.4),

$$\begin{aligned} R_{21} &\lesssim_{\alpha,p} \int_{t-1/n}^t \int_{\partial\mathcal{O}} g(s,y) d\sigma(y) ds \cdot \sup_{s\leqslant t} \left\| (I+\mathcal{B}_{p'})^{1+\alpha/2} P(t-s+\varepsilon) \right\|_{L^{p'}\to L^{p'}} \\ &\lesssim \frac{1}{n} \varepsilon^{-1-\alpha/2} \|g\|_{L^\infty([0,1];L^1(\partial\mathcal{O}))}. \end{aligned} \tag{3.31}$$

Finally, we turn to the bound on R_{22}. Similarly to the case of R_{21}, we notice that for each $i \in A_0(t)$,

$$\cup_{i:t_i^n \leqslant t \leqslant t_i^n + \frac{1}{2n}} Q_i^n \subseteq \left[t - 1/n, t + \frac{1}{2n}\right] \times \partial\mathcal{O}.$$

Hence, proceeding as for the term R_{21}, we get

$$\begin{aligned} R_{22} &\leqslant \sum_{i:t_i^n \leqslant t \leqslant t_i^n + \frac{1}{2n}} \int_{Q_i^n} g(s,y) ds d\sigma(y) \, \|P(t-t_i^n) p_\varepsilon(\cdot, \zeta_i^n)\|_{\mathcal{H}^{\alpha,p}} \\ &\leqslant \int_{t-1/n}^{t+1/(2n)} \int_{\partial\mathcal{O}} g(s,y) d\sigma(y) ds \sup_{s \leqslant t} \left\| (I + \mathcal{B}_{p'})^{1+\alpha/2} P(t-s+\varepsilon) \right\|_{L^{p'} \to L^{p'}} \\ &\lesssim \frac{1}{n} \varepsilon^{-1-\alpha/2} \|g\|_{L^\infty([0,1];L^1(\partial\mathcal{O}))}. \end{aligned} \tag{3.32}$$

Putting together the bounds (3.28), (3.29), (3.30), (3.31), (3.32), we conclude that

$$\left\| \sum_{i \in \mathsf{A}_0^n(t)} \omega_i^n P(t-t_i^n) p_\varepsilon(\cdot, \zeta_i^n) \right\|_{\mathcal{H}^{\alpha,p}} \lesssim_{\alpha,p} \|g\|_{L^\infty([0,1];L^p(\partial\mathcal{O}))} + \frac{1}{n} \varepsilon^{-1-\alpha/2} \|g\|_{L^\infty([0,1];L^1(\partial\mathcal{O}))}.$$

By the assumption $\varepsilon = \varepsilon(n) \gtrsim n^{-2/(2+\alpha)}$, we get the desired bound. □

We also estimate in a dedicated lemma the martingale terms appearing in the mild formulation of particle dynamics.

Lemma 3.20. *Let $p > 2$, $\frac{2}{p} < \alpha < 1$ and assume that $\varepsilon = \varepsilon(n) \gtrsim n^{-1/2}$. Then,*

$$\mathbb{E}\left[\sup_{t \in [0,1]} \left\| \int_0^t \sum_{i \in \mathsf{A}^n(s)} \omega_i^n P(t-s) \nabla_y p_\varepsilon(x, x_i^n(s)) dB_s^i \right\|_{\mathcal{H}^{\alpha,p}}^q \right] \leq C_{q,p,\alpha}. \tag{3.33}$$

Proof. By Sobolev embedding $\mathcal{H}^{1-2/p,2} \hookrightarrow L^p$ and Stochastic maximal regularity, see for example the discussion of subsection 2.1.4, it holds

$$\begin{aligned} &\mathbb{E}\left[\sup_{t \in [0,1]} \left\| \int_0^t \sum_{i \in \mathsf{A}^n(s)} \omega_i^n (I + \mathcal{B}_p)^{\alpha/2} P(t-s) \nabla_y p_\varepsilon(\cdot, x_i(s)) dB_s^i \right\|_{L^p}^q \right] \\ &\quad \lesssim_p \mathbb{E}\left[\sup_{t \in [0,1]} \left\| \int_0^t \sum_{i \in \mathsf{A}^n(s)} \omega_i^n (I + \mathcal{B}_2)^{(1+\alpha-2/p)/2} P(t-s) \nabla_y p_\varepsilon(\cdot, x_i(s)) dB_s^i \right\|^q \right] \\ &\qquad \lesssim_q \mathbb{E}\left[\left(\int_0^1 \sum_{i \in \mathsf{A}^n(s)} (\omega_i^n)^2 \left\| (I + \mathcal{B}_2)^{(1+\alpha-2/p)/2} \nabla_y p_\varepsilon(\cdot, x_i(s)) \right\|^2 ds \right)^{q/2} \right]. \end{aligned}$$

The inner integrand in the right-hand side can be controlled uniformly with respect to the particles' positions: by Lemma 3.5

$$\sup_{y\in\bar{\mathcal{O}}}\left\|(I+\mathcal{B}_2)^{(1+\alpha-2/p)/2}\nabla_y p_\varepsilon(\cdot,y)\right\|^2=\|\nabla P_\varepsilon\|^2_{\mathcal{H}^{-1-\alpha+2/p,2}\to C(\bar{\mathcal{O}})}\lesssim_{\alpha,p}\varepsilon^{-3-\alpha+2/p}.$$

Notice that the denominator is integrable in ds for all $\delta>0$. We now take into account the fact that $\omega_i^n\lesssim n^{-2}(\|\omega_0\|_{L^\infty(\mathcal{O})}+\|g\|_{L^\infty([0,1]\times\partial\mathcal{O})})$ uniformly in i, and that $N(n)\simeq n^2$. Hence, the left-hand side of (3.33) is bounded from above by

$$\text{L.H.S.}\lesssim_{p,q,\alpha} n^{-q}\varepsilon^{-(3+\alpha-2/p)\frac{q}{2}}(\|\omega_0\|_{L^\infty(\mathcal{O})}+\|g\|_{L^\infty([0,1]\times\partial\mathcal{O})})^q.$$

Taking $\varepsilon(n)$ as in the hypothesis, the last quantity is uniformly bounded in n. □

3.3.3 Estimates for Approximating Solutions

The following is the core estimate of our argument, providing uniform boundedness in strong norms for the approximating processes.

Proposition 3.21. *Let $p>2$, $\frac{2}{p}<\alpha<1$, and $\varepsilon=\varepsilon(n)\gtrsim n^{-1/2}$. For all $q\geq 2$ it holds*

$$\sup_n\mathbb{E}\left[\sup_{t\in[0,1]}\|\omega^n(t)\|^q_{\mathcal{H}^{\alpha,p}}\right]\leqslant C_{M,q,p,\alpha}. \tag{3.34}$$

Proof. In order to lighten notation we denote by $v_t^n(x):=F\left(\int_{\mathcal{O}}K_n(x,y)d\mathsf{S}^n(s,y)\right)$ the vector field acting on a single, smoothened particle. The mild formulation (3.25) allows to bound, by Minkowski inequality,

$$\begin{aligned}\mathbb{E}\left[\|\omega^n(t)\|^q_{\mathcal{H}^{\alpha,p}}\right]\lesssim_q\ &\mathbb{E}\left[\left\|\sum_{i\in\mathsf{A}^n(t)}\omega_i^nP(t-t_i^n)p_\varepsilon(\cdot,\zeta_i^n)\right\|^q_{\mathcal{H}^{\alpha,p}}\right]\\&+\mathbb{E}\left[\left\|\int_0^tP(t-s)\int_{\mathcal{O}}v^n(s,y)\nabla_yp_\varepsilon(x,y)d\mathsf{S}^n(s,y)ds\right\|^q_{\mathcal{H}^{\alpha,p}}\right]\\&+\mathbb{E}\left[\left\|\sqrt{2}\int_0^t\sum_{i\in\mathsf{A}^n(s)}\omega_i^nP(t-s)\nabla_yp_\varepsilon(x,x_i^n(s))dB_s^i\right\|^q_{\mathcal{H}^{\alpha,p}}\right]\\&=:I_1+I_2+I_3.\end{aligned}$$

The initial and boundary creation terms I_1 and the martingale term I_3 are uniformly bounded (both in n and in $t\in[0,1]$) respectively thanks to Lemma 3.18, Lemma 3.19 and Lemma 3.20.

We thus focus on the nonlinear interaction term I_2, where the regularizing effect of the heat semigroup is essential. Using the gradient estimate (3.6) we thus bound, for $f \in L^{p'}$,

$$
\begin{aligned}
&\left| \sum_{i \in \mathsf{A}^n(s)} \omega_i v^n(s, x_i(s)) \int_{\mathcal{O}} f(x)(I + \mathcal{B}_p)^{\alpha/2} P(t-s) \nabla_y p_\varepsilon(x, x_i(s)) dx \right| \\
&= \left| \sum_{i \in \mathsf{A}^n(s)} \omega_i v^n(s, x_i(s)) \int_{\mathcal{O}} (I + \mathcal{B}_{p'})^{\alpha/2} P(t-s) f(x) \nabla_y p_\varepsilon(x, x_i(s)) dx \right| \\
&= \left| \sum_{i \in \mathsf{A}^n(s)} \omega_i v^n(s, x_i(s)) \nabla P(\varepsilon)[(I + \mathcal{B}_{p'})^{\alpha/2} P(t-s) f](x_i(s)) \right| \\
&\leq M \sum_{i \in \mathsf{A}^n(s)} \omega_i \left| \nabla P(\varepsilon)[(I + \mathcal{B}_{p'})^{\alpha/2} P(t-s) f](x_i(s)) \right| \\
&\leq M \sum_{i \in \mathsf{A}^n(s)} \omega_i e^{C\varepsilon} P_\varepsilon \left| \nabla (I + \mathcal{B}_{p'})^{\alpha/2} P(t-s) f \right| (x_i(s)) \\
&= M e^{C\varepsilon} \sum_{i \in \mathsf{A}^n(s)} \omega_i \int_{\mathcal{O}} p_\varepsilon(x, x_i(s)) \left| \nabla (I + \mathcal{B}_{p'})^{\alpha/2} P(t-s) f \right| (x) dx \\
&= M e^{C\varepsilon} \int_{\mathcal{O}} P(\varepsilon) \mathsf{S}^n(s, x) \left| \nabla (I + \mathcal{B}_{p'})^{\alpha/2} P(t-s) f \right| (x) dx \\
&\leq M e^{C\varepsilon} \|\omega^n(s)\|_{L^p} \left\| \nabla (I + \mathcal{B}_{p'})^{\alpha/2} P(t-s) f \right\|_{L^{p'}} .
\end{aligned}
$$

By duality, this implies that

$$
\begin{aligned}
&\left\| \sum_{i \in \mathsf{A}^n(s)} \omega_i v^n(s, x_i(s)) (I + \mathcal{B}_p)^{\alpha/2} P(t-s) \nabla_y p_\varepsilon(\cdot, x_i(s)) \right\|_{L^p} \\
&\qquad \leq \frac{M e^{C\varepsilon}}{(t-s)^{(1+\alpha)/2}} \|\omega^n(s)\|_{L^p} \leq \frac{M e^{C\varepsilon}}{(t-s)^{(1+\alpha)/2}} \|\omega^n(s)\|_{\mathcal{H}^{\alpha,p}} ,
\end{aligned}
$$

so that

$$
\begin{aligned}
I_2^{1/q} &\leq \int_0^t \mathbb{E} \left[\left\| \sum_{i \in \mathsf{A}^n(s)} \omega_i v^n(s, x_i(s)) P(t-s) \nabla_y p_\varepsilon(\cdot, x_i(s)) \right\|_{\mathcal{H}^{\alpha,p}}^q \right]^{1/q} ds \\
&\leq C_\varepsilon \int_0^t \frac{M}{(t-s)^{(1+\alpha)/2}} \mathbb{E} \left[\|\omega^n(s)\|_{\mathcal{H}^{\alpha,p}}^q \right]^{1/q} \, ds
\end{aligned}
$$

with the constant C_ε decreasing as $\varepsilon \to 0$, in particular uniformly bounded in n. Altogether, we arrive to the integral inequality

$$\mathbb{E}\left[\|\omega^n(s)\|^q_{\mathcal{H}^{\alpha,p}}\right]^{1/q} \le C_{p,q,\alpha,\omega_0} + C\int_0^t \frac{M}{(t-s)^{(1+\alpha)/2}}\mathbb{E}\left[\|\omega^n(s)\|^q_{\mathcal{H}^{\alpha,p}}\right]^{1/q}\,ds$$

which, by Grönwall's lemma (as $\alpha < 1$), implies

$$\sup_n \sup_{t\in[0,1]} \mathbb{E}\left[\|\omega^n(t)\|^q_{\mathcal{H}^{\alpha,p}}\right] \leqslant C_{M,q,p,\alpha}. \tag{3.35}$$

In order to complete the proof, we apply again the mild formulation of the dynamics (3.25) to estimate

$$\begin{aligned}
\mathbb{E}\left[\sup_{t\in[0,1]}\|\omega^n(t)\|^q_{\mathcal{H}^{\alpha,p}}\right] \lesssim_q\ &\mathbb{E}\left[\sup_{t\in[0,1]}\left\|\sum_{i\in\mathsf{A}^n(t)}\omega^n_i P(t-t^n_i)p_\varepsilon(\cdot,\zeta^n_i)\right\|^q_{\mathcal{H}^{\alpha,p}}\right]\\
&+\mathbb{E}\left[\sup_{t\in[0,1]}\left\|\int_0^t P(t-s)\int_{\mathcal{O}} v^n(s,y)\nabla_y p_\varepsilon(x,y)d\mathsf{S}^n(s,y)ds\right\|^q_{\mathcal{H}^{\alpha,p}}\right]\\
&+\mathbb{E}\left[\sup_{t\in[0,1]}\left\|\sqrt{2}\int_0^t\sum_{i\in\mathsf{A}^n(s)}\omega^n_i P(t-s)\nabla_y p_\varepsilon(x,x^n_i(s))dB^i_s\right\|^q_{\mathcal{H}^{\alpha,p}}\right]\\
&=: J_1+J_2+J_3.
\end{aligned}$$

Once again, the initial and boundary creation terms J_1 and the martingale term J_3 are uniformly bounded in n respectively thanks to Lemma 3.18, Lemma 3.19 and Lemma 3.20. As for J_2, since

$$J_2 \le \mathbb{E}\left[\left(\sup_{t\in[0,1]}\int_0^t\left\|P(t-s)\int_{\mathcal{O}} v^n(s,y)\nabla_y p_\varepsilon(x,y)d\mathsf{S}^n(s,y)\right\|_{\mathcal{H}^{\alpha,p}}ds\right)^q\right],$$

we can repeat the computations performed to control I_2 obtaining

$$J_2 \lesssim_{p,q,\alpha} \mathbb{E}\left[\sup_{t\in[0,1]}\left(\int_0^t\frac{Me^{C\varepsilon}}{(t-s)^{(1+\alpha)/2}}\|\omega^n(s)\|_{\mathcal{H}^{\alpha,p}}\,ds\right)^q\right],$$

where we now apply Hölder's inequality,

$$\int_0^t\|\omega^n(s)\|_{\mathcal{H}^{\alpha,p}}\frac{ds}{(t-s)^{(1+\alpha)/2}} \le \left(\int_0^t\frac{ds}{(t-s)^{r(1+\alpha)/2}}\right)^{1/r}\left(\int_0^t\|\omega^n(s)\|^{r'}_{\mathcal{H}^{\alpha,p}}\,ds\right)^{1/r'},$$

in which we can choose a small enough $r>0$ (depending on α) so that the first factor on the right-hand side is finite and $q/r' < 1$, hence

$$J_2 \lesssim_{p,q,\alpha} \mathbb{E}\left[\int_0^1\|\omega^n(s)\|^{r'}_{\mathcal{H}^{\alpha,p}}\,ds\right]^{q/r'} \le \sup_{t\in[0,1]}\mathbb{E}\left[\|\omega^n(t)\|^{r'}_{\mathcal{H}^{\alpha,p}}\right]^{q/r'},$$

which is uniformly bounded in n by (3.35). □

It is worth noticing that the gradient estimates for $P(t)$ are crucial in controlling the nonlinear term of the dynamics, but the regularizing effect of $P(t)$ was neglected in estimating the initial empirical measure in Lemma 3.18 and the martingale terms in Lemma 3.20. The singularity of ∇p_ε appearing in the initial empirical measure and in the stochastic integrals is *not* improved by the heat semigroup and produces a restriction on the asymptotic behavior of $\varepsilon = \varepsilon(n)$.

Thanks to the good control provided by Proposition 3.21, we can estimate time increments –which we need for the equicontinuity part of our compactness argument– in a much weaker norm, thus allowing to exploit the weak formulation of the dynamics, easier to deal with compared to the variation-of-constants form but producing bounds in weaker norms.

Proposition 3.22. *Let $p > 2$, $\frac{2}{p} < \alpha < 1$, $\varepsilon = \varepsilon(n) \gtrsim n^{-1/2}$, $q \geq 2$, and $(\tau_n)_{n\in\mathbb{N}_0}$ be a sequence of $\mathcal{F}$-stopping times in $[0,1]$. It holds, for all $r \in (0,1)$,*

$$\mathbb{E}\left[\|\omega^n\left((\tau_n + r)\wedge 1\right) - \omega^n(\tau_n)\|^q_{\mathcal{H}^{-2,p}}\right] \leq \left(r^{q/2} + \frac{1}{n^q}\right) C_{M,q,p,\alpha}. \tag{3.36}$$

Proof. Substituting the weak formulation of particle dynamics (3.24), we estimate by Minkowski inequality:

$$\begin{aligned}
&\mathbb{E}\left[\|\omega^n\left((\tau_n + r)\wedge 1\right) - \omega^n(\tau_n)\|^q_{\mathcal{H}^{-2,p}}\right]\\
&\quad\lesssim_q \mathbb{E}\left[\left\|\sum_{i\in\mathsf{A}^n_0((\tau_n+r)\wedge 1)\setminus\mathsf{A}^n_0(\tau_n)} \omega^n_i p_\varepsilon(\cdot,\zeta^n_i)\right\|^q_{\mathcal{H}^{-2,p}}\right]\\
&\quad+\mathbb{E}\left[\left\|\int_{\tau_n}^{(\tau_n+r)\wedge 1} \Delta\omega^n(s)ds\right\|^q_{\mathcal{H}^{-2,p}}\right]\\
&\quad+\mathbb{E}\left[\left\|\int_{\tau_n}^{(\tau_n+r)\wedge 1} \int_{\mathcal{O}} F\left(K[\omega^n](y)\right)\nabla_y p_\varepsilon(\cdot,y)d\mathsf{S}^n(s,y)ds\right\|^q_{\mathcal{H}^{-2,p}}\right]\\
&\quad+\mathbb{E}\left[\left\|\int_{\tau_n}^{\tau_n+r} \sqrt{2}\sum_{i\in\mathsf{A}^n(s)} \omega^n_i\nabla_y p_\varepsilon(\cdot,x^n_i(s))dB^i_s\right\|^q_{\mathcal{H}^{-2,p}}\right]\\
&\quad=: I_1 + I_2 + I_3 + I_4.
\end{aligned}$$

We now proceed estimating each term separately.

Estimates on generation term I_1.

$$\begin{aligned}I_1 &\le \mathbb{E}\left[\left(\sum_{i\in \mathsf{A}^n(1)} \omega_i^n \mathbf{1}_{\tau_n<t_i\le(\tau_n+r)\wedge 1}\left\|(I+\mathcal{B}_p)^{-1}p_\varepsilon(\cdot,\zeta_i^n)\right\|_{L^p}\right)^q\right]\\ &\le \mathbb{E}\left[\left(\sum_{i\in \mathsf{A}^n(1)} \omega_i^n \mathbf{1}_{\tau_n<t_i\le(\tau_n+r)\wedge 1}\right)^q\right]\sup_{y\in\partial\mathcal{O}}\left\|(I+\mathcal{B}_p)^{-1}p_\varepsilon(\cdot,y)\right\|_{L^p}^q .\end{aligned}$$

The first factor on the right-hand side of the latter is controlled by

$$\begin{aligned}&\mathbb{E}\left[\left(\sum_{i\in \mathsf{A}^n(1)} \omega_i^n \mathbf{1}_{\tau_n<t_i\le(\tau_n+r)\wedge 1}\right)^q\right]\\ &\qquad\le \mathbb{E}\left[\left(\int_{\tau_n-\frac{1}{n}\vee 0}^{\tau_n+\frac{1}{n}+r\wedge 1}\int_{\partial\mathcal{O}} g(s,y)dyds\right)^q\right]\le\left(r+\frac{2}{n}\right)^q\|g\|^q_{L^\infty([0,1];L^1(\partial\mathcal{O}))},\end{aligned}$$

so we are left to control the second factor. For $f\in L^{p'}$, $y\in\partial\mathcal{O}$ and a sequence $y_k\in\mathcal{O}$ converging to y, it holds

$$\begin{aligned}\int_{\mathcal{O}}(I+\mathcal{B}_p)^{-1}p_\varepsilon(x,y)f(x)dx &= \int_{\mathcal{O}} p_\varepsilon(x,y)(I+\mathcal{B}_{p'})^{-1}f(x)dx\\ &=\lim_{k\to\infty}\int_{\mathcal{O}} p_\varepsilon(x,y_k)(I+\mathcal{B}_{p'})^{-1}f(x)dx,\end{aligned}$$

the exchange between the limit and the integral being allowed thanks to the fact that, by (3.5), $p_\varepsilon(x,y)\le\frac{C}{\varepsilon}e^{-|x-y|^2/(c\varepsilon)}$ $\forall x,y\in\mathcal{O}$. We can thus estimate:

$$\begin{aligned}\sup_{y\in\partial\mathcal{O}}\left\|(I+\mathcal{B}_p)^{-1}p_\varepsilon(\cdot,y)\right\|_{L^p} &= \sup_{y\in\partial\mathcal{O},\|f\|_{L^{p'}}=1}\left|\int_{\mathcal{O}}(I+\mathcal{B}_p)^{-1}p_\varepsilon(x,y)f(x)dx\right|\\ &\le \sup_{y\in\partial\mathcal{O},\|f\|_{L^{p'}}=1}\lim_{k\to\infty}\left|\int_{\mathcal{O}}p_\varepsilon(x,y_k)(I+\mathcal{B}_{p'})^{-1}f(x)dx\right|\\ &\le \sup_{y\in\bar{\mathcal{O}},\|f\|_{L^{p'}}=1}|(I+\mathcal{B}_{p'})^{-1}P(\varepsilon)f(y)|\le\left\|(I+\mathcal{B}_{p'})^{-1}P(\varepsilon)\right\|_{L^{p'}\to C(\bar{\mathcal{O}})}\\ &\le\left\|(I+\mathcal{B}_{p'})^{-1}\right\|_{L^{p'}\to C(\bar{\mathcal{O}})}\|P(\varepsilon)\|_{L^{p'}\to L^{p'}}\lesssim_p 1,\end{aligned}$$

where $\left\|(I+\mathcal{B}_{p'})^{-1}\right\|_{L^{p'}\to C(\bar{\mathcal{O}})}\lesssim_p\left\|(I+\mathcal{B}_{p'})^{-1}\right\|_{L^{p'}\to\mathcal{H}^{2,p'}}\lesssim_p 1$ follows by Morrey's inequality.

Estimates on the diffusion term I_2 follow from a straightforward application of Hölder's inequality,

$$
\begin{aligned}
I_2 &\leq \mathbb{E}\left[\left(\int_{\tau_n}^{(\tau_n+r)\wedge 1} \|\Delta\omega^n(s)\|_{\mathcal{H}^{-2,p}}\, ds\right)^q\right] \\
&\lesssim r^{q-1}\mathbb{E}\left[\int_{\tau_n}^{(\tau_n+r)\wedge 1} \|\omega^n(s)\|_{L^p}^q\, ds\right] \leq r^q\mathbb{E}\left[\sup_{t\in[0,1]} \|\omega^n(t)\|_{L^p}^q\right].
\end{aligned}
$$

Estimates on the nonlinear term I_3. We have:

$$
\begin{aligned}
I_3 &\leq \mathbb{E}\left[\left(\int_{\tau_n}^{(\tau_n+r)\wedge 1} \left\|\int_{\mathcal{O}} F\left(K[\omega^n](y)\right)\nabla_y p_\varepsilon(\cdot,y) d\mathsf{S}^n(s,y)\right\|_{\mathcal{H}^{-2,p}} ds\right)^q\right] \\
&\leq r^{q-1}\mathbb{E}\left[\int_{\tau_n}^{(\tau_n+r)\wedge 1} \left\|\int_{\mathcal{O}} F\left(K[\omega^n](y)\right)\nabla_y p_\varepsilon(\cdot,y) d\mathsf{S}^n(s,y)\right\|_{\mathcal{H}^{-2,p}}^q ds\right],
\end{aligned}
$$

in which we control, somewhat analogously to Proposition 3.21,

$$
\begin{aligned}
&\left\|\int_{\mathcal{O}} F\left(K[\omega^n](y)\right)\nabla_y p_\varepsilon(\cdot,y) d\mathsf{S}^n(s,y)\right\|_{\mathcal{H}^{-2,p}} \\
&\quad= \left\|\int_{\mathcal{O}} F\left(K[\omega^n](y)\right)(I+\mathcal{B}_p)^{-1}\nabla_y p_\varepsilon(\cdot,y) d\mathsf{S}^n(s,y)\right\|_{L^p} \\
&\quad= \sup_{f\in L^{p'}, \|f\|_{L^{p'}}=1} \int_{\mathcal{O}} F(K[\omega^n](y))\langle (I+\mathcal{B}_p)^{-1}\nabla_y p_\varepsilon(\cdot,y), f\rangle d\mathsf{S}^n(s,y) \\
&\quad\lesssim_M \sup_{f\in L^{p'}, \|f\|_{L^{p'}}=1} \int_{\mathcal{O}} |\nabla P(\varepsilon)(I+\mathcal{B}_{p'})^{-1} f(y)| d\mathsf{S}^n(s,y) \\
&\quad\lesssim_p e^{C\varepsilon} \sup_{f\in L^{p'}, \|f\|_{L^{p'}}=1} \int_{\mathcal{O}} d\mathsf{S}^n(s,y)\int_{\mathcal{O}} p_\varepsilon(y,x)|\nabla(I+\mathcal{B}_{p'})^{-1}f(x)|dx \\
&\quad= \sup_{f\in L^{p'}, \|f\|_{L^{p'}}=1} \left\langle \omega^n(s), \left|\nabla(I+\mathcal{B}_{p'})^{-1}f\right|\right\rangle \lesssim \|\omega^n(s)\|_{L^p}.
\end{aligned}
$$

Therefore I_3 is bounded, up to some constant independent of n, by

$$
r^q\mathbb{E}\left[\sup_{t\in[0,1]} \|\omega^n(t)\|_{L^p}^q\right].
$$

Estimates on the martingale term I_4. By Sobolev embedding and Burkholder–Davis–

Gundy inequality, see for example [160],

$$
\begin{aligned}
I_4 &\lesssim_p \mathbb{E}\left[\left\|\int_{\tau_n}^{\tau_n+r}\sum_{i\in\mathsf{A}^n(s)}\omega_i^n(I+\mathcal{B}_2)^{-1}\nabla_y p_\varepsilon(\cdot,x_i^n(s))dB_s^i\right\|_{\mathcal{H}^{1-2/p,2}}^q\right]\\
&=\mathbb{E}\left[\left\|\sum_{i=1}^N\omega_i^n\int_0^1\mathbf{1}_{t_i^n\geq s}\mathbf{1}_{\tau_n\leq s\leq(\tau_n+r)\wedge 1}(I+\mathcal{B}_2)^{-1/2-1/p}\nabla_y p_\varepsilon(\cdot,x_i^n(s))dB_s^i\right\|^q\right]\\
&\lesssim_q \mathbb{E}\left[\left(\sum_{i=1}^N(\omega_i^n)^2\int_{\tau_n}^{(\tau_n+r)\wedge 1}\mathbf{1}_{t_i^n\geq s}\left\|(I+\mathcal{B}_2)^{-1/2-1/p}\nabla_y p_\varepsilon(\cdot,x_i^n(s))\right\|^2ds\right)^{\frac{q}{2}}\right].
\end{aligned}
$$

Since $N=N(n)\simeq n^2$ and $\omega_i^n\lesssim n^{-2}\|g\|_{L^\infty([0,1]\times\partial\mathcal{O})}$ uniformly in i, and replacing the expectation involving particles's positions with a supremum over $\bar{\mathcal{O}}$, we obtain

$$
I_4\lesssim_{p,q}\frac{r^{q/2}}{n^q}\sup_{y\in\bar{\mathcal{O}}}\left\|(I+\mathcal{B}_2)^{-1/2-1/p}\nabla_y p_\varepsilon(\cdot,y)\right\|^q,
$$

which we combine with a consequence of Lemma 3.5,

$$
\sup_{y\in\bar{\mathcal{O}}}\left\|(I+\mathcal{B}_2)^{-1/2-1/p}\nabla_y p_\varepsilon(\cdot,y)\right\|\leq\sup_{y\in\bar{\mathcal{O}}}\|\nabla_y p_\varepsilon(\cdot,y)\|=\|\nabla P(\varepsilon)\|_{L^2\to C(\bar{\mathcal{O}})}\lesssim\varepsilon^{-1},
$$

and the assumption $\varepsilon(n)\gtrsim n^{-1/2}$, concluding that $I_4\lesssim_{p,q}r^{q/2}$. □

3.4 Proof of the Main Result

The proof of Theorem 3.17 proceeds as follows. We combine the estimates of the previous section with Aldous' Lemma in order to obtain tightness on the space of *càdlàg* functions taking values in Sobolev spaces. In order to verify that the limiting dynamics coincides with the Navier-Stokes equations (3.1) we need almost sure convergence in such a space, which we obtain by changing the underlying probability space by Skorohod theorem. Convergence in the original probability space is recovered by uniqueness of the deterministic limit.

In what follows, according to the hypothesis of Theorem 3.17, we tacitly assume that $p>2$ and $\frac{2}{p}<\alpha<1$ are fixed, as well as $\omega_0\in\mathrm{Lip}(\bar{\mathcal{O}})$ and $g\in\mathcal{B}_b([0,1]\times\partial\mathcal{O})$. We denote by ω the unique solution of (3.1) given by Proposition 3.11 with initial datum ω_0 and Neumann source g.

3.4.1 Compactness Argument

Let $(\Omega,\mathcal{F},\mathbb{P})$ be a complete, filtered probability space satisfying the standard hypothesis, on which it is defined a sequence $(B^i)_{i\in\mathbb{N}}$ of independent $\mathcal{F}$-Brownian motions. For all $n\geq 1$, *on this same* probability space we can consider the well-posed dynamics of Proposition 3.14, since the latter is a probabilistically strong existence and uniqueness result. In other words, we can consider for all n the dynamics of particles $(x_1^n(t), \dots, x_N^n(t))_{t\in[0,1]}$ and the one of the regularized empirical measure $\omega^n(t) = \sum_{i\in \mathsf{A}^n(t)} \omega_i^n p_\varepsilon(\cdot, x_i^n(t))$ as stochastic processed defined on $(\Omega, \mathcal{F}, \mathbb{P})$ and $\mathcal{F}$-adapted.

We denote by $\mathcal{L}_n$ the law of ω_n on $\mathbb{D}([0,1], \mathcal{H}^{\alpha,p})$; we can actually consider any parameters α, p since samples of the process ω^n are smooth in the space variable, but time dependence is at best *càdlàg* due to the creation of new particles.

We need first to show tightness of the laws $(\mathcal{L}_n)_{n\in\mathbb{N}}$:

Lemma 3.23. *For all $\frac{2}{p} < \eta < \alpha$, the sequence of laws $(\mathcal{L}_n)_{n\in\mathbb{N}}$ is tight on $\mathbb{D}([0,T];\mathcal{H}^{\eta,p})$.*

We then need to show that the limiting dynamics is actually a solution of (3.1).

Lemma 3.24. *For all $\frac{2}{p} < \eta < \alpha$, any weak limit point of the sequence $(\mathcal{L}_n)_{n\in\mathbb{N}}$ of measures on $\mathbb{D}([0,T];\mathcal{H}^{\eta,p})$ is concentrated on the unique solution ω of (3.1) given by Proposition 3.11; in other words there exists a unique limit point, δ_ω.*

These two lemmas imply Theorem 3.17.

Proof of Theorem 3.17. By Lemma 3.23 and Lemma 3.24, every subsequence $\mathcal{L}_{n(k)}$ admits a sub-subsequence which converges to the unique limit point δ_ω, where ω is the deterministic solution of (3.1). Then, for example by [26, Theorem 2.6], the whole sequence $\mathcal{L}_n$ converges weakly to δ_ω, and then also in probability as the limit point is a Dirac delta (see e.g. the argument in [26, page 27]). The proof is complete. □

3.4.2 Proof of Lemma 3.23

Thanks to the uniform estimates of section 3.3, we can straightforwardly apply Aldous' criterion, Proposition 3.8, to the processes $\omega^n(t)$ on $\mathbb{D}([0,T];\mathcal{H}^{\eta,p})$.

Condition (1), that is tightness of laws at fixed time in $\mathcal{H}^{\eta,p}$, actually holds for *all* $t \in [0,1]$. This is a trivial consequence of Proposition 3.21, Markov's inequality and the fact that $\mathcal{H}^{\eta',p} \hookrightarrow \mathcal{H}^{\eta,p}$ with $\eta' > \eta$ is a compact embedding.

Condition (2) of Proposition 3.8 is the harder one. Let us show that (3.15) holds: given a sequence $(\tau_n)_n$ of $\mathcal{F}$-stopping times we estimate, for $q > 2$,

$$\begin{aligned}
\mathbb{P}\left(\|\omega^n\left((\tau_n+r)\wedge 1\right) - \omega^n(\tau_n)\|_{\mathcal{H}^{\eta,p}} > \delta\right) &\leq \delta^{-q}\mathbb{E}\left[\|\omega^n\left((\tau_n+r)\wedge 1\right) - \omega^n(\tau_n)\|^q_{\mathcal{H}^{\eta,p}}\right] \\
&\lesssim_{\alpha,q} \delta^{-q}\mathbb{E}\left[\|\omega^n\left((\tau_n+r)\wedge 1\right) - \omega^n(\tau_n)\|^{\frac{q(\alpha-\eta)}{2+\alpha}}_{\mathcal{H}^{-2,p}} \cdot \sup_{t\in[0,1]} \|\omega^n(t)\|^{\frac{q(2+\eta)}{2+\alpha}}_{\mathcal{H}^{\alpha,p}}\right] \\
&\lesssim_{\alpha,q} \delta^{-q}\mathbb{E}\left[\|\omega^n\left((\tau_n+r)\wedge 1\right) - \omega^n(\tau_n)\|^{\frac{uq(\alpha-\eta)}{2+\alpha}}_{\mathcal{H}^{-2,p}}\right] \mathbb{E}\left[\sup_{t\in[0,1]} \|\omega^n(t)\|^{\frac{u'q(2+\eta)}{2+\alpha}}_{\mathcal{H}^{\alpha,p}}\right],
\end{aligned}$$

where the second step is the interpolation inequality between $\mathcal{H}^{-2,p}$ and $\mathcal{H}^{\alpha,p}$, and the third is Hölder's inequality with exponents $1 < u, u' < \infty$. A simple computation reveals that, since $q > 2$, one can choose u so that the exponents of the norms in the expected values are strictly larger than 2, therefore we can now apply the uniform estimates of Proposition 3.21 and Proposition 3.22, obtaining

$$\mathbb{P}\left(\|\omega^n\left((\tau_n + r)\wedge 1\right) - \omega^n\left(\tau_n\right)\|_{\mathcal{H}^{\eta,p}} > \delta\right) \lesssim_{M,\alpha,p,q} \delta^{-q}\left(r^{\frac{uq(\alpha-\eta)}{2(2+\alpha)}} + n^{-\frac{uq(\alpha-\eta)}{2+\alpha}}\right).$$

It is now possible, given $\delta > 0$, to choose $r_0 > 0$ small enough and n_0 large enough (taking then $r \leq r_0$ and $n \geq n_0$) so that the right-hand side of the latter inequality is arbitrarily small, thus satisfying Aldous' condition and implying tightness of $(\mathcal{L}_n)_{n\in\mathbb{N}}$.

3.4.3 Proof of Lemma 3.24

We consider a weak limit point $\mathcal{L}$ in $\mathbb{D}([0,T];\mathcal{H}^{\eta,p})$; for simplicity, we still denote by $\mathcal{L}_n$ the convergent subsequence.

Consider the joint law of ω^n and the sequence of Brownian motions $(B^i)_{i\in\mathbb{N}_0}$ on the product space $(\mathbb{R}^2)^{\mathbb{N}} \times \mathbb{D}([0,T];\mathcal{H}^{\eta,p})$ (which we regard as a product metric space, considering a distance on $(\mathbb{R}^2)^{\mathbb{N}}$ that makes it separable). In particular, the relation between Brownian motions and ω^n is encoded in the weak formulation of the dynamics:

$$\begin{aligned}\langle\omega^n(t),\varphi\rangle - \langle\omega^n_0,\varphi\rangle = &\sum_{i\in\mathsf{A}^n_0(t)}\omega^n_i\langle p_\varepsilon(\cdot,\zeta^n_i),\varphi\rangle + \int_0^t\langle\omega^n(s),\Delta\varphi\rangle ds \\ &+\int_0^t\int_{\mathcal{O}} F\left(K[\omega^n(s)](y)\right)\cdot\langle\nabla_y p_\varepsilon(\cdot,y),\varphi\rangle d\mathsf{S}^n(s,y)ds \\ &+\sqrt{2}\int_0^t\sum_{i\in\mathsf{A}^n(s)}\omega^n_i\langle\nabla_y p_\varepsilon(\cdot,x^n_i(s)),\varphi\rangle\cdot dB^i_s.\end{aligned} \tag{3.37}$$

$\mathbb{P}$-a.s. for each $t \in [0,1]$ and $\varphi \in \mathcal{H}^{s+2,p'}$ with $sp' > 2$ (and $1 < p' < 2$ since $p > 2$).

By Skorohod representation theorem on $(\mathbb{R}^2)^{\mathbb{N}} \times \mathbb{D}([0,T];\mathcal{H}^{\eta,p})$, [107, Theorem 3.30], there exists a complete probability space $(\tilde\Omega,\mathcal{A},\tilde{\mathbb{P}})$ with random elements $(\tilde\omega_n)_{n\in\mathbb{N}}$, $(\tilde B^{i,n})_{i\in\mathbb{N}}$ having the same joint laws of $(\omega_n)_{n\in\mathbb{N}}$, $(B^i)_{i\in\mathbb{N}}$, and $\tilde\omega$ with law $\tilde{\mathcal{L}} = \mathcal{L}$, such that $\tilde\omega_n$ converges $\tilde{\mathbb{P}}$-almost surely to $\tilde\omega$. Moreover we can take the filtration $\tilde{\mathcal{F}}^{n,0} = (\tilde{\mathcal{F}}^{n,0}_t)_t$ generated by $(\tilde\omega^n,(\tilde B^{i,n})_i)$ and the $\mathbb{P}$-null sets on $(\Omega,\mathcal{A})$ and we define the filtration $\tilde{\mathcal{F}}^n$ by $\tilde{\mathcal{F}}^n_t = \cap_{s>t}\tilde{\mathcal{F}}^{n,0}_t$; by a standard argument, see e.g. the proof of [16, Proposition 2.5, part 1], we get that ($\tilde\omega^n$ and $(\tilde B^{i,n})_i$ are progressively measurable with respect to $\tilde{\mathcal{F}}^n$ and) $(\tilde B^{i,n})_i$ is still a cylindrical Brownian motion with respect to $\tilde{\mathcal{F}}^n$. Arguing for example as in [19, Section 4.3.4] (see also [60, Proposition 4.1] or [76, Section 2.4.4]), we can show that (3.37) is satisfied by $\tilde\omega_n$, $(\tilde B^{i,n})_{i=1,\dots,N}$ on the filtered probability space $(\tilde\Omega,\tilde{\mathcal{F}}^n,\mathbb{P})$.

If we now show that $\tilde{\mathcal{L}} = \delta_\omega$, so that also $\mathcal{L} = \delta_\omega$, we obtain the first statement of the lemma. In proving such claim, we drop tilde signs from $\tilde{\omega}^n, (\tilde{B}^{i,n})_i$ to lighten notation. The tilde on our limit point $\tilde{\omega}$ is retained since a priori it can differ from the solution ω of (3.1). The aim is to pass to the limit (3.37), we do so term by term.

Linear Terms

It is clear that, $\mathbb{P}$-almost surely,

$$\langle \omega^n(t), \varphi \rangle \to \langle \tilde{\omega}(t), \varphi \rangle , \qquad \int_0^t \langle \omega^n(s), \Delta\varphi \rangle ds \to \int_0^t \langle \tilde{\omega}(s), \Delta\varphi \rangle ds.$$

We then need to prove that $\langle \omega_0^n, \varphi \rangle \to \langle \omega_0, \varphi \rangle$, that is, $\tilde{\omega}_0 = \omega_0$; we do so by showing that as $n \to \infty$, the following quantities vanish:

$$\begin{aligned} I_1 &:= \left| \langle \omega_0^n, \varphi \rangle - \int_{\mathcal{O}} dx \int_{\mathcal{O}} dy p_\varepsilon(x,y) \omega_0(y) \varphi(x) \right| , \\ I_2 &:= \left| \langle \omega_0, \varphi \rangle - \int_{\mathcal{O}} dx \int_{\mathcal{O}} dy p_\varepsilon(x,y) \omega_0(y) \varphi(x) \right| . \end{aligned}$$

We treat I_1 exploiting the fact that φ is Lipschitz continuous,

$$\begin{aligned} I_1 &= \left| \sum_{i:t_i=0} \omega_i^n \langle p_\varepsilon(\cdot, \zeta_i^n), \varphi \rangle - \int_{\mathcal{O}} dx \int_{\mathcal{O}} dy p_\varepsilon(x,y) \omega_0(y) \varphi(x) \right| \\ &= \left| \sum_{i:t_i=0} \omega_i^n P(\varepsilon)\varphi(\zeta_i^n) - \langle \omega_0, P(\varepsilon)\varphi \rangle \right| \le \sum_{i:t_i=0} \int_{Q_i^n} \omega_0(y) |P(\varepsilon)\varphi(\zeta_i^n) - P(\varepsilon)\varphi(y)| dy \\ &\le \sum_{i:t_i=0} \int_{Q_i^n} \omega_0(y) |y - \zeta_i^n| \|P(\varepsilon)\varphi\|_{C^1(\bar{\mathcal{O}})} dy \lesssim \frac{1}{n} \|\varphi\|_{\mathcal{H}^{s+2,p'}} \downarrow 0. \end{aligned}$$

As for I_2, using the regularity of ω_0 we assumed by hypothesis,

$$I_2 = |\langle \omega_0, \varphi \rangle - \langle P(\varepsilon)\omega_0, \varphi \rangle| \le \|\varphi\| \|(I - P(\varepsilon))\omega_0\| \downarrow 0.$$

Boundary Generation Terms

They converge to the PDE Neumann source, conveniently represented by $\int_0^t \langle g(s), \varphi \rangle_{\partial\mathcal{O}} ds$. The argument is analogous to parts of the proofs of Lemma 3.19 and Proposition 3.22, therefore we omit some details. We split:

$$\begin{aligned}
&\left|\sum_{i\in\mathsf{A}_0^n(t)}\omega_i^n\langle p_\varepsilon(\cdot,\zeta_i^n),\varphi\rangle-\int_0^t\langle g(s),\varphi\rangle_{\partial\mathcal{O}}ds\right|\\
&\qquad\le\left|\sum_{i\in\mathsf{A}_0^n(t)}\omega_i^n\langle p_\varepsilon(\cdot,\zeta_i^n),\varphi\rangle-\int_0^t\int_{\partial\mathcal{O}}g(s,y)\langle p_\varepsilon(\cdot,y),\varphi\rangle dyds\right|\\
&\qquad\quad+\left|\int_0^t\langle g(s),\varphi\rangle_{\partial\mathcal{O}}ds-\int_0^t\int_{\partial\mathcal{O}}g(s,y)\langle p_\varepsilon(\cdot,y),\varphi\rangle dyds\right|=J_1+J_2.
\end{aligned}$$

Let us define $I^n(t)=(t_j^n-1/(2n),t_j^n+1/2n]$ with t_j^n such that $t\in I^n(t)$. We control J_1 exploiting again the fact that φ is Lipschitz,

$$\begin{aligned}
J_1&=\left|\sum_{i\in\mathsf{A}_0^n(t)}\omega_i^nP(\varepsilon)\varphi(\zeta_i^n)-\int_0^t\int_{\partial\mathcal{O}}g(s,y)P(\varepsilon)\varphi(y)dyds\right|\\
&\le\sum_{i\in\mathsf{A}_0^n(t)}\int_{Q_i^n}g(s,y)|P(\varepsilon)\varphi(\zeta_i^n)-P(\varepsilon)\varphi(y)|dsdy+\int_{I^n(t)}\int_{\partial\mathcal{O}}g(s,y)|P(\varepsilon)\varphi(y)|dyds\\
&\lesssim\sum_{i\in\mathsf{A}_0^n(t)}\int_{Q_i^n}g(s,y)|\zeta_i^n-y|\|P(\varepsilon)\varphi\|_{C^1(\bar{\mathcal{O}})}dsdy+\frac{1}{n}\|g\|_{L^\infty([0,1]\times\partial\mathcal{O})}\|P_\varepsilon\varphi\|_{C(\bar{\mathcal{O}})}\\
&\lesssim_{s,p}\frac{1}{n}\|\varphi\|_{\mathcal{H}^{s+2,p'}}\downarrow 0.
\end{aligned}$$

Once again, the second term J_2 is the easier one:

$$\begin{aligned}
J_2&\le\|g\|_{L^1([0,1];L^\infty(\partial\mathcal{O}))}\|(I-P_\varepsilon)\varphi\|_{C(\bar{\mathcal{O}})}\\
&\qquad\le\|g\|_{L^1([0,1];L^\infty(\partial\mathcal{O}))}\left\|(I-P_\varepsilon)(I+\mathcal{B}_{p'})\varphi\right\|_{L^{p'}(\mathcal{O})}\downarrow 0.
\end{aligned}$$

Martingale Term

By Itô isometry,

$$\begin{aligned}
&\mathbb{E}\left[\left|\int_0^t\sum_{i\in\mathsf{A}^n(s)}\omega_i^n\langle\nabla_y p_\varepsilon(\cdot,x_i^n(s)),\varphi\rangle\cdot dB_s^{i,n}\right|^2\right]\\
&\qquad=\mathbb{E}\left[\sum_{i\in\mathsf{A}^n(t)}(\omega_i^n)^2\int_{t_i^n}^t|\nabla P(\varepsilon)\varphi(x_i^n(s))|^2ds\right]\\
&\qquad\lesssim\frac{1}{n^2}\sup_{x\in\bar{\mathcal{O}}}|\nabla P(\varepsilon)\varphi(x)|\lesssim_{s,p}\frac{1}{n^2}\|\varphi\|_{\mathcal{H}^{s+2,p'}}\downarrow 0.
\end{aligned}$$

Nonlinear Term

Observe first that, since we are assuming that $\omega^n \to \tilde{\omega}$, $\mathbb{P}$-almost surely in $\mathbb{D}([0,1];\mathcal{H}^{\eta,p})$ for all $\frac{2}{p} < \eta$, by Lemma 3.7 we also have that $K[\omega^n] \to K[\tilde{\omega}]$, $\mathbb{P}$-almost surely in $\mathbb{D}([0,1];C^1(\bar{\mathcal{O}};\mathbb{R}^2))$. We now proceed to show that

$$L := \left| \int_0^t \int_{\mathcal{O}} F\left(K[\omega^n](s,y)\right) \nabla P(\varepsilon)\varphi(y) d\mathsf{S}^n(s,y) ds \right. \\ \left. - \int_0^t \int_{\mathcal{O}} F(K[\tilde{\omega}(s)](y)) \nabla \varphi(y) \tilde{\omega}(s,y) dy ds \right| \xrightarrow{\mathbb{P}-a.s.} 0. \quad (3.38)$$

To do so, we add and subtract the same quantity three times, apply the triangular inequality and estimate differences. For the sake of shorter formulas and clearer passages we will write

$$\mathbf{K}^n_t(x) = F(K[\omega^n](t,x)), \quad \mathbf{K}_t(x) = F(K[\tilde{\omega}](t,x)), \quad x \in \bar{\mathcal{O}};$$

$$T_1 = \int_0^t \int_{\mathcal{O}} \mathbf{K}^n_t(y) \nabla P(\varepsilon)\varphi(y) d\mathsf{S}^n(s,y) ds, \quad T_2 = \int_0^t \int_{\mathcal{O}} \mathbf{K}^n_t(y) \nabla P(\varepsilon)\varphi(y) \omega^n(s,y) dy ds,$$

$$T_3 = \int_0^t \int_{\mathcal{O}} \mathbf{K}^n_t(y) \nabla P(\varepsilon)\varphi(y) \tilde{\omega}(s,y) dy ds, \quad T_4 = \int_0^t \int_{\mathcal{O}} \mathbf{K}_t(y) \nabla P(\varepsilon)\varphi(y) \tilde{\omega}(s,y) dy ds,$$

$$T_5 = \int_0^t \int_{\mathcal{O}} \mathbf{K}_t(y) \nabla \varphi(y) \tilde{\omega}(s,y) dy ds.$$

Notice that T_1 and T_5 coincide with the terms of the difference in (3.38), therefore the latter vanishes if $|T_1 - T_2|, |T_2 - T_3|, |T_3 - T_4|, |T_4 - T_5|$ do so. The first difference is the harder one, since it involves the pointwise difference between the empirical measure and its regularization:

$$\begin{aligned} |T_1 - T_2| &= \left| \int_0^t \sum_{i \in \mathsf{A}^n(s)} \omega^n_i \mathbf{K}^n_t(x^n_i(s)) \nabla P(\varepsilon)\varphi(x^n_i(s)) ds \right. \\ &\qquad \left. - \int_0^t \sum_{i \in \mathsf{A}^n(s)} \omega^n_i \int_{\mathcal{O}} \mathbf{K}^n_t(z) \nabla P(\varepsilon)\varphi(z) p_\varepsilon(z, x^n_i(s)) dz ds \right| \\ &\leq (\|\omega_0\|_{L^1(\mathcal{O})} + \|g\|_{L^1([0,1]\times\partial\mathcal{O})}) \cdot \\ &\quad \cdot \sup_{\substack{t \in [0,1], \\ y \in \bar{\mathcal{O}}}} \int_{\mathcal{O}} \left| \mathbf{K}^n_t(y) \nabla P(\varepsilon)\varphi(y) - \mathbf{K}^n_t(z) \nabla P(\varepsilon)\varphi(z) \right| p_\varepsilon(z,y) dz, \\ &\lesssim \sup_{\substack{t \in [0,1], \\ y \in \bar{\mathcal{O}}}} \int_{\mathcal{O}} \left| \mathbf{K}^n_t(y) \nabla P(\varepsilon)\varphi(y) - \mathbf{K}^n_t(z) \nabla P(\varepsilon)\varphi(y) \right| p_\varepsilon(z,y) dz \\ &\qquad + \sup_{\substack{t \in [0,1], \\ y \in \bar{\mathcal{O}}}} \int_{\mathcal{O}} \left| \mathbf{K}^n_t(z) \nabla P(\varepsilon)\varphi(y) - \mathbf{K}^n_t(z) \nabla P(\varepsilon)\varphi(z) \right| p_\varepsilon(z,y) dz. \end{aligned}$$

From there, since F is Lipschitz continuous, and applying the pointwise estimate on heat kernel (3.5),

$$\begin{aligned}
|T_1 - T_2| &\lesssim \sup_{\substack{t\in[0,1],\\ y\in\bar{\mathcal{O}}}} \int_{\mathcal{O}} \|\nabla P(\varepsilon)\varphi\|_{C(\bar{\mathcal{O}})} \, |y-z| \, \|K[\omega^n(t)]\|_{C^1(\bar{\mathcal{O}})} \, \frac{1}{\varepsilon} e^{|z-y|^2/(c\varepsilon)} dz \\
&\quad + \sup_{\substack{t\in[0,1],\\ y\in\bar{\mathcal{O}}}} \int_{\mathcal{O}} \|\nabla P(\varepsilon)\varphi\|_{C^1(\bar{\mathcal{O}})} \, |y-z| \, \|K[\omega^n(t)]\|_{C(\bar{\mathcal{O}})} \, \frac{1}{\varepsilon} e^{|z-y|^2/(c\varepsilon)} dz \\
&\lesssim_{s,p} \sup_{\substack{t\in[0,1],\\ y\in\bar{\mathcal{O}}}} \int_{\mathcal{O}} \|\varphi\|_{\mathcal{H}^{s+2,p'}} \, |y-z| \, \|K[\omega^n(t)]\|_{C^1(\bar{\mathcal{O}})} \, \frac{1}{\varepsilon} e^{|z-y|^2/(c\varepsilon)} dz \\
&\leq \|\varphi\|_{\mathcal{H}^{s+2,p'}} \sup_{y\in\bar{\mathcal{O}},n\in\mathbb{N}} \|\omega^n\|_{\mathbb{D}([0,1];\mathcal{H}^{\eta,p})} \int_{\mathcal{O}} |y-z| \frac{1}{\varepsilon} e^{|z-y|^2/(c\varepsilon)} dz \xrightarrow{\mathbb{P}-a.s.} 0.
\end{aligned}$$

The other differences now are controlled by the almost sure convergence assumption and functional analytic estimates already repeatedly applied. Bounding

$$\begin{aligned}
|T_2 - T_3| &\leq M \int_0^t \int_{\mathcal{O}} \|\nabla P(\varepsilon)\varphi\|_{C(\bar{\mathcal{O}})} \, |\omega^n(s,y) - \tilde{\omega}(s,y)| dy ds \\
&\lesssim_{s,p,M} \|\varphi\|_{\mathcal{H}^{s+2,p'}} \, \|\omega^n - \tilde{\omega}\|_{\mathbb{D}([0,1];\mathcal{H}^{\eta,p})} , \\
|T_3 - T_4| &\lesssim_M \int_0^t \int_{\mathcal{O}} \|\nabla P(\varepsilon)\varphi\|_{C(\bar{\mathcal{O}})} \, \|\tilde{\omega}(s)\|_{C(\bar{\mathcal{O}})} \, |K[\omega^n](s,y) - K[\tilde{\omega}](s,y)| dy ds \\
&\lesssim_{\eta,p} \|\nabla P(\varepsilon)\varphi\|_{C(\bar{\mathcal{O}})} \, \|\tilde{\omega}\|_{\mathbb{D}([0,1];\mathcal{H}^{\eta,p})} \, \|K\|_{H^{\eta,p}\to H^{\eta+1,p}} \, \|\omega^n - \tilde{\omega}\|_{\mathbb{D}([0,1];\mathcal{H}^{\eta,p})} , \\
|T_4 - T_5| &\leqslant M \, \|\tilde{\omega}\|_{\mathbb{D}([0,1];C(\bar{\mathcal{O}}))} \, \|\nabla(I - P(\varepsilon))\varphi\|_{C(\bar{\mathcal{O}})} \\
&\lesssim_{\eta,p,M} \|\tilde{\omega}\|_{\mathbb{D}([0,1];\mathcal{H}^{\eta,p})} \, \|(I - P(\varepsilon))\varphi\|_{\mathcal{H}^{s+2,p'}} \\
&= M \, \|\tilde{\omega}\|_{\mathbb{D}([0,1];\mathcal{H}^{\eta,p})} \left\| (I - P(\varepsilon))(I + \mathcal{B}_{p'})^{(s+2)/2} \varphi \right\|_{L^{p'}} ,
\end{aligned}$$

it is clear that differences on the right-hand side converge $\mathbb{P}$-almost surely to 0 as $n \to \infty$.

Removing the cutoff

The argument detailed so far implies that the limit $\tilde{\omega}$ is a weak solution, thus *the unique* weak solution of the cutoff PDE (1.22). However, by Proposition 3.12 (second statement) it is now possible to choose M large enough so that $\|K[\tilde{\omega}]\|_{L^\infty([0,1],L^\infty)} < M$, so that $F(K[\tilde{\omega}]) = K[\tilde{\omega}]$ and thus $\tilde{\omega} = \omega$ is actually the unique weak solution of the PDE (3.1) without cutoff. This concludes the proof of the first part of Lemma 3.24.

3.5 General Initial and Boundary data

In this section we provide a brief description of changes to the above arguments required to deal with general boundary data $\omega_0 \in \mathrm{Lip}(\bar{\mathcal{O}})$ and $g \in \mathcal{B}_b([0,1] \times \partial\mathcal{O})$, dropping the non-negativity assumption. The idea is simply to divide boundary data into positive and negative parts, and to consider particle and PDE dynamics for the two parts, properly taking into account their interactions.

3.5.1 Splitting of PDE and particle dynamics

Let us set

$$\omega_0^+ := \omega_0 \vee 0, \quad \omega_0^- := (-\omega_0) \vee 0, \quad g^+(t,x) := g(t,x) \vee 0, \quad g^-(t,x) := (-g(t,x)) \vee 0.$$

The limit dynamics (3.1) is then equivalent to a coupled system of PDEs,

$$\begin{cases} \partial_t \omega^+ + K[\omega^+ - \omega^-] \cdot \nabla \omega^+ = \Delta \omega^+, & \text{in } \mathcal{O} \times [0,T], \\ \partial_t \omega^- + K[\omega^+ - \omega^-] \cdot \nabla \omega^- = \Delta \omega^-, & \text{in } \mathcal{O} \times [0,T], \\ \nabla \omega^+ \cdot \hat{n} = g^+, \quad \nabla \omega^- \cdot \hat{n} = g^-, & \text{in } \partial\mathcal{O} \times (0,T), \\ \omega^+(0) = \omega_0^+, \quad \omega^-(0) = \omega_0^-, \quad \text{in } \mathcal{O}. \end{cases} \tag{3.39}$$

The notion of weak solution to the latter system is completely analogous to the one of Definition 3.9. Moreover, a couple (ω^+, ω^-) is a weak solution of the system above if and only if the couple $(\omega = \omega^+ - \omega^-, \omega^+)$ is a weak solution of

$$\begin{cases} \partial_t \omega + K[\omega] \cdot \nabla \omega = \Delta \omega, & \text{in } \mathcal{O} \times [0,T], \\ \partial_t \omega^+ + K[\omega] \cdot \nabla \omega^+ = \Delta \omega^+, & \text{in } \mathcal{O} \times [0,T], \\ \nabla \omega \cdot \hat{n} = g, \quad \nabla \omega^+ \cdot \hat{n} = g^+, & \text{in } \partial\mathcal{O} \times (0,T), \\ \omega(0) = \omega_0, \quad \omega^+(0) = \omega_0^+, \quad \text{in } \mathcal{O}. \end{cases} \tag{3.40}$$

Well-posedness of (3.40) follows easily from Proposition 3.11. Therefore, also (3.39) is well posed in $\mathbb{D}([0,T]; L^2(\mathcal{O})^2)$.
As for the approximating dynamics, coherently with notation of subsection 3.2.1, we define

$$\omega_i^{n,\pm} = \begin{cases} \int_{Q_i^n} \omega_0^\pm(x) dx & t_i^n = 0, \\ \int_{Q_i^n} g^\pm(s,x) ds d\sigma(x) & t_i^n > 0. \end{cases} \tag{3.41}$$

From our assumptions on the boundary data and on the mesh of the grid, it follows that $\omega_i^{n,\pm} \lesssim \frac{1}{n^2}$ uniformly in $i = 1, \dots, N$, for all n. We introduce two mutually independent sequences of independent $\mathcal{F}$-Brownian motions $(B^{i,\pm})_{i \in \mathbb{N}}$ and particles with positions $x_i^{n,\pm}(t)$

satisfying the following evolution equations: for $t > t_i^n$,

$$
\begin{aligned}
x_i^{n,\pm}(t) - \zeta_i^n = \int_{t_i^n}^t F\Bigg(&\sum_{j \in \mathsf{A}^n(s)} \omega_j^{n,+} K_n(x_i^{n,\pm}(s), x_j^{n,+}(s)) \\
&- \omega_j^{n,-} K_n(x_i^{n,\pm}(s), x_j^{n,-}(s)) \Bigg) ds + \sqrt{2} \int_{t_i^n}^t dB_s^{i,\pm} - k_i^{n,\pm}(t), \\
k_i^{n,\pm}(t) = \int_{t_i^n}^t \hat{n}(x_i^{n,\pm}(s)) d|k_i^{n,\pm}|(s), &\quad |k_i^{n,\pm}|(t) = \int_{t_i^n}^t \mathbf{1}_{x_i^{n,\pm}(s) \in \partial\mathcal{O}} d|k_i^{n,\pm}|(s).
\end{aligned}
$$

where $k_i^{n,\pm}$ are continuous, adapted, $\mathbb{R}^2$-valued processes with bounded variation trajectories, whereas for $t \leq t_i^n$, $x_i^{n,\pm}(t) = \zeta_i^n$ and $k_i^{n,\pm}(t) = 0$. Well-posedness of this particle system follows from the same arguments proving Proposition 3.14. Similarly to section 3.3 we introduce the empirical measures

$$
\mathsf{S}^{n,\pm}(t) = \sum_{i \in \mathsf{A}^n(t)} \omega_i^{n,\pm} \delta_{x_i^{n,\pm}(t)},
$$

and the regularized empirical measures

$$
\omega^{n,\pm}(t)(x) = P_\varepsilon \mathsf{S}^{n,\pm}(t) = \sum_{i \in \mathsf{A}^n(t)} \omega_i^{n,\pm} p_\varepsilon(x, x_i^{n,\pm}(t)), \quad x \in \bar{\mathcal{O}}.
$$

The following statement extends Theorem 3.17 to the general case of ω_0, g not necessarily non-negative.

Theorem 3.25. *Let $p > 2$, $\frac{2}{p} < \alpha < 1$, $\varepsilon = \varepsilon(n) \gtrsim n^{-1/2}$, and $\omega_0 \in \mathrm{Lip}(\bar{\mathcal{O}})$, $g \in \mathcal{B}_b([0,1] \times \partial\mathcal{O})$. There exists $M > 0$ (only depending on ω_0, g) such that for every $\eta \in (2/p, \alpha)$, as $n \to \infty$, the stochastic process $(\omega^{n,+}, \omega^{n,-})$ converges in probability on $\mathbb{D}([0,T]; \mathcal{H}^{\eta,p} \times \mathcal{H}^{\eta,p})$ to the unique weak solution of (3.39), therefore $\omega^{n,+} - \omega^{n,-}$ converges in the same topology to the unique solution of (3.1).*

3.5.2 Uniform Estimates for split dynamics

The regularized empirical measures satisfy the integral equations and the mild formulations below:

$$
\begin{aligned}
\omega^{n,\pm}(t,x) - \omega_0^{n,\pm}(x) = &\sum_{i \in \mathsf{A}_0^n(t)} \omega_i^{n,\pm} p_\varepsilon(x, \zeta_i^n) + \int_0^t \Delta \omega^{n,\pm}(s,x) ds \\
&+ \int_0^t \int_{\mathcal{O}} F\left(K[\omega^{n,+} - \omega^{n,-}](s,y)\right) \nabla_y p_\varepsilon(x,y) d\mathsf{S}^{n,\pm}(s,y) ds \\
&+ \sqrt{2} \int_0^t \sum_{i \in \mathsf{A}^n(s)} \omega_i^{n,\pm} \nabla_y p_\varepsilon(x, x_i^{n,\pm}(s)) dB_s^{i,\pm}.
\end{aligned} \tag{3.42}
$$

$$\begin{aligned}\omega^{n,\pm}(t,x) = P(t)\omega_0^{n,\pm}(x) + \sum_{i\in \mathsf{A}_0^n(t)} \omega_i^{n,\pm} P(t-t_i^n) p_\varepsilon(x,\zeta_i^n) \\ + \int_0^t P(t-s) \int_{\mathcal{O}} F\left(K[\omega^{n,+}-\omega^{n,-}](s,y)\right) \nabla_y p_\varepsilon(x,y) d\mathsf{S}^{n,\pm}(s,y) ds \\ + \sqrt{2}\int_0^t \sum_{i\in \mathsf{A}^n(s)} \omega_i^{n,\pm} P(t-s)\nabla_y p_\varepsilon(x, x_i^{n,\pm}(s)) dB_s^{i,\pm}.\end{aligned} \tag{3.43}$$

In order to obtain the required compactness we need to show that

$$\sup_n \mathbb{E}\left[\sup_{t\in[0,1]} \left\|\omega^{n,+}(t)\right\|^q_{\mathcal{H}^{\alpha,p}}\right] + \sup_n \mathbb{E}\left[\sup_{t\in[0,1]} \left\|\omega^{n,-}(t)\right\|^q_{\mathcal{H}^{\alpha,p}}\right] < C_{M,q,p,\alpha}, \tag{3.44}$$

$$\begin{aligned}\mathbb{E}\left[\left\|\omega^{n,+}(\tau_n+r) - \omega^{n,+}(\tau_n)\right\|^q_{\mathcal{H}^{-2,p}}\right] + \mathbb{E}\left[\left\|\omega^{n,-}(\tau_n+r) - \omega^{n,-}(\tau_n)\right\|^q_{\mathcal{H}^{-2,p}}\right] \\ \lesssim_{M,q,p,\alpha} \left(r^{q/2} + \frac{1}{n^q}\right).\end{aligned} \tag{3.45}$$

The proof of these inequalities follows the same strategy described in section 3.3. Indeed, linear terms, stochastic integrals and generation terms can be treated exactly as in the case of positive data. In fact, nonlinear terms present no additional difficulties and the crucial estimate for the nonlinear term of Proposition 3.21 is easily adapted. For $f \in L^{p'}$ such that $\|f\|_{L^{p'}} = 1$ the following chain of inequalities holds:

$$\begin{aligned}&\Bigg\| \sum_{i\in \mathsf{A}^n(s)} \omega_i^{n,\pm} F\left(K[\omega^{n,+}(s) - \omega^{n,-}(s)](x_i^{n,\pm}(s))\right) \\ &\qquad \cdot \int_{\mathcal{O}} f(x)(I+\mathcal{B}_p)^{\alpha/2} P(t-s)\nabla_y p_\varepsilon(x, x_i^{n,\pm}(s)) dx \Bigg\| \\ &\lesssim_M \sum_{i\in \mathsf{A}^n(s)} \omega_i^{n,\pm} \left|\nabla P_\varepsilon[(I+\mathcal{B}_{p'})^{\alpha/2} P(t-s) f](x_i^{n,\pm}(s))\right| \\ &\lesssim \sum_{i\in \mathsf{A}^n(s)} \omega_i^{n,\pm} P_\varepsilon \left|\nabla (I+\mathcal{B}_{p'})^{\alpha/2} P(t-s) f\right| (x_i^{n,\pm}(s)) \\ &= \sum_{i\in \mathsf{A}^n(s)} \omega_i^{n,\pm} \int_{\mathcal{O}} p_\varepsilon(x, x_i^{n,\pm}(s)) \left|\nabla (I+\mathcal{B}_{p'})^{\alpha/2} P(t-s) f\right|(x) dx \\ &= \int_{\mathcal{O}} P_\varepsilon(\mathsf{S}^{n,\pm})(x) \left|\nabla (I+\mathcal{B}_{p'})^{\alpha/2} P(t-s) f\right|(x) dx \\ &\le \left\|\omega^{n,\pm}(s)\right\|_{L^p} \left\|\nabla (I+\mathcal{B}_{p'})^{\alpha/2} P(t-s) f\right\|_{L^{p'}} \lesssim \frac{1}{(t-s)^{(1+\alpha)/2}} \left\|\omega^{n,\pm}(s)\right\|_{\mathcal{H}^{\alpha,p}}.\end{aligned}$$

Then the proof of (3.44) follows exactly as in Proposition 3.21. Similarly, in the proof of (3.45) the only differences concern the nonlinear terms, and they can be handled with analogous simple modifications.

Once uniform bounds are recovered, one can proceed to replicate the tightness argument of subsection 3.4.1. Passing to the limit dynamics is similar to that explained in subsection 3.4.3, once again we only outline the (slightly) different treatment required by nonlinear terms.

We consider the weak formulation satisfied by $\omega^{n,\pm}(t)$, $t \in [0,1]$ and test functions $(\varphi^+, \varphi^-) \in \mathcal{H}^{s+2,p'} \times \mathcal{H}^{s+2,p'}$ with $sp' > 2,\ 1 < p' < 2$,

$$\begin{aligned}
&\Big\langle \omega^{n,\pm}(t), \varphi^\pm \rangle - \langle \omega_0^{n,\pm}, \varphi^\pm \Big\rangle \qquad (3.46)\\
&\quad = \sum_{i \in \mathsf{A}_0^n(t)} \omega_i^{n,\pm} \langle p_\varepsilon(\cdot, \zeta_i^n), \varphi^\pm \rangle + \int_0^t \langle \omega^{n,\pm}(s), \Delta\varphi^\pm \rangle ds \\
&\quad + \int_0^t \int_{\mathcal{O}} F\left(K[\omega^{n,+} - \omega^{n,-}](s,y)\right) \cdot \langle \nabla_y p_\varepsilon(\cdot, y), \varphi^\pm \rangle d\mathsf{S}^{n,\pm}(s,y) ds \\
&\quad + \sqrt{2} \int_0^t \sum_{i \in \mathsf{A}^n(s)} \omega_i^{n,\pm} \langle \nabla_y p_\varepsilon(\cdot, x_i^{n,\pm}(s)), \varphi^\pm \rangle \cdot dB_s^{i,\pm}.
\end{aligned}$$

We stress once again that linear terms, stochastic integrals and generation terms present no changes with respect to subsection 3.4.3. Since (by means of a Skorohod argument) we can assume that $\omega^{n,\pm} \overset{\mathbb{P}-a.s.}{\to} \omega^\pm$ in $\mathbb{D}([0,1]; \mathcal{H}^{\eta,p})$ for all $\frac{2}{p} < \eta$, and by the properties of the Biot-Savart Kernel,

$$u^n := K[\omega^{n,+} - \omega^{n,-}] \overset{\mathbb{P}-a.s.}{\to} K[\omega^+ - \omega^-] =: u \quad \text{in } \mathbb{D}([0,1]; C^1(\bar{\mathcal{O}}; \mathbb{R}^2)).$$

A careful analysis of computations in subsection 3.4.3 reveals that only the terms $|T_1^\pm - T_2^\pm|$, and $|T_3^\pm - T_4^\pm|$ are affected by the generalization to the coupled positive-negative system. However, only little changes are needed:

$$\begin{aligned}
|T_1^\pm - T_2^\pm| &= \Bigg| \int_0^t \sum_{i \in \mathsf{A}^n(s)} \omega_i^{n,\pm} F\left(u^n(s, x_i^{n,\pm}(s))\right) \nabla P(\varepsilon) \varphi^\pm(x_i^{n,\pm}(s)) ds \\
&\quad - \int_0^t \sum_{i \in \mathsf{A}^n(s)} \omega_i^{n,\pm} \int_{\mathcal{O}} F\left(u^n(s,z)\right) \nabla P(\varepsilon) \varphi^\pm(z) p_\varepsilon(z, x_i^{n,\pm}(s)) dz ds \Bigg| \\
&\lesssim \sup_{\substack{t \in [0,1], \\ y \in \bar{\mathcal{O}}}} \int_{\mathcal{O}} \left| F\left(u^n(t,y)\right) \nabla P(\varepsilon) \varphi^\pm(y) - F\left(u^n(t,z)\right) \nabla P(\varepsilon) \varphi^\pm(z) \right| p_\varepsilon(z,y) dz \\
&\leq \sup_{\substack{t \in [0,1], \\ y \in \bar{\mathcal{O}}}} \int_{\mathcal{O}} \left| F\left(u^n(t,y)\right) \nabla P(\varepsilon) \varphi^\pm(y) - F\left(u^n(t,z)\right) \nabla P(\varepsilon) \varphi^\pm(y) \right| p_\varepsilon(z,y) dz
\end{aligned}$$

$$
\begin{aligned}
&+\sup_{\substack{t\in[0,1],\\ y\in\bar{\mathcal{O}}}}\int_{\mathcal{O}}\left|F\left(u^n(t,z)\right)\nabla P(\varepsilon)\varphi^{\pm}(y)-F\left(u^n(t,z)\right)\nabla P(\varepsilon)\varphi^{\pm}(z)\right|p_{\varepsilon}(z,y)dz\\
\lesssim&\sup_{\substack{t\in[0,1],\\ y\in\bar{\mathcal{O}}}}\int_{\mathcal{O}}\left\|\nabla P(\varepsilon)\varphi^{\pm}\right\|_{C(\bar{\mathcal{O}})}|y-z|\left\|u^n(t)\right\|_{C^1(\bar{\mathcal{O}})}\frac{1}{\varepsilon}e^{|z-y|^2/(c\varepsilon)}dz\\
&+\sup_{\substack{t\in[0,1],\\ y\in\bar{\mathcal{O}}}}\int_{\mathcal{O}}\left\|\nabla P(\varepsilon)\varphi^{\pm}\right\|_{C^1(\bar{\mathcal{O}})}|y-z|\left\|u^n(t)\right\|_{C(\bar{\mathcal{O}})}\frac{1}{\varepsilon}e^{|z-y|^2/(c\varepsilon)}dz\\
\lesssim&\sup_{\substack{t\in[0,1],\\ y\in\bar{\mathcal{O}}}}\int_{\mathcal{O}}\left\|\varphi^{\pm}\right\|_{\mathcal{H}^{s+2,p'}}|y-z|\left\|u^n(t)\right\|_{C^1(\bar{\mathcal{O}})}\frac{1}{\varepsilon}e^{|z-y|^2/(c\varepsilon)}dz\\
\lesssim&\left\|\varphi^{\pm}\right\|_{\mathcal{H}^{s+2,p'}}\sup_{y\in\bar{\mathcal{O}},n\in\mathbb{N}}\left\|\omega^{n,+}-\omega^{n,-}\right\|_{\mathbb{D}([0,1];\mathcal{H}^{\eta,p})}\xrightarrow{\mathbb{P}-a.s.}0,
\end{aligned}
$$

$$
\begin{aligned}
|T_3^{\pm}-T_4^{\pm}|\lesssim&\int_0^t\int_{\mathcal{O}}\left\|\nabla P(\varepsilon)\varphi^{\pm}\right\|_{C(\bar{\mathcal{O}})}\left\|\omega^{\pm}(s)\right\|_{C(\bar{\mathcal{O}})}|u^n(s,y)-u(s,y)|dyds\\
\lesssim&\left\|\nabla P(\varepsilon)\varphi^{\pm}\right\|_{C(\bar{\mathcal{O}})}\left\|\omega^{\pm}\right\|_{\mathbb{D}([0,1];\mathcal{H}^{\eta,p})}\|K\|_{H^{\eta,p}\to H^{\eta+1,p}}\\
&\left(\left\|\omega^{n,+}-\omega^{+}\right\|_{\mathbb{D}([0,1];\mathcal{H}^{\eta,p})}+\left\|\omega^{n,-}-\omega^{-}\right\|_{\mathbb{D}([0,1];\mathcal{H}^{\eta,p})}\right)\xrightarrow{\mathbb{P}-a.s.}0.
\end{aligned}
$$

The remaining passages then coincide with the non-negative case discussed above.

Chapter 4

The Problem of the Inviscid Limit for Stochastic Fluids: Convergence

This chapter and the following one present several issues related to problem of the inviscid limit for stochastic fluids in domains with boundaries. The study of the inviscid limit is a classical topic in fluid mechanics. Indeed, the Euler equations have very large classes of weak solutions, including non-dissipative ones [11], but the inviscid limit can in some cases furnish a selection principle [12]. As discussed in section 1.5, the case of no-slip boundary conditions has additional difficulties due to the different boundary conditions for the Euler equations. The difference of the boundary conditions produces the emergence of a boundary layer: close to the boundary the fluid presents a turbulent behavior for $\nu \to 0$. The thickness of the boundary layer and some control on the behavior of the fluid in this region are very challenging and mostly open questions and we refer to [14], [113], [114] for some reviews. In this chapter, following [136] and [135], we provide some results on the validity of the inviscid limit in the stochastic framework for the Navier-Stokes equations with additive noise and the Second-Grade fluid equations with either additive or transport noise. While we discussed extensively in section 1.1 on the modeling interest of additive noise in fluid dynamical models in domains with boundaries, we mention again that the introduction of transport noise in fluid dynamical models has some physical intuitions, see [120], [121], [122], and can be rigorously justified basing on the idea of separation of scales as in [101], [80], [61], [144]. Without having any ambition on furnish new results on the analysis of the boundary layer of a turbulent fluid, or providing some stochastic regimes which help in the analysis of the inviscid limit, our goal is to provide conditions under which the unique weak solution of

$$\begin{cases} du^\nu = (\nu\Delta u^\nu - u^\nu\cdot\nabla u^\nu + \nabla P^\nu)dt + \sqrt{\nu}\sum_{k\in\mathcal{K}}\sigma_k dW^k_t, \\ \operatorname{div} u^\nu = 0, \\ u^\nu|_{\partial\mathcal{O}} = 0, \\ u^\nu(0) = u^\nu_0 \end{cases} \tag{4.1}$$

$$\left(\text{resp.}\quad \begin{cases} d\mathsf{v}^\alpha = (\nu\Delta\mathsf{u}^\alpha - \operatorname{curl}(\mathsf{v}^\alpha)\times\mathsf{u}^\alpha + \nabla\mathsf{P}^\alpha)dt + \sqrt{\alpha}\sum_{k\in\mathcal{K}}(\sigma_k\cdot\nabla\mathsf{u}^\alpha + \nabla\tilde{\mathsf{P}}^\alpha_k)\circ dW^k_t \\ \operatorname{div}\mathsf{u}^\alpha = 0 \\ \mathsf{v}^\alpha = \mathsf{u}^\alpha - \alpha\Delta\mathsf{u}^\alpha \\ \mathsf{u}^\alpha|_{\partial\mathcal{O}} = 0 \\ \mathsf{u}^\alpha(0) = \mathsf{u}^\alpha_0, \end{cases}\right) \tag{4.2}$$

converge in mean in $L^\infty(0,T;H)$ to a solution of the deterministic Euler equations

$$\begin{cases} \partial_t\bar{u} = -\bar{u}\cdot\nabla\bar{u} + \nabla\bar{P}, \\ \operatorname{div}\bar{u} = 0, \\ \bar{u}\cdot\hat{n}|_{\partial\mathcal{O}} = 0, \\ \bar{u}(0) = \bar{u}_0. \end{cases} \tag{4.3}$$

In particular, we prove that the convergence of u^ν is guaranteed under some (usually unproved) conditions on the behaviour of the solution in the boundary layer related to anomalous dissipation phenomena, see section 1.5 and the discussion below Theorem 4.19 and Theorem 4.20. On the contrary, the convergence of u^α holds under a particular scaling between the elasticity and the viscosity of the system. Such scaling seems natural to us in the contest of (4.2) as discussed in Remark 4.26 below. The scaling between the viscosity and the noise introduced in (4.1), (4.2) is, perhaps, the more reasonable one in stochastic contexts, see [124, Chapter 10]. However, recently [178], [94] considered also different scaling.
The content of this chapter is as follows: in section 4.1 we recall some results on the well-posedness of the equations which are our object of study and some tools in order to study the inviscid limit. Secondly we state our main results in section 4.2. Then, section 4.3 is devoted to prove the validity of our conditioned results on the inviscid limit for the Navier-Stokes equations with additive noise. We provide unconditioned result for the validity of the inviscid limit for the second grade fluid equations in section 4.4. In particular the main part of the section is devoted to the case of transport noise, while in subsection 4.4.2 we add some remarks in order to understand how to generalize the strategy for the analysis of (4.2) to the case of additive noise.

4.1 Preliminaries

The plan of this section is the following: in subsection 4.1.1 we introduce the function spaces needed in our analysis, secondly in subsection 4.1.2 we introduce our set of assumptions on the coefficients, the notions of solutions for (4.1), (4.2), (4.3) and state some results about the well-posedness of the equations. Lastly subsection 4.1.3 is devoted to providing some additional tools to study the problem of the inviscid limit.

4.1.1 Functional Analytic Setup

Let us start, recalling some notation of section 1.2:

$$H = \{u \in L^2(\mathcal{O};\mathbb{R}^2),\ \operatorname{div} u = 0,\ u \cdot \hat{n}|_{\partial\mathcal{O}} = 0\},\ V = H^1_0(\mathcal{O};\mathbb{R}^2) \cap H,\ \mathsf{D}(\mathcal{A}) = H^2(\mathcal{O};\mathbb{R}^2) \cap V.$$

We endow V with the norm $\|u\|^2_V = \|\nabla u\|^2$ and $\mathsf{D}(\mathcal{A})$ with the norm $\|u\|^2_{\mathsf{D}(\mathcal{A})} = \|\mathcal{A}u\|^2$. It is well-known, see for example [168], that the unbounded linear operator $-\mathcal{A} : \mathsf{D}(\mathcal{A}) \subseteq H \to H$ defined by the identity

$$-\langle \mathcal{A}v, w\rangle = \langle \Delta v, w\rangle \tag{4.4}$$

for all $v \in \mathsf{D}(\mathcal{A}),\ w \in H$ is self-adjoint, generates an analytic semigroup of negative type on H and moreover $V = \mathsf{D}\left(\mathcal{A}^{1/2}\right)$. The definition of b and B of subsection 2.1.3 can be easily adapted to this framework. Actually, from a philosophical viewpoint, the definitions of this section are classical, see [167], while the ones of subsection 2.1.3 are an easy generalization. Define the trilinear form $b : \mathbb{L}^4 \times V \times \mathbb{L}^4 \to \mathbb{R}$ as

$$b\left(u, v, w\right) = \sum_{i,j=1}^{2} \int_{\mathcal{O}} u_i \partial_i v_j w_j \, dx = \int_{\mathcal{O}} \left(u \cdot \nabla v\right) \cdot w \, dx$$

which is well–defined and continuous on $\mathbb{L}^4 \times V \times \mathbb{L}^4$ by the Hölder's inequality. Since by Sobolev embedding theorem $V \subset \mathbb{L}^4$, b is also defined and continuous on $V \times V \times V$. Integrating by parts, the standard oddity relation below holds

$$b\left(u, v, w\right) = -b\left(u, w, v\right)$$

if $u \in \mathbb{L}^4$, $v, w \in V$. Analogously, we introduce the operator

$$B : \mathbb{L}^4 \times \mathbb{L}^4 \to V^*$$

defined by the identity

$$\langle B\left(u, v\right), \varphi\rangle = -b\left(u, \varphi, v\right) = -\int_{\mathcal{O}} \left(u \cdot \nabla \varphi\right) \cdot v \, dx$$

for all $\varphi \in V$. When $v \in V$, we may also write

$$\langle B\left(u, v\right), \varphi\rangle = b(u, v, \varphi).$$

Moreover, when $u \cdot \nabla v \in L^2\left(\mathcal{O};\mathbb{R}^2\right)$, it is explicitly given by

$$B\left(u, v\right) = \mathbf{P}(u \cdot \nabla v).$$

We will use also some properties of the projection operator $\mathbf{P}$ and the solution map of the Stokes operator. We refer to [168] for the proof of the lemmas below.

Lemma 4.1. *For $r > 0$, the restriction of the projection operator $\mathbf{P} : L^2(\mathcal{O};\mathbb{R}^2) \to H$ to $H^r(\mathcal{O};\mathbb{R}^2)$ is a continuous and linear map between $H^r(\mathcal{O};\mathbb{R}^2)$ and itself.*

Lemma 4.2. *The injection of V in H is compact. Thus, there exists a sequence e_i of elements of H which forms an orthonormal basis in H and an orthogonal basis in V. This sequence verifies*

$$\mathcal{A}e_i = \lambda_i e_i$$

where $\lambda_{i+1} > \lambda_i > 0,\ i = 1, 2, \dots$. Moreover $\lambda_i \to +\infty$. Lastly $e_i \in C^\infty(\bar{\mathcal{O}};\mathbb{R}^2)$ under our assumptions on $\mathcal{O}$

The tools introduced above are the standard ingredients in order to deal with the Navier-Stokes equations with no-slip boundary conditions. We end this Subsection recalling some other facts in order to treat Second-Grade fluid equations. We refer to [46],[151],[152],[87] for the proof of the various statements. We introduce the vector space

$$W = \{u \in V : \ \operatorname{curl}(u - \alpha^2 \Delta u) \in L^2(\mathcal{O};\mathbb{R}^2)\}$$

with norm $\|u\|_W^2 = \|u\|^2 + \alpha^2\|\nabla u\|^2 + \|\operatorname{curl}(u - \Delta u)\|^2$. It is well-known, see for example [46], that we can identify W with the space $\mathcal{E}_0^{SG}$ introduced in section 1.2, moreover there exists a constant such that

$$\|u\|_{H^3}^2 \le C\left(\|u\|^2 + \|\nabla u\|^2 + \|\operatorname{curl}(u - \Delta u)\|^2\right). \tag{4.5}$$

We will shortly denote by $\|u\|_* = \|\operatorname{curl}(u - \alpha\Delta u)\|$. We end this subsection recalling several properties on the nonlinearity of the second grade fluid equations.

Lemma 4.3. *For any smooth, divergence free $\varphi,\ v,\ w$ the following relation holds*

$$\langle \operatorname{curl}\varphi \times v, w\rangle = b(v, \varphi, w) - b(w, \varphi, v). \tag{4.6}$$

Moreover for $u,\ v,\ w$ the following inequalities hold

$$|\langle \operatorname{curl}(u - \alpha\Delta u) \times v, w\rangle| \le C\|u\|_{H^3}\|v\|_V\|w\|_W, \tag{4.7}$$

$$|\langle \operatorname{curl}(u - \alpha\Delta u) \times u, w\rangle| \le C\|u\|_V^2\|w\|_W. \tag{4.8}$$

Therefore there exists a bilinear operator $\hat{B} : W \times V \to W^*$ such that

$$\langle \hat{B}(u, v), w\rangle_{W^*,W} = \langle \mathbf{P}(\operatorname{curl}(u - \alpha\Delta u) \times v), w\rangle \tag{4.9}$$

which satisfies for $u \in V,\ v \in W$

$$\|\hat{B}(v, u)\|_{W^*} \le C\|u\|_V\|v\|_W, \tag{4.10}$$

$$\|\hat{B}(u, u)\|_{W^*} \le C\|u\|_V^2. \tag{4.11}$$

Lastly, for $u \in W,\ v \in V,\ w \in W$

$$\langle \hat{B}(u, v), w\rangle_{W^*,W} = -\langle \hat{B}(u, w), v\rangle_{W^*,W}. \tag{4.12}$$

4.1.2 Well-Posedness of the Systems

Now we are ready to introduce some assumptions on the stochastic part of systems (4.1), (4.2) in order to study the inviscid limit. The assumptions below are redundant for the scope of the current chapter, and we refer to the original results [136], [135] for some more natural set of assumptions. We present the assumptions in this way in order to avoid different settings of the noise when considering different systems, or simply the study of the convergence of this chapter and the fluctuactions of the next one.

Hypothesis 4.4. $W_t = \sum_{k\in\mathcal{K}} \sigma_k W_t^k$ where

- $\mathcal{K}$ is a (possibly countable) set of indexes, $\gamma \geq 2$.
- $\sigma_k \in D(\mathcal{A}^\gamma)$ satisfying
$$\sum_{k\in\mathcal{K}} \|\sigma_k\|^2_{D(\mathcal{A}^\gamma)} < +\infty.$$
- $\{W_t^k\}_{k\in\mathcal{K}}$ is a sequence of real, independent Brownian motions adapted to $\mathcal{F}_t$.

We start defining our notion of solution for (4.1).

Definition 4.5. A stochastic process u^ν is a weak solution of equation (4.1) if

$$u^\nu \in C_{\mathcal{F}}([0,T]\,;H) \cap L^2_{\mathcal{F}}(0,T;V)$$

and for every $\varphi \in \mathsf{D}(\mathcal{A})$, we have

$$\langle u^\nu(t),\varphi\rangle - \int_0^t b(u^\nu(s),\varphi,u^\nu(s))\ ds = \langle u_0^\nu,\varphi\rangle - \nu\int_0^t \langle u^\nu(s),\mathcal{A}\varphi\rangle\ ds + \sqrt{\nu}\sum_{k\in\mathcal{K}}\langle\sigma_k,\varphi\rangle W_t^k,$$

for every $t\in[0,T]\,,\ \mathbb{P}-a.s.$

Under previous assumptions on the coefficient σ_k, equation (4.1) is well posed in the sense of the definition. Indeed the following theorem holds, see [76].

Theorem 4.6. *If $u_0^\nu \in L^2(\mathcal{F}_0,H)$ and Hypothesis 4.4 is satisfied, there exists a unique weak solution of equation (4.1) in the sense of Definition 4.5. Moreover the following relations hold:*

$$\mathbb{E}\left[\|u^\nu(t)\|^2\right] + 2\nu\int_0^t \mathbb{E}\left[\|\nabla u^\nu(s)\|^2\right]\ ds = \mathbb{E}\left[\|u_0^\nu\|^2\right] + t\nu\sum_{k\in\mathcal{K}}\|\sigma_k\|^2, \tag{4.13}$$

$$\mathbb{E}\left[\sup\nolimits_{t\in[0,T]}\|u^\nu(t)\|^2\right] \leq \mathbb{E}\left[\|u_0^\nu\|^2\right] + T\nu\sum_{k\in\mathcal{K}}\|\sigma_k\|^2 + 2\sqrt{\nu\sum_{k\in\mathcal{K}}\mathbb{E}\left[\int_0^T\langle u^\nu(s),\sigma_k\rangle^2 ds\right]}, \tag{4.14}$$

$$\|u^\nu(t)\|^2 + 2\nu\int_0^t\|\nabla u^\nu(s)\|^2 ds = \|u_0^\nu\|^2 + t\nu\sum_{k\in\mathcal{K}}\|\sigma_k\|^2 + 2\sqrt{\nu}\sum_{k\in\mathcal{K}}\int_0^t\langle u^\nu(s),\sigma_k\rangle dW_s^k. \tag{4.15}$$

We call (4.13) the energy equality for the Navier-Stokes equations in the following, instead we call equation (4.15) Itô formula.

Concerning (4.2) we start with few comments. First we needed to add the additional pressure term $\sum_{k\in\mathcal{K}} \nabla \tilde{\mathsf{P}}^{\alpha}_k \circ dW^k_t$, the so-called turbulent pressure, in the system in order to deal with the fact that $\sum_{k\in\mathcal{K}} \sigma_k \cdot \nabla \mathsf{u}^{\alpha} \circ dW^k_t$ is not divergence free, therefore an additional martingale term orthogonal to H must be added to make the system feasible. Secondly we do not deal actually with equation (4.2), but we rewrite it, formally in Itô form, then we deal with the equation obtained. As an intermediate step, introducing the Stokes operator, equation (4.2) can be rewritten as

$$\begin{cases} d(\mathsf{u}^{\alpha}+\alpha\mathcal{A}\mathsf{u}^{\alpha}) = -(\nu\mathcal{A}\mathsf{u}^{\alpha}+\mathbf{P}(\mathrm{curl}(\mathsf{v}^{\alpha})\times\mathsf{u}^{\alpha}))dt+\sum_{k\in\mathcal{K}}\mathbf{P}(\sigma_k\cdot\nabla\mathsf{u}^{\alpha})\circ dW^k_t \\ \mathsf{v}^{\alpha}=\mathsf{u}^{\alpha}-\alpha\Delta\mathsf{u}^{\alpha} \\ \mathsf{u}^{\alpha}(0)=\mathsf{u}^{\alpha}_0, \end{cases} \tag{4.16}$$

therefore the corresponding Itô form is

$$\begin{cases} d(\mathsf{u}^{\alpha}+\alpha\mathcal{A}\mathsf{u}^{\alpha}) & = -(\nu\mathcal{A}\mathsf{u}^{\alpha}+\mathbf{P}(\mathrm{curl}(\mathsf{v}^{\alpha})\times\mathsf{u}^{\alpha}))dt+\sum_{k\in\mathcal{K}}\mathbf{P}(\sigma_k\cdot\nabla\mathsf{u}^{\alpha})dW^k_t \\ & \quad +\frac{1}{2}\sum_{k\in\mathcal{K}}\mathbf{P}(\sigma_k\cdot\nabla((I+\alpha\mathcal{A})^{-1}\mathbf{P}(\sigma_k\cdot\nabla\mathsf{u}^{\alpha})))dt \\ v=u-\alpha\Delta u \\ u(0)=u_0. \end{cases} \tag{4.17}$$

Indeed each of the Stratonovich integrals in equation (4.16) can be rewritten, thanks to the Stratonovich-Itô corrector associated to previous equation, in the following form:

$$\begin{aligned} \mathbf{P}(\sigma_k\cdot\nabla\mathsf{u}^{\alpha})\circ dW^k_t &= \mathbf{P}(\sigma_k\cdot\nabla\mathsf{u}^{\alpha})dW^k_t+\frac{1}{2}d[\mathbf{P}(\sigma_k\cdot\nabla\mathsf{u}^{\alpha}),W^k]_t \\ &= \mathbf{P}(\sigma_k\cdot\nabla\mathsf{u}^{\alpha})dW^k_t+\frac{1}{2}\mathbf{P}\left(\sigma_k\cdot\nabla d[\mathsf{u}^{\alpha},W^k]_t\right) \\ &= \mathbf{P}(\sigma_k\cdot\nabla\mathsf{u}^{\alpha})dW^k_t \\ &\quad +\frac{1}{2}\mathbf{P}\left(\sigma_k\cdot\nabla d\left[\int_0^{\cdot}(I+\alpha\mathcal{A})^{-1}\sum_{j\in\mathcal{K}}P(\sigma_j\cdot\nabla\mathsf{u}^{\alpha})dW^j_s,W^k\right]_t\right) \\ &= \mathbf{P}(\sigma_k\cdot\nabla\mathsf{u}^{\alpha})dW^k_t+\frac{1}{2}\mathbf{P}\left(\sigma_k\cdot\nabla\left((I+\alpha\mathcal{A})^{-1}\mathbf{P}(\sigma_k\cdot\nabla\mathsf{u}^{\alpha})\right)\right)dt. \end{aligned} \tag{4.18}$$

In order to shorter the notation, we denote by $F(u)=\frac{1}{2}\sum_{k\in\mathcal{K}}\mathbf{P}(\sigma_k\cdot\nabla((I+\alpha\mathcal{A})^{-1}\mathbf{P}(\sigma_k\cdot\nabla\mathsf{u})))$ and $G^k(u)=\mathbf{P}(\sigma_k\cdot\nabla u)$. By Lemma 4.16 below

$$F\in\mathcal{L}(V;H\cap H^1(\mathcal{O};\mathbb{R}^2)),\quad G^k\in\mathcal{L}(V;H)$$

and we are ready to introduce our notion of weak solution for (4.17).

Definition 4.7. A stochastic process with weakly continuous trajectories with values in W is a weak solution of equation (4.17) if

$$\mathsf{u}^\alpha \in L^p(\Omega, \mathcal{F}, \mathbb{P}; L^\infty(0,T;W)), \ \forall p \geq 2$$

and $\mathbb{P} - a.s.$ for every $t \in [0,T]$ and $\varphi \in W$ we have

$$\begin{aligned}&\langle \mathsf{u}^\alpha(t) - \mathsf{u}^\alpha_0, (I + \alpha\mathcal{A})\varphi\rangle + \int_0^t \nu\langle \nabla \mathsf{u}^\alpha(s), \nabla\varphi\rangle + \langle \operatorname{curl}(\mathsf{u}^\alpha(s) + \alpha\Delta \mathsf{u}^\alpha(s)) \times \mathsf{u}^\alpha(s), \varphi\rangle ds\\ &= \int_0^t \langle F(\mathsf{u}^\alpha(s)), \varphi\rangle ds + \sum_{k\in\mathcal{K}} \int_0^t \langle G^k(\mathsf{u}^\alpha(s)), \varphi\rangle dW^k_s.\end{aligned}$$

The well-posedness and some a priori estimates stable in the inviscid limit for (4.17) are nontrivial facts investigated in [135, Section 3, 4]. Before stating the well-posedness result for (4.17) we introduce some assumptions on the coefficients of the system in order to obtain the stable apriori estimates.

Hypothesis 4.8. The following hold:

- $\nu = O(\alpha)$.
- $\bar{u}_0 \in H^s(\mathcal{O}; \mathbb{R}^2) \cap H$ for some $s \geq 3$.
-

$$\mathbb{E}\left[\|\mathsf{u}^\alpha_0 - \bar{u}_0\|^2\right] \to 0; \tag{4.19}$$

$$\mathbb{E}\left[\alpha\|\nabla \mathsf{u}^\alpha_0\|^2\right] = o(1); \tag{4.20}$$

$$\mathbb{E}\left[\alpha^3\|\mathsf{u}^\alpha_0\|^2_{H^3(\mathcal{O};\mathbb{R}^2)}\right] = O(1). \tag{4.21}$$

Remark 4.9. The assumptions above are not meaningless, indeed if $\bar{u}_0 \in H \cap H^1(\mathcal{O}; \mathbb{R}^2)$, the existence of a family u^α_0 satisfying equations (4.19),(4.20),(4.21) is guaranteed by [132, Proposition 1].

For the scope of this chapter we limit to state the following well-posedness result, referring to [135] for its proof.

Theorem 4.10. *If $\mathsf{u}^\alpha_0 \in \cap_{p\geq 2} L^p(\mathcal{F}_0, W)$ and Hypothesis 4.4 holds, equation (4.17) has a unique solution in the sense of Definition 4.7. Moreover, almost surely, the stochastic process u^α has V continuous paths and*

$$\|\mathsf{u}^\alpha(t)\|^2 + \alpha\|\nabla \mathsf{u}^\alpha(t)\|^2 + \nu\int_0^t \|\nabla \mathsf{u}^\alpha(s)\|^2 ds = \|\mathsf{u}^\alpha_0\|^2 + \alpha\|\nabla \mathsf{u}^\alpha_0\|^2 \quad \mathbb{P} - a.s. \tag{4.22}$$

If, in addition, Hypothesis 4.8 is satisfied, then

$$\mathbb{E}\left[\alpha^3 \sup_{t\in[0,T]} \|\mathsf{u}^\alpha(t)\|^2_{H^3}\right] = O(1). \tag{4.23}$$

We end the section recalling some well-known results about existence and uniqueness of solutions for the deterministic Euler equations.

Definition 4.11. Given $\bar{u}_0 \in H$, we say that $\bar{u} \in C([0,T];H)$ is a weak solution of equation (4.3) if for every $\varphi \in C_0^\infty([0,T]\times\mathcal{O}) \cap C^1([0,T];H)$

$$\langle \bar{u}(t), \varphi(t)\rangle = \langle \bar{u}_0, \varphi(0)\rangle + \int_0^t \langle \bar{u}(s), \partial_s\varphi(s)\rangle ds + \int_0^t b(\bar{u}(s), \varphi(s), \bar{u}(s))ds$$

for every $t \in [0,T]$ and the energy inequality

$$\|\bar{u}(t)\|^2 \leq \|\bar{u}_0\|^2$$

holds.

The well-posedness of (4.3) in regular spaces is a classical result, see for example [112], [166], [110], [130].

Theorem 4.12. *If $\bar{u}_0 \in C^{1,\varepsilon}(\bar{\mathcal{O}}) \cap H$, then there exist $\bar{u}, \bar{P}$ classical solutions of equation (4.3). Moreover, $\bar{u}$, $\nabla\bar{u}$, $\bar{P}$, $\nabla\bar{P} \in C([0,T]\times\bar{\mathcal{O}})$ and*

$$\|\bar{u}(t)\| = \|\bar{u}_0\|, \ \forall t \in [0,T].$$

In such regularity class, $\bar{u}$ is unique and $\bar{P}$ is unique up to an arbitrary function of t which can be added to $\bar{P}$.
If moreover $\bar{u}_0 \in H^s(\mathcal{O};\mathbb{R}^2) \cap H$, for some $s \geq 3$, then

$$\bar{u} \in C([0,T];H^s(\mathcal{O};\mathbb{R}^2)) \cap C^1([0,T];H^{s-1}(\mathcal{O};\mathbb{R}^2)).$$

The following theorem of [130] provides stability estimates for the solutions of (4.3). Moreover it guarantees existence and uniqueness for the solutions of (4.3) in a extended class.

Theorem 4.13. *If u is a weak solution of the Euler equations with initial condition $u_0 \in H$ and $\bar{u}$ is the unique weak solution of the Euler equations with initial condition $\bar{u}_0 \in H \cap C^{1,\varepsilon}(\bar{\mathcal{O}})$, then*

$$\|u(t) - \bar{u}(t)\|^2 \leq e^{2t\|\nabla\bar{u}\|_{L^\infty([0,T]\times\mathcal{O})}}\|u_0 - \bar{u}_0\|^2.$$

Calling

$$O_n = \left\{u_0 \in H: \ \exists \bar{u}_0 \in H \cap C^{1,\varepsilon}(\bar{\mathcal{O}}), \ \|u_0 - \bar{u}_0\| < \frac{1}{n}e^{-3T\|\nabla\bar{u}\|_{L^\infty([0,T]\times\mathcal{O})}}\right\},$$

where $\bar{u}$ is the solution of the Euler equations with initial condition $\bar{u}_0$, then for each $u_0 \in \bigcap_{n\geq 1} O_n =: \tilde{O}$ there exists a unique $u \in C([0,T];H)$ weak solution of the Euler equations with initial condition u_0. Moreover the energy equality

$$\|u(t)\|^2 = \|u_0\|^2$$

holds.

For each $\bar{u}_0 \in \tilde{O}$ we say that the sequence $\{\bar{u}_0^m\}_{m\in\mathbb{N}}$ approximates $\bar{u}_0$ in the sense of Theorem 4.13 if $\bar{u}_0^m \in H \cap C^{1,\varepsilon}(\bar{\mathcal{O}})$ and

$$\|\bar{u}_0 - \bar{u}_0^m\| < \frac{1}{m} e^{-3T\|\nabla \bar{u}^m\|_{L^\infty([0,T]\times\mathcal{O})}}$$

where $\bar{u}^m$ is the solution of the Euler equations with initial condition $\bar{u}_0^m$.

4.1.3 Some Tools for Studying the Inviscid Limit

We start this section providing the classical tool in order to study the inviscid limit in bounded domains: the boundary layer corrector. Indeed, given $\bar{u}$ solution of (4.3), $\bar{u} \notin V$, however it is possible to find a vector field v supported in a boundary layer, with the same regularity of $\bar{u}$ such that $\bar{u} - v \in V$. The support of v can be made arbitrarily small, this guarantees also some estimates on the norms of v. The following proposition of [111] describe in a quantitative way the content of the previous sentences.

Proposition 4.14. *Under the assumptions of Theorem 4.12:*

- *it exists a smooth skew-symmetric matrix a such that $\bar{u} = \operatorname{div} a$ on $\partial\mathcal{O}$ and $a = 0$ on $\partial\mathcal{O}$;*
- *let $\xi : \mathbb{R}^+ \to \mathbb{R}^+$ be a smooth function such that $\xi(0) = 1,\ \xi(r) = 0$ if $r \geq 1$,*

  $$z : \mathcal{O} \to \mathbb{R}^+,\ z(x) = \xi(\rho/\delta) \quad \textit{with} \quad \rho = dist(x, \partial\mathcal{O})$$

  and δ a parameter which goes to 0 when ν goes to 0. Let, moreover, $v = \operatorname{div}(za)$. Then,

  $$\bar{u} - v \in C([0,T]\,; V) \cap C^1([0,T]\,; H).$$

  $\{x \in \mathcal{O} : \rho(x) \leq \delta\}$ is the boundary strip of width δ that we denote by Γ_δ;
- *the following estimates hold true*

  $$\begin{aligned} &\|v(t)\|_{L^\infty} \leq C,\ \|v(t)\| \leq C\delta^{\frac{1}{2}},\ \|\partial_t v(t)\| \leq C\delta^{\frac{1}{2}}, \\ &\|\nabla v(t)\|_{L^\infty} \leq C\delta^{-1},\ \|\nabla v(t)\| \leq C\delta^{-1/2},\ \|\rho\nabla v(t)\|_{L^\infty} \leq C, \\ &\|\rho^2\nabla v(t)\|_{L^\infty} \leq C\delta,\ \|\rho\nabla v(t)\| \leq C\delta^{\frac{1}{2}},\ \|\partial_t\nabla v(t)\| \leq C\delta^{-1/2}. \end{aligned}$$

  where the coefficient C depends from $\bar{u}$ and it is independent from t and δ.

In order to exploit completely the strength of the boundary layer corrector, we need a different notion of weak formulation satisfied by u^ν that is clarified by the following lemma.

Lemma 4.15. *Under the same assumptions of Theorem 4.6, if u^ν is a weak solution of equation (4.1), then for each $\varphi \in C([0,T];V) \cap C^1([0,T];H)$*

$$\begin{aligned}\langle u^\nu(t), \varphi(t)\rangle &= \langle u_0^\nu, \varphi(0)\rangle + \int_0^t \langle u^\nu(s), \partial_s\varphi(s)\rangle \, ds \\ &- \nu \int_0^t \langle \mathcal{A}^{1/2} u^\nu(s), \mathcal{A}^{\frac{1}{2}}\varphi(s)\rangle \, ds + \int_0^t b(u^\nu(s), \varphi(s), u^\nu(s)) \, ds \\ &+ \sqrt{\nu} \sum_{k\in\mathcal{K}} \langle \sigma_k, \varphi(t)\rangle W_t^k - \sqrt{\nu} \sum_{k\in\mathcal{K}} \int_0^t \langle \sigma_k, \varphi(s)\rangle W_s^k \, ds\end{aligned}$$

for every $t \in [0,T]$, $\mathbb{P}-a.s.$

Proof. Thanks to the regularity of the weak solution u^ν, by density we have that for each $\varphi \in V$

$$\begin{aligned}\langle u^\nu(t), \varphi\rangle - \int_0^t b(u^\nu(s), \varphi, u^\nu(s)) \, ds &= \langle u_0^\nu, \varphi\rangle + \sqrt{\nu} \sum_{k\in\mathcal{K}} \langle \sigma_k, \varphi\rangle W_t^k \\ &- \nu \int_0^t \langle \mathcal{A}^{\frac{1}{2}} u^\nu(s), \mathcal{A}^{\frac{1}{2}}\varphi\rangle \, ds,\end{aligned}$$

for every $t \in [0,T]$, $\mathbb{P}-a.s.$ Therefore, due to previous relation and the regularity properties of u^ν and $\{W^k\}_{k\in\mathcal{K}}$, it exists a measurable set $N \subseteq \Omega$ such that

1. $\mathbb{P}(N) = 0$.
2. For each $\omega \in N^c$, $u^\nu(t)$ and $\sum_{k\in\mathcal{K}} \sigma_k W_t^k$ have continuous trajectories taking values in H.
3. $$\begin{aligned}\langle u^\nu(t), \varphi\rangle - \int_0^t b(u^\nu(s), \varphi, u^\nu(s)) \, ds &= \langle u_0^\nu, \varphi\rangle + \sqrt{\nu} \sum_{k\in\mathcal{K}} \langle \sigma_k, \varphi\rangle W_t^k \\ &- \nu \int_0^t \langle \mathcal{A}^{\frac{1}{2}} u^\nu(s), \mathcal{A}^{\frac{1}{2}}\varphi\rangle \, ds \\ &\forall t \in [0,T], \ \varphi \in V, \ \omega \in N^c.\end{aligned}$$
4. $$\sup_{t\in[0,T]} \|u^\nu(t)\| + \int_0^T \|u^\nu(s)\|_V^2 \, ds + \sup_{t\in[0,T]} \left\| \sum_{k\in\mathcal{K}} \sigma_k W_t^k \right\| \le C(\omega,\nu) < +\infty \ \forall \omega \in N^c,$$

   where $C(\omega,\nu)$ is a constant possibly depending from $\omega \in \Omega$, $\nu > 0$.

The relation stated in the lemma then holds for each $\omega \in N^c$. Indeed, let $\varphi(t) \in C^1([0,T]\,;H)\cap C([0,T]\,;V)$. Let, moreover, $\pi = \{0 = t_0 < t_1 < \cdots < T_n = T\}$ be a partition of $[0,T]$. Thus, using the identities

$$\langle u^\nu(t_{i+1}), \varphi(t_{i+1})\rangle - \langle u^\nu(t_{i+1}), \varphi(t_i)\rangle = \int_{t_i}^{t_{i+1}} \langle u^\nu(t_{i+1}), \partial_s\varphi(s)\rangle \, ds,$$

$$\langle \sigma_k W^k_{t_{i+1}}, \varphi(t_{i+1})\rangle - \langle \sigma_k W^k_{t_{i+1}}, \varphi(t_i)\rangle = \int_{t_i}^{t_{i+1}} \langle \sigma_k W^k_{t_{i+1}}, \partial_s\varphi(s)\rangle \, ds,$$

we get

$$\begin{aligned}
\langle u^\nu(t_{i+1}), \varphi(t_{i+1})\rangle = \langle u^\nu(t_i), \varphi(t_i)\rangle &- \int_{t_i}^{t_{i+1}} \langle \mathcal{A}^{\frac{1}{2}} u^\nu(s), \mathcal{A}^{\frac{1}{2}}\varphi(t_i)\rangle \, ds \\
&+ \int_{t_i}^{t_{i+1}} b(u^\nu(s), \varphi(t_i), u^\nu(s)) \, ds \\
&+ \int_{t_i}^{t_{i+1}} \langle u^\nu(t_{i+1}), \partial_s\varphi(s)\rangle \, ds \\
&- \sqrt{\nu} \sum_{k\in\mathcal{K}} \int_{t_i}^{t_{i+1}} \langle \sigma_k W^k(t_{i+1}), \partial_s\varphi(s)\rangle \, ds \\
&+ \sqrt{\nu} \sum_{k\in\mathcal{K}} \Big(\langle \sigma_k W^k_{t_{i+1}}, \varphi(t_{i+1})\rangle - \langle \sigma_k W^k_{t_i}, \varphi(t_i)\rangle \Big).
\end{aligned}$$

It implies

$$\begin{aligned}
\langle u^\nu(T), \varphi(T)\rangle = \langle u^\nu_0, \varphi(0)\rangle &- \int_0^T \langle \mathcal{A}^{\frac{1}{2}} u^\nu(s), \mathcal{A}^{\frac{1}{2}}\varphi(s^-_\pi)\rangle \, ds \\
&+ \int_0^T b(u^\nu(s), \varphi(s^-_\pi), u^\nu(s)) \, ds \\
&+ \int_0^T \langle u^\nu(s^+_\pi), \partial_s\varphi(s)\rangle \, ds - \sqrt{\nu} \sum_{k\in\mathcal{K}} \int_0^T \langle \sigma_k W^k_{s^+_\pi}, \partial_s\varphi(s)\rangle \, ds \\
&+ \sqrt{\nu} \sum_{k\in\mathcal{K}} \Big(\langle \sigma_k W^k_T, \varphi(T)\rangle - \langle \sigma_k W^k_0, \varphi(0)\rangle \Big).
\end{aligned}$$

where $s^-_\pi(s) = t_i$ if $s \in [t_i, t_{i+1}]$ and $s^+_\pi(s) = t_{i+1}$ if $s \in [t_i, t_{i+1}]$. Taking the limit over a sequence of partitions π_l with size going to zero, we get, thanks to the regularity of u, φ,

$\sum_{k\in\mathcal{K}}\sigma_k W^k$ and dominated convergence theorem,

$$\begin{aligned}\langle u^\nu(T),\varphi(T)\rangle &= \langle u_0^\nu,\varphi(0)\rangle - \int_0^T \langle \mathcal{A}^{\frac{1}{2}}u^\nu(s),\mathcal{A}^{\frac{1}{2}}\varphi(s)\rangle\, ds\\ &+\int_0^T b(u^\nu(s),\varphi(s),u^\nu(s))\, ds\\ &+\int_0^T \langle u^\nu(s),\partial_s\varphi(s)\rangle\, ds - \sqrt{\nu}\sum_{k\in\mathcal{K}}\int_0^T\langle\sigma_k W_s^k,\partial_s\varphi(s)\rangle\, ds\\ &+\sqrt{\nu}\sum_{k\in\mathcal{K}}\Big(\langle\sigma_k W_T^k,\varphi(T)\rangle - \langle\sigma_k W_0^k,\varphi(0)\rangle\Big).\end{aligned}$$

The assumptions for applying dominated convergence theorem being satisfied for each $\omega\in N^c$. Indeed, thanks to property 2. we have

$$\begin{aligned}&b(u^\nu(s),\varphi(s_{\pi_l}^-),u^\nu(s))+\langle u^\nu(s_{\pi_l}^+),\partial_s\varphi(s)\rangle\ -\sqrt{\nu}\sum_{k\in\mathcal{K}}\langle\sigma_k W^k_{s^+_{\pi_l}},\partial_s\varphi(s)\rangle\\ &\overset{|\pi_l|\to 0}{\to} b(u^\nu(s),\varphi(s),u^\nu(s))+\langle u^\nu(s),\partial_s\varphi(s)\rangle\ -\sqrt{\nu}\sum_{k\in\mathcal{K}}\langle\sigma_k W^k_s,\partial_s\varphi(s)\rangle\end{aligned}$$

and by property 4.

$$\begin{aligned}&\left|b(u^\nu(s),\varphi(s_{\pi_l}^-),u^\nu(s))+\langle u^\nu(s_{\pi_l}^+),\partial_s\varphi(s)\rangle\ -\sqrt{\nu}\sum_{k\in\mathcal{K}}\langle\sigma_k W^k_{s^+_{\pi_l}},\partial_s\varphi(s)\rangle\right|\\ &\le C_{\omega,\nu,\varphi}\left(\|u^\nu\|_V^2+\sup_{t\in[0,T]}\|u^\nu(t)\|+\sup_{t\in[0,T]}\left\|\sum_{k\in\mathcal{K}}\sigma_k W_t^k\right\|\right)\in L^1(0,T),\end{aligned}$$

where $C_{\omega,\nu,\varphi}$ is a constant depending by ω,ν,φ. The argument applies to a generic $t\in[0,T]$, hence we have the thesis. □

As anticipated in the previous section, we need to study the regularity of the operators G^k and F. This is the object of the following lemma.

Lemma 4.16. *Let $h\in H,\ u\in V,\ w\in V\cap H^2$, then*

$$\|G^k(u)\|\le\|\sigma_k\|_{L^\infty}\|\nabla u\|, \tag{4.24}$$

$$\|\nabla G^k(w)\|\le C\|\sigma_k\|_{W^{1,\infty}}\|w\|_{H^2}, \tag{4.25}$$

$$\|(I+\alpha\mathcal{A})^{-1}h\|\le\|h\|, \tag{4.26}$$

$$\|\mathcal{A}^{1/2}(I+\alpha\mathcal{A})^{-1}h\|\le\frac{1}{2\sqrt{\alpha}}\|h\|, \tag{4.27}$$

$$\|\mathcal{A}(I+\alpha\mathcal{A})^{-1}h\|\le\frac{1}{\alpha}\|h\|, \tag{4.28}$$

$$\|(I+\alpha\mathcal{A})^{-1}(\mathbf{P}(\sigma_k\cdot\nabla w))\|_W\le C\|\sigma_k\|_{W^{1,\infty}}\|w\|_{H^2}. \tag{4.29}$$

Therefore, if $u \in V$ the following inequalities hold true

$$\|(I+\alpha\mathcal{A})^{-1}\mathbf{P}(\sigma_k \cdot \nabla u)\| \leq \|\sigma_k\|_{L^\infty}\|\nabla u\|, \tag{4.30}$$

$$\|\nabla(I+\alpha\mathcal{A})^{-1}\mathbf{P}(\sigma_k \cdot \nabla u)\| \leq \frac{\|\sigma_k\|_{L^\infty}}{2\sqrt{\alpha}}\|\nabla u\|, \tag{4.31}$$

$$\|\mathbf{P}(\sigma_k \cdot \nabla((I+\alpha\mathcal{A})^{-1}\mathbf{P}(\sigma_k \cdot \nabla u)))\| \leq \frac{\|\sigma_k\|^2_{L^\infty}}{2\sqrt{\alpha}}\|\nabla u\|. \tag{4.32}$$

$$\|\mathbf{P}(\sigma_k \cdot \nabla((I+\alpha\mathcal{A})^{-1}\mathbf{P}(\sigma_k \cdot \nabla u)))\|_{H^1} \leq \frac{C\|\sigma_k\|_{L^\infty}\|\sigma_k\|_{W^{1,\infty}}\|\nabla u\|}{\alpha}. \tag{4.33}$$

In particular,

$$G^k \in \mathcal{L}(V;H), \quad F \in \mathcal{L}(V; H \cap H^1(\mathcal{O};\mathbb{R}^2)).$$

Proof. Inequalities (4.24), (4.25) are trivial. Indeed, by Lemma 4.1 it holds

$$\|G^k(u)\| = \|P(\sigma_k \cdot \nabla u)\| \leq \|\sigma_k \cdot \nabla u\| \leq \|\sigma_k\|_{L^\infty}\|\nabla u\|.$$

$$\begin{aligned}\|\nabla G^k(w)\| &= \|\nabla \mathbf{P}(\sigma_k \cdot \nabla w)\| \\ &\leq C\|\sigma_k \cdot \nabla w\|_{H^1} \\ &\leq C\|\sigma_k\|_{W^{1,\infty}}\|\nabla w\|_{H^1} \\ &\leq C\|\sigma_k\|_{W^{1,\infty}}\|w\|_{H^2}.\end{aligned}$$

In order to prove inequalities (4.26), (4.27), (4.28) we exploit the Fourier decomposition $h = \sum_{i\in\mathbb{N}}\langle h, e_i\rangle e_i$. Therefore it holds

$$\|(I+\alpha\mathcal{A})^{-1}h\|^2 = \sum_{i\in\mathbb{N}}\frac{\langle h, e_i\rangle^2}{(1+\alpha\lambda_i)^2} \leq \|h\|^2,$$

$$\|\mathcal{A}^{1/2}(I+\alpha\mathcal{A})^{-1}h\|^2 = \sum_{i\in\mathbb{N}}\frac{\lambda_i}{(1+\alpha\lambda_i)^2}\langle h, e_i\rangle^2 \leq \frac{1}{4\alpha}\|h\|^2,$$

$$\|\mathcal{A}(I+\alpha\mathcal{A})^{-1}h\|^2 = \sum_{i\in\mathbb{N}}\frac{\lambda_i^2}{(1+\alpha\lambda_i)^2}\langle h, e_i\rangle^2 \leq \frac{1}{\alpha^2}\|h\|^2.$$

For what concerns inequality (4.29), by definition of the norm in the space W it holds

$$\begin{aligned}\|(I+\alpha\mathcal{A})^{-1}(\mathbf{P}(\sigma_k \cdot \nabla w))\|^2_W &= \|(I+\alpha\mathcal{A})^{-1/2}(\mathbf{P}(\sigma_k \cdot \nabla w))\|^2 \\ &\quad + \|\operatorname{curl}((I-\alpha\Delta)(I+\alpha\mathcal{A})^{-1}(\mathbf{P}(\sigma_k \cdot \nabla w)))\|^2.\end{aligned}$$

For the first term in the relation above we have

$$\|(I+\alpha\mathcal{A})^{-1/2}(\mathbf{P}(\sigma_k\cdot\nabla w))\|^2 \leq \|\sigma_k\|^2_{L^\infty}\|\nabla w\|^2, \tag{4.34}$$

while for the second one

$$\|\mathrm{curl}\left((I-\alpha\Delta)(I+\alpha\mathcal{A})^{-1}(\mathbf{P}(\sigma_k\cdot\nabla w))\right)\| = \|\mathrm{curl}(\mathbf{P}(\sigma_k\cdot\nabla w))\| \leq C\|\sigma_k\|_{W^{1,\infty}}\|w\|_{H^2} \tag{4.35}$$

Combining (4.34) and (4.35), inequality (4.29) follows.

Combining relation (4.24) with relations (4.26) and (4.27), inequalities (4.30) and (4.31) follow immediately. Let us now prove equation (4.32). By Hölder's inequality and relation (4.25) we have

$$\begin{aligned}\|\mathbf{P}(\sigma_k\cdot\nabla((I+\alpha\mathcal{A})^{-1}\mathbf{P}(\sigma_k\cdot\nabla u)))\| &\leq \|\sigma_k\cdot\nabla((I+\alpha\mathcal{A})^{-1}P(\sigma_k\cdot\nabla u))\|\\ &\leq \|\sigma_k\|_{L^\infty}\|\nabla((I+\alpha\mathcal{A})^{-1}\mathbf{P}(\sigma_k\cdot\nabla u))\|\\ &\leq \frac{\|\sigma_k\|^2_{L^\infty}}{2\sqrt{\alpha}}\|\nabla u\|.\end{aligned}$$

For what concerns the last one, by Lemma 4.1, the equivalence of the H^2 norm and the norm in $\mathsf{D}(\mathcal{A})$ for vector fields in $\mathsf{D}(\mathcal{A})$ and relations (4.28) it holds

$$\begin{aligned}\|\mathbf{P}(\sigma_k\cdot\nabla((I+\alpha\mathcal{A})^{-1}\mathbf{P}(\sigma_k\cdot\nabla u)))\|_{H^1} &\leq C\|\sigma_k\cdot\nabla((I+\alpha\mathcal{A})^{-1}\mathbf{P}(\sigma_k\cdot\nabla u))\|_{H^1}\\ &\leq C\|\sigma_k\|_{W^{1,\infty}}\|\nabla((I+\alpha\mathcal{A})^{-1}\mathbf{P}(\sigma_k\cdot\nabla u))\|_{H^1}\\ &\leq C\|\sigma_k\|_{W^{1,\infty}}\|(I+\alpha\mathcal{A})^{-1}\mathbf{P}(\sigma_k\cdot\nabla u)\|_{H^2}\\ &\leq C\|\sigma_k\|_{W^{1,\infty}}\|\mathcal{A}(I+\alpha\mathcal{A})^{-1}\mathbf{P}(\sigma_k\cdot\nabla u)\|\\ &\leq \frac{C}{\alpha}\|\sigma_k\|_{W^{1,\infty}}\|\mathbf{P}(\sigma_k\cdot\nabla u)\|\\ &\leq \frac{C\|\sigma_k\|_{L^\infty}\|\sigma_k\|_{W^{1,\infty}}\|\nabla u\|}{\alpha}.\end{aligned}$$

□

Lastly we recall two technical tools used in the proof of Theorem 4.22. We refer to [87] for the proof of the interpolation inequality and to [156] for the proof of the stochastic Grönwall's lemma.

Theorem 4.17. *Each function $f\in H^2$ satisfies the following inequality:*

$$\|f\|_{H^1} \leq C\|f\|^{1/2}\|f\|^{1/2}_{H^2}. \tag{4.36}$$

Theorem 4.18. *Let $Z(t)$ and $H(t)$ be continuous, nonnegative, adapted processes, $\psi(t)$ a nonnegative deterministic function and $M(t)$ a continuous local martingale such that*

$$Z(t) \le \int_0^t \psi(s)Z(s)ds + M(t) + H(t) \quad \forall t \in [0,T].$$

Then $Z(t)$ satisfies the following inequality

$$\mathbb{E}[Z(t)] \le exp\left(\int_0^t \psi(s)ds\right)\mathbb{E}\left[\sup_{r\in[0,s]} H(s)\right]. \tag{4.37}$$

4.2 Main results

Now we are ready to state the main results of this chapter.
Since the stochastic forcing appearing in equation (4.1) goes to 0 as $\nu \to 0$, we assume that the external force in the Euler equations is identically 0, i.e. (4.3). Theorem 4.19 is a generalization of the results in [111] to this stochastic framework and also the idea of the proof is similar. In Theorem 4.20 we consider a wider set of initial conditions and provide some sufficient conditions for the validity of the inviscid limit in this framework. As it would be clear from the computations in subsection 4.3.2, its proof follows by a combinations of the argument of Theorem 4.20 and some ideas related to Theorem 4.13.

Theorem 4.19. *If $\bar{u}_0 \in C^{1,\varepsilon}(\bar{\mathcal{O}})$ and under previous assumptions on u_0^ν and σ_k, if*

$$\lim_{\nu\to 0}\mathbb{E}\left[\|u_0^\nu - \bar{u}_0\|^2\right] = 0,$$

then the following conditions are equivalent:

1. $\lim_{\nu\to 0}\mathbb{E}\left[\sup_{t\in[0,T]}\|u^\nu(t) - \bar{u}(t)\|^2\right] = 0.$
2. $u^\nu(t) \rightharpoonup \bar{u}(t)$ *in* $L^2(\Omega\times\mathcal{O})$ *for each* $t \in [0,T]$.
3. $\lim_{\nu\to 0}\nu\int_0^T \mathbb{E}\left[\|\nabla u^\nu(t)\|^2\right]\, dt = 0.$
4. $\lim_{\nu\to 0}\nu\int_0^T \mathbb{E}\left[\|\nabla u^\nu(t)\|^2_{L^2(\Gamma_{c\nu})}\right]\, dt = 0.$

Theorem 4.20. *Assume $\bar{u}_0 \in \tilde{O}$, $u_0^n \in L^2(\mathcal{F}_0, H)$, $\lim_{n\to+\infty}\mathbb{E}\left[\|u_0^n - \bar{u}_0\|^2\right] = 0$. Let $\bar{u}$ be the solution of the Euler equations with initial condition $\bar{u}_0$, u^n be the solution of the stochastic Navier-Stokes equations with viscosity ν_n and initial condition u_0^n. If*

$$\lim_{n\to+\infty}\nu_n = 0, \quad \lim_{n\to+\infty}\nu_n\int_0^T \mathbb{E}\left[\|\nabla u^n(t)\|^2_{L^2(\Gamma_{c\nu_n})}\right]\, dt = 0,$$

then

$$\lim_{n\to+\infty}\mathbb{E}\left[\sup_{t\in[0,T]}\|u^n(t) - \bar{u}(t)\|^2\right] = 0.$$

Theorem 4.19 means that if the convergence does not take place, the energy dissipation within the boundary layer of width $c\nu$ remains finite as $\nu \to 0$. This suggests that something violent must have happened. As discussed in section 1.5, this possibility is linked with the phenomenon of the anomalous dissipation, whose validity in the 2D case with domains with boundaries being a open problems also in the deterministic framework. Due to Theorem 1.4 and Theorem 4.19 the validity of the inviscid limit is equivalent to the Kolmogorov's zeroth law of turbulence. Therefore, unconditioned results on the inviscid limit in bounded domains are in contrast to the latter. In particular, the results of [13], [133], [154] provide specific deterministic frameworks in bounded domains where no anomalous dissipations happens. We will come back on this topic in the next chapter, when providing an explicit stochastic framework related to [133] where no anomalous dissipation occurs and, therefore, the validity of the inviscid limit can be proven.

After [136], following the same ideas of Theorem 4.19, similar results has been proven. In particular, conditioned results for the validity of the inviscid limit under more general stochastic forcings than additive noise have been proven in [94]. In [178] the case of additive noise has been considered in a very special case, namely if the noise does not scale with respect to the viscosity, therefore the limit Euler equation remains a stochastic partial differential equation.

Remark 4.21. Theorem 4.20 is new also in the deterministic framework, namely considering $\sigma_k = 0 \ \forall \ k \in \mathcal{K}$. More discussions on the deterministic framework can be found in [136, Section 5].

Concerning the convergence of (4.2), according to the analysis started in [88] discussed shortly in subsection 1.6.2, the influence of the transport noise in the limit behaviour is related to the ℓ^2 norm of its coefficients, therefore we expect that the solution of equation (4.2) converges to the solution of the Euler equations with no forcing terms. This is, indeed, the case and we can prove the following result.

Theorem 4.22. *Under the same assumptions of Theorem 4.10, calling u^α the solution of (4.2) and $\bar{u}$ the unique smooth solution of (4.3), it holds*

$$\lim_{\alpha\to 0} \mathbb{E}\left[\sup_{t\in[0,T]} \|\mathsf{u}^\alpha(t) - \bar{u}(t)\|^2\right] = 0.$$

An analogous statement can be proven also in the case of additive noise, see Theorem 4.37 below, we prefer to concentrate to Theorem 4.22 in section 4.4, providing only few remarks for the additive noise case in subsection 4.4.2, since it will be more deeply treated in the next chapter when dealing with large deviation principles.

Remark 4.23. Either Theorem 4.10 and Theorem 4.22 continue to hold for $\nu = 0$ and consider slightly different scaling between the noise and the viscosity. We refer to [135] for this more general framework.

Remark 4.24. As a byproduct of the proof of Theorem 4.22, we obtain that it holds

$$\alpha \mathbb{E}\left[\sup_{t\in[0,T]} \|\nabla \mathsf{u}^\alpha(t)\|^2\right] \to 0.$$

Remark 4.25. Theorem 4.20 is in a certain sense complementary to Theorem 4.22. In Theorem 4.20 we require poor regularity on the initial conditions of the Euler equations and the Navier-Stokes equations but we get a conditioned result. On the contrary, in Theorem 4.22 we require strong regularity on the initial conditions of the two problems and a special type of convergence of the initial conditions but we arrive at a not conditioned result.

Remark 4.26. The assumption on $\nu = O(\alpha)$ is hidden in equation (4.17). For high frequencies $\Delta \mathsf{u}^\alpha$ is a damping term in equation (4.17). In fact, for high frequencies $\mathsf{v}^\alpha \approx -\alpha\Delta\mathsf{u}^\alpha$, thus the equation becomes, formally,

$$-\alpha\partial_t \Delta \mathsf{u}^\alpha - \nu\Delta\mathsf{u}^\alpha + \cdots = 0.$$

Asking $\nu = O(\alpha)$, means requiring that the damping coefficient does not blow-up.

4.3 Inviscid Limit for the Stochastic Navier-Stokes Equations

4.3.1 Proof of Theorem 4.19

1. $\Rightarrow$ 2. and 3. $\Rightarrow$ 4. are obvious implications. We need only prove that 2. $\Rightarrow$ 3. and 4. $\Rightarrow$ 1.

2. $\Rightarrow$ 3. By the energy equality (4.13) for $t = T$ we have

$$\nu\mathbb{E}\left[\int_0^T \|\nabla u^\nu(s)\|^2\right] = \frac{1}{2}\mathbb{E}\left[\|u_0^\nu\|^2\right] - \frac{1}{2}\mathbb{E}\left[\|u^\nu(T)\|^2\right] + T\nu\sum_{k\in\mathcal{K}}\|\sigma_k\|^2.$$

Taking the limsup of this expression and exploiting the fact that under our assumptions it follows that

$$\mathbb{E}\left[\|u_0^\nu - \bar{u}_0\|^2\right] \to 0$$

$$\|\bar{u}(T)\|^2 \le \liminf_{\nu\to 0}\mathbb{E}\left[\|u^\nu(T)\|^2\right],$$

we get the thesis.

4. $\Rightarrow$ 1. The proof of Theorem 4.19 follows from a preliminary weaker result, namely that

$$\lim_{\nu\to 0}\sup_{t\in[0,T]}\mathbb{E}\left[\|u^\nu(t) - \bar{u}(t)\|^2\right] = 0. \tag{4.38}$$

First, note that, starting from equation (4.14), we can easily show that

$$\begin{aligned}\mathbb{E}\left[\sup_{t\in[0,T]}\|u^\nu(t)\|^2\right] &\leq \mathbb{E}\left[\|u_0^\nu\|^2\right] + T\nu\sum_{k\in\mathcal{K}}\|\sigma_k\|^2 + \sqrt{\nu}\sqrt{\sum_{k\in\mathcal{K}}\mathbb{E}\left[\int_0^T\langle u^\nu(s),\sigma_k\rangle^2\,ds\right]}\\ &\lesssim 1 \end{aligned}\tag{4.39}$$

uniformly in ν. Now we can start the main argument. For each time $t\in[0,T]$ we have, thanks to the Itô formula for the weak solutions of the stochastic Navier-Stokes equations and the energy relation satisfied by classical solutions of the Euler equations

$$\begin{aligned}\|u^\nu(t)-\bar{u}(t)\|^2 &= \|u^\nu(t)\|^2+\|\bar{u}(t)\|^2-2\langle u^\nu(t),\bar{u}(t)\rangle\\ &= \|u_0^\nu\|^2+t\nu\sum_{k\in\mathcal{K}}\|\sigma_k\|^2+2\sqrt{\nu}\sum_{k\in\mathcal{K}}\int_0^t\langle u^\nu(s),\sigma_k\rangle dW_s^k+\|\bar{u}_0\|^2\\ &- 2\langle u^\nu(t),\bar{u}(t)\rangle\\ &= \|u_0^\nu\|^2+t\nu\sum_{k\in\mathcal{K}}\|\sigma_k\|^2+2\sqrt{\nu}\sum_{k\in\mathcal{K}}\int_0^t\langle u^\nu(s),\sigma_k\rangle dW_s^k+\|\bar{u}_0\|^2\\ &- 2\langle u^\nu(t),\bar{u}(t)-v(t)\rangle-2\langle u^\nu(t),v(t)\rangle,\end{aligned}$$

$v(t)$ being the corrector of the boundary layer. Let us rewrite $\langle u^\nu(t),\bar{u}(t)-v(t)\rangle$ thanks to the weak formulation satisfied by u^ν

$$\begin{aligned}-2\langle u^\nu(t),\bar{u}(t)-v(t)\rangle &= -2\langle u_0^\nu,\bar{u}_0-v_0\rangle-2\int_0^t\langle u^\nu(s),\partial_s(\bar{u}-v)(s)\rangle\,ds\\ &+2\nu\int_0^t\langle\mathcal{A}^{\frac{1}{2}}u^\nu(s),\mathcal{A}^{\frac{1}{2}}(\bar{u}-v)(s)\rangle\,ds-2\int_0^t b(u^\nu,\bar{u}-v,u^\nu)(s)\,ds\\ &-2\sqrt{\nu}\sum_{k\in\mathcal{K}}\langle\sigma_k,(\bar{u}-v)\rangle W_t^k+2\sqrt{\nu}\sum_{k\in\mathcal{K}}\int_0^t\langle\sigma_k,(\bar{u}-v)(s)\rangle W_s^k\,ds.\end{aligned}\tag{4.40}$$

Combining (4.40) with the following identities

$$-2\langle u^\nu(s),\partial_s(\bar{u}-v)(s)\rangle = 2\langle u^\nu(s),\partial_s v(s)\rangle+2b(\bar{u}(s),\bar{u}(s),u^\nu(s)),$$

$$b(u^\nu,\bar{u},u^\nu)-b(\bar{u},\bar{u},u^\nu)=b(u^\nu-\bar{u},\bar{u},u^\nu-\bar{u}),$$

we obtain

$$\begin{aligned}\|u^\nu(t)-\bar{u}(t)\|^2 = &\left(\|u_0^\nu-\bar{u}_0\|^2+2\langle u_0^\nu, v_0\rangle-2\langle u^\nu(t), v(t)\rangle+2\int_0^t\langle u^\nu(s), \partial_s v(s)\rangle ds\right)\\ &+2\sqrt{\nu}\left(\sum_{k\in\mathcal{K}}\int_0^t\langle u^\nu(s),\sigma_k\rangle dW_s^k-\sum_{k\in\mathcal{K}}\langle\sigma_k,(\bar{u}-v)(t)\rangle W_t^k\right.\\ &\qquad\left.+\sum_{k\in\mathcal{K}}\int_0^t\langle\sigma_k,(\bar{u}-v)(s)\rangle W_s^k ds+\frac{\sqrt{\nu}t}{2}\sum_{k\in\mathcal{K}}\|\sigma_k\|^2\right)\\ &-2\int_0^t b((u^\nu-\bar{u})(s),\bar{u}(s),(u^\nu-\bar{u})(s))ds\\ &+2\left(\nu\int_0^t\langle\mathcal{A}^{\frac12}u^\nu(s),\mathcal{A}^{\frac12}(\bar{u}-v)(s)\rangle ds+\int_0^t b(u^\nu(s),v(s),u^\nu(s))ds\right)\\ &=I_1(t)+I_2(t)+I_3(t)+I_4(t),\end{aligned}\tag{4.41}$$

where

$$\begin{aligned}I_1(t)&=\|u_0^\nu-\bar{u}_0\|^2+2\langle u_0^\nu, v_0\rangle-2\langle u^\nu(t), v(t)\rangle+2\int_0^t\langle u^\nu(s),\partial_s v(s)\rangle ds,\\ I_2(t)&=2\sqrt{\nu}\left(\sum_{k\in\mathcal{K}}\int_0^t\langle u^\nu(s),\sigma_k\rangle dW_s^k-\sum_{k\in\mathcal{K}}\langle\sigma_k,(\bar{u}-v)(t)\rangle W_t^k\right.\\ &\qquad\left.+\sum_{k\in\mathcal{K}}\int_0^t\langle\sigma_k,(\bar{u}-v)(s)\rangle W_s^k ds+\frac{\sqrt{\nu}t}{2}\sum_{k\in\mathcal{K}}\|\sigma_k\|^2\right),\\ I_3(t)&=-2\int_0^t b((u^\nu-\bar{u})(s),\bar{u}(s),(u^\nu-\bar{u})(s))ds,\\ I_4(t)&=2\left(\nu\int_0^t\langle\mathcal{A}^{\frac12}u^\nu(s),\mathcal{A}^{\frac12}(\bar{u}-v)(s)\rangle ds+\int_0^t b(u^\nu(s),v(s),u^\nu(s))ds\right).\end{aligned}$$

By simple manipulations and Proposition 4.14 we have

$$I_1(t)\le\|u_0^\nu-\bar{u}_0\|^2+C(1+T)\delta^{1/2}\sup\nolimits_{t\in[0,T]}\|u^\nu(t)\|.\tag{4.42}$$

Concerning $I_2(t)$, we have, by Hölder's inequality, Proposition 4.14 and Hypothesis 4.4

$$\begin{aligned}I_2(t)\le o(1)+\sqrt{\nu}&\left(\sup\nolimits_{t\in[0,T]}\left|\sum_{k\in\mathcal{K}}\int_0^t\langle u^\nu(s),\sigma_k\rangle dW_s^k\right|\right.\\ &\qquad\left.+(1+T)\left(\|\bar{u}_0\|+C\delta^{1/2}\right)\sup_{t\in[0,T]}\|W(t)\|\right).\end{aligned}\tag{4.43}$$

$I_3(t)$ can be treated easily by Hölder's inequality obtaining

$$I_3(t) \le 2\|\nabla \bar{u}\|_{L^\infty([0,T]\times\mathcal{O})} \int_0^t \|u^\nu(s) - \bar{u}(s)\|^2 ds. \tag{4.44}$$

For what concern $I_4(t)$ we observe that

$$\begin{aligned} |b(u^\nu(s), v(s), u^\nu(s))| &\le \int_{\Gamma_\delta} |\nabla v(s)|\rho^2 \frac{|u^\nu(s)|^2}{\rho^2}\, dx \\ &\le C\delta \left\| \frac{u^\nu(s)}{\rho} \right\|^2_{L^2(\Gamma_\delta)} \\ &\le C\delta \|\nabla u^\nu(s)\|^2_{L^2(\Gamma_\delta)}, \end{aligned} \tag{4.45}$$

where in the first step we use Proposition 4.14 and in the second Hardy's inequality. Secondly

$$\begin{aligned} &\left| \nu \langle \mathcal{A}^{\frac{1}{2}} u^\nu(s), \mathcal{A}^{\frac{1}{2}} (\bar{u} - v)(s) \rangle \right| \\ &\qquad \le \nu \|\nabla u^\nu(s)\| \|\nabla \bar{u}(s)\| + \nu \|\nabla u^\nu(s)\|_{L^2(\Gamma_\delta)} \|\nabla v(s)\|_{L^2(\Gamma_\delta)} \\ &\qquad\qquad \le C\nu \|\nabla u^\nu(s)\| + C\nu\delta^{-1/2} \|\nabla u^\nu(s)\|_{L^2(\Gamma_\delta)}, \end{aligned}$$

thanks to Proposition 4.14. As a consequence

$$I_4(t) \le C\delta \int_0^t \|\nabla u^\nu(s)\|^2_{L^2(\Gamma_\delta)} ds + C\nu\delta^{-1/2} \int_0^t \|\nabla u^\nu(s)\|_{L^2(\Gamma_\delta)} ds + C\nu \int_0^t \|\nabla u^\nu(s)\| ds. \tag{4.46}$$

Combining (4.42),(4.43),(4.44),(4.46) for $\delta = c\nu$ we have

$$\begin{aligned} \|u^\nu(t) - \bar{u}(t)\|^2 &\le o(1) + \|u_0^\nu - \bar{u}_0\|^2 + C(1+T)\sqrt{\nu}\, \sup_{t\in[0,T]} \|u^\nu(t)\| \\ &\quad + \sqrt{\nu} \Bigg(\sup_{t\in[0,T]} \left| \sum_{k\in\mathcal{K}} \int_0^t \langle u^\nu(s), \sigma_k \rangle dW_s^k \right| \\ &\qquad + (1+T)\left(\|\bar{u}_0\| + C\sqrt{\nu}\right) \sup_{t\in[0,T]} \|W(t)\| \Bigg) \\ &\quad + C\sqrt{\nu} \int_0^t \|\nabla u^\nu(s)\|_{L^2(\Gamma_\delta)} ds + C\nu \int_0^t \|\nabla u^\nu(s)\| ds \\ &\quad + C\nu \int_0^t \|\nabla u^\nu(s)\|^2_{L^2(\Gamma_\delta)} ds + 2\|\nabla \bar{u}\|_{L^\infty([0,T]\times\mathcal{O})} \int_0^t \|u^\nu(s) - \bar{u}(s)\|^2 ds. \end{aligned} \tag{4.47}$$

Considering the expected value of relation (4.47), we get by Grönwall's lemma

$$\mathbb{E}\left[\|u^\nu(t)-\bar{u}(t)\|^2\right] \le e^{2T\|\nabla \bar{u}\|_{L^\infty([0,T]\times\mathcal{O})}}(o(1)+R(t)),$$

$R(t)$ being given by

$$\begin{aligned}
R(t) = {} & \mathbb{E}\left[\|u_0^\nu-\bar{u}_0\|^2\right] + C(1+T)\sqrt{\nu}\,\mathbb{E}\left[\sup\nolimits_{t\in[0,T]}\|u^\nu(t)\|\right] \\
& + \sqrt{\nu}\left(\mathbb{E}\left[\sup\nolimits_{t\in[0,T]}\left|\sum_{k\in\mathcal{K}}\int_0^t \langle u^\nu(s),\sigma_k\rangle dW_s^k\right|\right]\right. \\
& \qquad \left. +(1+T)\left(\|\bar{u}_0\|+C\sqrt{\nu}\right)\mathbb{E}\left[\sup_{t\in[0,T]}\|W(t)\|\right]\right) \\
& + C\sqrt{\nu}\,\mathbb{E}\left[\int_0^t \|\nabla u^\nu(s)\|_{L^2(\Gamma_\delta)}ds\right] + C\nu\,\mathbb{E}\left[\int_0^t \|\nabla u^\nu(s)\|ds\right] \\
& + C\nu\,\mathbb{E}\left[\int_0^t \|\nabla u^\nu(s)\|^2_{L^2(\Gamma_\delta)}ds\right]. \qquad (4.48)
\end{aligned}$$

We are left to show that $\sup_{t\in[0,T]} R(t) = R(T) \to 0$ in order to prove the validity of relation (4.38). This is the case. Indeed, due to our assumptions on the convergence of the initial conditions, the Kato type hypothesis on the behaviour of $\nu\mathbb{E}\left[\int_0^T \|\nabla u^\nu(t)\|^2_{L^2(\Gamma_{c\nu})}dt\right]$, (4.39) and Hypothesis 4.4 we have by Burkholder-Davis-Gundy inequality

$$\begin{aligned}
R(T) \lesssim {} & \mathbb{E}\left[\|u_0^\nu-\bar{u}_0\|^2\right] + C(1+T)\sqrt{\nu} \\
& + \sqrt{\nu}\left(\left(\sum_{k\in\mathcal{K}}\|\sigma_k\|^2\right)^{1/2} + (1+T)\left(\|\bar{u}_0\|+C\sqrt{\nu}\right)\left(\sum_{k\in\mathcal{K}}\|\sigma_k\|^2\right)^{1/2}\right) \\
& + C\sqrt{\nu}\left(\mathbb{E}\left[\nu\int_0^T \|\nabla u^\nu(s)\|^2 ds\right]\right)^{1/2} \\
& + C\nu\,\mathbb{E}\left[\int_0^T \|\nabla u^\nu(s)\|^2_{L^2(\Gamma_\delta)}ds\right] + C\left(\mathbb{E}\left[\nu\int_0^T \|\nabla u^\nu(s)\|^2_{L^2(\Gamma_\delta)}ds\right]\right)^{1/2} \\
& \to 0. \qquad (4.49)
\end{aligned}$$

This concludes the proof of relation (4.38). In order to conclude the proof of Theorem 4.19, let us restart by relation (4.47). Considering the expected value of the supremum in time of (4.47) we obtain

$$\begin{aligned}
\mathbb{E}\left[\sup\nolimits_{t\in[0,T]}\|u^\nu(t)-\bar{u}(t)\|^2\right] \le {} & o(1)+R(T) \\
& + 2\|\nabla\bar{u}\|_{L^\infty([0,T]\times\mathcal{O})}\mathbb{E}\left[\int_0^T \|u^\nu(s)-\bar{u}(s)\|^2 ds\right],
\end{aligned}$$

$R(t)$ being the one defined in (4.48). Thanks to (4.49) we already know that

$$R(T) \to 0.$$

Therefore we are left to show

$$2\|\nabla \bar{u}\|_{L^\infty([0,T]\times\mathcal{O})}\mathbb{E}\left[\int_0^T \|u^\nu(s) - \bar{u}(s)\|^2 ds\right] \to 0,$$

but this follows easily combining Fubini's theorem and (4.38).

4.3.2 Proof of Theorem 4.20

Similarly to subsection 4.3.1, we start with a weaker result with the supremum in time outside the expected value to obtain the stronger one with the supremum in time inside the expected value. The idea behind the proof is simply to introduce an approximation of $\bar{u}_0$ in the sense of Theorem 4.13, then

$$\|u^n - \bar{u}\|^2 \leq 2\|u^n - \bar{u}^m\|^2 + 2\|\bar{u}^m - \bar{u}\|^2,$$

where $\bar{u}^m$ is the solution of the Euler Equations with initial condition $\bar{u}_0^m \in H \cap C^{1,\varepsilon}(\bar{\mathcal{O}})$. Thus, the second term can be estimate via Theorem 4.13, the first one is analyzed exploiting techniques similar to the ones of the previous section.

Remark 4.27. If $\bar{u}_0 \in \tilde{O}$ and $\{\bar{u}_0^m\}_{m\in\mathbb{N}}$ approximates $\bar{u}_0$ in the sense of Theorem 4.13, then

$$\|\bar{u}_0 - \bar{u}_0^m\|^2 e^{2T\|\nabla\bar{u}^m\|_{L^\infty([0,T]\times\mathcal{O})}} \leq \frac{1}{m^2} e^{-4T\|\nabla\bar{u}^m\|_{L^\infty([0,T]\times\mathcal{O})}}$$
$$\|\bar{u}_0 - \bar{u}_0^m\|^2 e^{2T\|\nabla\bar{u}^m\|_{L^\infty([0,T]\times\mathcal{O})}} T\|\nabla\bar{u}^m\|_{L^\infty([0,T]\times\mathcal{O})} \leq \frac{1}{4m^2}.$$

After these preliminaries observations, we are ready to prove Theorem 4.20.
Let $\{\bar{u}_0^m\}_{m\in\mathbb{N}}$ approximating $\bar{u}_0$ in the sense of Theorem 4.13 and $\{\bar{u}^m\}_{m\in\mathbb{N}}$ the corresponding solutions of the Euler equations, then for each $t \in [0,T]$, n, m we have thanks to Theorem 4.13 and Remark 4.27

$$\begin{aligned}\|u^n(t) - \bar{u}(t)\|^2 &\leq 2\|u^n(t) - \bar{u}^m(t)\|^2 + 2\|\bar{u}^m(t) - \bar{u}(t)\|^2 \\ &\leq \frac{2}{m^2} + 2\|u^n(t) - \bar{u}^m(t)\|^2.\end{aligned} \tag{4.50}$$

We adapt the computations of the proof of Theorem 4.19 to analyze the second term, hence some explanations will be omitted. For each m and $\delta > 0$ fixed, let us introduce the corrector of the boundary layer v^m. For each $m \in \mathbb{N}$, the corrector of the boundary layer satisfies the estimates of Proposition 4.14 with respect to a constant depending on m and independent on t and δ, namely

$$\begin{aligned}
&\|v^m(t)\|_{L^\infty} \le C_m,\ \|v^m(t)\| \le C_m\delta^{\frac{1}{2}},\ \|\partial_t v^m(t)\| \le C_m\delta^{\frac{1}{2}},\\
&\|\nabla v^m(t)\|_{L^\infty} \le C_m\delta^{-1},\ \|\nabla v^m(t)\| \le C_m\delta^{-1/2},\ \|\rho\nabla v^m(t)\|_{L^\infty} \le C_m,\\
&\|\rho^2\nabla v^m(t)\|_{L^\infty} \le C_m\delta,\ \|\rho\nabla v^m(t)\| \le C_m\delta^{\frac{1}{2}}. \qquad (4.51)
\end{aligned}$$

For each time $t \in [0,T]$ we have, thanks to Itô formula for the weak solutions of the stochastic Navier-Stokes equations and the energy relation for classical solutions of the Euler equations with initial condition $\bar{u}_0^m$,

$$\begin{aligned}
\|u^n(t) - \bar{u}^m(t)\|^2 &= \|u^n(t)\|^2 + \|\bar{u}^m(t)\|^2 - 2\langle u^n(t), \bar{u}^m(t)\rangle\\
&\le \|u_0^n\|^2 + t\nu_n \sum_{k\in\mathcal{K}} \|\sigma_k\|^2 + 2\sum_{k\in\mathcal{K}} \sqrt{\nu_n}\int_0^t \langle u^n(s), \sigma_k\rangle dW_s^k\\
&\quad + \|\bar{u}_0^m\|^2 - 2\langle u^n(t), (\bar{u}^m - v^m)(t)\rangle - 2\langle u^n(t), v^m(t)\rangle. \qquad (4.52)
\end{aligned}$$

To analyze the second-last term we use the weak formulation satisfied by u^n, taking $\bar{u}^m - v^m$ as test function, obtaining

$$\begin{aligned}
-2\langle u^n(t), (\bar{u}^m - v^m)(t)\rangle &= -2\langle u_0^n, \bar{u}_0^m - v_0^m\rangle - 2\int_0^t \langle u^n(s), \partial_s(\bar{u}^m - v^m)(s)\rangle\, ds\\
&\quad + 2\nu_n \int_0^t \langle \mathcal{A}^{\frac{1}{2}} u^n(s), \mathcal{A}^{\frac{1}{2}}(\bar{u}^m - v^m)(s)\rangle\, ds\\
&\quad - 2\int_0^t b(u^n(s), (\bar{u}^m - v^m)(s), u^n(s))\, ds\\
&\quad - 2\sqrt{\nu_n}\sum_{k\in\mathcal{K}} \langle \sigma_k, (\bar{u}^m - v^m)(t)\rangle W_t^k\\
&\quad + 2\sqrt{\nu_n}\sum_{k\in\mathcal{K}} \int_0^t \langle \sigma_k, (\bar{u}^m - v^m)(s)\rangle W_s^k\, ds.
\end{aligned}$$

Moreover

$$-2\langle u^n(s), \partial_s(\bar{u}^m - v^m)(s)\rangle = 2\langle u^n(s), \partial_s v^m(s)\rangle + 2b(\bar{u}^m(s), \bar{u}^m(s), u^n(s)).$$

Thanks to the relations above and noting that

$$b(u^n(s), \bar{u}^m(s), u^n(s)) - b(\bar{u}^m(s), \bar{u}^m(s), u^n(s)) = b((u^n - \bar{u}^m)(s), \bar{u}^m(s), (u^n - \bar{u}^m)(s))$$

we have

$$\begin{aligned}
&\|u^n(t)-\bar{u}^m(t)\|^2\\
&=\left(\|u_0^n-\bar{u}_0^m\|^2+2\langle u_0^n,v_0^m\rangle-2\langle u^n(t),v^m(t)\rangle+2\int_0^t\langle u^n(s),\partial_s v^m(s)\rangle\,ds\right)\\
&+\Bigg(t\nu_n\sum_{k\in\mathcal{K}}\|\sigma_k\|^2-2\sqrt{\nu_n}\sum_{k\in\mathcal{K}}\langle\sigma_k,(\bar{u}^m-v^m)(t)\rangle W_t^k\\
&\quad+2\sqrt{\nu_n}\sum_{k\in\mathcal{K}}\int_0^t\langle u^n(s),\sigma_k\rangle dW_s^k+2\sqrt{\nu_n}\sum_{k\in\mathcal{K}}\int_0^t\langle\sigma_k,(\bar{u}^m-v^m)(s)\rangle W_s^k\,ds\Bigg)\\
&-2\int_0^t b((u^n-\bar{u}^m)(s),\bar{u}^m(s),(u^n-\bar{u}^m)(s))\,ds\\
&+2\left(\nu_n\int_0^t\langle\mathcal{A}^{\frac12}u^n(s),\mathcal{A}^{\frac12}(\bar{u}^m-v^m)(s)\rangle\,ds+\int_0^t b(u^n(s),v^m(s),u^n(s))\,ds\right)\\
&=I_1(t)+I_2(t)+I_3(t)+I_4(t),
\end{aligned}$$

where

$$\begin{aligned}
I_1(t)&=\|u_0^n-\bar{u}_0^m\|^2+2\langle u_0^n,v_0^m\rangle-2\langle u^n(t),v^m(t)\rangle+2\int_0^t\langle u^n(s),\partial_s v^m(s)\rangle\,ds,\\
I_2(t)&=t\nu_n\sum_{k\in\mathcal{K}}\|\sigma_k\|^2-2\sqrt{\nu_n}\sum_{k\in\mathcal{K}}\langle\sigma_k,(\bar{u}^m-v^m)(t)\rangle W_t^k\\
&\quad+2\sqrt{\nu_n}\sum_{k\in\mathcal{K}}\int_0^t\langle u^n(s),\sigma_k\rangle dW_s^k+2\sqrt{\nu_n}\sum_{k\in\mathcal{K}}\int_0^t\langle\sigma_k,(\bar{u}^m-v^m)(s)\rangle W_s^k\,ds,\\
I_3(t)&=-2\int_0^t b((u^n-\bar{u}^m)(s),\bar{u}^m(s),(u^n-\bar{u}^m)(s))\,ds,\\
I_4(t)&=2\left(\nu_n\int_0^t\langle\mathcal{A}^{\frac12}u^n(s),\mathcal{A}^{\frac12}(\bar{u}^m-v^m)(s)\rangle\,ds+\int_0^t b(u^n(s),v^m(s),u^n(s))\,ds\right).
\end{aligned}$$

Thanks to (4.51) we can treat $I_1(t)$ obtaining

$$I_1(t)\le 2\|u_0^n-\bar{u}_0\|^2+2\|\bar{u}_0^m-\bar{u}_0\|^2+C_m(1+T)\delta^{1/2}\sup_{t\in[0,T]}\|u^n(t)\|. \tag{4.53}$$

Concerning $I_2(t)$ we have thanks to (4.51), Hölder's inequality and Hypothesis 4.4

$$\begin{aligned}
I_2(t)\le o(1)+\sqrt{\nu_n}\Bigg(&\sup_{t\in[0,T]}\left|\sum_{k\in\mathcal{K}}\int_0^t\langle u^n(s),\sigma_k\rangle dW_s^k\right|\\
&+(1+T)\left(\|\bar{u}_0^m\|+C_m\delta^{1/2}\right)\sup_{t\in[0,T]}\|W(t)\|\Bigg).
\end{aligned}\tag{4.54}$$

$I_3(t)$ can be treated easily by Hölder's inequality obtaining

$$I_3(t) \leq 2\|\nabla \bar{u}^m\|_{L^\infty([0,T]\times\mathcal{O})} \int_0^t \|u^n(s) - \bar{u}^m(s)\|^2 ds. \tag{4.55}$$

For what concern $I_4(t)$, arguing as in the proof of Theorem 4.19 we have

$$\begin{aligned} I_4(t) \leq\ & C_m\delta \int_0^t \|\nabla u^n(s)\|^2_{L^2(\Gamma_\delta)} ds + C_m\nu_n\delta^{-1/2} \int_0^t \|\nabla u^n(s)\|_{L^2(\Gamma_\delta)} ds \\ & + C_m\nu_n \int_0^t \|\nabla u^n(s)\| ds. \end{aligned} \tag{4.56}$$

Combining (4.53), (4.54), (4.55), (4.56) for $\delta = c\nu_n$ we obtain

$$\begin{aligned} \|u^n(t) - \bar{u}^m(t)\|^2 \leq\ & o(1) + 2\|u^n_0 - \bar{u}_0\|^2 + 2\|\bar{u}^m_0 - \bar{u}_0\|^2 + C_m(1+T)\sqrt{\nu_n}\,\sup\nolimits_{t\in[0,T]}\|u^n(t)\| \\ & + \sqrt{\nu_n}\left(\sup\nolimits_{t\in[0,T]}\left|\sum_{k\in\mathcal{K}} \int_0^t \langle u^n(s), \sigma_k\rangle dW^k_s\right|\right. \\ & \qquad \left. + (1+T)\left(\|\bar{u}^m_0\| + C_m\sqrt{\nu_n}\right) \sup_{t\in[0,T]} \|W(t)\|\right) \\ & + C_m\sqrt{\nu_n}\int_0^t \|\nabla u^n(s)\|_{L^2(\Gamma_\delta)} ds + C_m\nu_n \int_0^t \|\nabla u^n(s)\| ds \\ & + C_m\nu_n \int_0^t \|\nabla u^n(s)\|^2_{L^2(\Gamma_\delta)} ds + 2\|\nabla \bar{u}^m\|_{L^\infty([0,T]\times\mathcal{O})} \int_0^t \|u^n(s) - \bar{u}^m(s)\|^2 ds. \end{aligned} \tag{4.57}$$

Considering the expected value of (4.57), we get by Grönwall's lemma

$$\mathbb{E}\left[\|u^n(t) - \bar{u}^m(t)\|^2\right] \leq e^{2T\|\nabla \bar{u}^m\|_{L^\infty([0,T]\times\mathcal{O})}} \left(o(1) + R^m(t)\right), \tag{4.58}$$

$R^m(t)$ being given by

$$\begin{aligned} R^m(t) =\ & 2\mathbb{E}\left[\|u^n_0 - \bar{u}_0\|^2\right] + 2\|\bar{u}^m_0 - \bar{u}_0\|^2 + C_m(1+T)\sqrt{\nu_n}\mathbb{E}\left[\sup\nolimits_{t\in[0,T]}\|u^n(t)\|\right] \\ & + \sqrt{\nu_n}\mathbb{E}\left[\left(\sup\nolimits_{t\in[0,T]}\left|\sum_{k\in\mathcal{K}} \int_0^t \langle u^n(s), \sigma_k\rangle dW^k_s\right|\right]\right. \\ & \qquad \left. + (1+T)\left(\|\bar{u}^m_0\| + C_m\sqrt{\nu_n}\right)\mathbb{E}\left[\sup_{t\in[0,T]} \|W(t)\|\right]\right) \\ & + C_m\sqrt{\nu_n}\mathbb{E}\left[\int_0^t \|\nabla u^n(s)\|_{L^2(\Gamma_\delta)} ds\right] + C_m\nu_n\mathbb{E}\left[\int_0^t \|\nabla u^n(s)\| ds\right] \\ & + C_m\nu_n\mathbb{E}\left[\int_0^t \|\nabla u^n(s)\|^2_{L^2(\Gamma_\delta)} ds\right]. \end{aligned}$$

Therefore, arguing as in the proof of Theorem 4.19, taking the limsup with respect to n of (4.58) for m fixed we obtain

$$\begin{aligned}\text{limsup}_{n\to+\infty}\sup_{t\in[0,T]}\mathbb{E}\left[\|u^n(t)-\bar{u}^m(t)\|^2\right] &\le 2e^{2T\|\nabla\bar{u}^m\|_{L^\infty([0,T]\times\mathcal{O})}}\|\bar{u}_0^m-\bar{u}_0\|^2\\ &\le 2\frac{e^{-4T\|\nabla\bar{u}^m\|_{L^\infty([0,T]\times\mathcal{O})}}}{m^2}, \end{aligned}\tag{4.59}$$

where the last inequality follows by Remark 4.27. Therefore, let us consider (4.50). If we fix $\varepsilon>0$ and choose $\bar{m}$ such that $\frac{6}{\bar{m}^2}<\varepsilon$, then taking the limsup with respect to n of (4.58) for $m=\bar{m}$ we have

$$\limsup_{n\to+\infty}\sup_{t\in[0,T]}\mathbb{E}\left[\|u^n(t)-\bar{u}(t)\|^2\right]\le\varepsilon$$

and the arbitrariness of ε implies

$$\lim_{n\to+\infty}\sup_{t\in[0,T]}\mathbb{E}\left[\|u^n(t)-\bar{u}(t)\|^2\right]=0.\tag{4.60}$$

In order to end the proof of Theorem 4.20, let us restart from (4.57) considering the expected value of its supremum in time, obtaining

$$\begin{aligned}\mathbb{E}\left[\text{sup}_{t\in[0,T]}\|u^n(t)-\bar{u}^m(t)\|^2\right] &\le o(1)+R^m(T)\\ &\quad+2\|\nabla\bar{u}^m\|_{L^\infty([0,T]\times\mathcal{O})}\mathbb{E}\left[\int_0^T\|u^n(s)-\bar{u}^m(s)\|^2ds\right].\end{aligned}$$

Considering the limsup with respect to n for m fixed of the expression above, thanks to (4.59) and Remark 4.27 we obtain

$$\begin{aligned}\text{limsup}_{n\to+\infty}\mathbb{E}\left[\text{sup}_{t\in[0,T]}\|u^n(t)-\bar{u}^m(t)\|^2\right] &\le\frac{2}{m^2}\\ &\quad+4T\|\nabla\bar{u}^m\|_{L^\infty([0,T]\times\mathcal{O})}\frac{e^{-4T\|\nabla\bar{u}^m\|_{L^\infty([0,T]\times\mathcal{O})}}}{m^2}\\ &\le\frac{3}{m^2}.\end{aligned}\tag{4.61}$$

Let us fix $\varepsilon>0$ and choose $\bar{m}$ such that $\frac{8}{\bar{m}^2}<\varepsilon$, then considering the expected value of the supremum in time of (4.50) for $m=\bar{m}$, thanks to (4.61) we obtain

$$\text{limsup}_{n\to+\infty}\mathbb{E}\left[\sup_{t\in[0,T]}\|u^n(t)-\overline{u}(t)\|^2\right]<\varepsilon.$$

This concludes the proof from the arbitrariness of ε.

4.4 Inviscid Limit for the Stochastic Second-Grade Fluid Equations

4.4.1 Proof of Theorem 4.22

In order to prove Theorem 4.22, we follow the ideas of [134] and of the previous section. Therefore, again, we start with a weaker result with the supremum in time outside the expected value and then we move to the stronger one with the supremum in time inside the expected value.

Proof of Theorem 4.22. Let $W^\alpha = \mathsf{u}^\alpha - \bar{u}$, it satisfies $\mathbb{P}-a.s.$ for each $\varphi \in H$ and $t \in [0,T]$

$$\begin{aligned}\langle W^\alpha(t),\varphi\rangle - \langle W^\alpha_0,\varphi\rangle &= -\alpha\langle \mathcal{A}\mathsf{u}^\alpha(t),\varphi\rangle + \alpha\langle \mathcal{A}\mathsf{u}^\alpha_0,\varphi\rangle - \nu\int_0^t \langle \mathcal{A}\mathsf{u}^\alpha(s),\varphi\rangle ds\\ &\quad - \int_0^t b(\mathsf{u}^\alpha(s),W^\alpha(s),\varphi)ds - \int_0^t b(W^\alpha(s),\bar{u}(s),\varphi)ds\\ &\quad - \alpha\int_0^t b(\varphi,\Delta\mathsf{u}^\alpha(s),\mathsf{u}^\alpha(s))ds + \alpha\int_0^t b(\mathsf{u}^\alpha(s),\Delta\mathsf{u}^\alpha(s),\varphi)ds\\ &\quad + \nu\int_0^t \langle F(\mathsf{u}^\alpha)(s),\varphi\rangle ds + \sqrt{\nu}\sum_{k\in\mathcal{K}}\int_0^t \langle G^k(\mathsf{u}^\alpha(s)),\varphi\rangle dW^k_s.\end{aligned}$$

Following the idea of [111], let v the corrector of the boundary layer of width δ, for $\bar{u}$ as described in Proposition 4.14. Let $\delta=\delta(\alpha)$ such that

$$\lim_{\alpha\to 0}\delta = 0,\quad \lim_{\alpha\to 0}\frac{\alpha}{\delta} = 0. \tag{4.62}$$

We want to write the Itô's formula for $\|W^\alpha(t)\|^2$. Let us take an orthonormal basis of H, $\{e_i\}$ made by eigenvectors of $-\mathcal{A}$, let $\{-\lambda_i\}$ the corresponding eigenvalues. Let us consider the weak formulation with test functions $\varphi = e_i$ and call $W^{\alpha,n} = \sum_{i=1}^n\langle W^\alpha, e_i\rangle e_i$, $\mathsf{u}^{\alpha,n} = \sum_{i=1}^n\langle \mathsf{u}^\alpha, e_i\rangle e_i$, $\bar{u}^n = \sum_{i=1}^n\langle \bar{u}, e_i\rangle e_i$ e $v^n = \sum_{i=1}^n\langle v, e_i\rangle e_i$, then we get

$$\begin{aligned}W^{\alpha,n}(t) - W^{\alpha,n}_0 &= -\alpha\mathcal{A}\mathsf{u}^{\alpha,n}(t) + \alpha\mathcal{A}\mathsf{u}^{\alpha,n}_0 - \nu\int_0^t \mathcal{A}\mathsf{u}^{\alpha,n}(s)ds\\ &\quad - \int_0^t\sum_{i=1}^n b(\mathsf{u}^\alpha,W^\alpha(s),e_i)e_i ds - \int_0^t\sum_{i=1}^n b(W^\alpha(s),\bar{u}(s),e_i)e_i ds\\ &\quad - \alpha\int_0^t\sum_{i=1}^n b(e_i,\Delta\mathsf{u}^\alpha(s),\mathsf{u}^\alpha(s))ds + \alpha\int_0^t\sum_{i=1}^n b(\mathsf{u}^\alpha(s),\Delta\mathsf{u}^\alpha(s),e_i)ds\\ &\quad + \nu\int_0^t\sum_{i=1}^n\langle F(\mathsf{u}^\alpha(s)),e_i\rangle e_i ds + \sqrt{\nu}\sum_{k\in\mathcal{K}}\int_0^t\sum_{i=1}^n\langle G^k(\mathsf{u}^\alpha(s)),e_i\rangle e_i dW^k_s.\end{aligned} \tag{4.63}$$

Therefore

$$\begin{aligned} d\|W^{\alpha,n}\|^2 &= 2\langle W^{\alpha,n}, dW^{\alpha,n}\rangle + \alpha^2 d\langle\langle -\mathcal{A}\mathsf{u}^{\alpha,n}, -\mathcal{A}\mathsf{u}^{\alpha,n}\rangle\rangle_t + \nu \sum_{k\in\mathcal{K}}\sum_{i=1}^{n}\langle G^k(\mathsf{u}^\alpha), e_i\rangle^2 dt \\ &\quad + \alpha\sqrt{\nu}\sum_{k\in\mathcal{K}}\sum_{i=1}^{n}\langle G^k(\mathsf{u}^\alpha), e_i\rangle d\langle\langle e_i W^k, -\mathcal{A}\mathsf{u}^{\alpha,n}\rangle\rangle_t. \end{aligned} \tag{4.64}$$

In the same way, considering the weak formulation satisfied by u^α, we get

$$\begin{aligned} -d\mathcal{A}\mathsf{u}^{\alpha,n} &= \left(\nu\mathcal{A}(I+\alpha\mathcal{A})^{-1}\mathcal{A}\mathsf{u}^{\alpha,n} + \sum_{i=1}^{n} b(\mathsf{u}^{\alpha,n}, \mathsf{u}^{\alpha,n}, e_i)\mathcal{A}(I+\alpha\mathcal{A})^{-1}e_i\right) dt \\ &\quad + \alpha\sum_{i=1}^{n} b(e_i, \Delta\mathsf{u}^\alpha, \mathsf{u}^\alpha)\mathcal{A}(I+\alpha\mathcal{A})^{-1}e_i dt \\ &\quad - \nu\sum_{i=1}^{n}\langle F(\mathsf{u}^\alpha), e_i\rangle\mathcal{A}(I+\alpha\mathcal{A})^{-1}e_i dt \\ &\quad - \sqrt{\nu}\sum_{k\in\mathcal{K}}\sum_{i=1}^{n}\langle G^k(\mathsf{u}^\alpha), e_i\rangle\mathcal{A}(I+\alpha\mathcal{A})^{-1}e_i dW_t^k. \end{aligned} \tag{4.65}$$

Combining relation (4.63), (4.64), (4.65) we obtain

$$\begin{aligned} d\|W^{\alpha,n}\|^2 &= -2\alpha\langle W^{\alpha,n}, d\mathcal{A}\mathsf{u}^{\alpha,n}\rangle - 2\nu\langle W^{\alpha,n}, \mathcal{A}\mathsf{u}^{\alpha,n}\rangle dt \\ &\quad - 2\langle W^{\alpha,n}, \sum_{i=1}^{n} b(W^\alpha, \bar{u}, e_i)e_i\rangle dt - 2\langle W^{\alpha,n}, \sum_{i=1}^{n} b(\mathsf{u}^\alpha, W^\alpha, e_i)e_i\rangle dt \\ &\quad - 2\alpha\langle W^{\alpha,n}, \sum_{i=1}^{n} b(e_i, \Delta\mathsf{u}^\alpha, \mathsf{u}^\alpha)e_i\rangle dt + 2\alpha\langle W^{\alpha,n}, \sum_{i=1}^{n} b(\mathsf{u}^\alpha, \Delta\mathsf{u}^\alpha, e_i)e_i\rangle\ dt \\ &\quad + 2\nu\sum_{i=1}^{n}\langle F(\mathsf{u}^\alpha), e_i\rangle\langle W^{\alpha,n}, e_i\rangle dt + 2\sqrt{\nu}\sum_{k\in\mathcal{K}}\sum_{i=1}^{n}\langle G^k(\mathsf{u}^\alpha), e_i\rangle\langle W^{\alpha,n}, e_i\rangle dW_t^k \\ &\quad + \alpha^2\nu\sum_{k\in\mathcal{K}}\sum_{i=1}^{n}\langle G^k(\mathsf{u}^\alpha), e_i\rangle^2\|\mathcal{A}(I+\alpha\mathcal{A})^{-1}e_i\|^2\ dt \\ &\quad + \nu\sum_{k\in\mathcal{K}}\sum_{i=1}^{n}\langle G^k(\mathsf{u}^\alpha), e_i\rangle^2 dt + \alpha\nu\sum_{k\in\mathcal{K}}\sum_{i=1}^{n}\langle G^k(\mathsf{u}^\alpha), e_i\rangle^2\langle -\mathcal{A}(I+\alpha\mathcal{A})^{-1}e_i, e_i\rangle dt. \end{aligned}$$

Let us rewrite $-\langle W^{\alpha,n}, d\mathcal{A}\mathsf{u}^{\alpha,n}\rangle$ in a different way

$$\begin{aligned}
-\langle W^{\alpha,n}, d\mathcal{A}\mathsf{u}^{\alpha,n}\rangle &= \langle \bar{u}^n - \mathsf{u}^{\alpha,n}, d\mathcal{A}\mathsf{u}^{\alpha,n}\rangle \\
&= -\langle \mathsf{u}^{\alpha,n}, d\mathcal{A}\mathsf{u}^{\alpha,n}\rangle + \langle \bar{u}^n - v^n, d\mathcal{A}\mathsf{u}^{\alpha,n}\rangle + \langle v^n, d\mathcal{A}\mathsf{u}^{\alpha,n}\rangle \\
&= -\langle \mathcal{A}^{1/2}\mathsf{u}^{\alpha,n}, d\mathcal{A}^{1/2}\mathsf{u}^{\alpha,n}\rangle + \langle \mathcal{A}^{1/2}(\bar{u}^n - v^n), d\mathcal{A}^{1/2}\mathsf{u}^{\alpha,n}\rangle + \langle v^n, d\mathcal{A}\mathsf{u}^{\alpha,n}\rangle \\
&= -\frac{d\|\mathcal{A}^{1/2}\mathsf{u}^{\alpha,n}\|^2}{2} + \frac{d\langle\langle \mathcal{A}^{1/2}\mathsf{u}^{\alpha,n}, \mathcal{A}^{1/2}\mathsf{u}^{\alpha,n}\rangle\rangle_t}{2} + d\langle \mathcal{A}^{1/2}(\bar{u}^n - v^n), \mathcal{A}^{1/2}\mathsf{u}^{\alpha,n}\rangle \\
&\quad - \langle \mathcal{A}^{1/2}\partial_t(\bar{u}^n - v^n), \mathcal{A}^{1/2}\mathsf{u}^{\alpha,n}\rangle + d\langle v^n, \mathcal{A}\mathsf{u}^{\alpha,n}\rangle - \langle \partial_t v^n, \mathcal{A}\mathsf{u}^{\alpha,n}\rangle.
\end{aligned}$$

Therefore, we arrive to this final expression

$$\begin{aligned}
\|W^{\alpha,n}(t)\|^2 &= \|W_0^{\alpha,n}\|^2 - \alpha\|\nabla\mathsf{u}^{\alpha,n}(t)\| + \alpha\|\nabla\mathsf{u}_0^{\alpha,n}\|^2 \\
&\quad + \nu\alpha\sum_{k\in\mathcal{K}}\int_0^t\sum_{i=1}^n \langle G^k(\mathsf{u}^\alpha(s)), e_i\rangle^2 \|\mathcal{A}^{1/2}(I+\alpha\mathcal{A})^{-1}e_i\|^2 ds \\
&\quad + 2\alpha\bigg(\langle \nabla(\bar{u}^n - v^n)(t), \nabla\mathsf{u}^{\alpha,n}(t)\rangle - \langle \nabla(\bar{u}^n - v^n)_0, \nabla\mathsf{u}_0^{\alpha,n}\rangle - \langle v^n(t), \Delta\mathsf{u}^{\alpha,n}(t)\rangle \\
&\quad + \langle v_0^n, \Delta\mathsf{u}_0^{\alpha,n}\rangle - \int_0^t \langle \nabla\partial_s(\bar{u}^n(s) - v^n(s)), \nabla\mathsf{u}^{\alpha,n}(s)\rangle ds + \int_0^t \langle \partial_s v^n(s), \Delta\mathsf{u}^{\alpha,n}(s)\rangle ds\bigg) \\
&\quad - 2\nu\int_0^t \langle W^{\alpha,n}(s), \mathcal{A}\mathsf{u}^{\alpha,n}(s)\rangle ds - 2\int_0^t \langle W^{\alpha,n}(s), \sum_{i=1}^n b(W^\alpha(s), \bar{u}(s), e_i)e_i\rangle ds \\
&\quad - 2\int_0^t \langle W^{\alpha,n}(s), \sum_{i=1}^n \left(b(\mathsf{u}^\alpha(s), W^\alpha(s), e_i) + \alpha b(e_i, \Delta\mathsf{u}^\alpha(s), \mathsf{u}^\alpha(s))\right) e_i\rangle ds \\
&\quad + 2\alpha\int_0^t \langle W^{\alpha,n}(s), \sum_{i=1}^n b(\mathsf{u}^\alpha(s), \Delta\mathsf{u}^\alpha(s), e_i)e_i\rangle ds \\
&\quad + 2\nu\sum_{i=1}^n\int_0^t \langle F(\mathsf{u}^\alpha(s)), e_i\rangle\langle W^{\alpha,n}(s), e_i\rangle ds \\
&\quad + 2\sqrt{\nu}\sum_{k\in\mathcal{K}}\int_0^t\sum_{i=1}^n \langle G^k(\mathsf{u}^\alpha(s)), e_i\rangle\langle W^{\alpha,n}(s), e_i\rangle dW_s^k \\
&\quad + \nu\sum_{k\in\mathcal{K}}\int_0^t\sum_{i=1}^n \left(\alpha^2\langle G^k(\mathsf{u}^\alpha(s)), e_i\rangle^2\|\mathcal{A}(I+\alpha\mathcal{A})^{-1}e_i\|^2 + \langle G^k(\mathsf{u}^\alpha(s)), e_i\rangle^2\right) ds \\
&\quad - \alpha\nu\sum_{k\in\mathcal{K}}\int_0^t\sum_{i=1}^n \langle G^k(\mathsf{u}^\alpha(s)), e_i\rangle^2\langle \mathcal{A}(I+\alpha\mathcal{A})^{-1}e_i, e_i\rangle ds.
\end{aligned}$$

Now, letting $n \to +\infty$, exploiting the regularity of u^α, $\bar{u}$, v and the continuity of the trilinear form b we arrive to the formula below

$$\begin{aligned}
\|W^\alpha(t)\|^2 + \alpha\|\nabla \mathsf{u}^\alpha(t)\| &= \|W_0^\alpha\|^2 + \alpha\|\nabla \mathsf{u}_0^\alpha\| \\
&\quad + \alpha\nu \sum_{k\in\mathcal{K}} \int_0^t \|\mathcal{A}^{1/2}(I+\alpha\mathcal{A})^{-1} G^k(\mathsf{u}^\alpha(s))\|^2 ds \\
&\quad + 2\alpha\langle \nabla(\bar{u}-v)(t), \nabla \mathsf{u}^\alpha(t)\rangle - 2\alpha\langle \nabla(\bar{u}-v)_0, \nabla \mathsf{u}_0^\alpha\rangle \\
&\quad - 2\alpha \int_0^t \langle \nabla\partial_s(\bar{u}(s)-v(s)), \nabla \mathsf{u}^\alpha(s)\rangle ds - 2\alpha\langle v(t), \Delta \mathsf{u}^\alpha(t)\rangle + 2\alpha\langle v_0, \Delta \mathsf{u}_0^\alpha\rangle \\
&\quad + 2\alpha \int_0^t \langle \partial_s v(s), \Delta \mathsf{u}^\alpha(s)\rangle ds - 2\nu \int_0^t \langle W^\alpha(s), \mathcal{A}\mathsf{u}^\alpha(s)\rangle ds \\
&\quad - 2\int_0^t b(W^\alpha(s), \bar{u}(s), W^\alpha(s))\rangle ds \\
&\quad - 2\alpha \int_0^t b(W^\alpha(s), \Delta \mathsf{u}^\alpha(s), \mathsf{u}^\alpha(s)) ds + 2\alpha \int_0^t b(\mathsf{u}^\alpha(s), \Delta \mathsf{u}^\alpha(s), W^\alpha(s)) ds \\
&\quad + 2\nu \int_0^t \langle F(\mathsf{u}^\alpha(s)), W^\alpha(s)\rangle ds + 2\sqrt{\nu} \sum_{k\in\mathcal{K}} \int_0^t \langle G^k(\mathsf{u}^\alpha(s)), W^\alpha(s)\rangle dW_s^k \\
&\quad + \alpha^2\nu \sum_{k\in\mathcal{K}} \int_0^t \|\mathcal{A}(I+\alpha\mathcal{A})^{-1} G^k(\mathsf{u}^\alpha(s))\|^2 ds + \nu \sum_{k\in\mathcal{K}} \int_0^t \|G^k(\mathsf{u}^\alpha(s))\|^2 ds \\
&\quad - \alpha\nu \sum_{k\in\mathcal{K}} \int_0^t \langle \mathcal{A}(I+\alpha\mathcal{A})^{-1} G^k(\mathsf{u}^\alpha(s)), G^k(\mathsf{u}^\alpha(s))\rangle ds \\
&= I_1(t) + I_2(t) + I_3(t) + I_4(t) + I_5(t) + I_6(t) + M(t),
\end{aligned}$$

where:

$$\begin{aligned}
I_1(t) &= \|W_0^\alpha\|^2 + \alpha\|\nabla \mathsf{u}_0^\alpha\|^2 + 2\alpha\langle \nabla(\bar{u}-v)(t), \nabla \mathsf{u}^\alpha(t)\rangle - 2\alpha\langle \nabla(\bar{u}-v)_0, \nabla \mathsf{u}_0^\alpha\rangle \\
&\quad - 2\alpha\langle v(t), \Delta \mathsf{u}^\alpha(t)\rangle + 2\alpha\langle v_0, \Delta \mathsf{u}_0^\alpha\rangle,
\end{aligned}$$

$$\begin{aligned}
I_2(t) &= \alpha\nu \sum_{k\in\mathcal{K}} \int_0^t \|\mathcal{A}^{1/2}(I+\alpha\mathcal{A})^{-1} G^k(\mathsf{u}^\alpha)\|^2 ds + \alpha^2\nu \sum_{k\in\mathcal{K}} \int_0^t \|\mathcal{A}(I+\alpha\mathcal{A})^{-1} G^k(\mathsf{u}^\alpha(s))\|^2 ds \\
&\quad + \nu \sum_{k\in\mathcal{K}} \int_0^t \|G^k(\mathsf{u}^\alpha(s))\|^2 ds - \alpha\nu \sum_{k\in\mathcal{K}} \int_0^t \langle \mathcal{A}(I+\alpha\mathcal{A})^{-1} G^k(\mathsf{u}^\alpha(s)), G^k(\mathsf{u}^\alpha(s))\rangle ds \\
&\quad + 2\nu \int_0^t \langle F(\mathsf{u}^\alpha(s)), W^\alpha(s)\rangle ds,
\end{aligned}$$

$$I_3(t) = -2\alpha \int_0^t \langle \nabla\partial_s(\bar{u}(s)-v(s)), \nabla \mathsf{u}^\alpha(s)\rangle ds + 2\alpha \int_0^t \langle \partial_s v(s), \Delta \mathsf{u}^\alpha(s)\rangle ds,$$

$$I_4(t) = -2\nu \int_0^t \langle W^\alpha(s), \mathcal{A}\mathsf{u}^\alpha(s)\rangle ds,$$

$$I_5(t) = -2 \int_0^t b(W^\alpha(s), \bar{u}(s), W^\alpha(s))\rangle ds,$$

$$I_6(t) = -2\alpha \int_0^t b(W^\alpha(s), \Delta\mathsf{u}^\alpha(s), \mathsf{u}^\alpha(s))ds + 2\alpha \int_0^t b(\mathsf{u}^\alpha(s), \Delta\mathsf{u}^\alpha(s), W^\alpha(s))ds,$$

$$M(t) = 2\sqrt{\nu} \sum_{k\in\mathcal{K}} \int_0^t \langle G^k(\mathsf{u}^\alpha(s)), W^\alpha(s)\rangle dW_s^k.$$

Our approach is almost completely pathwise. Therefore we need to estimate the terms $I_i(t),\ t \in \{1,\dots,6\}$. The analysis of $I_1(t)$ follows by Young's inequality, the estimates on the boundary layer corrector Proposition 4.14 and the interpolation estimate (4.36)

$$\begin{aligned}
I_1(t) &\le \|W_0^\alpha\|^2 + \alpha\|\nabla\mathsf{u}_0^\alpha\|^2 + C\alpha\delta^{1/2}\|\nabla\mathsf{u}_0^\alpha\|^{1/2}\|\mathsf{u}_0^\alpha\|_{H^3}^{1/2} \\
&\quad + C\alpha(1+\delta^{-1/2})\|\nabla\mathsf{u}^\alpha(t)\| + C\alpha\delta^{1/2}\|\nabla\mathsf{u}^\alpha(t)\|^{1/2}\|\mathsf{u}^\alpha(t)\|_{H^3}^{1/2} \\
&\quad + C\alpha(1+\delta^{-1/2})\|\nabla\mathsf{u}_0^\alpha\| \\
&\le \|W_0^\alpha\|^2 + C\alpha\|\nabla\mathsf{u}_0^\alpha\|^2 + C\alpha(1+\delta^{-1}) \\
&\quad + C\alpha^3\delta(\|\mathsf{u}_0^\alpha\|_{H^3}^2 + \|\mathsf{u}^\alpha(t)\|_{H^3}^2) + C\delta^{1/2} + \frac{\alpha}{2}\|\nabla\mathsf{u}^\alpha(t)\|^2.
\end{aligned} \tag{4.66}$$

The analysis of $I_2(t)$ follows by Young's inequality and the results of Lemma 4.16. Indeed it holds

$$\begin{aligned}
I_2(t) &\le C\nu \sum_{k\in\mathcal{K}} \|\sigma_k\|_{L^\infty}^2 \int_0^t \|\nabla\mathsf{u}^\alpha(s)\|^2 ds + \frac{\nu}{\sqrt{\alpha}} \sum_{k\in\mathcal{K}} \|\sigma_k\|_{L^\infty}^2 \int_0^t \|W^\alpha(s)\|\|\nabla\mathsf{u}^\alpha(s)\| ds \\
&\le C\nu \sum_{k\in\mathcal{K}} \|\sigma_k\|_{L^\infty}^2 \int_0^t \|\nabla\mathsf{u}^\alpha(s)\|^2 ds + \sum_{k\in\mathcal{K}} \|\sigma_k\|_{L^\infty}^2 \int_0^t \|W^\alpha(s)\|^2 ds \\
&\quad + \left(\frac{\nu}{\sqrt{\alpha}}\right)^2 \sum_{k\in\mathcal{K}} \|\sigma_k\|_{L^\infty}^2 \int_0^t \|\nabla\mathsf{u}^\alpha(s)\|^2 ds.
\end{aligned} \tag{4.67}$$

The analysis of $I_3(t)$ follows by Young's inequality, the estimates on the boundary layer corrector Proposition 4.14 and the interpolation estimate (4.36)

$$\begin{aligned}
I_3(t) &\le C\alpha(1+\delta^{-1/2}) \int_0^t \|\nabla\mathsf{u}^\alpha(s)\| ds + C\alpha\delta^{1/2} \int_0^t \|\nabla\mathsf{u}^\alpha(s)\|^{1/2}\|\mathsf{u}^\alpha(s)\|_{H^3}^{1/2} \\
&\le C\delta^{1/2} + C\alpha(1+\delta^{-1}) + C\alpha \int_0^t \|\nabla\mathsf{u}^\alpha(s)\|^2 ds + C\alpha^3\delta \int_0^t \|\mathsf{u}^\alpha(s)\|_{H^3}^2 ds.
\end{aligned} \tag{4.68}$$

The analysis of $I_4(t)$ is analogous to equations (3.20)-(3.21)-(3.22) in [134], it implies:

$$\begin{aligned}-2\nu\int_0^t \langle W^\alpha(s),\mathcal{A}\mathsf{u}^\alpha(s)\rangle ds &\le -2\nu\int_0^t \|\nabla \mathsf{u}^\alpha(s)\|^2 ds + C\frac{\nu}{\sqrt{\alpha}}(1+\delta^{-1/2})\int_0^t \sqrt{\alpha}\|\nabla \mathsf{u}^\alpha(s)\|ds\\ &+\frac{C\nu\delta^{1/2}}{\alpha}\int_0^t \alpha\|\Delta \mathsf{u}^\alpha(s)\|ds.\end{aligned}$$

Therefore by the interpolation inequality (4.36) and Young's inequality we have

$$\begin{aligned}-2\nu\int_0^t \langle W^\alpha(s),\mathcal{A}\mathsf{u}^\alpha(s)\rangle ds &\le -2\nu\int_0^t \|\nabla \mathsf{u}^\alpha(s)\|^2 ds + C\alpha\int_0^t \|\nabla \mathsf{u}^\alpha(s)\|^2 ds\\ &+C\alpha^3\delta\int_0^t \|\mathsf{u}^\alpha(s)\|_{H^3}^2 ds + C\left(\frac{\nu}{\alpha}\right)^2\delta^{1/2} + C\left(\frac{\nu}{\sqrt{\alpha}}\right)^2(1+\delta^{-1}).\end{aligned} \tag{4.69}$$

The analysis of $I_5(t)$ follows immediately by Hölder's inequality:

$$I_5(t)\le \|\bar{u}\|_{L^\infty(0,T;H^3)}\int_0^t \|W^\alpha(s)\|^2 ds. \tag{4.70}$$

For what concerns the analysis of $I_6(t)$, preliminary we observe that

$$\begin{aligned}-2\alpha b(W^\alpha,\Delta \mathsf{u}^\alpha,\mathsf{u}^\alpha)+2\alpha b(\mathsf{u}^\alpha,\Delta \mathsf{u}^\alpha,W^\alpha) &= 2\alpha b(\bar{u},\Delta \mathsf{u}^\alpha,\mathsf{u}^\alpha)-2\alpha b(\mathsf{u}^\alpha,\Delta \mathsf{u}^\alpha,\mathsf{u}^\alpha)\\ &+2\alpha b(\mathsf{u}^\alpha,\Delta \mathsf{u}^\alpha,\mathsf{u}^\alpha)-2\alpha b(\mathsf{u}^\alpha,\Delta \mathsf{u}^\alpha,\bar{u}).\end{aligned}$$

Arguing as in [132], equations (4.18)-(4.19) we get

$$I_6(t)\le C\alpha\left(1+\|\bar{u}\|_{L^\infty(0,T;H^3)}\right)\int_0^t \|\nabla \mathsf{u}^\alpha(s)\|^2 ds + C\alpha\|\bar{u}\|_{L^\infty(0,T;H^3)}^4\int_0^t \|\mathsf{u}^\alpha(s)\|^2 ds. \tag{4.71}$$

Combining equations (4.66),(4.67),(4.68),(4.69),(4.70),(4.71) and exploiting our assumptions on the behavior of ν, α, see Hypothesis 4.8, we have the integral relation below:

$$\begin{aligned}\|W^\alpha(t)\|^2+\frac{\alpha}{2}\|\nabla \mathsf{u}^\alpha(t)\|^2 &\le M(t)+C\alpha(1+\delta^{-1})+C\delta^{1/2}+\|W_0^\alpha\|^2+C\alpha\|\nabla \mathsf{u}_0^\alpha\|^2\\ &+C\alpha^3\delta(\|\mathsf{u}_0^\alpha\|_{H^3}^2+\|\mathsf{u}^\alpha(t)\|_{H^3}^2)\\ &+C\int_0^t \|W^\alpha(s)\|^2+\alpha\|\nabla \mathsf{u}^\alpha(s)\|^2 ds\\ &+C\alpha^3\delta\int_0^t \|\mathsf{u}^\alpha(s)\|_{H^3}^2 ds + C\alpha\int_0^t \|\mathsf{u}^\alpha(s)\|^2 ds.\end{aligned} \tag{4.72}$$

The relation above, implies by Theorem 4.18:

$$\begin{aligned}
&\sup_{t\in[0,T]}\mathbb{E}\left[\|W^\alpha(t)\|^2\right]+\alpha\sup_{t\in[0,T]}\mathbb{E}\left[\|\nabla\mathsf{u}^\alpha(t)\|^2\right]\\
&\leq C\left(\alpha\mathbb{E}\left[\int_0^T\|\mathsf{u}^\alpha(s)\|^2ds\right]+\alpha^3\delta\int_0^T\|\mathsf{u}^\alpha(s)\|^2_{H^3}ds+\alpha^3\delta\mathbb{E}\left[\sup_{t\in[0,T]}\|\mathsf{u}^\alpha(t)\|^2_{H^3}\right]\right)\\
&+C\left(\alpha(1+\delta^{-1})+\delta^{1/2}+\mathbb{E}\left[\|W^\alpha_0\|^2+\alpha\|\nabla\mathsf{u}^\alpha_0\|^2+\alpha^3\delta\|\mathsf{u}^\alpha_0\|^2_{H^3}\right]\right). \qquad (4.73)
\end{aligned}$$

Thanks to Hypothesis 4.8 and our assumptions on δ, see equation (4.62), we have that

$$\alpha(1+\delta^{-1})+\delta^{1/2}+\mathbb{E}\left[\|W^\alpha_0\|^2+\alpha\|\nabla\mathsf{u}^\alpha_0\|^2+\alpha^3\delta\|\mathsf{u}^\alpha_0\|^2_{H^3}\right]\to 0. \qquad (4.74)$$

Thanks to (4.22) and (4.23), we have that

$$\alpha\mathbb{E}\left[\int_0^T\|\mathsf{u}^\alpha(s)\|^2ds\right]+\alpha^3\delta\int_0^T\|\mathsf{u}^\alpha(s)\|^2_{H^3}ds+\alpha^3\delta\mathbb{E}\left[\sup_{t\in[0,T]}\|\mathsf{u}^\alpha(t)\|^2_{H^3}\right]\to 0. \qquad (4.75)$$

Therefore

$$\sup_{t\in[0,T]}\mathbb{E}\left[\|W^\alpha(t)\|^2\right]+\alpha\sup_{t\in[0,T]}\mathbb{E}\left[\|\nabla\mathsf{u}^\alpha(t)\|^2\right]\to 0. \qquad (4.76)$$

Restarting from equation (4.72) and considering the expected value of the supremum of both the terms in the left hand side we have

$$\begin{aligned}
&\mathbb{E}\left[\sup_{t\in[0,T]}\|W^\alpha(t)\|^2\right]+\alpha\mathbb{E}\left[\sup_{t\in[0,T]}\|\nabla\mathsf{u}^\alpha(t)\|^2\right]\\
&\leq C\left(\alpha\mathbb{E}\left[\int_0^T\|\mathsf{u}^\alpha(s)\|^2ds\right]+\alpha^3\delta\int_0^T\|\mathsf{u}^\alpha(s)\|^2_{H^3}ds+\alpha^3\delta\mathbb{E}\left[\sup_{t\in[0,T]}\|\mathsf{u}^\alpha(t)\|^2_{H^3}\right]\right)\\
&+C\left(\alpha(1+\delta^{-1})+\delta^{1/2}+\mathbb{E}\left[\|W^\alpha_0\|^2+\alpha\|\nabla\mathsf{u}^\alpha_0\|^2+\alpha^3\delta\|\mathsf{u}^\alpha_0\|^2_{H^3}\right]\right)\\
&+C\mathbb{E}\left[\sup_{t\in[0,T]}M(t)\right]+C\mathbb{E}\left[\int_0^T\|W^\alpha(s)\|^2+\alpha\|\nabla\mathsf{u}^\alpha(s)\|^2ds\right]. \qquad (4.77)
\end{aligned}$$

We already proved that almost all the terms in the right hand side of equation (4.77) go to 0. Therefore in order to complete the proof we left to show that

$$\mathbb{E}\left[\sup_{t\in[0,T]}M(t)\right]+\mathbb{E}\left[\int_0^T\|W^\alpha(s)\|^2+\alpha\|\nabla\mathsf{u}^\alpha(s)\|^2ds\right]\to 0.$$

By the weaker convergence described by equation (4.76) and Fubini's Theorem

$$\mathbb{E}\left[\int_0^t\|W^\alpha(s)\|^2+\alpha\|\nabla\mathsf{u}^\alpha(s)\|^2ds\right]\to 0.$$

For what concerns the other, the convergence follows by Burkholder-Davis-Gundy inequality, Hypothesis 4.8, equation (4.76), Fubini's Theorem and relation (4.24). Indeed

$$\begin{aligned}
\mathbb{E}\left[\sup_{t\in[0,T]} M(t)\right] &\leq C\sqrt{\nu}\mathbb{E}\left[\left(\sum_{k\in\mathcal{K}}\int_0^T \|G^k(\mathsf{u}^\alpha(s))\|^2\|W^\alpha(s)\|^2 ds\right)^{1/2}\right]\\
&\leq C\left(\sum_{k\in\mathcal{K}}\|\sigma_k\|_{L^\infty}^2\right)^{1/2}\sqrt{\nu}\mathbb{E}\left[\left(\int_0^T\|\nabla \mathsf{u}^\alpha(s)\|^2\|W^\alpha(s)\|^2 ds\right)^{1/2}\right]\\
&\leq C\left(\sum_{k\in\mathcal{K}}\|\sigma_k\|_{L^\infty}^2\right)^{1/2}\sqrt{\nu}\mathbb{E}\left[\sup_{t\in[0,T]}\|\nabla \mathsf{u}^\alpha(t)\|\left(\int_0^T\|W^\alpha(s)\|^2 ds\right)^{1/2}\right]\\
&\leq C\left(\mathbb{E}\left[\int_0^T\|W^\alpha(s)\|^2 ds\right]\mathbb{E}\left[\alpha\sup_{t\in[0,T]}\|\nabla \mathsf{u}^\alpha(t)\|^2\right]\sum_{k\in\mathcal{K}}\|\sigma_k\|_{L^\infty}^2\right)^{1/2}\\
&\to 0.
\end{aligned}$$

Now the proof is complete. □

Remark 4.28. Combining Theorem 4.10 and Theorem 4.22 we understand that, if $\nu = O(\alpha)$, the assumptions on the behavior of the initial conditions u_0^α in norm H, H^1 and H^3 are satisfied also for $t \in [0,T]$.

4.4.2 The Case of Additive Noise

For what concerns the case with additive noise, the well-posedness is a well-known fact for which we refer to [152] and we can prove a result completely analogous to Theorem 4.22, following exactly the same argument. The energy estimates of Theorem 4.10 were the crucial tool in the proof of Theorem 4.22. Thus, in this section we want to explain an approach to prove these energy estimates in the case of additive noise. This way of proceed cannot be applied to the case of transport noise due to the presence of the operators G^k and F, for which a more involved proof is needed, see [135, Section 4]. These computations are more similar to what happens in the deterministic framework. We keep previous assumptions on the coefficients σ_k and the Brownian motions W^k. Thus we consider

$$\begin{cases}
d\mathsf{v}^\alpha = (\nu\Delta\mathsf{u}^\alpha - \operatorname{curl}(\mathsf{v}^\alpha)\times\mathsf{u}^\alpha + \nabla\mathsf{P}^\alpha)dt + \sqrt{\nu}\sum_{k\in\mathcal{K}}\sigma_k dW_t^k\\
\operatorname{div}\mathsf{u}^\alpha = 0\\
\mathsf{v}^\alpha = \mathsf{u}^\alpha - \alpha\Delta\mathsf{u}^\alpha\\
\mathsf{u}^\alpha|_{\partial\mathcal{O}} = 0\\
\mathsf{u}^\alpha(0) = \mathsf{u}_0^\alpha.
\end{cases} \tag{4.78}$$

Before going on, we need to recall a result of [132].

Lemma 4.29. *Let $q \in L^2(\mathcal{O})$, there exists a unique $\varphi \in H_0^2(\mathcal{O})$ solution of*

$$\begin{cases} \Delta\varphi - \alpha\Delta^2\varphi = q \\ \varphi|_{\partial\mathcal{O}} = \hat{n}\cdot\nabla\varphi|_{\partial\mathcal{O}} = 0 \end{cases}$$

which satisfies

$$\langle\nabla\varphi, \nabla v\rangle + \alpha\langle\Delta\varphi, \Delta v\rangle = -\langle q, v\rangle \text{ for each } v \in H_0^2.$$

Moreover, the solution map is continuous from $L^2(\mathcal{O})$ to $H_0^2(\mathcal{O}) \cap H^4(\mathcal{O})$.

Thanks to this lemma, we can define an operator $\mathbb{K} : L^2(\mathcal{O}) \to H^3(\mathcal{O}) \cap W_0^{1,\infty}(\mathcal{O})$ which associates to each $q \in L^2(\mathcal{O})$ the vector field $u = \nabla^\perp\varphi$, where φ is the solution of the equation of Lemma 4.29.

Definition 4.30. A stochastic process u^α weakly continuous with values in W and continuous with values in V is a weak solution of equation (4.78) if

$$\mathsf{u}^\alpha \in L^p(\Omega, \mathcal{F}, \mathbb{P}; L^\infty(0,T;W)), \ \forall p \geq 2.$$

and $\mathbb{P}-a.s.$ for every $t \in [0,T]$ and $\varphi \in \mathsf{D}(\mathcal{A})$ we have

$$\begin{aligned}\langle\mathsf{u}^\alpha(t), (I+\alpha\mathcal{A})\varphi\rangle - \langle\mathsf{u}_0^\alpha, (I+\alpha\mathcal{A})\varphi\rangle &= -\nu\int_0^t \langle\mathsf{u}^\alpha(s), \mathcal{A}\varphi\rangle ds \\ &\quad - \int_0^t b(\mathsf{u}^\alpha(s), \mathsf{u}^\alpha(s) - \alpha\Delta\mathsf{u}^\alpha(s), \varphi)ds \\ &\quad - \alpha\int_0^t b(\varphi, \Delta\mathsf{u}^\alpha(s), \mathsf{u}^\alpha(s))ds + \sqrt{\nu}\sum_{k\in\mathcal{K}}\langle\sigma_k, \varphi\rangle W_t^k.\end{aligned}$$

Arguing as in the first part of the proof of Theorem 4.22 we can prove the following result.

Lemma 4.31. *Let u^α be a weak solution of problem (4.78) in the sense of Definition 4.30, then the following relations hold true*

1.

$$\begin{aligned} d\|\mathsf{u}^\alpha\|^2 + \alpha d\|\nabla\mathsf{u}^\alpha\|^2 = (-2\nu\|\nabla\mathsf{u}^\alpha\|^2 + \nu\sum_{k\in\mathcal{K}}\langle\sigma_k, (I+\alpha\mathcal{A})^{-1}\sigma_k\rangle)dt \\ + 2\sqrt{\nu}\sum_{k\in\mathcal{K}}\langle\sigma_k, \mathsf{u}^\alpha\rangle dW_t^k \end{aligned}$$

2.

$$\begin{aligned}&\mathbb{E}\left[\|\mathsf{u}^\alpha(t)\|^2\right]+\alpha\mathbb{E}\left[\|\nabla\mathsf{u}^\alpha(t)\|^2\right]+2\nu\int_0^t\mathbb{E}\left[\|\nabla\mathsf{u}^\alpha(s)\|^2\right]ds\\&=\mathbb{E}\left[\|\mathsf{u}_0^\alpha\|^2\right]+\alpha\mathbb{E}\left[\|\nabla\mathsf{u}_0^\alpha\|^2\right]+t\nu\sum_{k\in\mathcal{K}}\langle\sigma_k,(I+\alpha\mathcal{A})^{-1}\sigma_k\rangle\end{aligned}$$

3.

$$\begin{aligned}&\mathbb{E}\left[\sup_{t\in[0,T]}\|\mathsf{u}^\alpha(t)\|^2\right]+\alpha\mathbb{E}\left[\sup_{t\in[0,T]}\|\nabla\mathsf{u}^\alpha(t)\|^2\right]+2\nu\int_0^T\mathbb{E}\left[\|\nabla\mathsf{u}^\alpha(s)\|^2\right]ds\\&\lesssim\mathbb{E}\left[\|\mathsf{u}_0^\alpha\|^2\right]+\alpha\mathbb{E}\left[\|\nabla\mathsf{u}_0^\alpha\|^2\right]+T\nu\sum_{k\in\mathcal{K}}\langle\sigma_k,(I+\alpha\mathcal{A})^{-1}\sigma_k\rangle\\&+\sqrt{\nu}\mathbb{E}\left[\left(\int_0^T\sum_{k\in\mathcal{K}}\langle\sigma_k,\mathsf{u}^\alpha(s)\rangle^2ds\right)^{1/2}\right]\end{aligned}$$

Let us introduce the vorticity formulation of (4.78), we denote $s_k=\operatorname{curl}\sigma_k$

$$\begin{cases}d\mathsf{q}^\alpha+\left(\frac{\nu}{\alpha}(\mathsf{q}^\alpha-\operatorname{curl}\mathsf{u}^\alpha)+\mathsf{u}^\alpha\cdot\nabla\mathsf{q}^\alpha\right)dt=\sqrt{\nu}\sum_{k\in\mathcal{K}}s_k dW_t^k\\\operatorname{div}\mathsf{u}^\alpha=0\\\mathsf{q}^\alpha=\operatorname{curl}(\mathsf{u}^\alpha-\alpha\Delta\mathsf{u}^\alpha)\\\mathsf{q}^\alpha(0)=\mathsf{q}_0^\alpha:=\operatorname{curl}(\mathsf{u}_0^\alpha-\alpha\Delta\mathsf{u}_0^\alpha)\\\mathsf{u}^\alpha|_{\partial\mathcal{O}}=0\end{cases}\tag{4.79}$$

Definition 4.32. A stochastic process q^α, which is weakly continuous with values in $L^2(\mathcal{O})$ and continuous with values in $H^{-1}(\mathcal{O})$, is a weak solution of equation (4.79) if

$$\mathsf{q}^\alpha\in L^p(\Omega,\mathcal{F},\mathbb{P};L^\infty(0,T;L^2(D))),\ \forall p\geq 2.$$

and $\mathbb{P}-a.s.$ for every $t\in[0,T]$ and $\varphi\in H_0^2(\mathcal{O})$ we have

$$\begin{aligned}\langle\mathsf{q}^\alpha(t),\varphi\rangle-\langle\mathsf{q}_0^\alpha,\varphi\rangle&=\int_0^t\int_{\mathcal{O}}\mathsf{u}^\alpha(s)\cdot\nabla\varphi\mathsf{q}^\alpha(s)\ dxds-\frac{\nu}{\alpha}\int_0^t\int_{\mathcal{O}}(\mathsf{q}^\alpha(s)-\operatorname{curl}\mathsf{u}^\alpha(s))\varphi\ dxds\\&+\sqrt{\nu}\sum_{k\in\mathcal{K}}\langle s_k,\varphi\rangle W_t^k\quad\mathbb{P}-a.s.\end{aligned}$$

$\mathsf{u}^\alpha=\nabla^\perp\mathbb{K}\mathsf{q}^\alpha.$

Let us obtain a result about the equivalence between the solutions of these two problems. Since we know from the results of [152] that problem (4.78) is well-posed, then problem (4.79) is well-posed as well.

Proposition 4.33. *Let* u^α *be a solution of* (4.78) *in the sense of Definition 4.30, then* $\mathsf{q}^\alpha := \operatorname{curl}(\mathsf{u}^\alpha - \alpha\Delta\mathsf{u}^\alpha)$ *is a solution of* (4.79) *in the sense of Definition 4.32. Conversely, if* q^α *is a solution of* (4.79) *in the sense of Definition 4.32 then* $\mathsf{u}^\alpha = \nabla^\perp\mathbb{K}\mathsf{q}^\alpha$ *is a solution of* (4.78) *in the sense of Definition 4.30.*

Proof. Definition 4.30 $\Longrightarrow$ Definition 4.32 is immediate taking $\varphi = -\nabla^\perp\tilde{\varphi}$, $\tilde{\varphi} \in H^2_0(D)$ as test function for problem (4.78).
Therefore it remains to show that Definition 4.32 $\Longrightarrow$ Definition 4.30. We take $\mathsf{u}^\alpha = \nabla^\perp\mathbb{K}\mathsf{q}^\alpha$, $\mathsf{v}^\alpha = \mathsf{u}^\alpha - \alpha\Delta\mathsf{u}^\alpha$, where φ is obtained by Lemma 4.29 and $\varphi = -\nabla^\perp\tilde{\varphi}$, where $\tilde{\varphi} \in H^2_0(\mathcal{O})$. Then integrating by parts and exploiting that $\operatorname{curl}\nabla^\perp = \Delta$, $\Delta\mathbb{K}\mathsf{q}^\alpha - \alpha\Delta^2\mathbb{K}\mathsf{q}^\alpha = \mathsf{q}^\alpha$ and q^α is a solution of (4.79) in the sense of Definition 4.32 we get

$$\begin{aligned}
&-\langle (I-\alpha\Delta)\mathsf{u}^\alpha(t), \nabla^\perp\tilde{\varphi}\rangle + \langle (I-\alpha\Delta)u_0, \nabla^\perp\tilde{\varphi}\rangle - \frac{\nu}{\alpha}\int_0^t \langle v(s) - \mathsf{u}^\alpha(s), \nabla^\perp\tilde{\varphi}\rangle ds\\
&+\int_0^t\int_{\mathcal{O}} (\mathsf{u}^\alpha(s)\cdot\nabla)\nabla^\perp\tilde{\varphi}\mathsf{v}^\alpha(s)\ dxds - \alpha\int_0^t\int_{\mathcal{O}} (\nabla^\perp\tilde{\varphi}\cdot\nabla)\Delta\mathsf{u}^\alpha(s)\mathsf{u}^\alpha(s)dxds\\
&+\sqrt{\nu}\sum_{k\in\mathcal{K}}\langle\sigma_k, \nabla^\perp\tilde{\varphi}\rangle W^k_t = 0 \quad \mathbb{P}-a.s.
\end{aligned}$$

From the last relation the result follows if we are able to prove the continuity properties of u^α. The weak continuity of u^α with values in W follows immediately from the regularity of q^α and Lemma 4.29. Again by Lemma 4.29 we get the strong continuity of u^α with values in V. In fact, via Lax-Milgram lemma we get the regularity of the solution mapping of the problem described in Lemma 4.29 between $H^{-2}(\mathcal{O})$ and $H^2_0(\mathcal{O})$. Via interpolation techniques we recover the regularity of the solution mapping between $H^{-1}(\mathcal{O})$ and $H^3(\mathcal{O}) \cap H^2_0(\mathcal{O})$, therefore the required regularity for u^α.

□

Approximating the process $\mathsf{q}^\alpha(t)$ solution of (4.79) by the eigenvectors of the Laplacian with Dirichlet boundary conditions and then arguing as in the first part of the proof of Theorem 4.22, we can obtain some Itô's formula and energy estimates. Moreover, if $\mathsf{u}^\alpha \in V$ we have $\|\nabla\mathsf{u}^\alpha\|^2 = \|\operatorname{curl}\mathsf{u}^\alpha\|^2$. Thanks to Proposition 4.33, we know that u^α appearing in problem (4.79) is a solution of problem (4.78). Therefore, thanks to Lemma 4.31 we know that

$$2\nu\int_0^t \mathbb{E}\left[\|\nabla\mathsf{u}^\alpha(s)\|^2\right] ds \le \mathbb{E}\left[\|\mathsf{u}^\alpha_0\|^2\right] + \alpha\mathbb{E}\left[\|\nabla\mathsf{u}^\alpha_0\|^2\right] + t\nu\sum_{k\in\mathcal{K}}\langle\sigma_k, (I+\alpha\mathcal{A})^{-1}\sigma_k\rangle, \tag{4.80}$$

$$\alpha\mathbb{E}\left[\|\nabla\mathsf{u}^\alpha(t)\|^2\right] \le \mathbb{E}\left[\|\mathsf{u}^\alpha_0\|^2\right] + \alpha\mathbb{E}\left[\|\nabla\mathsf{u}^\alpha_0\|^2\right] + t\nu\sum_{k\in\mathcal{K}}\langle\sigma_k, (I+\alpha\mathcal{A})^{-1}\sigma_k\rangle \tag{4.81}$$

and we can obtain the following energy relations.

Lemma 4.34. *Let q^α be a weak solution of problem (4.79) in the sense of Definition 4.32, then the following relations hold true*

1.

$$d\|\mathsf{q}^\alpha\|^2 = -\frac{2\nu}{\alpha}\langle \mathsf{q}^\alpha - \operatorname{curl}\mathsf{u}^\alpha, \mathsf{q}^\alpha\rangle dt + \nu\sum_{k\in\mathcal{K}}\|s_k\|^2\ dt + 2\sqrt{\nu}\sum_{k\in\mathcal{K}}\langle s_k, \mathsf{q}^\alpha\rangle dW_t^k$$

2.

$$\begin{aligned}\mathbb{E}\left[\|\mathsf{q}^\alpha(t)\|^2\right] \le e^{-\frac{\nu}{\alpha}t}\mathbb{E}\left[\|\mathsf{q}_0^\alpha\|^2\right] + \alpha(1-e^{-\frac{\nu t}{\alpha}})\sum_{k\in\mathcal{K}}\|s_k\|^2 \\ + \frac{1}{2\nu}\left(\mathbb{E}\left[\|\mathsf{u}_0^\alpha\|^2\right] + \alpha\mathbb{E}\left[\|\nabla\mathsf{u}_0^\alpha\|^2\right] + T\nu\sum_{k\in\mathcal{K}}\langle\sigma_k, (I+\alpha\mathcal{A})^{-1}\sigma_k\rangle\right)\end{aligned}$$

3.

$$\begin{aligned}\mathbb{E}\left[\sup\nolimits_{t\in[0,T]}\|\mathsf{q}^\alpha(t)\|^2\right] \le \mathbb{E}\left[\|\mathsf{q}_0^\alpha\|^2\right] + \nu\sum_{k\in\mathcal{K}}\|s_k\|^2 T \\ + C\sqrt{\nu}\mathbb{E}\left[\left(\sum_{k\in\mathcal{K}}\int_0^T\langle s_k, \mathsf{q}^\alpha(s)\rangle^2 ds\right)^{1/2}\right] \\ + \frac{1}{2\alpha}\left(\mathbb{E}\left[\|\mathsf{u}_0^\alpha\|^2\right] + \alpha\mathbb{E}\left[\|\nabla\mathsf{u}_0^\alpha\|^2\right] + T\nu\sum_{k\in\mathcal{K}}\langle\sigma_k, (I+\alpha\mathcal{A})^{-1}\sigma_k\rangle\right).\end{aligned}$$

Remark 4.35. We can control the H^3 norm of u^α via the H^1 norm of u and the L^2 norm of q^α in the following way

$$\begin{aligned}\|\mathsf{u}^\alpha(t)\|_{H^3} &\le C\left(\|\nabla\mathsf{u}^\alpha(t)\| + \|\operatorname{curl}\Delta\mathsf{u}^\alpha(t)\|\right) \\ &\le C\left(\frac{\|\mathsf{q}^\alpha(t)\|}{\alpha} + \frac{\|\operatorname{curl}\mathsf{u}^\alpha(t)\|}{\alpha} + \|\nabla\mathsf{u}^\alpha(t)\|\right) \\ &= C\left(\frac{\|\mathsf{q}^\alpha(t)\|}{\alpha} + \frac{\|\nabla\mathsf{u}^\alpha(t)\|}{\alpha} + \|\nabla\mathsf{u}^\alpha(t)\|\right). \qquad (4.82)\end{aligned}$$

Therefore, thanks to Lemma 4.34 it holds

$$\begin{aligned}\mathbb{E}\left[\|\mathsf{u}^\alpha(t)\|_{H^3}^2\right] \lesssim \frac{e^{-\frac{\nu}{\alpha}t}}{\alpha^2}\mathbb{E}\left[\|\mathsf{q}_0^\alpha\|^2\right] + \frac{1}{\alpha}(1-e^{-\frac{\nu t}{\alpha}})\sum_{k\in\mathcal{K}}\|s_k\|^2 \\ + \left(\frac{1}{\alpha} + \frac{1}{\alpha^3} + \frac{1}{\nu\alpha^2}\right)\left(\mathbb{E}\left[\|\mathsf{u}_0^\alpha\|^2\right] + \alpha\mathbb{E}\left[\|\nabla\mathsf{u}_0^\alpha\|^2\right] + T\nu\sum_{k\in\mathcal{K}}\langle\sigma_k, (I+\alpha\mathcal{A})^{-1}\sigma_k\rangle\right) \\ (4.83)\end{aligned}$$

$$\begin{aligned}\mathbb{E}\left[\sup_{t\in[0,T]}\|\mathsf{u}^\alpha(t)\|_{H^3}^2\right] \lesssim{}& \frac{1}{\alpha^2}\mathbb{E}\left[\|\mathsf{q}_0^\alpha\|^2\right] + \frac{\nu}{\alpha^2}\sum_{k\in\mathcal{K}}\|s_k\|^2 \\ &+ \frac{1}{\alpha^2}\mathbb{E}\left[\left(\sum_{k\in\mathcal{K}}\int_0^T \langle s_k, \mathsf{q}^\alpha(s)\rangle^2 ds\right)^{1/2}\right] \\ &+ \left(\frac{1}{\alpha^3}+\frac{1}{\alpha^2}\right)\left(\mathbb{E}\left[\|\mathsf{u}_0^\alpha\|^2\right] + \alpha\mathbb{E}\left[\|\nabla \mathsf{u}_0^\alpha\|^2\right] + \nu\sum_{k\in\mathcal{K}}\langle\sigma_k, (I+\alpha\mathcal{A})^{-1}\sigma_k\rangle\right) \\ &+ \sqrt{\nu}\left(\frac{1}{\alpha^3}+\frac{1}{\alpha^2}\right)\left(\sum_{k\in\mathcal{K}}\mathbb{E}\left[\int_0^T\langle\sigma_k, u(s)\rangle^2 ds\right]^{1/2}\right) \qquad (4.84)\end{aligned}$$

Thanks to Remark 4.35, if we consider the scaled equation with additive noise and initial condition u_0^α satisfying Hypothesis 4.8, then the following result follows immediately.

Lemma 4.36. *Under the same assumptions of Theorem 4.22, if* u^α *is the unique solution of* (4.78) *in the sense of Definition 4.30 of the problem with initial condition* u_0^α*, then*

$$\begin{aligned}&\mathbb{E}\left[\sup_{t\in[0,T]}\|\mathsf{u}^\alpha(t)\|^2\right] + \alpha\mathbb{E}\left[\sup_{t\in[0,T]}\|\nabla\mathsf{u}^\alpha(t)\|^2\right] + 2\nu\int_0^T\mathbb{E}\left[\|\nabla\mathsf{u}^\alpha(s)\|^2\right]ds = O(1), \\ &\mathbb{E}\left[\alpha^3\sup_{t\in[0,T]}\|\mathsf{u}^\alpha(t)\|_{H^3}^2\right] = O(1).\end{aligned}$$

Looking carefully at the proof of Theorem 4.22, Lemma 4.36 contains the crucial bounds on the norm of the solutions to obtain the inviscid limit. Therefore, following the same ideas of subsection 4.4.1, one can prove that the inviscid limit holds:

Theorem 4.37. *Under the same assumptions of Theorem 4.22, calling* u^α *the solution of the stochastic second-grade fluid equations with additive noise* (4.78) *and* $\bar{u}$ *the solution of* (4.3)*, then*

$$\lim_{\alpha\to 0}\mathbb{E}\left[\sup_{t\in[0,T]}\|\mathsf{u}^\alpha(t)-\bar{u}(t)\|^2\right] = 0.$$

Chapter 5

The Problem of the Inviscid Limit for Stochastic Fluids: Large Deviations Principle

The results of the previous chapter open to several possibilities in the analysis of the inviscid limit for stochastic fluid dynamical systems. Other stochastic regimes have been investigated by different authors in [94] and [178] assuming a Kato-type condition. While the possibility to find a stochastic regimes which provides some help in the study of the inviscid limit with no-slip boundary conditions is still an open problem of primary interest either from the mathematical and the physical viewpoint, in this chapter, following [37], we focus on a different problem with applicative consequences: the asymptotic behavior for the probability of large fluctuations away from the zero-noise and zero-viscosity limit. We are interested in establishing Large Deviations principles for

$$\begin{cases} du^\nu = (\nu\Delta u^\nu - u^\nu\cdot\nabla u^\nu + \nabla P^\nu)dt + \sqrt{\nu}dW(t)\\ \operatorname{div} u^\nu = 0\\ u^\nu|_{\partial\mathcal{O}} = 0\\ u^\nu(0) = u_0, \end{cases} \tag{5.1}$$

and

$$\begin{cases} d\mathsf{v}^\alpha = (\nu\Delta\mathsf{u}^\alpha - \operatorname{curl}(\mathsf{v}^\alpha)\times\mathsf{u}^\alpha + \nabla\mathsf{P}^\alpha)dt + \sqrt{\alpha}dW_t\\ \operatorname{div}\mathsf{u}^\alpha = 0\\ \mathsf{v}^\alpha = \mathsf{u}^\alpha - \alpha\Delta\mathsf{u}^\alpha\\ \mathsf{u}^\alpha|_{\partial\mathcal{O}} = 0\\ \mathsf{u}^\alpha(0) = \mathsf{u}_0, \end{cases} \tag{5.2}$$

focusing, in case, on their relations with some form of Kato-type condition. At the same time, the validity of a Freidlin-Wentzell kind of Large Deviation Principle allows, at least in principle, to characterize rare phenomena arising in these systems in the small noise small viscosity regime, by translating these problems into an optimal control one. This might have several applications, and constitutes one of the main strength of the theory of Large Deviations (see for example the book by Freidlin and Wentzell [83]). We will prove the validity of a Large Deviation Principle for the inviscid limit of Navier-Stokes equations and Second-Grade Fluid Equations with additive Gaussian noise in two-dimensional bounded domains and no-slip boundary conditions, presenting, for the first system, a natural Kato-type condition that closely resembles the ones from classical conditioned results of [111] and the previous chapter. The interplay between Kato-type conditions, i.e. some controls on the dissipation of the energy in the solutions of the stochastic Navier-Stokes equations within the boundary layer, and the large fluctuations away from the zero-noise and zero-viscosity limit is a technical issue that is addressed in this chapter.

Large Deviation for fluid dynamic models in 2D have been established in the case of Navier-Stokes with additive noise, see [42], and multiplicative noise, see [162]. While the first result is based on a technique developed by Freidlin and Wentzell, based on a discretization of the equation and the application of the so-called contraction principle, the second one resorts to the the Weak Convergence Approach developed in [36]. While the Freidlin-Wentzell technique is best suited for equations with additive noise, the Weak Convergence Approach has proved to be much more flexible in many other situations. As an example, in [89], the authors proved a large deviations principle for the convergence of the Euler equation with transport noise on the 2D torus to a deterministic Navier-Stokes system using the weak convergence approach. We adopt this approach as well, even if our equations have additive noise, as the vanishing of the viscosity together with the noise constitutes a technical issue that cannot be addressed via a classical contraction argument.

As already mentioned in subsection 1.5.1, the validity of a Large Deviation Principle for the inviscid limit of the Navier-Stokes equations with periodic or free boundary conditions has been shown in [25] using the weak convergence approach. Similarly to other results with these kind of boundary conditions, the result of [25] is based on the validity of the enstrophy equality, which allows to obtain stable estimates in the limit $\nu \to 0$ stronger than the one guaranteed by the energy equality. These relations are not available in the case of no-slip boundary conditions, due to the generation of vorticity close the boundary. Therefore, the introduction of some Kato-type hypothesis, see [111], is required in order to show the validity of the Large Deviation Principle, similarly to the validity of the inviscid limit. On the contrary, as showed also in the previous chapter, there are fluid dynamic frameworks where the inviscid limit holds in bounded domains without any assumption on the behavior of the solution in the boundary layer. This is the case of the second grade fluid equations with the scaling of the parameters introduced in [134] and the case of the Navier-Stokes equations in the open ball with forcing and initial conditions with radial symmetry to which part of this chapter is devoted.

In section 5.1 we introduce some facts used repeatedly among the chapter. In particular, in subsection 5.1.1, we recall several information about Large Deviation Principle, presenting its classical formulation and some equivalent formulations. We state the main theorems of [37] in section 5.2 . The analysis of the validity of a Large Deviation Principle for the inviscid limit of the Navier-Stokes equations is the object of section 5.3. In particular we prove Theorem 5.17 in subsection 5.3.1, namely the validity of the Large Deviation Principle assuming a Kato condition. In subsection 5.3.2 we show that the Large Deviation Principle holds unconditionally assuming initial conditions and forcing terms with radial symmetry. As discussed in section 4.2, this provides informations about the Kolmogorov's zeroth law of turbulence, see Remark 5.27 and Remark 5.33 below. In section 5.4 we prove the validity of the Large Deviation Principle for the Second-Grade fluid equations under the scaling of the parameter introduced in [134]. Lastly, we add some comments on the Kato conditions assumed in this chapter and the previous one in section 5.5.

5.1 Preliminaries

In this section we provide some tools in order to study the fluctuactions of our fluid dynamical models. First, we recall in subsection 5.1.1 the abstract framework of the weak convergence approach to Large Deviations developed in [36]. Secondly, in subsection 5.1.2 we expand the discussion of section 4.1 providing some specific well posedness results we will need in the following.

5.1.1 Large Deviations Principle

We begin with an usual filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$, $t \in [0, T]$. Let $\mathcal{U}$ be a separable Hilbert space and $\mathcal{Q}$ a trace-class operator on $\mathcal{U}$. As discussed in section 1.2, we denote by $\mathcal{H}_0$ the reproducing kernel Hilbert space of a Wiener process $\{W_t\}_{t\in[0,T]}$ on $\mathcal{U}$ with covariance operator $\mathcal{Q}$. It is a separable Hilbert space if endowed with the inner product

$$\langle g, h\rangle_{\mathcal{H}_0} = \langle \mathcal{Q}^{-1/2}g, \mathcal{Q}^{-1/2}h\rangle_{\mathcal{U}}.$$

For the convenience of the reader we also recall the space

$$S^N := S^N(\mathcal{H}_0) := \left\{v \in L^2(0,T;\mathcal{H}_0) : \quad \int_0^T \|v(s)\|^2_{\mathcal{H}_0} ds \le N\right\},$$

which is a Polish space when endowed with the weak topology and $\mathcal{P}_2 := \mathcal{P}_2(\mathcal{H}_0)$, the space of $\mathcal{H}_0$-valued, $\mathcal{F}_t$-predictable and $\mathbb{P}$-a.s. square integrable processes. Lastly we recall the following notation

$$\mathcal{P}_2^N := \{\varphi \in \mathcal{P}_2 : \ \varphi(\omega) \in S^N \quad \mathbb{P} - a.s.\}$$

Let $\mathcal{E}$ and $\mathcal{E}_0$ be Polish spaces.

Definition 5.1. A function $I : \mathcal{E} \to [0, \infty]$ is called a rate function if for any $M < \infty$, the level set $\{f \in \mathcal{E} : I(f) \leqslant M\}$ is a compact subset of $\mathcal{E}$. A family of rate functions I_x on $\mathcal{E}$, parametrized by $x \in \mathcal{E}_0$, is said to have compact level sets on compacts if for all compact subsets K of $\mathcal{E}_0$ and each $M < \infty$, $\cup_{x \in K}\{f \in \mathcal{E} : I_x(f) \leqslant M\}$ is a compact subset of $\mathcal{E}$.

Let us give now the definition of LDP in the original formulation by Varadhan (see [175])

Definition 5.2. We say that a Large Deviation Principle holds for a family μ^ε of probability measures on a metric space $(\mathcal{E}, d)$ with rate function I and speed ε if for every borel set Γ of $\mathcal{E}$

$$I(\mathring{\Gamma}) \leqslant \liminf_{\varepsilon \to 0} \varepsilon \log(\mu^\varepsilon(\Gamma)) \leqslant \limsup_{\varepsilon \to 0} \varepsilon \log(\mu^\varepsilon(\Gamma)) \leqslant I(\bar{\Gamma}) \tag{5.3}$$

where $I(A) := -\inf_A I$.

This condition has been proved equivalent by Bryc in [33] to the so called Laplace principle. Here, we state a uniform version of this principle, that is, we let I and μ^ε depend also on some parameter $x \in \mathcal{E}_0$.

Definition 5.3. (Uniform Laplace Principle) Let I_x be a family of rate functions on $\mathcal{E}$ parameterized by $x \in \mathcal{E}_0$ and assume that this family has compact level sets on compacts. The family of random variables $\{X^{x,\varepsilon}\}$ distributed according to $\mu^{x,\varepsilon}$ are said to satisfy the Laplace principle on $\mathcal{E}$ with rate function I_x, uniformly on compacts, if for all compact subsets $K \subset \mathcal{E}_0$ and all bounded continuous functions h mapping $\mathcal{E}$ into $\mathbb{R}$,

$$\lim_{\varepsilon \to 0} \sup_{x \in K} \left| \varepsilon \log \mathbb{E}_x \left[\exp\big(-\varepsilon^{-1} h(X^{x,\varepsilon}) \big) \right] + \inf_{f \in \mathcal{E}} \{h(f) - I_x(f)\} \right| = 0 \tag{5.4}$$

We are interested in the case when the family of measures μ^ε is given by the laws of some stochastic process $X^{x,\varepsilon}$ solving some SPDE and driven by $\varepsilon^{1/2} W_t$. In this case, we can often represent $X^{x,\varepsilon} = \mathcal{G}^\varepsilon(x, \varepsilon^{1/2} W)$ for some measurable map $\mathcal{G}^\varepsilon : \mathcal{E}_0 \times C([0,T]; \mathcal{U}) \to \mathcal{E}$. In this setting, in [36] the authors provided a handy criterium that allows to deduce the uniform Laplace principle. This is known as the *weak convergence approach* to Large deviations. The criterium goes as follows.

Hypothesis 5.4. There exists a measurable map $\mathcal{G}^0 : \mathcal{E}_0 \times C([0,T]; \mathcal{U}) \to \mathcal{E}$ such that:

1. For any $N < \infty$ and compact set $K \subset \mathcal{E}_0$, $\Gamma_{K,N} := \{\mathcal{G}^0(x, \int_0^\cdot v(s)ds) : v \in S^N,\ x \in K\}$ is a compact subset of $\mathcal{E}$.
2. Consider $N < \infty$ and families $\{x^\varepsilon\} \subset \mathcal{E}_0$, $\{u^\varepsilon\} \subset \mathcal{P}_2^N$ such that, as $\varepsilon \to 0$, $x^\varepsilon \to x$ and u^ε converge in law to u as S^N-valued random element, then $\mathcal{G}^\varepsilon\big(x^\varepsilon, \varepsilon^{1/2} W + \int_0^\cdot u^\varepsilon(s)ds\big)$ converges in law to $\mathcal{G}^0(x, \int_0^\cdot u(s)ds)$ in the topology of $\mathcal{E}$.

Theorem 5.5. *Let* $X^{\varepsilon,x} = \mathcal{G}^{\varepsilon}(x, \varepsilon^{1/2}W)$ *and suppose Hypothesis 5.4 holds. Define, for* $x \in \mathcal{E}_0$ *and* $f \in \mathcal{E}$

$$I_x(f) := \inf_{\{w \in L^2(0,T;\mathcal{H}_0):\ f = \mathcal{G}^0(x, \int_0^{\cdot} w(s)ds)\}} \int_0^T \|w(s)\|^2_{\mathcal{H}_0} ds$$

with the convention that $\inf \emptyset = +\infty$. *Assume that for all* $f \in \mathcal{E}$, $x \to I_x(f)$ *is a lower semicontinuous map from* $\mathcal{E}_0$ *to* $[0,\infty]$. *Then for all* $x \in \mathcal{E}_0$, $f \to I_x(f)$ *is a rate function on* E *and the family* $I_x x \in \mathcal{E}_0$ *of rate functions has compact level sets on compacts. Furthermore, the family* $\{X^{\varepsilon,x}\}$ *satisfies the Laplace principle on* $\mathcal{E}$, *with the rate functions* $\{I_x\}$, *uniformly on compact subsets of* $\mathcal{E}_0$.

5.1.2 Well-Known Facts on fluid dynamic models

First we observe that, if $c > 0$ and we denote by $\mathcal{S}^c(t)$ the strongly continuous semigroup generated by $-c\mathcal{A}$, by scaling the equation satisfied by $\mathcal{S}^c(t)u_0$, $u_0 \in H$, it follows immediately that

$$\mathcal{S}^c(t) = \mathcal{S}(ct) \quad \forall t \geq 0,\ c > 0.$$

We continue to assume Hypothesis 4.4 in this chapter and denote by H_0 the RKHS associated to W_t.

Remark 5.6. Hypothesis 4.4 implies in particular that $H_0 \hookrightarrow \mathsf{D}(\mathcal{A}^{\gamma})$ and that W is a process with continuous paths with values in $\mathsf{D}(\mathcal{A}^{\gamma})$. Since $\lambda_i \sim Ci$, see [104], a simple example of noise satisfying Hypothesis 4.4 is $W_t = \mathcal{A}^{-\gamma-1/2-\delta}W_H(t)$, $\delta > 0$ and $W_H(t)$ being the cylindrical Wiener process on H. With this particular choice of the coefficients σ_k, $H_0 = \mathsf{D}(\mathcal{A}^{\gamma+1/2+\delta})$.

Since we are going to prove the validity of the Large Deviation Principle via the weak convergence approach, we need to analyze the well-posedness of some partial differential equations, slightly more general than the ones considered in previous chapter. Therefore, let $\beta \geq 0$ and $f \in \mathcal{P}_2^N$, $N \geq 0$ arbitrary we consider the stochastic partial differential equations below

$$\begin{cases} du^{\nu} = (\nu\Delta u^{\nu} - u^{\nu}\cdot\nabla u^{\nu} + \nabla P^{\nu} + f)dt + \sqrt{\beta}dW(t) \\ \operatorname{div} u^{\nu} = 0 \\ u^{\nu}|_{\partial\mathcal{O}} = 0 \\ u^{\nu}(0) = u_0^{\nu}, \end{cases} \tag{5.5}$$

$$\begin{cases} d\mathsf{v}^{\alpha} = (\nu\Delta\mathsf{u}^{\alpha} - \operatorname{curl}(\mathsf{v}^{\alpha}) \times \mathsf{u}^{\alpha} + \nabla\mathsf{P}^{\alpha} + f)dt + \sqrt{\beta}dW(t) \\ \operatorname{div} \mathsf{u}^{\alpha} = 0 \\ \mathsf{v}^{\alpha} = \mathsf{u}^{\alpha} - \alpha\Delta\mathsf{u}^{\alpha} \\ \mathsf{u}^{\alpha}|_{\partial\mathcal{O}} = 0 \\ \mathsf{u}^{\alpha}(0) = \mathsf{u}_0^{\alpha}. \end{cases} \tag{5.6}$$

Definition 5.7. A stochastic process with continuous trajectories with values in H is a weak solution of equation (5.5) if

$$u^\nu \in C_{\mathcal{F}}([0,T];H) \cap L^2_{\mathcal{F}}(0,T;V)$$

and $\mathbb{P}-a.s.$ for every $t \in [0,T]$ and $\varphi \in \mathsf{D}(\mathcal{A})$ we have

$$\begin{aligned}\langle u^\nu_t - u_0, \varphi\rangle + \int_0^t \nu\langle \nabla u^\nu(s), \nabla\varphi\rangle = \int_0^t b(u^\nu(s), \varphi, u^\nu(s))ds + \int_0^t \langle f(s), \varphi\rangle ds \\ + \sqrt{\beta}\langle W(t), \varphi\rangle.\end{aligned}$$

Definition 5.8. A stochastic process with weakly continuous trajectories with values in W is a weak solution of equation (5.6) if

$$\mathsf{u}^\alpha \in L^2(\Omega, \mathcal{F}, \mathbb{P}; L^\infty(0,T;W))$$

and $\mathbb{P}-a.s.$ for every $t \in [0,T]$ and $\varphi \in W$ we have

$$\begin{aligned}&\langle \mathsf{u}^\alpha(t) - \mathsf{u}^\alpha_0, (I+\alpha\mathcal{A})\varphi\rangle + \int_0^t \nu\langle \nabla \mathsf{u}^\alpha(s), \nabla\varphi\rangle + \langle \mathrm{curl}(\mathsf{u}^\alpha(s) - \alpha\Delta\mathsf{u}^\alpha(s)) \times \mathsf{u}^\alpha, \varphi\rangle ds \\ &= \int_0^t \langle f(s), \varphi\rangle ds + \sqrt{\beta}\langle W(t), \varphi\rangle.\end{aligned}$$

The well-posedness of (5.5) (resp. (5.6)) in the sense of Definition 5.7 (resp. Definition 5.8) is guaranteed by Theorem 5.9 below, see [77] (resp. Theorem 5.10, see [152], [135, Section 6] and previous chapter).

Theorem 5.9. *For each $u^\nu_0 \in H^3(\mathcal{O};\mathbb{R}^2) \cap H$, $f \in \mathcal{P}^N_2$ there exists a unique weak solution of (5.5) in the sense of Definition 5.7. Moreover the following relation holds true*

$$\begin{aligned}\|u^\nu_t\|^2 + 2\nu\int_0^t \|\nabla u^\nu(s)\|^2 ds = \|u^\nu_0\|^2 + t\beta\sum_{k\in\mathcal{K}}\|\sigma_k\|^2 + 2\sqrt{\beta}\int_0^t \langle u^\nu(s), dW(s)\rangle \\ + 2\int_0^t \langle f(s), u^\nu(s)\rangle ds \quad \mathbb{P}-a.s. \end{aligned} \tag{5.7}$$

Theorem 5.10. *For each $\mathsf{u}^\alpha_0 \in W$, $f \in \mathcal{P}^N_2$ there exists a unique weak solution of (5.6) in the sense of Definition 5.8. Moreover u^α has continuous paths with values in V and it holds*

$$\begin{aligned}\|\mathsf{u}^\alpha(t)\|^2 + \alpha\|\nabla\mathsf{u}^\alpha(t)\|^2 + 2\nu\int_0^t \|\nabla\mathsf{u}^\alpha(s)\|^2 ds = \|\mathsf{u}^\alpha_0\|^2 + \alpha\|\nabla\mathsf{u}^\alpha_0\|^2 + t\beta\sum_{k\in\mathcal{K}}\|(I+\alpha\mathcal{A})^{-1/2}\sigma_k\|^2 \\ + 2\int_0^t \langle f(s), \mathsf{u}^\alpha(s)\rangle ds \\ + 2\sqrt{\beta}\int_0^t \langle \mathsf{u}^\alpha(s), dW(s)\rangle \quad \mathbb{P}-a.s. \end{aligned} \tag{5.8}$$

Calling $\mathsf{q}^\alpha = \operatorname{curl}(\mathsf{u}^\alpha - \alpha\Delta \mathsf{u}^\alpha),\ \ s_k = \operatorname{curl}\sigma_k,\ \ \mathsf{q}_0^\alpha = \operatorname{curl}(\mathsf{u}_0^\alpha - \alpha\Delta \mathsf{u}_0^\alpha)$ *it holds*

$$\begin{aligned}\|\mathsf{q}^\alpha(t)\| = \|\mathsf{q}_0^\alpha\|^2 - \frac{2\nu}{\alpha}\int_0^t \langle \mathsf{q}^\alpha(s) - \operatorname{curl}\mathsf{u}^\alpha(s), \mathsf{q}^\alpha(s)\rangle ds + t\beta\sum_{k\in\mathcal{K}}\|s_k\|^2 \\ + 2\sqrt{\beta}\sum_{k\in\mathcal{K}}\int_0^t \langle s_k, \mathsf{q}^\alpha(s)\rangle dW_s^k + 2\int_0^t \langle \operatorname{curl} f(s), \mathsf{q}^\alpha(s)\rangle ds \quad \mathbb{P}-a.s. \qquad (5.9)\end{aligned}$$

Lastly we need to recall some results about the well-posedness of Euler equations with forcing term $f \in \mathcal{P}_2^N$, namely solutions of

$$\begin{cases}\partial_t \bar{u} + \bar{u}\cdot\nabla\bar{u} = \nabla\bar{p} + f\\ \operatorname{div}\bar{u} = 0\\ \bar{u}\cdot n|_{\partial\mathcal{O}} = 0\\ \bar{u}(0) = \bar{u}_0.\end{cases} \qquad (5.10)$$

Definition 5.11. Given $\bar{u}_0 \in H,\ f \in C([0,T];H)$, we say that a stochastic process adapted to the filtration $(\mathcal{F}_t)_{t\in[0,T]}$, with continuous trajectories with values in H is a weak solution of equation (5.10) if $\mathbb{P}-a.s.$ for every $\varphi \in C_0^\infty([0,T]\times\mathcal{O})\cap C^1([0,T];H)$

$$\langle \bar{u}(t), \varphi(t)\rangle = \langle \bar{u}_0, \varphi(0)\rangle + \int_0^t \langle \bar{u}(s), \partial_s\varphi(s)\rangle ds + \int_0^t b(\bar{u}(s), \varphi(s), \bar{u}(s)) ds + \int_0^t \langle f(s), \bar{u}(s)\rangle ds$$

for every $t \in [0,T]$ and the energy inequality

$$\|\bar{u}(t)\|^2 \le \|\bar{u}_0\|^2 + 2\int_0^t \langle f(s), \bar{u}(s)\rangle ds$$

holds.

The well-posedness of (5.10) in regular spaces is a classical result, we refer for example to [24],[23],[92] for details.

Theorem 5.12. *For each* $\bar{u}_0 \in H\cap H^3(\mathcal{O};\mathbb{R}^2),\ f \in \mathcal{P}_2^N$ *there exists a unique weak solution of* (5.10) *with trajectories in* $C([0,T];W^{2,4}(\mathcal{O};\mathbb{R}^2))$. *Moreover*

$$\|\bar{u}(t)\|^2 = \|\bar{u}_0\|^2 + 2\int_0^t \langle f(s), \bar{u}(s)\rangle ds \quad \mathbb{P}-a.s. \qquad (5.11)$$

$$\sup\nolimits_{t\in[0,T]}\|\bar{u}(t)\|_{W^{2,4}} \le C(\|u_0\|_{W^{2,4}}, N) \quad \mathbb{P}-a.s. \qquad (5.12)$$

Remark 5.13. The well-posedness of equation (5.5), equation (5.6) holds under weaker assumptions on the noise than Hypothesis 4.4. We need to assume a noise so regular in order to guarantee that there exists a unique solution of (5.10) which belongs to $C([0,T];W^{2,4}(\mathcal{O};\mathbb{R}^2))\cap C([0,T];H)$. This is the reason why we introduced this particular setting of assumptions.

5.2 Main Results

Our goal is to prove a Large Deviation Principle via the weak convergence approach introduced in subsection 5.1.1. Therefore we need to introduce some maps $\mathcal{G}^{NS,\nu}$, $\mathcal{G}^{SG,\alpha}$, $\mathcal{G}^0$. Following the notation of subsection 5.1.1, let

$$\mathcal{E}_0^{NS} := H^3(\mathcal{O};\mathbb{R}^2)\cap H, \quad \mathcal{E}_0^{SG} := W, \quad \mathcal{E} := C([0,T];H).$$

According to Theorem 5.12 we can introduce the measurable map

$$\mathcal{G}^{NS,0} : \mathcal{E}_0^{NS}\times C([0,T];H)\to\mathcal{E} \quad \Big(resp\ \mathcal{G}^{SG,0} : \mathcal{E}_0^{SG}\times C([0,T];H)\to\mathcal{E}\Big)$$

which associates to each $u_0\in\mathcal{E}_0^{NS}$ (resp. $u_0\in\mathcal{E}_0^{SG}$) and $\int_0^\cdot f(s)ds,\ f\in L^2(0,T;H_0)$ the unique regular solution of (5.10) with initial condition u_0 and forcing term f guaranteed by Theorem 5.12, 0 otherwise. Analogously thanks to Theorem 5.9 (resp. Theorem 5.10) we can introduce the measurable map

$$\mathcal{G}^{NS,\nu} : \mathcal{E}_0^{NS}\times C([0,T];H)\to\mathcal{E} \quad \Big(resp\ \mathcal{G}^{SG,\alpha} : \mathcal{E}_0^{SG}\times C([0,T];H)\to\mathcal{E}\Big)$$

such that for each $u_0\in\mathcal{E}_0^{NS}$ (resp. $\mathsf{u}_0\in\mathcal{E}_0^{SG}$), $\mathcal{G}^{NS,\nu}(u_0,\sqrt{\nu}W.)$ (resp. $\mathcal{G}^{SG,\alpha}(\mathsf{u}_0,\sqrt{\alpha}W.)$ is the unique weak solution of (5.5) (resp. (5.6)) with $\beta=\nu$ (resp. $\beta=\alpha$), initial condition u_0 (resp. u_0) and null forcing term guaranteed by Theorem 5.9 (resp. Theorem 5.10). More generally, it follows that, if $f\in\mathcal{P}_2^N$, $\mathcal{G}^{NS,\nu}(u_0,\sqrt{\nu}W.+\int_0^\cdot f(s)ds)$ (resp. $\mathcal{G}^{SG,\alpha}(\mathsf{u}_0,\sqrt{\alpha}W.+\int_0^\cdot f(s)ds)$) is the unique solution of (5.5) (resp. (5.6)) $\beta=\nu$ (resp. $\beta=\alpha$), initial condition u_0 (resp. u_0) and forcing term f. When dealing with the inviscid limit for Navier-Stokes equations and no-slip boundary conditions one can choose either to assume a Kato-type hypothesis or to require strong assumptions on the regularity of the domain, initial conditions and forcing term. We will follow both these lines. As in the previous chapter, given $c>0$, we will denote $\Gamma_c=\{x\in\mathcal{O}:\quad d(x,\partial\mathcal{O})\le c\}$ in the following.

Hypothesis 5.14 (Strong Kato Hypothesis)**.** For each $N\in\mathbb{N}$, $u_0^\nu,\ u_0\in\mathcal{E}_0^{NS}$ and $f^\nu,\ f\in\mathcal{P}_2^N$ such that $u_0^\nu\to u_0$ in $\mathcal{E}_0^{NS}$ and $f^\nu\to_{\mathcal{L}} f$ in S^N, if $(\Omega,\mathcal{F},\mathcal{F}_t,\mathbb{P})$ is a filtered probability space where all f^ν, f are defined together and $f^\nu\to f$ $\mathbb{P}-a.s.$ in S^N, then, it exists $c>0$ such that for every $\delta>0$

$$\mathbb{P}\left(\nu\int_0^T\left\|\nabla\mathcal{G}^{NS,\nu}\left(u_0^\nu,\sqrt{\nu}W.+\int_0^\cdot f^\nu(s)ds\right)\right\|^2_{L^2(\Gamma_{c\nu})}ds>\delta\right)\to 0.$$

Remark 5.15. In the previous condition, the set $\mathcal{E}_0^{NS}$ need not to be the full space $H^3(\mathcal{O};\mathbb{R}^2)\cap H$, but it can be a closed subset of it, even consisting of a singleton. Of course, the LDP that we will be able to prove will then only be uniform with respect to initial conditions belonging to such subset (cf. Definition 5.3). In contrast, the space in which the forcing f varies cannot be restricted as easily, without substantially weakening the strength of the LDP (see the discussion in section 5.5).

Remark 5.16. By Skorokhod's representation theorem, given f^ν, $f \in \mathcal{P}_2^N$ such that $f^\nu \to_{\mathcal{L}} f$ in S^N there exists at least a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ where all f^ν, f are defined together and $f^\nu \to f$ $\mathbb{P}-a.s.$ in S^N.

Now we are ready to state our main result on the validity of a Large Deviation Principle under Hypothesis 5.14.

Theorem 5.17. *Assuming Hypothesis 5.14, the family* $\{u^{NS,\nu} = \mathcal{G}^{NS,\nu}(u_0, \sqrt{\nu}W.)\}_{u_0 \in \mathcal{E}_0^{NS}}$ *satisfies the uniform Laplace principle with the rate function*

$$\begin{aligned} I_{u_0}^{NS}(w) &= \inf_{\{f \in L^2(0,T;H_0):\ w = \mathcal{G}^{NS,0}(u_0, \int_0^\cdot f(s)ds)\}} \frac{1}{2}\int_0^T \|f(s)\|_{H_0}^2 ds \\ &= \frac{1}{2}\int_0^T \|\partial_s w(s) + \mathbf{P}(w(s)\cdot\nabla w(s))\|_{H_0}^2 ds. \end{aligned}$$

where $u_0 \in \mathcal{E}_0^{NS}$, $w \in C([0,T];H)$, *with the convention that* $I_{u_0}^{NS}(w) = +\infty$ *anytime* w *is not in the range of* $\mathcal{G}^{NS,0}(u_0,\cdot)$.

The last equality is guaranteed by the injectivity of the map $\mathcal{G}^{NS,0}$ in the second component, which in turn is a consequence of the uniqueness for the Euler system in our setting, see Theorem 5.12.
The validity of Hypothesis 5.14 is in contrast to the so-called *Kolmogorov's zeroth law of turbulence*, see [115], [116], [117]. The latter describing the physical evidence that the anomalous dissipation of the kinetic energy holds for three dimensional fluids at high Reynolds number. As already mentioned in section 1.5, nowadays, the *Kolmogorov's zeroth law of turbulence* is a well-accepted assumption for three dimensional fluids where counterexamples to Hypothesis 5.14 has been shown in the case of deterministic forcing and domains without boundaries, see for example [32] for an explicit counterexample and [108], [147] for some numerical discussions, the situation is less clear in the two dimensional case. This is due the fact that either Navier-Stokes and Euler's flows preserve smooth solutions, see for example [168] and [130]. We refer to [49] and the references therein for further discussions on this topic. Indeed, even if Hypothesis 5.14 may look too restrictive, we will provide, thanks to Theorem 5.21, an explicit example where it is satisfied, see also Remark 5.27 and Remark 5.33 below. Moreover, we will come back on the meaning of Hypothesis 5.14 and its relation with a more classical version of the Kato Hypothesis, i.e. non depending on f^ν, in section 5.5.

As pointed out in previous chapter, in order to obtain an unconditioned result for the Second-Grade fluid equations, we cannot take any scaling of $\nu \to 0$ but it is necessary to assume:

Hypothesis 5.18. $\nu = O(\alpha)$.

Now we can state our main result on the Second Grade Fluid equations.

Theorem 5.19. *Assuming Hypothesis 5.18, the family* $\{u^{SG,\alpha} = \mathcal{G}^{SG,\alpha}(\mathsf{u}_0, \sqrt{\alpha}W.)\}_{\mathsf{u}_0\in\mathcal{E}_0^{SG}}$ *satisfies the uniform Laplace principle with the rate function*

$$\begin{aligned} I^{SG}_{\mathsf{u}_0}(w) &= \frac{1}{2}\inf_{f\in L^2(0,T;H_0):\ w=\mathcal{G}^{SG,0}(\mathsf{u}_0,\int_0^{\cdot} f(s)ds)}\int_0^T \|f(s)\|^2_{H_0}ds \\ &= \frac{1}{2}\int_0^T \|\partial_s w(s) + \mathbf{P}(w(s)\cdot\nabla w(s))\|^2_{H_0}ds \end{aligned}$$

where $\mathsf{u}_0 \in \mathcal{E}_0^{SG},\ w\in C([0,T];H)$, *with the convention that* $I^{SG}_{\mathsf{u}_0}(w) = +\infty$ *anytime* w *is not in the range of* $\mathcal{G}^{SG,0}(\mathsf{u}_0,\cdot)$.

Lastly we want to consider the case of fluids with radial symmetry. In such case, the inviscid limit usually holds without any assumptions on the behavior of the fluid in the boundary layer as observed in [133]. Therefore, calling B the open ball in $\mathbb{R}^2$, centered in 0 with radius 1, we introduce

$$\mathcal{H}^{RS} = \overline{\left\{\frac{x^{\perp}}{|x|}\bar{u}(|x|),\quad \bar{u}\in C_c^{\infty}(0,1)\right\}}^{\mathsf{D}(\mathcal{A}^{\gamma})}$$

endowed with the $\mathsf{D}(\mathcal{A}^{\gamma})$ norm and

$$\mathcal{E}_0^{RS} = \mathcal{E}_0^{NS}\cap\left\{u = \frac{x^{\perp}}{|x|}\bar{u}(|x|),\quad \bar{u}\in L^2(0,1)\right\}$$

endowed with the H^3 norm. As above we need to introduce a particular forced Navier-Stokes systems:

$$\begin{cases} du^{RS,\nu} = (\nu\Delta u^{RS,\nu} - u^{RS,\nu}\cdot\nabla u^{RS,\nu} + \nabla p^{RS,\nu} + f)dt + \sqrt{\nu}dW^{RS}(t) \\ \operatorname{div} u^{RS,\nu} = 0 \\ u^{RS,\nu}|_{\partial\mathcal{O}} = 0 \\ u^{RS,\nu}(0) = u_0, \end{cases} \tag{5.13}$$

Now can introduce the assumptions in order to deal the case with radial symmetry and study the Large Deviation Principle in this framework:

Hypothesis 5.20. $\mathcal{O} = B$, $W^{RS}(t) = \sum_{k\in\mathcal{K}}\sigma_k W_t^k$ where

- $\mathcal{K}$ is a (possibly countable) set of indexes, $\gamma \geq 2$.
- $\sigma_k \in \mathcal{H}^{RS}$ satisfying

$$\sum_{k\in\mathcal{K}}\|\sigma_k\|^2_{\mathsf{D}(\mathcal{A}^{\gamma})} < +\infty.$$

- $\{W_t^k\}_{k\in\mathcal{K}}$ is a sequence of real, independent Brownian motions adapted to $\mathcal{F}_t$.

We denote by H_0^{RS} the RKHS associated to $W^{RS}(t)$.
Since Theorem 5.9, Theorem 5.12 continue to hold considering

$$u_0\in\mathcal{E}_0^{RS},\quad f\in L^2(0,T;H_0^{RS})$$

and assuming Hypothesis 5.20, we can define the measurable maps $\mathcal{G}^{RS,\nu}$ and $\mathcal{G}^{RS,0}$ as above for $\mathcal{G}^{NS,\nu}$ and $\mathcal{G}^{NS,0}$ considering $\mathcal{E}_0^{RS}$ instead of $\mathcal{E}_0^{NS}$.

Theorem 5.21. *Assuming Hypothesis 5.20, the family* $\{u^{RS,\nu}=\mathcal{G}^{RS,\nu}(u_0,\sqrt{\nu}W_{\cdot}^{RS})\}_{u_0\in\mathcal{E}_0^{RS}}$ *satisfies the uniform Laplace principle with the rate function*

$$\begin{aligned}I_{u_0}^{RS}(w)&=\frac{1}{2}\inf_{f\in L^2(0,T;H_0^{RS}):\ w=\mathcal{G}^{RS,0}(u_0,\int_0^{\cdot}f(s)ds)}\int_0^T\|f(s)\|_{H_0^{RS}}^2ds\\&=\frac{1}{2}\int_0^T\|\partial_s w(s)+\mathbf{P}(w(s)\cdot\nabla w(s))\|_{H_0^{RS}}^2ds\end{aligned}$$

where $u_0\in\mathcal{E}_0^{RS}$, $w\in C([0,T];H)$, *with the convention that* $I_{u_0}^{SG}(w)=+\infty$ *anytime* w *is not in the range of* $\mathcal{G}^{RS,0}(u_0,\cdot)$.

Let us observe that all of the three functionals have the same integral representation, the only difference being the space where u_0 and the noise take values. This phenomenon roughly means that the LDPs we are able to establish are not strong enough to distinguish between the models, at the level of the asymptotics of their fluctuations. In principle, a sharper LDP could display such differences. This kind of results, which are out of the scope of this chapter, would be of incredible interest also in the periodic framework, acting as selection principles for solutions coming from convex integration schemes.

Remark 5.22. Theorem 5.17, Theorem 5.19 and Theorem 5.21 continue to hold also if we add a deterministic forcing term g in $L^2(0,T;H_0)$ or $L^2(0,T;H_0^{RS})$ in equations (4.1), (4.78), up to re-defining the maps $\mathcal{G}^{\nu}$, $\mathcal{G}^{\alpha}$ and $\mathcal{G}^{0}$ accordingly. Indeed the computations below can be easily adapted to this framework. Moreover, it is enough to assume the validity of Hypothesis 5.14 for equation (1.32) without any forcing g in order to prove the validity of Theorem 5.17 also if we add the forcing term g.

Remark 5.23. With particular choices of the noise coefficients, the Large deviations functional can be made more explicit, as a relevant example, in the framework of Remark 5.6, $I_{u_0}^{NS}(w)$ reduces to

$$I_{u_0}^{NS}(w)=\frac{1}{2}\int_0^T\|\mathcal{A}^{\gamma+1/2+\delta}\left(\partial_s w(s)+\mathbf{P}(w(s)\cdot\nabla w(s))\right)\|^2ds,$$

similarly for the other functionals.

5.3 Navier-Stokes Equations

5.3.1 Proof of Theorem 5.17.

Condition 1

Let us fix $N > 0$, K a compact subset of $\mathcal{E}_0^{NS}$, we want to show that the set

$$K_N = \{\mathcal{G}^{NS,0}(u_0, \int_0^{\cdot} f(s)ds) \quad v \in S^N, u_0 \in K\} \overset{c}{\hookrightarrow} \mathcal{E}.$$

Therefore let us fix two sequences $\{u_0^n\}_{n\in\mathbb{N}} \subset K$, $\{f^n\}_{n\in\mathbb{N}} \subset S^N$. Since K is compact subset of $\mathcal{E}_0^{NS}$, $\|f^n\|^2_{L^2(0,T;H_0)} \leq N$ we can find a subsequence $\{n_k\}_{k\in\mathbb{N}}$, $u_0 \in K$, $f \in S^N$ such that $u_0^{n_k} \to u_0$ in $\mathcal{E}_0^{NS}$, $f^{n_k} \rightharpoonup f$ in $L^2(0,T;H_0)$. Let $u^{n_k} := \mathcal{G}^{NS,0}(u_0^{n_k}, \int_0^{\cdot} f^{n_k}(s)ds)$ (resp. $u := \mathcal{G}^{NS,0}(u_0, \int_0^{\cdot} f(s)ds)$). According to Theorem 5.12, u^{n_k} (resp. u) is the unique regular weak solution of (5.10) with initial condition $u_0^{n_k}$ (resp. u_0) and forcing term f^{n_k} (resp. f). Our goal is to show that $u^{n_k} \to u$ in $\mathcal{E}$. We emphasize, once for all, that the weak convergence of f^{n_k} to f is not directly sufficient in order to show the validity of *Condition 1* in Hypothesis 5.4. Therefore, we need to move to some integrated-in-time version of f^{n_k} and f in order to gain strong convergence, uniform in time, in weaker Sobolev spaces. We will adopt similar arguments also in the forthcoming sections in order to establish the validity of *Condition 2* in Hypothesis 5.4. Fix $\theta > 0$ arbitrarily small and define $F^{n_k} = \int_0^{\cdot} f^{n_k}(s)ds$, $F = \int_0^{\cdot} f(s)ds$. By Hypothesis 4.4, $H_0 \hookrightarrow \mathsf{D}(\mathcal{A}^2)$. This implies, see for example [77, Proposition 2.10], that

$$F^{n_k} \to F \quad in\ C([0,T];\mathsf{D}(\mathcal{A}^{2-\theta})). \tag{5.14}$$

Obviously

$$\sup_{k\geq 1}\|F^{n_k}\|_{C([0,T];\mathsf{D}(\mathcal{A}^2))} + \|F\|_{C([0,T];\mathsf{D}(\mathcal{A}^2))} \leq C(N). \tag{5.15}$$

Lastly, since $u_0^{n_k} \to u_0$ in $\mathcal{E}_0^{NS}$, from (5.12) we can find a constant $C = C(N, u_0)$ only depending on N and $\|u_0\|_{\mathcal{E}_0^{NS}}$ such that

$$\sup_{k\geq 1}\|u^{n_k}\|_{C([0,T];W^{2,4})} + \|u\|_{C([0,T];W^{2,4})} \leq C(\|u_0\|_{W^{2,4}}, N). \tag{5.16}$$

We introduce

$$v^{n_k}(t) = u^{n_k}(t) - F^{n_k}(t), \quad v(t) = u(t) - F(t).$$

By triangle inequality v^{n_k}, v satisfy relation (5.16), too. Since $F^{n_k} \to F$ in $\mathcal{E}$, it is enough to show that $v^{n_k} \to v$ in $\mathcal{E}$ in order to prove the validity of Condition 1 in Hypothesis 5.4. This is the aim of Lemma 5.24 below. We will follow the idea introduced in [180] to show uniqueness of the solutions with bounded vorticity of the Euler equations. However, in order to prove the continuous dependence from the data we exploit the higher regularity and the uniform bounds guaranteed by relation (5.16).

Lemma 5.24. *$v^{n_k} \to v$ in $C([0,T];H)$.*

Proof. Let

$$\begin{aligned}
&\zeta^{n_k} = \operatorname{curl} u^{n_k}(t), \quad \zeta(t) = \operatorname{curl} u(t),\\
&\varphi^{n_k}(t) = \operatorname{curl} F^{n_k}(t), \quad \varphi(t) = \operatorname{curl} F(t),\\
&h^{n_k}(t) = \operatorname{curl} v^{n_k}(t) = \zeta^{n_k}(t) - \varphi^{n_k}(t), \quad h(t) = \operatorname{curl} v(t) = \zeta(t) - \varphi(t).
\end{aligned}$$

$h^{n_k}(t)$ (resp. $h(t)$) is a weak solution of the vorticity equation

$$\begin{cases} \partial_t h^{n_k} + u^{n_k} \cdot \nabla (h^{n_k} + \varphi^{n_k}) = 0 \\ h_0^{n_k} = \operatorname{curl} u_0^{n_k} \end{cases} \qquad \left(\text{resp. } \begin{cases} \partial_t h + u \cdot \nabla (h + \varphi) = 0 \\ h_0 = \operatorname{curl} u_0 \end{cases} \right). \tag{5.17}$$

Thanks to (5.14), (5.15), (5.16) $h^{n_k},\ h,\ \varphi^{n_k},\ \varphi$ satisfy

$$\varphi^{n_k} \to \varphi \quad \text{in } C([0,T];H^{3-\theta}), \tag{5.18}$$

$$\sup_{k\geq 1}\|\varphi^{n_k}\|_{C([0,T];H^3)} + \|\varphi\|_{C([0,T];H^3)} \leq C(N), \tag{5.19}$$

$$\sup_{k\geq 1}\|h^{n_k}\|_{C([0,T];W^{1,4})} + \|h\|_{C([0,T];W^{1,4})} \leq C(\|u_0\|_{W^{2,4}}, N). \tag{5.20}$$

We need to introduce the stream function $\psi^{n_k}(t)$ (resp. $\psi(t)$) which is the weak solution of

$$\begin{cases} -\Delta \psi^{n_k}(t) = h^{n_k}(t) \\ \psi^{n_k}(t)|_{\partial\mathcal{O}} = 0 \end{cases} \qquad \left(\text{resp. } \begin{cases} -\Delta \psi(t) = h(t) \\ \psi(t)|_{\partial\mathcal{O}} = 0 \end{cases} \right). \tag{5.21}$$

By standard elliptic regularity theory, see for example [8], and the uniform bound (5.20), it holds

$$\sup_{k\geq 1}\|\psi^{n_k}\|_{C([0,T];W^{3,4})} + \|\psi\|_{C([0,T];W^{3,4})} \leq C(\|u_0\|_{W^{2,4}}, N). \tag{5.22}$$

Lastly, introducing

$$\vartheta^{n_k}(t) = \psi^{n_k}(t) - \psi(t), \quad g^{n_k}(t) = \varphi^{n_k}(t) - \varphi(t), \quad G^{n_k}(t) = F^{n_k}(t) - F(t),$$

It is well-known that $v^{n_k} = -\nabla^{\perp}\psi^{n_k},\ v = -\nabla^{\perp}\psi$, see for example [142]. With these notations in mind, thanks to equations (5.17) and (5.21), $-\Delta\vartheta^{n_k}$ solves in a weak sense

$$\begin{aligned}
\partial_t(-\Delta\vartheta^{n_k}) + \Big[(-\nabla^{\perp}\psi^{n_k} + F^{n_k}) \cdot \nabla\Big]\Big(-\Delta\vartheta^{n_k} + g^{n_k}\Big) &= \\
-\Big[-\nabla^{\perp}\vartheta^{n_k} + G^{n_k} \cdot \nabla\Big](-\Delta\psi + \varphi).
\end{aligned} \tag{5.23}$$

Therefore, arguing as in [180, Theorem 3.1], we use ϑ^{n_k} itself as a test function in (5.23), obtaining

$$\begin{aligned}\frac{1}{2}\|\nabla\vartheta^{n_k}(t)\|^2 &= \frac{1}{2}\|\nabla\vartheta_0^{n_k}\|^2 + \int_0^t\int_{\mathcal{O}}\Big((-\nabla^{\perp}\psi^{n_k}(s)\cdot\nabla\vartheta^{n_k}(s))\big(-\Delta\vartheta^{n_k}(s)+g^{n_k}(s)\big)\\ &\quad + F^{n_k}(s)\cdot\nabla\vartheta^{n_k}(s)(-\Delta\vartheta^{n_k}(s)+g^{n_k}(s)) \qquad (5.24)\\ &\quad + \big(-\Delta\psi(s)+\varphi(s)\big)\big(G^{n_k}(s)\cdot\nabla\vartheta^{n_k}(s)\big)\Big)dxds\\ &= \frac{1}{2}\|\nabla\vartheta_0^{n_k}\|^2 + I_1(t)+I_2(t)+I_3(t)+I_4(t)+I_5(t), \qquad (5.25)\end{aligned}$$

where

$$\begin{aligned}I_1(t) &= \int_0^t\int_{\mathcal{O}}\nabla^{\perp}\psi^{n_k}(s)\cdot\nabla\vartheta^{n_k}(s)\Delta\vartheta^{n_k}(s)dxds,\\ I_2(t) &= -\int_0^t\int_{\mathcal{O}}\nabla^{\perp}\psi^{n_k}(s)\cdot\nabla\vartheta^{n_k}(s)g^{n_k}(s)dxds,\\ I_3(t) &= -\int_0^t\int_{\mathcal{O}}F^{n_k}(s)\cdot\nabla\vartheta^{n_k}(s)\Delta\vartheta^{n_k}(s)dxds,\\ I_4(t) &= \int_0^t\int_{\mathcal{O}}F^{n_k}(s)\cdot\nabla\vartheta^{n_k}(s)g^{n_k}(s)dxds,\\ I_5(t) &= \int_0^t\int_{\mathcal{O}}(-\Delta\psi(s)+\varphi(s))\big(G^{n_k}(s)\cdot\nabla\vartheta^{n_k}(s)\big)dxds\end{aligned}$$

Therefore we need to understand the behavior of $I_1(t)$, $I_2(t)$, $I_3(t)$, $I_4(t)$, $I_5(t)$ in equation (5.24). $I_1(t)$ can be estimated easily integrating by parts, thanks to the uniform bound (5.16) and Hölder's inequality. Indeed it holds:

$$\begin{aligned}-\int_{\mathcal{O}}v_i^{n_k}\partial_i\vartheta^{n_k}\partial_{j,j}\vartheta^{n_k}dx &= -\int_{\mathcal{O}}\partial_j v_i^{n_k}\partial_i\vartheta^{n_k}\partial_j\vartheta^{n_k}dx - \int_{\mathcal{O}}v_i^{n_k}\partial_{i,j}\vartheta^{n_k}\partial_j\vartheta^{n_k}dx\\ &\le \|v^{n_k}\|_{W^{1,\infty}}\|\nabla\vartheta^{n_k}\|^2 - \frac{1}{2}\int_{\mathcal{O}}v_i^{n_k}\partial_i|\partial_j\vartheta^{n_k}|^2dx\\ &\le C(\|u_0\|_{W^{2,4}},N)\|\nabla\vartheta^{n_k}\|^2.\end{aligned}$$

Therefore

$$I_1(t)\le C(\|u_0\|_{W^{2,4}},N)\int_0^t\|\nabla\vartheta^{n_k}(s)\|^2ds. \qquad (5.26)$$

I_3 can be estimated similarly integrating by parts, thanks to the uniform bound (5.15) and

Hölder's inequality:

$$\begin{aligned}\int_{\mathcal{O}} F_i^{n_k}\partial_i\vartheta^{n_k}\partial_{j,j}\vartheta^{n_k}dx &= -\int_{\mathcal{O}}\partial_j F_i^{n_k}\partial_i\vartheta^{n_k}\partial_j\vartheta^{n_k}dx - \int_{\mathcal{O}} F_i^{n_k}\partial_{i,j}\vartheta^{n_k}\partial_j\vartheta^{n_k}dx\\ &\le \|F^{n_k}\|_{W^{1,\infty}}\|\nabla\vartheta^{n_k}\|^2 - \frac{1}{2}\int_{\mathcal{O}} F_i^{n_k}\partial_i|\partial_j\vartheta^{n_k}|^2dx\\ &\le C(N)\|\nabla\vartheta^{n_k}\|^2.\end{aligned}$$

Therefore

$$I_3(t) \le C(N)\int_0^t \|\nabla\vartheta^{n_k}(s)\|^2ds. \tag{5.27}$$

$I_2(t)$, $I_4(t)$ and $I_5(t)$ can be bounded easily by Hölder and Young's inequalities and the uniform bounds (5.15), (5.16), (5.19), (5.20), (5.22). Indeed it holds:

$$\begin{aligned}I_2(t)+I_4(t)+I_5(t) &\le \int_0^t \left(\|v^{n_k}(s)\|_{L^\infty} + \|F^{n_k}(s)\|_{L^\infty}\right)\|\nabla\vartheta^{n_k}(s)\|\|g^{n_k}(s)\|dsds\\ &+ \int_0^t \left(\|\varphi(s)\|_{L^\infty}\|h(s)\|_{L^\infty}\right)\|\nabla\vartheta^{n_k}(s)\|\|G^{n_k}(s)\|\\ &\le \int_0^t\|\nabla\vartheta^{n_k}(s)\|^2ds\\ &+ C\left(\|v^{n_k}\|_{C([0,T];W^{1,4})} + \|F^{n_k}\|_{C([0,T];W^{1,4})}\right)^2\|g^{n_k}\|^2_{C([0,T];L^2)}\\ &+ C\left(\|\varphi\|_{C([0,T];W^{1,4})} + \|h\|_{C([0,T];W^{1,4})}\right)^2\|G^{n_k}\|^2_{C([0,T];L^2)}\\ &\le \int_0^t\|\nabla\vartheta^{n_k}(s)\|^2ds\\ &+ C(\|u_0\|_{W^{2,4}},N)\left(\|g^{n_k}\|^2_{C([0,T];L^2)} + \|G^{n_k}\|^2_{C([0,T];L^2)}\right).\end{aligned} \tag{5.28}$$

Combining relations (5.26), (5.27) and (5.28) we get

$$\begin{aligned}\frac{1}{2}\|\nabla\vartheta^{n_k}(t)\|^2 &\le \frac{1}{2}\|\nabla\vartheta_0^{n_k}\|^2 + C(\|u_0\|_{W^{2,4}},N)\int_0^t\|\nabla\vartheta^{n_k}(s)\|^2ds\\ &+ C(\|u_0\|_{W^{2,4}},N)\left(\|g^{n_k}\|^2_{C([0,T];L^2)} + \|G^{n_k}\|^2_{C([0,T];L^2)}\right),\end{aligned}$$

which implies, by Grönwall's Lemma,

$$\begin{aligned}\|v^{n_k} - v\|_{C([0,T];H)} \le C(\|u_0\|_{W^{2,4}},N)\big(&\|u_0^{n_k} - u_0\| + \|g^{n_k}\|_{C([0,T];L^2)}\\ &+ \|G^{n_k}\|_{C([0,T];L^2)}\big).\end{aligned} \tag{5.29}$$

Thanks to our assumptions the result follows. □

Condition 2

Fix $N>0$, let $\tilde{f}^\nu,\tilde{f}\in\mathcal{P}_2^N,\ u_0^\nu,u_0\in\mathcal{E}_0^{NS}$ such that $\tilde{f}^\nu\rightarrow_{\mathcal{L}}\tilde{f}$ weakly in $L^2(0,T;H_0),\ u_0^\nu\rightarrow u_0$ in $\mathcal{E}_0^{NS}$. We will show that for each sequence $\nu_n\rightarrow 0$, $\mathcal{G}^{\nu_n,NS}\big(u_0^{\nu_n},\sqrt{\nu_n}W+\int_0^\cdot \tilde{f}^{\nu_n}(s)ds\big)$ converges in law to $\mathcal{G}^{0,NS}(u_0,\int_0^\cdot\tilde{f}(s)ds)$ in the topology of $\mathcal{E}$. This implies the validity of the second condition in Hypothesis 5.4. In order to simplify the notation, we will consider $\nu>0$ in the following dropping the subscript ν_n, having in mind it is a countable family. Since S^N is a Polish space, by Skorokhod's representation theorem we can introduce a further filtered probability space $(\tilde{\Omega},\tilde{\mathcal{F}},\tilde{\mathcal{F}}_t,\tilde{\mathbb{P}})$ and random variables f^ν,W^ν,f such that (f^ν,W^ν) has the same joint law of $(\tilde{f^\nu},W)$, f has the same law of $\tilde{f}$ and $f^\nu\rightarrow_{\tilde{\mathbb{P}}-a.s.}f$ in $L^2(0,T;H_0)$, see for example [77] for details. Thanks to Theorem 5.9 for each ν we can define u^ν as the unique solution of (5.5) with forcing term f^ν, initial condition u_0^ν and Brownian forcing term W^ν. The family $\{u^\nu\}_{\nu>0}$ satisfies Hypothesis 5.14. Moreover, by Theorem 5.12 we can define u^E as the unique regular solution of (5.10) with forcing term f and initial condition u_0. We will show that u_0^ν converges to u^E in probability in $C([0,T];H)$. This implies the validity of Condition 2.

Before starting with the computation we recall some facts. In the following, with some abuse of notation, we will simply use $\mathbb{P}$, $\mathbb{E}$ instead of $\tilde{\mathbb{P}}$, $\tilde{\mathbb{E}}$. Fix $\theta>0$ arbitrarily small and define $F^\nu(t)=\int_0^t f^\nu(s)ds,\ F(t)=\int_0^t f(s)ds$. By Hypothesis 4.4, $H_0\hookrightarrow \mathsf{D}(\mathcal{A}^\gamma)$. This implies, see for example [77, Proposition 2.10] that

$$F^\nu\rightarrow_{\mathbb{P}-a.s.}F\quad in\ C([0,T];\mathsf{D}(\mathcal{A}^{\gamma-\theta})). \tag{5.30}$$

Obviously

$$\sup_{\nu>0}\|F^\nu\|_{C([0,T];\mathsf{D}(\mathcal{A}^\gamma))}+\|F\|_{C([0,T];\mathsf{D}(\mathcal{A}^\gamma))}\leq C(N)\quad \mathbb{P}-a.s. \tag{5.31}$$

Starting from (5.7), Burkholder-Davis-Gundy inequality, Grönwall's lemma and the convergence of u_0^ν to u_0 imply

$$\sup_{\nu>0}\left\{\mathbb{E}\left[\sup_{t\in[0,T]}\|u^\nu(t)\|^2\right]+\nu\mathbb{E}\left[\int_0^T\|\nabla u^\nu(s)\|^2ds\right]\right\}\leq C(N,\|u_0\|). \tag{5.32}$$

In order to show the convergence of u^ν to u^E we introduce $z^\nu=\int_0^t\mathcal{S}(\nu(t-s))f^\nu(s)ds,\ v^\nu=u^\nu-z^\nu,\ v^E=u^E-F$ and show separately the convergence of z^ν to F and of v^ν to v^E. While the convergence of z^ν to F will be established by exploiting the properties of the Stokes semigroup, the convergence of v^ν to v^E will be the more demanding part of the argument relying on Hypothesis 5.14 and the introduction of a corrector of the boundary layer for v^E satisfying suitable estimates. As we have seen in the previous chapter, this way to proceed is typical in the case of the analysis of the inviscid limit for the Navier-Stokes equations with no-slip boundary conditions.

We start with the convergence of z^ν to F.

Lemma 5.25. *For each* $\theta > 0$, $z^\nu(t) \to F$ *in* $C([0,T];\mathsf{D}(\mathcal{A}^{\gamma-1/2-\theta})) \quad \mathbb{P}-a.s.$ *and in* $L^2(\Omega,\mathbb{P})$.

Proof. z^ν can be rewritten as

$$z^\nu(t) = \int_0^t f^\nu(s)ds - \nu\int_0^t \mathcal{A}\mathcal{S}(\nu(t-s))F^\nu(s)ds = I_1 + I_2.$$

$I_1 \to F \in C([0,T];\mathsf{D}(\mathcal{A}^{\gamma-1/2-\theta})\ \mathbb{P}-a.s.$ thanks to (5.30). Moreover, since (5.31) holds, previous convergence holds also in $L^2(\Omega,\mathbb{P})$ by Lebesgue theorem. It remains to show that $I_2 \to 0$ properly. The $\mathbb{P}-a.s.$ convergence can be obtained as follows

$$\begin{aligned}
&\nu\sup_{t\in[0,T]}\left\|\int_0^t \mathcal{A}\mathcal{S}(\nu(t-s))F^\nu(s)ds\right\|_{\mathsf{D}(\mathcal{A}^{\gamma-1/2-\theta})}\\
&\leq \nu\sup_{t\in[0,T]}\int_0^t\left\|\mathcal{A}^{1/2+\gamma-\theta}\mathcal{S}(\nu(t-s))F^\nu(s)ds\right\|\\
&\leq \nu\sup_{t\in[0,T]}\int_0^t\left\|\mathcal{A}^{1/2-\theta}\mathcal{S}(\nu(t-s))\mathcal{A}^\gamma F^\nu(s)\right\|ds\\
&\leq \nu^{1/2+\theta}\sup_{t\in[0,T]}\int_0^t\frac{\|F^\nu(s)\|_{\mathsf{D}(\mathcal{A}^\gamma)}ds}{(t-s)^{1/2-\theta}}\\
&\leq C(N)\nu^{1/2+\theta}\to 0 \quad \mathbb{P}-a.s.
\end{aligned}$$

Since previous bound is uniform in $\omega\in\Omega$ the convergence holds also in $L^2(\Omega,\mathbb{P})$ and the result follows. □

In order to prove the convergence of v^ν to v^E we observe that they solve in a sense analogous to Definition 5.7, Definition 5.11

$$dv^\nu + \mathbf{P}\left((v^\nu+z^\nu)\cdot\nabla\left(v^\nu+z^\nu\right)\right)dt + \nu\mathcal{A}v^\nu dt = \sqrt{\nu}dW(t), \tag{5.33}$$

$$\partial_t v^E + \mathbf{P}(u^E\cdot\nabla u^E) = 0. \tag{5.34}$$

By triangle inequality and the uniform bound guaranteed by Lemma 5.25, estimates analogous to (5.32), (5.12) hold for v^ν and v^E, too. We observe that, thanks to the regularity of u^E guaranteed by Theorem 5.12

$$\begin{aligned}
\|\partial_t v^E\|_{C([0,T];L^\infty(\mathcal{O})))} &\lesssim \|\partial_t v^E\|_{C([0,T];W^{1,4}(\mathcal{O}))} \lesssim \|\mathbf{P}(u^E\cdot\nabla u^E)\|_{C([0,T];W^{1,4}(\mathcal{O}))}\\
&\lesssim \|u^E\|^2_{C([0,T];W^{2,4}(\mathcal{O}))} \leq C(N,u_0) \quad \mathbb{P}-a.s.
\end{aligned}\tag{5.35}$$

Following the idea of [111], thanks to Proposition 4.14, let v be the corrector of the boundary layer of width $\delta=\delta(\nu)$, i.e. a divergence free vector field with support in a strip of the boundary of width δ such that $v^E - v \in V$ and $\mathbb{P} - a.s.$ uniformly in $t \in [0,T],\ \omega \in \Omega$

$$\begin{aligned}
&\|v(t)\|_{L^\infty} \le C(N,u_0),\ \|v(t)\| \le C(N,u_0)\delta^{\frac{1}{2}},\ \|\partial_t v(t)\| \le C(N,u_0)\delta^{\frac{1}{2}},\\
&\|\nabla v(t)\|_{L^\infty} \le C(N,u_0)\delta^{-1},\ \|\nabla v(t)\| \le C(N,u_0)\delta^{-1/2},\ \|\rho\nabla v(t)\|_{L^\infty} \le C(N,u_0),\\
&\|\rho^2\nabla v(t)\|_{L^\infty} \le C(N,u_0)\delta,\ \|\rho\nabla v(t)\| \le C(N,u_0)\delta^{\frac{1}{2}}, \qquad (5.36)
\end{aligned}$$

ρ being the distance function to $\partial\mathcal{O}$. Now we are ready to show the convergence of v^ν to v^E.

Lemma 5.26. *$v^\nu \to v^E$ in $C([0,T];H)$ in probability.*

Proof. Arguing as in the proof of Theorem 4.22 one can show that the following relations hold true.

$$\begin{aligned}
\|v^\nu(t)\|^2 + 2\nu\int_0^t \|\nabla v^\nu\|^2 ds = \|u_0^\nu\|^2 - 2\int_0^t b(v^\nu + z^\nu, z^\nu, v^\nu)ds + 2\sqrt{\nu}\int_0^t \langle dW^\nu(s), v^\nu\rangle\\
+ \nu t Tr(Q) \quad \mathbb{P} - a.s., \qquad (5.37)
\end{aligned}$$

$$\|v^E(t)\|^2 = \|u_0\|^2 - 2\int_0^t b(v^E + F, F, v^E)ds \quad \mathbb{P} - a.s. \qquad (5.38)$$

Exploiting relations (5.37), (5.38) we can study $\|v^\nu(t) - v^E(t)\|^2$. Indeed, it holds

$$\begin{aligned}
\|v^\nu(t) - v^E(t)\|^2 &= \|v^\nu(t)\|^2 + \|v^E(t)\|^2 - 2\langle v^\nu(t), v^E(t)\rangle\\
&\le \|u_0^\nu\|^2 - 2\int_0^t b(v^\nu(s) + z^\nu(s), z^\nu(s), v^\nu(s))ds + 2\sqrt{\nu}\int_0^t \langle dW^\nu(s), v^\nu(s)\rangle\\
&+ \nu t Tr(Q) + \|u_0\|^2 - 2\int_0^t b(v^E(s) + F(s), F(s), v^E(s))ds\\
&- 2\langle v^\nu(t), v^E(t) - v(t)\rangle - 2\langle v^\nu(t), v(t)\rangle. \qquad (5.39)
\end{aligned}$$

Thanks to the fact that $v^E - v \in C^1([0,T];V)$ we can rewrite $\langle v^\nu(t), v^E(t) - v(t)\rangle$ via Itô formula:

$$\begin{aligned}
\langle v^\nu(t), v^E(t) - v(t)\rangle = \langle u_0^\nu, u_0 - v_0\rangle + \int_0^t \langle v^\nu(s), \partial_s(v^E(s) - v(s))\rangle ds\\
+ \int_0^t \langle dv^\nu(s), v^E(s) - v(s)\rangle.
\end{aligned}$$

Therefore

$$\begin{aligned}\langle v^\nu(t), v^E(t)-v(t)\rangle &= \langle u_0^\nu, u_0 - v_0\rangle + \int_0^t \langle v^\nu(s), \partial_s(v^E(s)-v(s))\rangle ds\\ &\quad - \nu\int_0^t \langle \nabla v^\nu(s), \nabla(v^E(s)-v(s))\rangle ds\\ &\quad + \int_0^t b(v^\nu(s)+z^\nu(s), v^E(s)-v(s), v^\nu(s)+z^\nu(s))ds\\ &\quad + \sqrt{\nu}\langle W^\nu(t), v^E(t)-v(t)\rangle\\ &\quad - \sqrt{\nu}\int_0^t \langle W^\nu(s), \partial_s(v^E(s)-v(s))\rangle ds. \qquad (5.40)\end{aligned}$$

Let us observe that by assumptions

$$\|u_0^\nu\|^2 + \|u_0\|^2 - 2\langle u_0^\nu, u_0\rangle = \|u_0^\nu - u_0\|^2 = o(1), \quad \nu t Tr(Q) \le \nu T Tr(Q) = o(1).$$

Moreover, thanks to the properties of the boundary layer corrector (5.36), $\mathbb{P}-a.s.$ it holds

$$\begin{aligned}\langle u_0^\nu, v_0\rangle &\le C(u_0,N)\delta^{1/2} = o(1),\\ \langle v^\nu(t), v(t)\rangle &\le C(N,u_0)\delta^{1/2}\sup_{t\in[0,T]}\|v^\nu(t)\|,\\ \sqrt{\nu}\langle W^\nu(t), v^E(t)-v(t)\rangle &\le \sqrt{\nu}C(N,u_0)\sup_{t\in[0,T]}\|W^\nu(t)\|\\ &\quad + \sqrt{\nu}\int_0^t \langle W^\nu(s), \partial_s(v^E(s)-v(s))\rangle ds.\end{aligned}$$

Exploiting these facts, inserting relation (5.40) in (5.39) we obtain

$$\begin{aligned}\|v^\nu(t)-v^E(t)\|^2 &\le o(1) + \delta^{1/2}C(N,u_0)\sup_{t\in[0,T]}\|v^\nu(t)\|\\ &\quad + \sqrt{\nu}C(N,u_0)\sup_{t\in[0,T]}\|W^\nu(t)\| - 2\int_0^t b(v^\nu(s)+z^\nu(s), z^\nu(s), v^\nu(s))ds\\ &\quad + 2\sqrt{\nu}\int_0^t \langle dW^\nu(s), v^\nu(s)\rangle - 2\int_0^t b(v^E(s)+F(s), F(s), v^E(s))ds\\ &\quad - 2\int_0^t \langle v^\nu(s), \partial_s(v^E(s)-v(s))\rangle ds + 2\nu\int_0^t \langle \nabla v^\nu(s), \nabla(v^E(s)-v(s))\rangle ds\\ &\quad - 2\int_0^t b(v^\nu(s)+z^\nu(s), v(s)^E - v(s), v^\nu(s)+z^\nu(s))ds \quad \mathbb{P}-a.s. \qquad (5.41)\end{aligned}$$

In order to understand the behavior of $\int_0^t \langle v^\nu(s), \partial_s(v^E(s)-v(s))\rangle ds$, we observe that, thanks to (5.36),

$$\int_0^t \langle v^\nu(s), \partial_s v(s)\rangle ds \le \delta^{1/2}C(N,u_0)\sup_{t\in[0,T]}\|v^\nu(t)\| \quad \mathbb{P}-a.s.$$

Moreover, since v^E satisfies (5.34), we have $\int_0^t \langle v^\nu(s), \partial_s v^E(s)\rangle ds = -\int_0^t b(v^E(s)+F(s), v^E(s)+F(s), v^\nu(s))ds$. Let us rewrite the trilinear forms appearing in (5.41):

$$\begin{aligned}
&b(v^E+F, v^E+F, v^\nu) - b(v^\nu+z^\nu, z^\nu, v^\nu) - b(v^E+F, F, v^E)\\
&- b(v^\nu+z^\nu, v^E-v, v^\nu+z^\nu)\\
&= b(v^E, v^E, v^\nu-v^E) + b(v^E+F, F, v^\nu) + b(F, v^E+F, v^\nu)\\
&- b(v^\nu+z^\nu, z^\nu, v^\nu) - b(v^E+F, F, v^E) - b(v^\nu, v^E, v^\nu-v^E)\\
&- b(z^\nu, v^E-v, v^\nu+z^\nu) - b(v^\nu, v^E, z^\nu) + b(v^\nu, v, v^\nu+z^\nu).
\end{aligned} \tag{5.42}$$

By simple computations the terms in (5.42) can be rewritten as:

$$|b(v^E, v^E, v^\nu-v^E) - b(v^\nu, v^E, v^\nu-v^E)| \le \|\nabla v^E\|_{L^\infty}\|v^\nu-v^E\|^2. \tag{5.43}$$

$$b(v^\nu, v, v^\nu+z^\nu) - b(z^\nu, v^E-v, v^\nu+z^\nu) = b(u^\nu, v, u^\nu) - b(z^\nu, v^E, u^\nu). \tag{5.44}$$

$$-b(v^\nu+z^\nu, z^\nu, v^\nu) + b(v^\nu, v^E, z^\nu) = b(v^\nu, z^\nu, v^E-v^\nu) - b(z^\nu, z^\nu, v^\nu). \tag{5.45}$$

$$\begin{aligned}
&b(F, v^E+F, v^\nu) + b(v^E+F, F, v^\nu) - b(v^E+F, F, V^E)\\
&= b(v^E, F, v^\nu-v^E) + b(F, F, v^\nu-v^E) + b(F, v^E, v^\nu).
\end{aligned} \tag{5.46}$$

Preliminarily, let us rewrite the last terms in each of (5.44), (5.45) and (5.46) obtaining

$$-b(z^\nu, v^E, u^\nu) - b(z^\nu, z^\nu, v^\nu) + b(F, v^E, v^\nu) = b(F-z^\nu, v^E, v^\nu) + b(z^\nu, v^\nu-v^E, z^\nu). \tag{5.47}$$

Let us leave out $b(u^\nu, v, u^\nu)$ from our analysis for a moment. Indeed, it well be treated differently. Then considering the other terms appearing in (5.44), (5.45) and (5.46) and exploiting (5.47), we have

$$\begin{aligned}
&b(v^\nu, z^\nu, v^E-v^\nu) + b(v^E, F, v^\nu-v^E) + b(F, F, v^\nu-v^E)\\
&+ b(F-z^\nu, v^E, v^\nu) - b(z^\nu, z^\nu, v^\nu-v^E) \pm b(z^\nu, F, v^\nu-v^E)\\
&\pm b(v^E, z^\nu, v^E-v^\nu)\\
&= b(v^E+z^\nu, F-z^\nu, v^\nu-v^E) + b(v^\nu-v^E, z^\nu, v^E-v^\nu)\\
&+ b(F-z^\nu, v^E, v^\nu) + b(F-z^\nu, F, v^\nu-v^E).
\end{aligned} \tag{5.48}$$

Therefore we can simplify (5.41) and it holds

$$\begin{aligned}\|v^\nu(t)-v^E(t)\|^2 &\leq o(1)+\delta^{1/2}C(N,u_0)\sup\nolimits_{t\in[0,T]}\|v^\nu(t)\|+\sqrt{\nu}C(N,u_0)\sup\nolimits_{t\in[0,T]}\|W^\nu(t)\|\\ &+2\sqrt{\nu}\int_0^t\langle dW^\nu(s),v^\nu(s)\rangle+2\nu\int_0^t\langle\nabla v^\nu(s),\nabla(v^E(s)-v(s))\rangle ds\\ &+2\|\nabla v^E\|_{L^\infty(0,T;L^\infty}\int_0^t\|v^\nu(s)-v^E(s)\|^2ds+2\int_0^t b(u^\nu(s),v(s),u^\nu(s))ds\\ &+2\int_0^t\|v^\nu(s)-v^E(s)\|\|\nabla F(s)\|_{L^4}\|F(s)-z^\nu(s)\|_{L^4}ds\\ &+2\int_0^t\|v^\nu(s)-v^E(s)\|\|\nabla(F(s)-z^\nu(s))\|\|v^E(s)+z^\nu(s)\|_{L^\infty}ds\\ &+2\|\nabla z^\nu\|_{L^\infty(0,T;L^\infty)}\int_0^t\|v^\nu(s)-v^E(s)\|^2ds\\ &+2\int_0^t\|\nabla v^E(s)\|\|v^\nu(s)\|\|F(s)-z^\nu(s)\|_{L^\infty}ds. \end{aligned}\tag{5.49}$$

Now we can treat the term $\nu\int_0^t\langle\nabla v^\nu(s),\nabla(v^E(s)-v(s))\rangle ds$ exploiting the properties of the boundary layer corrector (5.36) and the convergence $z^\nu\to F$ in $C([0,T];\mathsf{D}(\mathcal{A}^{\gamma-\frac{1}{2}-\theta}))$ $\mathbb{P}$-a.s.

$$\begin{aligned}&2\nu\int_0^t\langle\nabla v^\nu(s),\nabla(v^E(s)-v(s))\rangle ds\\ &=2\nu\int_0^t\langle\nabla u^\nu,\nabla(v^E(s)-v(s))\rangle ds-2\nu\int_0^t\langle\nabla z^\nu(s),\nabla(v^E(s)-v(s))\rangle ds\\ &\leq 2\nu\int_0^t\|\nabla u^\nu(s)\|\|\nabla v^E(s)\|ds+2\nu\int_0^t\|\nabla u^\nu(s)\|_{L^2(\Gamma_\delta)}\|\nabla v(s)\|ds\\ &+2\nu\int_0^t\|\mathcal{A}^{1/2}z^\nu(s)\|\|\nabla(v^E(s)-v(s))\|ds\\ &\leq 2\nu\int_0^t\|\nabla u^\nu(s)\|\|\nabla v^E(s)\|ds+\delta^{-1/2}\nu C(N,u_0)\int_0^t\|\nabla u^\nu(s)\|_{L^2(\Gamma_\delta)}\\ &+\delta^{-1/2}\nu C(N,u_0)\int_0^t\|\mathcal{A}^{1/2}z^\nu(s)\|ds.\end{aligned}\tag{5.50}$$

The term $\int_0^t b(u^\nu,v,u^\nu)ds$ is the classical term in the analysis of the inviscid limit in the Kato's regime, it can be estimated by

$$\left|\int_0^t b(u^\nu(s),v(s),u^\nu(s))ds\right|\leq\delta C(N,u_0)\int_0^t\|\nabla u^\nu(s)\|^2_{L^2(\Gamma_\delta)}ds\tag{5.51}$$

see (4.45) and [111, Equation (5.8)]. Combining estimates (5.49), (5.50) and (5.51), choosing

$\delta = c\nu$, where c is the constant appearing in Hypothesis 5.14, it holds

$$\begin{aligned}
\|v^\nu(t) - v^E(t)\|^2 \le{}& o(1) + \sqrt{\nu} C(N, u_0) \sup_{t\in[0,T]} \|v^\nu(t)\| + 2\sqrt{\nu} \int_0^t \langle dW^\nu(s), v^\nu(s)\rangle \\
&+ \nu C(N, u_0) \int_0^t \|\nabla u^\nu(s)\| ds + \sqrt{\nu} C(N, u_0) \sup_{t\in[0,T]} \|W^\nu(t)\| \\
&+ \sqrt{\nu} C(N, u_0) \int_0^t \|\nabla u^\nu(s)\|_{L^2(\Gamma_{c\nu})} ds + \sqrt{\nu} C(N, u_0) \int_0^t \|\nabla u^\nu(s)\|_{L^2(\Gamma_{c\nu})} \\
&+ 2\|\nabla v^E\|_{L^\infty(0,T;L^\infty)} \int_0^t \|v^\nu - v^E\|^2 ds + \nu C(N, u_0) \int_0^t \|\nabla u^\nu(s)\|^2_{L^2(\Gamma_{c\nu})} ds \\
&+ 2(1 + \|\nabla z^\nu\|_{L^\infty(0,T;L^\infty)}) \int_0^t \|v^\nu(s) - v^E(s)\|^2 ds \\
&+ 2 \int_0^t \|v^\nu(s)\| \|\nabla v^E(s)\| \|F(s) - z^\nu(s)\|_{L^\infty} ds. \qquad (5.52)
\end{aligned}$$

Therefore, by Grönwall's inequality, equation (5.52) implies

$$\begin{aligned}
\sup_{t\in[0,T]} \|v^\nu(t) - v^E(t)\|^2 \le{}& e^{2T(1+\|\nabla v^E\|_{L^\infty(0,T;L^\infty)} + \|\nabla z^\nu\|_{L^\infty(0,T;L^\infty)})} \times \\
&\Big(o(1) + \sqrt{\nu} C(N, u_0) \sup_{t\in[0,T]} \|v^\nu(t)\| \\
&+ 2\sqrt{\nu} \sup_{t\in[0,T]} \left| \int_0^t \langle dW^\nu(s), v^\nu(s)\rangle \right| \\
&+ \sqrt{\nu} C(N, u_0) \sup_{t\in[0,T]} \|W^\nu(t)\| \\
&+ \nu C(N, u_0) \int_0^T \|\nabla u^\nu(s)\| ds \\
&+ \sqrt{\nu} C(N, u_0) \int_0^T \|\nabla u^\nu(s)\|_{L^2(\Gamma_{c\nu})} ds \\
&+ \nu C(N, u_0) \int_0^T \|\nabla u^\nu(s)\|^2_{L^2(\Gamma_{c\nu})} ds \\
&+ 2 \int_0^T \|v^\nu(s)\| \|\nabla v^E(s)\| \|F(s) - z^\nu(s)\|_{L^\infty} ds \Big). \qquad (5.53)
\end{aligned}$$

Since $\gamma \ge 2$ we can find $\theta > 0$ small enough such that $\mathsf{D}(\mathcal{A}^{\gamma-1/2-\theta}) \hookrightarrow W^{1,\infty}$. Therefore, $\|\nabla z^\nu\|_{L^\infty(0,T;L^\infty)}$ is $\mathbb{P} - a.s.$ bounded by $C(N, u_0)$ thanks to Lemma 5.25. Similarily, from Theorem 5.12, $\|\nabla v^E\|_{L^\infty(0,T;L^\infty)} \le C(N, u_0) \quad \mathbb{P} - a.s.$
Therefore $e^{2T(1+\|\nabla v^E\|_{L^\infty_t L^\infty_x} + \|\nabla z^\nu\|_{L^\infty_t L^\infty_x})} \le C(N, u_0) \quad \mathbb{P} - a.s.$ This means that, in order to

show that Lemma 5.26 holds, it is enough to prove that

$$\begin{aligned}
&\sqrt{\nu}C(N,u_0)\left(\sup\nolimits_{t\in[0,T]}\|W^\nu(t)\| + \sup\nolimits_{t\in[0,T]}\left|\int_0^t \langle dW^\nu(s), v^\nu(s)\rangle\right|\right)\\
&+\sqrt{\nu}C(N,u_0)\int_0^T \|\nabla u^\nu(s)\|_{L^2(\Gamma_{c\nu})} + \nu C(N,u_0)\int_0^T \|\nabla u^\nu(s)\|^2_{L^2(\Gamma_{c\nu})}ds\\
&+2\int_0^T \|v^\nu(s)\|\|\nabla v^E(s)\|\|F(s)-z^\nu(s)\|_{L^\infty}ds + \nu C(N,u_0)\int_0^T \|\nabla u^\nu(s)\|ds\\
&\to 0 \quad \text{in Probability.}
\end{aligned} \tag{5.54}$$

The terms

$$\sqrt{\nu}C(N,u_0)\int_0^T \|\nabla u^\nu(s)\|_{L^2(\Gamma_{c\nu})} + \nu C(N,u_0)\int_0^T \|\nabla u^\nu(s)\|^2_{L^2(\Gamma_{c\nu})}ds \to 0 \quad \text{in Probability}$$

thanks to Hypothesis 5.14. The terms

$$\begin{aligned}
&\sqrt{\nu}C(N,u_0)\left(\sup_{t\in[0,T]}\|W^\nu(t)\|\right) + \sqrt{\nu}\sup\nolimits_{t\in[0,T]}\left|\int_0^t \langle dW^\nu(s), v^\nu(s)\rangle\right|\\
&+\nu C(N,u_0)\int_0^T \|\nabla u^\nu(s)\|ds \to 0 \quad \text{in Probability}
\end{aligned}$$

since it holds by Burkholder-Davis-Gundy inequality, Hölder's inequality and (5.32)

$$\begin{aligned}
&\mathbb{E}\left[\sup_{t\in[0,T]}\|W^\nu(t)\|\right] + \mathbb{E}\left[\sup\nolimits_{t\in[0,T]}\left|\int_0^t \langle dW^\nu(s), v^\nu(s)\rangle\right|\right] + \sqrt{\nu}\mathbb{E}\left[\int_0^T \|\nabla u^\nu(s)\|ds\right]\\
&\leq C + C\mathbb{E}\left[\sup\nolimits_{t\in[0,T]}\|v^\nu(t)\|^2\right]^{1/2} + C(T)\mathbb{E}\left[\nu\int_0^T \|\nabla u^\nu(s)\|^2 ds\right]\\
&\leq C(N,u_0,T).
\end{aligned}$$

Lastly

$$\int_0^T \|v^\nu(s)\|\|\nabla v^E(s)\|\|F(s)-z^\nu(s)\|_{L^\infty}ds \to 0 \quad \text{in Probability}$$

thanks to Lemma 5.25, (5.32) and (5.12). Indeed it holds:

$$\begin{aligned}
&\mathbb{E}\left[\int_0^T \|v^\nu(s)\|\|\nabla v^E(s)\|\|F(s)-z^\nu(s)\|_{L^\infty}ds\right]\\
&\leq C(N,u_0)\mathbb{E}\left[\int_0^T \|v^\nu(s)\|\|F(s)-z^\nu(s)\|_{L^\infty}ds\right]\\
&\leq C(N,u_0)\mathbb{E}\left[\int_0^T \|v^\nu(s)\|^2\right]^{1/2}\mathbb{E}\left[\int_0^T \|F(s)-z^\nu(s)\|^2_{L^\infty}ds\right]^{1/2} \to 0.
\end{aligned}$$

Therefore (5.54) holds and the result follows. □

Combining Lemma 5.25 and Lemma 5.26 the second condition in Hypothesis 5.4 holds.

Proof of Theorem 5.17. Since we already checked the validity of condition 1 and condition 2 in Hypothesis 5.4, it remains to show that for each $w \in \mathcal{E}$ the map $u_0 \to I^{NS}_{u_0}(w)$ is a lower continuous map from $\mathcal{E}^{NS}_0$ to $[0, +\infty]$ in order to apply Theorem 5.5 and complete the proof. The arguments goes in this way. Fix $u_0 \in \mathcal{E}^{NS}_0$ and a family $\{u^n_0\}_{n\in\mathbb{N}} \subseteq \mathcal{E}^{NS}_0$ converging to u_0. Without loss of generality we may assume $\operatorname{liminf}_{n\to+\infty} I^{NS}_{u^n_0}(w) = M < +\infty$ otherwise we have nothing to prove. Therefore, thanks to the well-posedness of the Euler equations guaranteed by Theorem 5.12, there exists a subsequence n_k and family $\{f^{n_k}\}_{n_k\in\mathbb{N}} \subseteq S^{2M}$ such that $\mathcal{G}^{NS,0}(u^{n_k}_0, \int_0^\cdot f^{n_k}(s)ds) = w$. Moreover $f^{n_k} \in S^{2M}$ for all k. Up to passing to a further subsequence, which we continue to denote by f^{n_k} for simplicity of notation, there exists $f \in S^{2M}$ such that $f^{n_k} \rightharpoonup f$ in $L^2(0,T;H_0)$. Thanks to (5.14) and (5.29) it follows that $\mathcal{G}^{NS}_0 : \mathcal{E}^{NS}_0 \times L^2([0,T];H_0) \to \mathcal{E}$ is a continuous map endowing $L^2([0,T];H_0)$ with the weak topology. Therefore $\mathcal{G}^{NS,0}(u_0, f) = w$ and from the lower semicontinuity of the norm with respect to the weak convergence the result follows immediately:

$$I_{u_0}(w) \le \frac{1}{2}\int_0^T \|f(s)\|^2_{H_0} ds \le \operatorname{liminf}_{k\to+\infty} \frac{1}{2}\int_0^T \|f^{n_k}(s)\|^2_{H_0} ds \le M = \operatorname{liminf}_{n\to+\infty} I_{u^n_0}(w).$$

□

Remark 5.27. As it is classical in the analysis of the inviscid limit in bounded domains, Hypothesis 5.14 for the forcing terms f^ν, f is implied by the convergence in probability of u^ν to u^E in the probability space introduced by Skorokhod's representation theorem. Let us consider (5.7) for $t = T$ and take the limsup of this expression for $\nu \to 0$. It holds

$$\begin{aligned}2 \operatorname{limsup}_{\nu\to0} \nu \int_0^T \|\nabla u^\nu(s)\|^2 ds \le{}& \limsup_{\nu\to0} \|u^\nu_0\|^2 + \limsup_{\nu\to0} \nu T \sum_{k\in\mathcal{K}} \|\sigma_k\|^2 \\ &+ 2 \limsup_{\nu\to0} \sqrt{\nu} \int_0^T \langle u^\nu(s), dW^\nu(s)\rangle \\ &+ 2 \limsup_{\nu\to0} \int_0^T \langle f^\nu(s), u^\nu(s)\rangle ds + \limsup_{\nu\to0} \{-\|u^\nu_T\|^2\}.\end{aligned} \tag{5.55}$$

Under our assumptions it follows immediately that

$$\limsup_{\nu\to0} \|u^\nu_0\|^2 + \limsup_{\nu\to0} \nu T \sum_{k\in\mathcal{K}} \|\sigma_k\|^2 + \limsup_{\nu\to0} \{-\|u^\nu_T\|^2\} = \|u_0\|^2 - \|u^E_T\|^2$$

in Probability. Moreover

$$\limsup_{\nu\to0} \sqrt{\nu} \left| \int_0^T \langle u^\nu, dW^\nu(s)\rangle \right| = 0 \quad \text{in Probability}$$

since by (5.32)

$$\mathbb{E}\left[\left|\int_0^T \langle dW^\nu(s), u^\nu(s)\rangle\right|\right] \leq C\mathbb{E}\left[\sup_{t\in[0,T]}\|u^\nu(t)\|^2\right]^{1/2} \leq C(N, u_0, T).$$

Lastly

$$\lim_{\nu\to 0}\int_0^T \langle f^\nu(s), u^\nu(s)\rangle ds = \int_0^T \langle f(s), u^E(s)\rangle ds \quad \text{in Probability.}$$

Indeed

$$\begin{aligned}\left|\int_0^T \langle f^\nu(s), u^\nu(s)\rangle - \langle f(s), u^E(s)\rangle ds\right| &\leq \left|\int_0^T \langle f^\nu(s), u^\nu(s) - u^E(s)\rangle ds\right| \\ &+ \left|\int_0^T \langle f^\nu(s) - f(s), u^E(s)\rangle ds\right| \\ &= I_1 + I_2.\end{aligned}$$

$I_1 \to 0$ in Probability since we assumed that $u^\nu \to u^E$ in $C([0,T];H)$ in Probability. $I_2 \to 0$ $\mathbb{P}-a.s.$ since in the space introduced by Skorokhod's representation theorem $f^\nu \to f$ weakly in $L^2(0,T;H_0)$ $\mathbb{P}-a.s.$ and $u^E \in \left(L^2(0,T;H_0)\right)^*$ $\mathbb{P}-a.s.$ Therefore we proved that

$$\limsup_{\nu\to 0}\nu\int_0^T \|\nabla u^\nu(s)\|^2 ds = \|u_0\|^2 - \|u_T^E\|^2 - 2\int_0^T \langle f(s), u^E(s)\rangle ds \quad \text{in Probability}$$

which implies Hypothesis 5.14 by (5.12).

5.3.2 The Case of fluids with radial symmetry

We now prove that an unconditioned result holds in presence of strong assumpions on the domain and data, that is a circular symmetry, see Hypothesis 5.20. Therefore, $\mathcal{O} = B(0,1)$ in this section.

The reason why this particular geometry can be treated more easily lies in the fact that solutions of the Navier Stokes equations given by Theorem 5.9 posses radial symmetry, and in turn show that the nonlinear term in the equation vanishes. We will be able to represent the solution $u^\nu(t,x) = v^\nu(x)\frac{x^\perp}{|x|}$ where v^ν is a radial function satisfying an appropriate auxiliary equation. Then we will exploit this particular representation formula in order to prove the validity of Theorem 5.21.

By radial functions, we mean functions g such that $g(R_\theta x) = g(x)$ for a.e. $x \in \mathcal{O}$, for each $\theta \in [0, 2\pi]$, $R_\theta : \mathcal{O} \to \mathcal{O}$ being the counterclockwise rotation of the disk about its center by the angle θ. Any function $u(x)$ that can be written as $\bar{u}(|x|)\frac{x^\perp}{|x|}$ will be called circularly symmetric and the radial function $\bar{u}$ will be called its radial part.

We want to consider the following equation for the scalar function v^ν:

$$dv^\nu = \Big[\nu(\Delta v^\nu - \frac{v^\nu}{|x|^2}) + \bar{f}\Big]dt + \sqrt{\nu}\sum_{k\in K}\sigma_k(|x|)dW_t^k \tag{5.56}$$

where the forcing $\bar{f}(t)$ and the initial datum χ^ν are radial functions in $L^2(\mathcal{O})$. In order to study problem (5.56) we need to introduce some space of functions and operators, we refer to [133] and the references therein for the proof of these statements and some discussions on this topic.
Let $\mathsf{H}^1 := H_0^1(\mathcal{O}) \cap L^2(\mathcal{O}, \frac{dx}{|x|^2})$ endowed with the following scalar product

$$\langle u, v\rangle_{\mathsf{H}^1} = \langle \nabla u, \nabla v\rangle + \langle \frac{u}{|x|}, \frac{v}{|x|}\rangle$$

Define the operator $\bar{A} : \mathsf{D}(\bar{A}) \to L^2(\mathcal{O})$ as $\bar{A}u = g$ whenever there exist $g \in L^2(\mathcal{O})$ such that

$$\langle u, v\rangle_{\mathsf{H}^1} = \langle g, v\rangle.$$

Then the following statement holds.

Lemma 5.28. *The operator $-\bar{A}$ generates a self-adjoint analytic semigroup of negative type $\bar{S}(t)$ over $L^2(\mathcal{O})$, $\mathsf{D}(\bar{A}^{1/2}) = \mathsf{H}^1$. Moreover, if χ has radial symmetry then the same holds also for $\bar{S}(t)\chi$ $\forall t \geq 0$.*

Therefore, according to [55], problem (5.56) can be interpreted in mild form as

$$v^\nu(t) := \bar{S}(\nu t)\chi^\nu + \int_0^t \bar{S}(\nu(t-s))\bar{f}(s)ds + \sqrt{\nu}\sum_{k\in K}\int_0^t \bar{S}(\nu(t-s))\sigma_k(|x|)dW_s^k \tag{5.57}$$

We introduce the notion of weak solution of problem (5.56).

Definition 5.29. We say that v^ν is a weak solution of (5.56) if

$$v^\nu \in C_{\mathcal{F}}\left([0,T]; L^2(\mathcal{O})\right) \cap L^2_{\mathcal{F}}(0,T;\mathsf{H}^1)$$

and for every $\varphi \in \mathsf{D}(\bar{A})$, we have

$$\langle v^\nu(t), \varphi\rangle = \langle \chi^\nu, \varphi\rangle - \nu\int_0^t \langle v^\nu(s), \bar{A}\varphi\rangle\, ds + \int_0^t \langle \bar{f}(s), \varphi\rangle ds + \sqrt{\nu}\sum_{k\in K}\langle s_k, \varphi\rangle\, W_t^k$$

for every $t \in [0,T], \mathbb{P}-$ a.s., where $s_k = \sigma_k(|x|)$.

Thanks to [55, Theorem 5.4] the mild formula (5.57) gives the unique weak solution of (5.56). Indeed the following hold.

Lemma 5.30. *$v^\nu \in C_{\mathcal{F}}\big([0,T];L^2(\mathcal{O})\big) \cap L^2_{\mathcal{F}}(0,T;\mathsf{H}^1)$ and has radial symmetry. Moreover, it is a weak solution in the sense of Definition 5.29 of equation (5.56).*

We introduced the problem (5.56), because of the representation formula for the unique solution of (5.13) guaranteed by the following proposition.

Proposition 5.31. *The unique weak solution u^ν of the Navier-Stokes system (5.13) for $f(t,x) = \bar{f}(t,|x|)\frac{x^\perp}{|x|} \in H_0^{RS}$, $u_0(x) = \bar{u}_0(|x|)\frac{x^\perp}{|x|} \in \mathcal{E}_0^{RS}$ and with noise W^{RS} is given by $\bar{u}^\nu(t,|x|)\frac{x^\perp}{|x|}$ where $\bar{u}^\nu$ solves equation (5.56) with forcing $\bar{f}$ and initial datum $\bar{u}_0$*

Proof. We begin by proving that for every $\varphi \in C_c^\infty(\mathcal{O}\setminus\{0\};\mathbb{R}^2)$,

$$b\left(\bar{u}^\nu(s,x)\frac{x^\perp}{|x|}, \varphi, \bar{u}^\nu(s,x)\frac{x^\perp}{|x|}\right) = 0.$$

Indeed

$$\begin{aligned}\int_{\mathcal{O}} |\bar{u}^\nu(s,|x|)|^2 \frac{x^\perp}{|x|^2}\cdot(x^\perp\cdot\nabla\varphi)dx &= \int_{\mathcal{O}} |\bar{u}^\nu(s,|x|)|^2 \frac{x^\perp}{|x|^2}\cdot(\nabla(x^\perp\cdot\varphi)+\varphi^\perp)dx\\ &= \int_{\mathcal{O}} \frac{|\bar{u}^\nu(s,|x|)|^2}{|x|^2}(x^\perp\cdot\nabla(x^\perp\cdot\varphi)+x\cdot\varphi)dx = I_1+I_2\end{aligned}$$

Now we have

$$I_1 = \int_{\mathcal{O}} \frac{|\bar{u}^\nu(s,|x|)|^2}{|x|^2}\operatorname{div}(x^\perp(x^\perp\cdot\varphi))dx = -\int_{\mathcal{O}} \nabla\Big[\frac{|\bar{u}^\nu(s,|x|)|^2}{|x|^2}\Big]\cdot x^\perp(x^\perp\cdot\varphi)dx = 0$$

since the gradient of a radial function is always parallel to x. While, if we define $V(\rho) = \int_0^\rho \frac{|\bar{u}^\nu(s,r)|^2}{r}dr$, we have

$$I_2 = \int_{\mathcal{O}} \frac{|\bar{u}^\nu(s,|x|)|^2}{|x|^2} x\cdot\varphi dx = \int_{\mathcal{O}} \nabla V(|x|)\cdot\varphi dx = -\int_{\mathcal{O}} V(|x|)\operatorname{div}\varphi dx = 0.$$

Next, we observe that $\varphi\cdot\frac{x^\perp}{|x|} \in \mathsf{D}(\bar{A})$. Since we want to prove that, neglecting the non-linear term which is zero,

$$\langle u^\nu(t)-u_0,\varphi\rangle + \int_0^t \nu\langle\nabla u^\nu(s),\nabla\varphi\rangle ds = \int_0^t \langle f(s),\varphi\rangle ds + \sqrt{\nu}\sum_{k\in K}\langle\sigma_k\frac{x^\perp}{|x|},\varphi\rangle. \tag{5.58}$$

We rewrite this as

$$\begin{aligned}\int_{\mathcal{O}}[\bar{u}^\nu(t,|x|)-\bar{u}_0(|x|)]\varphi(x)\cdot\frac{x^\perp}{|x|}dx + \int_0^t\int_{\mathcal{O}} \nu\nabla[\bar{u}^\nu(s,|x|)\frac{x^\perp}{|x|}]\cdot\nabla\varphi(x)dxds&\\ = \int_0^t\int_{\mathcal{O}} \bar{f}(s,|x|)\varphi(x)\cdot\frac{x^\perp}{|x|}dxds&\\ + \sqrt{\nu}\Big(\sum_{k\in K}\int_{\mathcal{O}}\sigma_k(|x|)\varphi(x)\cdot\frac{x^\perp}{|x|}dx\Big)W_t^k.&\end{aligned}$$

which, comparing with the Definition 5.29, holds true if we prove that

$$\int_0^t \int_{\mathcal{O}} \nabla[\bar{u}^\nu(s,|x|)\frac{x^\perp}{|x|}] \cdot \nabla\varphi(x) dx ds = \int_0^t \int_{\mathcal{O}} \bar{A}^{1/2}\bar{u}^\nu(s,|x|)\bar{A}^{1/2}(\varphi(x)\cdot\frac{x^\perp}{|x|}) dx ds. \tag{5.59}$$

This can be proven with simple calculations, upon noticing that

$$\Delta(\varphi(x)\cdot\frac{x^\perp}{|x|}) = \Delta(\varphi(x))\cdot\frac{x^\perp}{|x|} + \varphi(x)\cdot\frac{x^\perp}{|x|^3} - 2\operatorname{div}\left(\frac{x\cdot\varphi}{|x|^3}x^\perp\right). \tag{5.60}$$

Finally, we obtain that u^ν is a weak solution of (5.13) by observing that the closure of $C_c^\infty(\mathcal{O}\setminus\{0\};\mathbb{R}^2)$ vectors field in the H^1 norm, is exactly $H_0^1(\mathcal{O};\mathbb{R}^2)$, which implies that (5.58) holds in particular for every $\varphi \in \mathsf{D}(A)$. ☐

In the same manner we can prove the analogous result for the Euler system, that is

Proposition 5.32. *Given* $\chi(|x|)\frac{x^\perp}{|x|} \in \mathcal{E}_0^{RS}$ *the unique solution of the system* (5.10) *in*

$$C([0,T];W^{2,4}(\mathcal{O};\mathbb{R}^2)) \cap C([0,T];H)$$

is given by $\bar{u}(t,|x|)\frac{x^\perp}{|x|}$ *where the radial function* $\bar{u}$ *is given by*

$$\bar{u}(t,|x|) := \chi(|x|) + \int_0^t \bar{f}(s,|x|)ds.$$

Condition 2

In this section we prove that the second condition in the weak convergence approach is easily fulfilled unconditionally in the case of fluids with radial symmetries. Since the proof of the validity of Condition 1 in Hypothesis 5.4 and the lower semicontinuity of the map $u_0 \to I_{u_0}^{RS}(w)$ presented in subsection 5.3.1 is valid also in this framework, this is enough to prove Theorem 5.21. Let $u_0^\nu(x) := \bar{u}_0^\nu(|x|)\frac{x^\perp}{|x|} \to u_0(x) := \bar{u}_0(|x|)\frac{x^\perp}{|x|}$ in $\mathcal{E}_0^{RS}$, which we recall being endowed with the H^3 norm and $f^\nu(t,x) := \bar{f}^\nu(t,|x|)\frac{x^\perp}{|x|} \to f(t,x) := \bar{f}(t,|x|)\frac{x^\perp}{|x|}$ in law as $S^N(H_0^{RS})$-random variables. We will show that for each sequence $\nu_n \to 0$, $\mathcal{G}^{\nu_n,RS}\big(u_0^{\nu_n}, \sqrt{\nu_n}W + \int_0^\cdot f^{\nu_n}(s)ds\big)$ converges in law to $\mathcal{G}^{0,RS}(u_0, \int_0^\cdot f(s)ds)$ in the topology of $\mathcal{E}$. This implies the validity of the second condition in Hypothesis 5.4. In order to simplify the notation, we will consider $\nu > 0$ in the following dropping the subscript ν_n, having in mind it is a countable family. Arguing as in section 5.3.1, up to passing to a different filtered probability space, by Skorokhod's representation theorem, we can assume that the convergence of the f^ν is a.s. in the weak topology of $L^2(0,T;\mathsf{D}(\mathcal{A}^{1/2}))$ (since, by construction, we have the embedding $H_0^{RS} \hookrightarrow \mathsf{D}(\mathcal{A}^{1/2})$). We call $u^\nu = \mathcal{G}^{\nu,RS}(u^\nu, \int_0^\cdot f^\nu(s)ds + \sqrt{\nu}W)$ and $u = \mathcal{G}^0(u, \int_0^\cdot f(s)ds)$.

From the results in the previous section, we have that both u^ν and u have circular symmetry, and their radial parts are given by

$$\bar{u}^\nu(t) := \bar{S}(\nu t)\bar{u}_0^\nu + \int_0^t \bar{S}(\nu(t-s))\bar{f}^\nu(s)ds + \sqrt{\nu}\sum_{k\in K}\int_0^t \bar{S}(\nu(t-s))\sigma_k(|\cdot|)dW_s^k$$

$$\bar{u}(t) = \bar{u}_0 + \int_0^t \bar{f}(s)ds$$

Actually we will prove the stronger result:

$$\mathbb{E}\left[\sup_{t\in[0,T]} \|u^\nu(t) - u(t)\|^2\right] \to 0 \tag{5.61}$$

in the probability space defined via Skorokhod's representation theorem. From our representation formula, it is sufficient to show

$$\mathbb{E}\left[\sup_{t\in[0,T]} ||\bar{u}^\nu - \bar{u}||^2\right] \to 0$$

To do so we write

$$\begin{aligned}\bar{u}^\nu(t) - \bar{u}(t) =& \bar{S}(\nu t)(\bar{u}_0^\nu - \bar{u}_0) + (\bar{S}(\nu t) - I)\bar{u}_0 + \int_0^t \bar{S}(\nu(t-s))(\bar{f}^\nu(s) - \bar{f}(s))ds \\ &+ \int_0^t (\bar{S}(\nu(t-s)) - I)\bar{f}(s) + \sqrt{\nu}\sum_{k\in K}\int_0^t \bar{S}(\nu(t-s))\sigma_k(|\cdot|)dW_s^k. \end{aligned} \tag{5.62}$$

Preliminarily we observe that in the proof of Proposition 5.31, we showed equality (5.59) for every vector field $\varphi(x) \in H_0^1$. If we disregard the time integration and let $\bar{u}^\nu(s)$ in (5.59) be any generic radial function in H^1 we obtain that the map $J : \mathsf{H}_R^1 \to \mathsf{D}(\mathcal{A}^{1/2})$ that sends any radial function $v(|x|)$ in H^1 to the circular symmetric vector field $v(|x|)\frac{x^\perp}{|x|}$ is an isometry (where we indicated with H_R^1 the set of radial function of H^1). We then obtain

$$\|f^\nu\|_{\mathsf{D}(\mathcal{A}^{1/2})} = \|\bar{f}^\nu(s,|x|)\frac{x^\perp}{|x|}\|_{\mathsf{D}(\mathcal{A}^{1/2})} = \|\bar{f}^\nu(s,|x|)\|_{\mathsf{H}^1},$$

$$\|f\|_{\mathsf{D}(\mathcal{A}^{1/2})} = \|\bar{f}(s,|x|)\frac{x^\perp}{|x|}\|_{\mathsf{D}(\mathcal{A}^{1/2})} = \|\bar{f}(s,|x|)\|_{\mathsf{H}^1}$$

which gives exactly

$$\int_0^T \|\bar{f}(s)\|_{\mathsf{H}^1}^2 ds + \sup_{\nu>0}\int_0^T \|\bar{f}^\nu(s)\|_{\mathsf{H}^1}^2 ds \le C(N). \tag{5.63}$$

Moreover $\bar{f}^\nu \rightharpoonup \bar{f}$ in $L^2(0,T;\mathsf{H}^1)$ $\mathbb{P}-a.s.$ Now we can treat $\bar{u}^\nu - \bar{u}$. The first and second terms in (5.62) go to zero in L^2 norm thanks to the strong convergence of $\bar{u}_0^\nu$ to $\bar{u}_0$ and the continuity of the semigroup. The convergence is uniform in time since, for the first term $\|\sup_{t\leqslant T}\bar{S}(\nu t)(\bar{u}_0^\nu - \bar{u}_0)\| \leqslant \|\bar{u}_0^\nu - \bar{u}_0\|$ while for the second, we choose for every ν, t_ν for which $\|(e^{\nu\bar{A}t_\nu} - I)\bar{u}_0\|$ achieves its maximum over $[0,T]$. Then by observing that $t_\nu \leq T$, we get that as $\nu\to 0$, $\nu t_\nu \to 0$, and we conclude by the continuity of the semigroup. The stochastic integral term can be easily controlled using the Itô formula and Burkholder-Davis-Gundy inequality for Stochastic Convolutions, see for example [160], obtaining

$$\mathbb{E}\left[\sup_{t\leqslant T}\|\sqrt{\nu}\sum_{k\in K}\int_0^t \bar{S}(\nu(t-s))\sigma_k(|\cdot|)dW_s^k\|^2\right] \leqslant 2T\sqrt{\nu}\,\mathbb{E}\left[\sup_{t\leqslant T}\|\bar{u}^\nu(s)\|^2\right]\left(\sum_{k\in K}\|\sigma_k\|^2\right)$$

which converges to zero as all the quantities are bounded. The remaining terms can be treated similarly to Lemma 5.25. In order to study the third term, call $\bar{F}^\nu(t) = \int_0^t \bar{f}^\nu(s)ds$ and $\bar{F}(t) = \int_0^t \bar{f}(s)ds$. By [77, Proposition 2.10],

$$\bar{F}^\nu \to \bar{F} \quad in\ C([0,T];L^2) \quad \mathbb{P}-a.s. \tag{5.64}$$

The third term in (5.62) can can be rewritten as

$$\int_0^t \bar{S}(\nu(t-s))(\bar{f}^\nu(s) - \bar{f}(s))ds = \nu\int_0^t \bar{A}\bar{S}(\nu(t-s))(\bar{F}^\nu(s) - \bar{F}(s))ds + (\bar{F}^\nu(t) - \bar{F}(t)).$$

The second term above converges to 0 $\mathbb{P}-a.s.$ and in $L^2(\Omega,\mathbb{P})$ thanks to (5.63) and (5.64). Concerning the other we use standard properties of analytic semigroup, see for example [146, Chapter 2], and (5.63) obtaining

$$\begin{aligned}&\mathbb{E}\left[\sup_{t\in[0,T]}\|\nu\int_0^t \bar{A}\bar{S}(\nu(t-s))(\bar{F}^\nu(s) - \bar{F}(s))ds\|^2\right]\\ &\leq C\nu\,\mathbb{E}\left[\sup\nolimits_{t\in[0,T]}\left(\int_0^t \frac{\|\bar{F}^\nu - \bar{F}\|_{C([0,T];\mathsf{H}^1)}}{(t-s)^{1/2}}ds\right)^2\right] \leq C(N,T)\nu \to 0.\end{aligned}$$

Finally, for the fourth term, by standard properties of analytic semigroups, see for example [146, Chapter 2], and our assumption on f, see (5.63),

$$\begin{aligned}\sup_{t\in[0,T]}\|\int_0^t(\bar{S}(\nu(t-s)) - I)\bar{f}(s)ds\|^2 &\leqslant \left(\sup_{t\in[0,T]}\int_0^t \|(\bar{S}(\nu(t-s)) - I)\bar{f}(s)\|ds\right)^2\\ &\leqslant C\nu\left(\sup_{t\in[0,T]}\int_0^t (t-s)^{1/2}\|\bar{f}(s)\|_{\mathsf{H}^1}ds\right)^2\\ &\leqslant C(N,T)\nu \quad \mathbb{P}-a.s.\end{aligned}$$

Therefore we get

$$\mathbb{E}\Big[\sup_{[0,T]}\|\int_0^t(\bar{S}(\nu(t-s))-I)\bar{f}(s)ds\|^2\Big]\to 0.$$

This prove the validity of relation (5.61).

Remark 5.33. According to Remark 5.27, computation above implies in particular that Hypothesis 5.14 are satisfied in this framework.

5.4 Second-Grade Fluid Equations

We observe that the proof of the validity of Condition 1 of Hypothesis 5.4 presented in subsection 5.3.1 holds also in this framework. Moreover the same is true also for the lower continuity of the map $I_{u_0}^{SG}(w)$ for $w\in\mathcal{E}$ fixed. Therefore, in order to prove Theorem 5.19, it is enough to show the validity of Condition 2 of Hypothesis 5.4.

5.4.1 Condition 2

We argue similarly to the proof of the validity Condition 2 in subsection 5.3.1. Fix $N>0$, let $\tilde{f}^\alpha,\tilde{f}\in\mathcal{P}_2^N,\ \mathsf{u}_0^\alpha,\ u_0\in\mathcal{E}_0^{SG}$ such that $\tilde{f}^\alpha\to_{\mathcal{L}}\tilde{f}$ weakly in $L^2(0,T;H_0),\ \mathsf{u}_0^\alpha\to u_0$ in $\mathcal{E}_0^{SG}$. We will show that for each sequence $\alpha_n\to 0,\nu_n\to 0$ s.t. $\nu_n=O(\alpha_n)$, $\mathcal{G}^{\alpha_n,SG}\big(\mathsf{u}_0^{\alpha_n},\sqrt{\alpha_n}W+\int_0^\cdot f^{\alpha_n}(s)ds\big)$ converges in law to $\mathcal{G}^{0,SG}(u_0,\int_0^\cdot u(s)ds)$ in the topology of $\mathcal{E}$. This implies the validity of the second condition in Hypothesis 5.4. In order to simplify the notation, we will consider $\alpha>0,\ \nu>0$ in the following dropping the subscript $\alpha_n,\ \nu_n$, having in mind they are countable families. Since S^N is a Polish space, by Skorokhod's representation theorem we can introduce a further filtered probability space $(\tilde{\Omega},\tilde{\mathcal{F}},\tilde{\mathcal{F}}_t,\tilde{\mathbb{P}})$ and random variables f^α,W^α,f such that (f^α,W^α) has the same joint law of $(\tilde{f}^\alpha,W)$, f has the same law of $\tilde{f}$ and $f^\alpha\to_{\tilde{\mathbb{P}}-a.s.}f$ in $L^2(0,T;H_0)$, see for example [77] for details. Thanks to Theorem 5.10 for each α we can define u^α as the unique solution of (5.6) with forcing term f^α, initial condition u_0^α and Brownian forcing term W^α. Moreover, by Theorem 5.12 we can define u^E as the unique regular solution of (5.10) with forcing term f and initial condition u_0. We will show that u^α converges to u^E in probability in $C([0,T];H)$. This implies the validity of Condition 2.
Before starting with the computation we recall some facts. In the following, with some abuse of notation, we will simply use $\mathbb{P},\ \mathbb{E}$ instead of $\tilde{\mathbb{P}},\ \tilde{\mathbb{E}}$. Fix $\theta>0$ arbitrarily small and define $F^\alpha(t)=\int_0^t f^\alpha(s)ds,\ F(t)=\int_0^t f(s)ds$. By Hypothesis 4.4, $H_0\hookrightarrow\mathsf{D}(\mathcal{A}^\gamma)$. This implies, see for example [77, Proposition 2.10] that

$$F^\alpha\to_{\mathbb{P}-a.s.}F\quad in\ C([0,T];\mathsf{D}(\mathcal{A}^{\gamma-\theta})).\tag{5.65}$$

Obviously

$$\sup_{\alpha>0}\|F^\alpha\|_{C([0,T];\mathsf{D}(\mathcal{A}^\gamma))}+\|F\|_{C([0,T];\mathsf{D}(\mathcal{A}^\gamma))}\leq C(N)\quad\mathbb{P}-a.s.\tag{5.66}$$

Starting from (5.8) and (5.9), under Hypothesis 5.18, Burkholder-Davis-Gundy inequality, Grönwall's lemma and the convergence of u_0^α to u_0 imply

$$\mathbb{E}\left[\sup_{t\in[0,T]}\|\mathsf{u}^\alpha(t)\|^2\right]+\alpha\mathbb{E}\left[\sup_{t\in[0,T]}\|\nabla\mathsf{u}^\alpha(t)\|^2\right]+2\nu\mathbb{E}\left[\int_0^T\|\nabla\mathsf{u}^\alpha(s)\|^2ds\right]\leq C(N,u_0), \tag{5.67}$$

$$\alpha^3\mathbb{E}\left[\sup_{t\in[0,T]}\|\mathsf{u}^\alpha(t)\|^2_{H^3}\right]\leq C(N,u_0), \tag{5.68}$$

see [135, Section 6] for the details. In order to show the convergence of u^α to u^E we introduce the semigroup, $\mathbf{S}(t)$, with infinitesimal generator $-(I+\alpha\mathcal{A})^{-1}\mathcal{A}$ and denote by

$$\mathsf{z}^\alpha=\int_0^t\mathbf{S}(\nu(t-s))(I+\alpha\mathcal{A})^{-1}f^\alpha(s)ds$$

which is the mild solution of

$$d(I+\alpha\mathcal{A})\mathsf{z}^\alpha=-\nu\mathcal{A}\mathsf{z}^\alpha+f^\alpha,$$

$\mathsf{v}^\alpha=\mathsf{u}^\alpha-\mathsf{z}^\alpha$, $v^E=u^E-F$ and show separately the convergence of z^α to F and of v^α to v^E. Once again, the convergence of z^α to F will be the easiest part of the argument, while the convergence of v^α to v^E will be the more demanding and its proof will be based on the introduction of a corrector of the boundary layer for v^E satisfying suitable properties. Before starting showing the convergence of z^α to F, we recall that the the operators $-\mathcal{A}$ and $(I+\alpha\mathcal{A})^{-1}$ commute on $\mathsf{D}(\mathcal{A}^\vartheta)$ for each $\vartheta\geq 0$. Moreover

$$-(I+\alpha\mathcal{A})^{-1}\mathcal{A}:\mathsf{D}(\mathcal{A}^\vartheta)\to\mathsf{D}(\mathcal{A}^\vartheta)$$

is a linear, self-adjoint, negative, bounded operator, for each $\vartheta\geq 0$ with operatorial norm equal to $\frac{1}{\alpha}$.

Lemma 5.34. *For each $\theta>0$, $\mathsf{z}^\alpha(t)\to F$ in $C([0,T];\mathsf{D}(\mathcal{A}^{\gamma-2\theta}))$ $\mathbb{P}-a.s.$ and in $L^2(\Omega,\mathbb{P})$.*

Proof. z^α can be rewritten as

$$\mathsf{z}^\alpha(t)=-\nu\int_0^t(I+\alpha\mathcal{A})^{-2}\mathcal{A}\mathbf{S}(\nu(t-s))F^\alpha(s)ds+(I+\alpha\mathcal{A})^{-1}F^\alpha(t)=I_1+I_2$$

Let us show that $I_2\to F$ and $I_1\to 0$ properly. We start from I_2 :

$$I_2=((I+\alpha\mathcal{A})^{-1}F^\alpha(t)-F^\alpha(t))+F^\alpha(t)=I_{2,1}+I_{2,2}$$

Now $I_{2,2} \to F(t)$ in $C([0,T];\mathsf{D}(\mathcal{A}^{\gamma-\theta}))$ $\mathbb{P}-a.s.$ and $L^2(\Omega,\mathbb{P})$ by (5.65) and (5.66). For what concerns $I_{2,1}$ we have

$$\begin{aligned}
\sup_{t\in[0,T]}\|\mathcal{A}^{\gamma-2\theta}I_{2,1}\| &= \sup_{t\in[0,T]}\|\mathcal{A}^{\gamma-2\theta}(F^\alpha(t)-(I+\alpha\mathcal{A})^{-1}F^\alpha(t))\|\\
&= \sup_{t\in[0,T]}\|\mathcal{A}^{\gamma-2\theta}\left(\frac{I}{\alpha}+\mathcal{A}\right)^{-1}\mathcal{A}F^\alpha(t)\|\\
&\le \sup_{t\in[0,T]}\|\mathcal{A}^{1-\theta}\left(\frac{I}{\alpha}+\mathcal{A}\right)^{-1}\|\|\mathcal{A}^{\gamma-\theta}F^\alpha(t)\|\\
&\le \alpha^\theta\frac{\left(\frac{1-\theta}{\theta}\right)^{1-\theta}}{1+\frac{1-\theta}{\theta}}C(N,T)\to 0 \ \mathbb{P}-a.s.
\end{aligned}$$

Since previous bound is uniform in $\omega\in\Omega$, previous inequalities imply also the convergence in $L^2(\Omega,\mathbb{P})$. For what concerns I_1 we have

$$\begin{aligned}
&\sup_{t\in[0,T]}\|\mathcal{A}^{\gamma-\theta}I_1\|\\
&\le \nu\sup_{t\in[0,T]}\int_0^t\|\mathcal{A}^{1-\theta}(I+\alpha\mathcal{A})^{-2}\mathbf{S}(\nu(t-s))\mathcal{A}^\gamma F^\alpha(s)\|ds\\
&\le \alpha^\theta\frac{\left(\frac{1-\theta}{\theta+1}\right)^{\frac{1-\theta}{2}}}{1+\frac{1-\theta}{\theta+1}}C(N,T)\to 0 \ \mathbb{P}-a.s.
\end{aligned}$$

Since previous bound is uniform in $\omega\in\Omega$, previous inequalities imply also the convergence in $L^2(\Omega,\mathbb{P})$. Combining the convergence of I_1, $I_{2,1}$ and $I_{2,2}$ the result follows. □

Remark 5.35. Since $H_0\hookrightarrow\mathsf{D}(\mathcal{A}^{3/2})\hookrightarrow W$, z^α satisfies relations (5.67), (5.68). We show a stronger relation. Indeed it holds:

$$\begin{aligned}
\sup_{t\in[0,T]}\|\mathcal{A}^{3/2}\mathsf{z}^\alpha(t)\| &\le \sup_{t\in[0,T]}\int_0^t\|\mathcal{A}^{3/2}\mathbf{S}(\nu(t-s))(I+\alpha\mathcal{A})^{-1}f^\alpha(s)\|ds\\
&\le \sup_{t\in[0,T]}\int_0^t\|\mathcal{A}^{3/2}f^\alpha(s)\|ds\\
&\le C(N,T) \quad \mathbb{P}-a.s.
\end{aligned}\tag{5.69}$$

Since previous bound is uniform in $\omega\in\Omega$, we have also

$$\mathbb{E}\left[\sup_{t\in[0,T]}\|\mathcal{A}^{3/2}\mathsf{z}^\alpha(t)\|^2\right]\le C(N,T).\tag{5.70}$$

In order to prove the convergence of v^α to v^E we observe that they solve in a sense analogous to Definition 5.8, Definition 5.11

$$d(\mathsf{v}^\alpha-\alpha\Delta\mathsf{v}^\alpha)=(\nu\Delta\mathsf{v}^\alpha-\operatorname{curl}(\mathsf{v}^\alpha+\mathsf{z}^\alpha-\alpha\Delta(\mathsf{v}^\alpha+\mathsf{z}^\alpha))\times(\mathsf{v}^\alpha+\mathsf{z}^\alpha)+\nabla q^\alpha)dt+\sqrt{\alpha}dW(t)\tag{5.71}$$

and (5.34). By triangle inequality and the uniform bound guaranteed by Remark 5.35, estimates analogous to (5.67), (5.68), (5.12) hold for v^α and v^E, too. Moreover v^E satisfies (5.35). Again, due to Proposition 4.14, we introduce the corrector of the boundary layer v of width $\delta = \delta(\alpha)$, i.e. a divergence free vector field with support in a strip of the boundary of width δ such that $v^E - v \in V$ and $\mathbb{P} - a.s.$ uniformly in $t \in [0,T],\ \omega \in \Omega$

$$\begin{aligned}&\|v(t)\|_{L^\infty} \le C(N,u_0),\ \|v(t)\| \le C(N,u_0)\delta^{\frac{1}{2}},\ \|\partial_t v(t)\| \le C(N,u_0)\delta^{\frac{1}{2}},\\&\|\nabla v(t)\|_{L^\infty} \le C(N,u_0)\delta^{-1},\ \|\nabla v(t)\| \le C(N,u_0)\delta^{-1/2},\ \|\partial_t \nabla v(t)\| \le C(N,u_0)\delta^{-\frac{1}{2}}.\end{aligned} \tag{5.72}$$

We choose δ such that

$$\lim_{\alpha\to 0}\delta = 0, \quad \lim_{\alpha\to 0}\frac{\alpha}{\delta} = 0. \tag{5.73}$$

Now we are ready to show the convergence of v^α to v^E.

Lemma 5.36. *$\mathsf{v}^\alpha \to v^E$ in $C([0,T];H)$ in probability.*

Proof. Let $\mathsf{w}^\alpha = \mathsf{v}^\alpha - v^E$. Arguing as in the proof of Theorem 4.22 one can show that the following relations hold true:

$$\begin{aligned}d\|\mathsf{w}^\alpha\|^2 = {}&\alpha Tr(Q)dt + \alpha^3 Tr(\mathcal{A}^2(I+\alpha\mathcal{A})^{-2}Q)dt + \alpha^2 Tr(-\mathcal{A}(I+\alpha\mathcal{A})^{-1}Q)dt\\&- 2\nu\langle \mathsf{w}^\alpha, \mathcal{A}\mathsf{v}^\alpha\rangle dt + 2\sqrt{\alpha}\langle \mathsf{w}^\alpha, dW^\alpha(t)\rangle + b(v^E + F, v^E + F, \mathsf{w}^\alpha)dt\\&- 2b(\mathsf{v}^\alpha + \mathsf{z}^\alpha, (I-\alpha\Delta)(\mathsf{v}^\alpha + \mathsf{z}^\alpha), \mathsf{w}^\alpha)dt - 2b(\mathsf{w}^\alpha, (I-\alpha\Delta)(\mathsf{v}^\alpha + \mathsf{z}^\alpha), \mathsf{v}^\alpha + \mathsf{z}^\alpha)dt\\&- 2\alpha\langle \mathsf{w}^\alpha, d\mathcal{A}\mathsf{v}^\alpha\rangle\end{aligned} \tag{5.74}$$

where

$$\begin{aligned}-\langle \mathsf{w}^\alpha, d\mathcal{A}\mathsf{v}^\alpha\rangle = {}&-\frac{d\|\mathcal{A}^{1/2}\mathsf{v}^\alpha\|^2}{2} + \frac{d\langle\langle \mathcal{A}^{1/2}\mathsf{v}^\alpha, \mathcal{A}^{1/2}\mathsf{v}^\alpha\rangle\rangle}{2}\\&+ d\langle \mathcal{A}^{1/2}(v^E - v), \mathcal{A}^{1/2}\mathsf{v}^\alpha\rangle - \langle \partial_t \mathcal{A}^{1/2}(v^E - v), \mathcal{A}^{1/2}\mathsf{v}^\alpha\rangle\\&+ d\langle v, \mathcal{A}\mathsf{v}^\alpha\rangle - \langle \partial_t v, \mathcal{A}\mathsf{v}^\alpha\rangle.\end{aligned}$$

First we observe that

$$\alpha Tr(Q) + \alpha^3 Tr(\mathcal{A}^2(I+\alpha\mathcal{A})^{-2}Q) - \alpha^2 Tr(\mathcal{A}(I+\alpha\mathcal{A})^{-1}Q) = o(1). \tag{5.75}$$

Secondly, we can rewrite the trilinear forms as

$$\begin{aligned}&b(v^E + F, v^E + F, \mathsf{w}^\alpha) - b(\mathsf{v}^\alpha + \mathsf{z}^\alpha, \mathsf{v}^\alpha + \mathsf{z}^\alpha, \mathsf{w}^\alpha) + \alpha b(\mathsf{v}^\alpha + \mathsf{z}^\alpha, \Delta(\mathsf{v}^\alpha + \mathsf{z}^\alpha), \mathsf{w}^\alpha)\\&- \alpha b(\mathsf{w}^\alpha, \Delta(\mathsf{v}^\alpha + \mathsf{z}^\alpha), \mathsf{v}^\alpha + \mathsf{z}^\alpha) \pm b(\mathsf{v}^\alpha, v^E, \mathsf{w}^\alpha) =\\&b(F, v^E + F, \mathsf{w}^\alpha) + b(v^E, F, \mathsf{w}^\alpha) - b(\mathsf{z}^\alpha, \mathsf{v}^\alpha + \mathsf{z}^\alpha, \mathsf{w}^\alpha) - b(\mathsf{v}^\alpha, \mathsf{z}^\alpha, \mathsf{w}^\alpha) - b(\mathsf{w}^\alpha, v^E, \mathsf{w}^\alpha)\\&+ \alpha b(\mathsf{u}^\alpha, \Delta\mathsf{u}^\alpha, \mathsf{w}^\alpha) - \alpha b(\mathsf{w}^\alpha, \Delta\mathsf{u}^\alpha, \mathsf{u}^\alpha).\end{aligned} \tag{5.76}$$

Integrating in time between 0 and t equation (5.74) and exploiting (5.75), (5.76), we get

$$\begin{aligned}
\|\mathsf{w}^\alpha(t)\|^2+\alpha\|\nabla\mathsf{v}^\alpha(t)\|^2&=o(1)+\|\mathsf{u}_0^\alpha-u_0\|^2+\alpha\|\nabla\mathsf{u}_0^\alpha\|^2+2\alpha\langle\nabla(v^E(t)-v(t)),\nabla\mathsf{v}^\alpha(t)\rangle\\
&-2\alpha\langle\nabla(u_0-v_0),\nabla\mathsf{u}_0^\alpha\rangle-2\alpha\int_0^t\langle\partial_s\nabla(v^E(s)-v(s)),\nabla\mathsf{v}^\alpha(s)\rangle ds\\
&-2\alpha\langle v(t),\Delta\mathsf{v}^\alpha(t)\rangle+2\alpha\langle v_0,\Delta\mathsf{u}_0^\alpha\rangle+2\alpha\int_0^t\langle\partial_s v(s),\Delta\mathsf{v}^\alpha(s)\rangle ds\\
&-2\nu\int_0^t\langle\mathsf{w}^\alpha(s),\mathcal{A}\mathsf{v}^\alpha(s)\rangle ds+2\sqrt{\alpha}\int_0^t\langle\mathsf{w}^\alpha(s),dW^\alpha(s)\rangle\\
&+2\int_0^t b(F(s),u^E(s),\mathsf{w}^\alpha(s))-b(\mathsf{z}^\alpha(s),\mathsf{u}^\alpha(s),\mathsf{w}^\alpha(s))ds\\
&+2\int_0^t b(v^E(s),F(s),\mathsf{w}^\alpha(s))-b(\mathsf{v}^\alpha(s),\mathsf{z}^\alpha(s),\mathsf{w}^\alpha(s))ds\\
&+2\alpha\int_0^t b(\mathsf{u}^\alpha(s),\Delta\mathsf{u}^\alpha(s),\mathsf{w}^\alpha(s))-b(\mathsf{w}^\alpha(s),\Delta\mathsf{u}^\alpha(s),\mathsf{u}^\alpha(s))ds\\
&-2\int_0^t b(\mathsf{w}^\alpha(s),v^E(s),\mathsf{w}^\alpha(s))ds. \qquad (5.77)
\end{aligned}$$

In order to reach our final expression for the evolution of $\|\mathsf{w}^\alpha(t)\|^2$ we rewrite better the terms related to the forcing terms $f^\alpha,\ f$ in equation (5.76)

$$\begin{aligned}
&b(F,u^E,\mathsf{w}^\alpha)+b(v^E,F,\mathsf{w}^\alpha)-b(\mathsf{v}^\alpha,\mathsf{z}^\alpha,\mathsf{w}^\alpha)-b(\mathsf{z}^\alpha,\mathsf{u}^\alpha,\mathsf{w}^\alpha)\\
&=b(F,v^E,\mathsf{v}^\alpha)+b(F,F,\mathsf{w}^\alpha)+b(v^E,F,\mathsf{w}^\alpha)-b(\mathsf{z}^\alpha,\mathsf{z}^\alpha,\mathsf{w}^\alpha)\\
&+b(\mathsf{z}^\alpha,\mathsf{v}^\alpha,v^E)-b(\mathsf{v}^\alpha,\mathsf{z}^\alpha,\mathsf{w}^\alpha)\pm b(F,\mathsf{z}^\alpha,\mathsf{w}^\alpha)\pm b(v^E,\mathsf{z}^\alpha,\mathsf{w}^\alpha)\\
&=b(F-\mathsf{z}^\alpha,v^E,\mathsf{v}^\alpha)+b(F,F-\mathsf{z}^\alpha,\mathsf{w}^\alpha)\\
&+b(v^E,F-\mathsf{z}^\alpha,\mathsf{w}^\alpha)+b(F-\mathsf{z}^\alpha,\mathsf{z}^\alpha,\mathsf{w}^\alpha)+b(\mathsf{w}^\alpha,\mathsf{z}^\alpha,\mathsf{w}^\alpha). \qquad (5.78)
\end{aligned}$$

Since $u_0^\alpha\to u_0$ in $\mathcal{E}_0^{SG}$, our choice of δ, see (5.73), and the properties of the boundary layer corrector (5.72) we have easily, see (4.66) for details,

$$\alpha\|\nabla\mathsf{u}_0^\alpha\|^2-2\alpha\langle\nabla(u_0-v_0),\nabla\mathsf{u}_0^\alpha\rangle+\alpha\langle v_0,\Delta\mathsf{u}_0^\alpha\rangle=o(1).$$

Inserting (5.78) in (5.77) we get

$$\begin{aligned}
\|\mathsf{w}^\alpha(t)\|^2+\alpha\|\nabla\mathsf{v}^\alpha(t)\|^2&=o(1)+2\alpha\langle\nabla(v^E(t)-v(t)),\nabla\mathsf{v}^\alpha(t)\rangle-2\alpha\langle v(t),\Delta\mathsf{v}^\alpha(t)\rangle\\
&-2\alpha\int_0^t\langle\partial_s\nabla(v^E(s)-v(s)),\nabla\mathsf{v}^\alpha(s)\rangle ds\\
&+2\alpha\int_0^t\langle\partial_s v(s),\Delta\mathsf{v}^\alpha(s)\rangle ds-2\nu\int_0^t\langle\mathsf{w}^\alpha(s),\mathcal{A}\mathsf{v}^\alpha(s)\rangle ds
\end{aligned}$$

$$
\begin{aligned}
&+2\int_0^t b(F(s)+v^E(s),F(s)-\mathsf{z}^\alpha(s),\mathsf{w}^\alpha(s))ds\\
&-2\int_0^t b(F(s)-\mathsf{z}^\alpha(s),v^E(s),\mathsf{v}^\alpha(s))ds\\
&+2\int_0^t b(F(s)-\mathsf{z}^\alpha(s),\mathsf{z}^\alpha(s),\mathsf{w}^\alpha(s))ds\\
&-2\int_0^t b(\mathsf{w}^\alpha(s),\mathsf{z}^\alpha(s),\mathsf{w}^\alpha(s))-b(\mathsf{w}^\alpha(s),v^E(s),\mathsf{w}^\alpha(s))ds\\
&+2\alpha\int_0^t b(\mathsf{u}^\alpha(s),\Delta\mathsf{u}^\alpha(s),\mathsf{w}^\alpha(s))-b(\mathsf{w}^\alpha(s),\Delta\mathsf{u}^\alpha(s),\mathsf{u}^\alpha)ds\\
&+2\sqrt{\alpha}\int_0^t\langle\mathsf{w}^\alpha(s),dW^\alpha(s)\rangle\\
&=I_1(t)+I_2(t)+I_3(t)+I_4(t)+I_5(t)+I_6(t)+M(t), \qquad (5.79)
\end{aligned}
$$

where

$$
\begin{aligned}
I_1(t)&=2\alpha\langle\nabla(v^E(t)-v(t)),\nabla\mathsf{v}^\alpha(t)\rangle-2\alpha\langle v(t),\Delta\mathsf{v}^\alpha(t)\rangle,\\
I_2(t)&=-2\alpha\int_0^t\langle\partial_s\nabla(v^E(s)-v(s)),\nabla\mathsf{v}^\alpha(s)\rangle ds+2\alpha\int_0^t\langle\partial_s v(s),\Delta\mathsf{v}^\alpha(s)\rangle ds,\\
I_3(t)&=-2\nu\int_0^t\langle\mathsf{w}^\alpha(s),\mathcal{A}\mathsf{v}^\alpha(s)\rangle ds,\\
I_4(t)&=-2\int_0^t b(\mathsf{w}^\alpha(s),\mathsf{z}^\alpha(s),\mathsf{w}^\alpha(s))-b(\mathsf{w}^\alpha(s),v^E(s),\mathsf{w}^\alpha(s))ds,\\
I_5(t)&=2\int_0^t b(F(s)+v^E(s),F(s)-\mathsf{z}^\alpha(s),\mathsf{w}^\alpha(s))-b(F(s)-\mathsf{z}^\alpha(s),v^E(s),\mathsf{v}^\alpha(s))ds\\
&\quad+2\int_0^t b(F(s)-\mathsf{z}^\alpha(s),\mathsf{z}^\alpha(s),\mathsf{w}^\alpha(s))ds,\\
I_6(t)&=2\alpha\int_0^t b(\mathsf{u}^\alpha(s),\Delta\mathsf{u}^\alpha(s),\mathsf{w}^\alpha(s))-b(\mathsf{w}^\alpha(s),\Delta\mathsf{u}^\alpha(s),\mathsf{u}^\alpha)ds,\\
M(t)&=2\sqrt{\alpha}\int_0^t\langle\mathsf{w}^\alpha(s),dW^\alpha(s)\rangle.
\end{aligned}
$$

Equation (5.79) is the final expression that we use in order to estimate the various terms and apply Grönwall's lemma. The analysis of $I_1(t)$ follows by the properties of the boundary layer corrector (5.72), our choice of δ (5.73) and the interpolation inequality (4.36). Therefore it holds:

$$
\begin{aligned}
I_1(t)&\leq\alpha C(N,u_0)(1+\delta^{-1/2})\|\nabla\mathsf{v}^\alpha(t)\|+\alpha\delta^{1/2}C(N,u_0)\|\mathsf{v}^\alpha(t)\|_{H^2}\\
&\leq\frac{\alpha}{10}\|\nabla\mathsf{v}^\alpha(t)\|^2+\alpha C(N,u_0)(1+\delta^{-1})+\alpha\delta^{1/2}C(N,u_0)\|\nabla\mathsf{v}^\alpha\|^{1/2}\|\mathsf{v}^\alpha\|_{H^3}^{1/2}
\end{aligned}
$$

$$\begin{aligned}
&\leq o(1) + \frac{\alpha}{5}\|\nabla \mathsf{v}^\alpha(t)\|^2 + \alpha^3\delta\|\mathsf{v}^\alpha(t)\|^2_{H^3} + \delta^{1/2}C(N,u_0)\\
&= o(1) + \frac{\alpha}{5}\|\nabla \mathsf{v}^\alpha(t)\|^2 + \alpha^3\delta\|\mathsf{v}^\alpha(t)\|^2_{H^3}. \qquad (5.80)
\end{aligned}$$

The analysis of $I_2(t)$ is analogous to the (5.80) and leads us to

$$I_2(t) \leq o(1) + \alpha\int_0^t \|\nabla \mathsf{v}^\alpha(s)\|^2 ds + \alpha^3\delta\int_0^T \|\mathsf{v}^\alpha(s)\|^2_{H^3} ds. \qquad (5.81)$$

In order to treat $I_3(t)$, we split w^α in v^α, $v^E - v$ and v. Then the first two terms are integrated by parts. Exploiting the properties of the boundary layer corrector (5.72), our choice of δ,ν, (5.73), (5.18) and the interpolation inequality (4.36) it holds

$$\begin{aligned}
I_3(t) &= -\nu\int_0^t \|\nabla \mathsf{v}^\alpha(s)\|^2 ds + \nu\int_0^t \langle v^E(s) - v(s), \mathcal{A}\mathsf{v}^\alpha(s)\rangle ds + \nu\int_0^t \langle v(s), \mathcal{A}\mathsf{v}^\alpha(s)\rangle ds\\
&= -\nu\int_0^t \|\nabla \mathsf{v}^\alpha(s)\|^2 ds + \nu\int_0^t \|\nabla(v^E(s) - v(s))\|\|\nabla \mathsf{v}^\alpha(s)\| ds + \nu\int_0^t \|v(s)\|\|\mathsf{v}^\alpha(s)\|_{H^2} ds\\
&\leq -\nu\int_0^t \|\nabla \mathsf{v}^\alpha(s)\|^2 ds + \nu(1+\delta^{-1/2})C(N,u_0)\int_0^t \|\nabla \mathsf{v}^\alpha(s)\| ds\\
&+ \nu\delta^{1/2}C(N,u_0)\int_0^t \|\nabla \mathsf{v}^\alpha(s)\|^{1/2}\|\mathsf{v}^\alpha(s)\|^{1/2}_{H^3} ds\\
&\leq \alpha\int_0^t \|\nabla \mathsf{v}^\alpha(s)\|^2 ds + \alpha^3\delta\int_0^T \|\mathsf{v}^\alpha(s)\|^2_{H^3} ds + \alpha(1+\delta^{-1})C(N,u_0) + \delta^{1/2}C(N,u_0)\\
&= o(1) + \alpha\int_0^t \|\nabla \mathsf{v}^\alpha(s)\|^2 ds + \alpha^3\delta\int_0^T \|\mathsf{v}^\alpha(s)\|^2_{H^3} ds. \qquad (5.82)
\end{aligned}$$

$I_4(t)$ can be bounded easily by Hölder's inequality, obtaining

$$I_4(t) \leq \left(\|\mathsf{z}^\alpha\|_{L^\infty(0,T;W^{1,\infty})} + \|v^E\|_{L^\infty(0,T;W^{1,\infty})}\right)\int_0^t \|\mathsf{w}^\alpha(s)\|^2 ds. \qquad (5.83)$$

$I_5(t)$ can be handle via Hölder's inequality, exploiting the bounds available on F and v^E, see (5.31) and (5.12):

$$\begin{aligned}
I_5(t) &\leq \int_0^t |b(F(s)+v^E(s), F(s)-\mathsf{z}^\alpha(s), \mathsf{w}^\alpha(s))| + |b(F(s)-\mathsf{z}^\alpha(s), v^E(s), \mathsf{v}^\alpha(s))| ds\\
&+ \int_0^t |b(F(s)-\mathsf{z}^\alpha(s), \mathsf{z}^\alpha(s), \mathsf{w}^\alpha(s))| ds\\
&\leq \int_0^t \|\mathsf{w}^\alpha(s)\|\|F(s)-\mathsf{z}^\alpha(s)\|_{H^1}\|F(s)+v(s)^E\|_{L^\infty} ds
\end{aligned}$$

$$
\begin{aligned}
&+\int_0^t \|\mathsf{v}^\alpha(s)\|\|F(s)-\mathsf{z}^\alpha(s)\|\|v(s)^E\|_{W^{1,\infty}}ds\\
&+\int_0^t \|\mathsf{w}^\alpha(s)\|\|F(s)-\mathsf{z}^\alpha(s)\|\|\mathsf{z}^\alpha(s)\|_{W^{1,\infty}}ds\\
&\leq C(N,u_0)\|F-\mathsf{z}^\alpha\|_{C([0,T];H^1)}\left(1+\|F-\mathsf{z}^\alpha\|_{C([0,T];H^1)}\right)+\int_0^t\|\mathsf{w}^\alpha(s)\|^2ds\\
&+\int_0^t \|\mathsf{w}^\alpha(s)\|\|F(s)-\mathsf{z}^\alpha(s)\|\|\mathsf{z}^\alpha(s)\|_{W^{1,\infty}}ds. \qquad (5.84)
\end{aligned}
$$

Now we can move to $I_6(t)$ which is the most difficult term. Preliminarily we observe that

$$
\begin{aligned}
\alpha b(\mathsf{u}^\alpha,\Delta\mathsf{u}^\alpha,\mathsf{w}^\alpha)-\alpha b(\mathsf{w}^\alpha,\Delta\mathsf{u}^\alpha,\mathsf{u}^\alpha) &= \alpha b(\mathsf{v}^\alpha,\Delta\mathsf{u}^\alpha,\mathsf{v}^\alpha)+\alpha b(\mathsf{z}^\alpha,\Delta\mathsf{u}^\alpha,\mathsf{w}^\alpha)\\
&-\alpha b(\mathsf{v}^\alpha,\Delta\mathsf{u}^\alpha,v^E)-\alpha b(\mathsf{v}^\alpha,\Delta\mathsf{u}^\alpha,\mathsf{v}^\alpha)\\
&-\alpha b(\mathsf{w}^\alpha,\Delta\mathsf{u}^\alpha,z^\alpha)+\alpha b(v^E,\Delta\mathsf{u}^\alpha,\mathsf{v}^\alpha)\\
&=\alpha b(\mathsf{z}^\alpha,\Delta\mathsf{u}^\alpha,\mathsf{w}^\alpha)-\alpha b(\mathsf{v}^\alpha,\Delta\mathsf{u}^\alpha,v^E)\\
&-\alpha b(\mathsf{w}^\alpha,\Delta\mathsf{u}^\alpha,z^\alpha)+\alpha b(v^E,\Delta\mathsf{u}^\alpha,\mathsf{v}^\alpha).
\end{aligned}
$$

We start considering $-\alpha\int_0^t b(\mathsf{v}^\alpha(s),\Delta\mathsf{u}^\alpha(s),v^E(s))ds+\alpha\int_0^t b(v^E(s),\Delta\mathsf{u}^\alpha(s),\mathsf{v}^\alpha(s))ds$. It can treated similarly to [132, Equations (4.18)-(4.19)]. $-\alpha\int_0^t b(\mathsf{v}^\alpha(s),\Delta\mathsf{u}^\alpha(s),v^E(s))ds$ can be integrated by parts, then it holds:

$$
\begin{aligned}
-\alpha b(\mathsf{v}^\alpha,\Delta\mathsf{u}^\alpha,v^E) &= \alpha\int_{\mathcal{O}}\mathsf{v}_i^\alpha\partial_i v_j^E\partial_{k,k}\mathsf{u}_j^\alpha\ dx\\
&=-\alpha\int_{\mathcal{O}}\partial_k\mathsf{v}_i^\alpha\partial_i v_j^E\partial_k(\mathsf{v}_j^\alpha+\mathsf{z}_j^\alpha)dx-\alpha\int_{\mathcal{O}}\mathsf{v}_i^\alpha\partial_{i,k}v_j^E\partial_k\mathsf{u}_j^\alpha dx\\
&\leq o(1)+2\alpha\|v^E\|_{W^{2,4}}\|\nabla\mathsf{v}^\alpha\|^2. \qquad (5.85)
\end{aligned}
$$

In the last step we use the fact that $\alpha\|v^E\|_{W^{2,4}}\|\nabla\mathsf{z}^\alpha\|^2=o(1)\quad\mathbb{P}-a.s.$ by Lemma 5.34. For what concerns $\alpha\int_0^t b(v^E(s),\Delta\mathsf{u}^\alpha(s),\mathsf{v}^\alpha(s))ds$, we split it in three terms:

$$
\begin{aligned}
\alpha b(v^E,\Delta\mathsf{u}^\alpha,\mathsf{v}^\alpha) &= -\alpha\int_{\mathcal{O}}v^E\cdot\nabla\mathsf{v}^\alpha\Delta\mathsf{u}^\alpha dx\\
&=-\alpha\int_{\mathcal{O}}\left((v^E-v)\cdot\nabla\mathsf{v}^\alpha\Delta\mathsf{v}^\alpha+v\cdot\nabla\mathsf{v}^\alpha\Delta\mathsf{v}^\alpha+v^E\cdot\nabla\mathsf{v}^\alpha\Delta\mathsf{z}^\alpha\right)dx\\
&=J_1+J_2+J_3. \qquad (5.86)
\end{aligned}
$$

J_3 is the easiest term and can be bounded by the right hand side of (5.85) arguing as above. Since $v^E-v|_{\partial\mathcal{O}},\ \mathsf{v}^\alpha|_{\partial\mathcal{O}}=0$, we can integrate by part J_1 repeatedly, obtaining via Hölder's inequality the following estimate:

$$\begin{aligned}
-\alpha\int_{\mathcal{O}}(v^E - v)\cdot\nabla\mathsf{v}^\alpha\Delta\mathsf{v}^\alpha dx &= \alpha\int_{\mathcal{O}}\partial_k(v_i^E - v_i)\partial_i\mathsf{v}_j^\alpha\partial_k\mathsf{v}_j^\alpha dx + \frac{\alpha}{2}\int_{\mathcal{O}}(v_i^E - v_i)\partial_i|\partial_k\mathsf{v}_j^\alpha|^2 dx\\
&= \alpha\int_{\mathcal{O}}\partial_k v_i^E\partial_i\mathsf{v}_j^\alpha\partial_k\mathsf{v}_j^\alpha dx - \alpha\int_{\mathcal{O}}\partial_k v_i\partial_i\mathsf{v}_j^\alpha\partial_k\mathsf{v}_j^\alpha dx\\
&\le \alpha\|v^E\|_{W^{2,4}}\|\nabla\mathsf{v}^\alpha\|^2 - \alpha\int_{\mathcal{O}}\partial_k v_i\partial_i\mathsf{v}_j^\alpha\partial_k\mathsf{v}_j^\alpha dx\\
&= \alpha\|v^E\|_{W^{2,4}}\|\nabla\mathsf{v}^\alpha\|^2 + \alpha\int_{\mathcal{O}}\partial_k v_i\mathsf{v}_j^\alpha\partial_{i,k}\mathsf{v}_j^\alpha dx\\
&= \alpha\|v^E\|_{W^{2,4}}\|\nabla\mathsf{v}^\alpha\|^2 - \frac{\alpha}{2}\int_{\mathcal{O}}v_i\partial_i|\partial_k\mathsf{v}_j^\alpha|^2 dx\\
&- \alpha\int_{\mathcal{O}}v_i\mathsf{v}_j^\alpha\partial_{i,k,k}\mathsf{v}_j^\alpha dx\\
&= \alpha\|v^E\|_{W^{2,4}}\|\nabla\mathsf{v}^\alpha\|^2 + \alpha\int_{\mathcal{O}}v_i\partial_i\mathsf{v}_j^\alpha\partial_{k,k}\mathsf{v}_j^\alpha dx\\
&= \alpha\|v^E\|_{W^{2,4}}\|\nabla\mathsf{v}^\alpha\|^2 - J_2.
\end{aligned}\tag{5.87}$$

Combining (5.85), (5.86) , (5.87) we get

$$\begin{aligned}
&-\alpha\int_0^t b(\mathsf{v}^\alpha(s),\Delta\mathsf{u}^\alpha(s),v^E(s))ds + \alpha\int_0^t b(v^E(s),\Delta\mathsf{u}^\alpha(s),\mathsf{v}^\alpha(s))ds\\
&\le o(1) + \alpha C(N,u_0)\|v^E\|_{L^\infty(0,T;W^{2,4})}\int_0^t\|\nabla\mathsf{v}^\alpha(s)\|^2 ds.
\end{aligned}\tag{5.88}$$

We left to estimate $\alpha\int_0^t b(\mathsf{z}^\alpha(s),\Delta\mathsf{u}^\alpha(s),\mathsf{w}^\alpha(s))ds - \alpha\int_0^t b(\mathsf{w}^\alpha(s),\Delta\mathsf{u}^\alpha(s),z^\alpha(s))ds$.
We start considering $\alpha\int_0^t b(\mathsf{z}^\alpha(s),\Delta\mathsf{u}^\alpha(s),\mathsf{w}^\alpha(s))ds$ integrating by parts repeatedly since $\mathsf{z}^\alpha|_{\partial\mathcal{O}} = 0$, we obtain by Hölder's inequality and the $\mathbb{P}-a.s.$ estimates on z^α and v^E guaranteed by equation (5.65) and (5.12)

$$\begin{aligned}
\alpha b(\mathsf{z}^\alpha,\Delta\mathsf{u}^\alpha,\mathsf{w}^\alpha) &= -\alpha\int_{\mathcal{O}}\mathsf{z}_i^\alpha\partial_i\mathsf{w}_j^\alpha\partial_{k,k}\mathsf{u}_j^\alpha dx\\
&= \alpha\int_{\mathcal{O}}\partial_k\mathsf{z}_i^\alpha\partial_i\mathsf{w}_j^\alpha\partial_k\mathsf{u}_j^\alpha dx + \alpha\int_{\mathcal{O}}\mathsf{z}_i^\alpha\partial_{i,k}\mathsf{w}_j^\alpha\partial_k\mathsf{u}_j^\alpha dx\\
&\le \alpha\|\mathsf{z}^\alpha\|_{W^{2,4}}(\|\nabla\mathsf{v}^\alpha\| + \|\nabla\mathsf{z}^\alpha\|)(\|\nabla\mathsf{v}^\alpha\| + \|\nabla v^E\|)\\
&+ \alpha\int_{\mathcal{O}}z_i^\alpha\partial_{i,k}(u_j^\alpha - z_j^\alpha - v_j^E)\partial_k\mathsf{u}_j^\alpha dx\\
&\le o(1) + \alpha\|\mathsf{z}^\alpha\|_{W^{2,4}}\|\nabla\mathsf{v}^\alpha\|^2\\
&+ \frac{\alpha}{2}\int_{\mathcal{O}}z_i^\alpha\partial_i|\partial_k\mathsf{u}_k^\alpha|^2 dx + \alpha\|\mathsf{z}^\alpha\|_{L^4}(\|\mathsf{z}^\alpha\|_{W^{2,4}} + \|v^E\|_{W^{2,4}})(\|\nabla\mathsf{v}^\alpha\| + \|\nabla\mathsf{z}^\alpha\|)\\
&\le o(1) + \alpha\left(\|\mathsf{z}^\alpha\|_{W^{2,4}} + \|v^E\|_{W^{2,4}}\right)\|\nabla\mathsf{v}^\alpha\|^2.
\end{aligned}\tag{5.89}$$

Lastly we consider $-\alpha\int_0^t b(\mathsf{w}^\alpha(s),\Delta\mathsf{u}^\alpha(s),z^\alpha(s))ds$. Here we want again integrate by parts repeatedly, for this reason we add and subtract $\alpha\int_0^t b(v(s),\Delta\mathsf{u}^\alpha(s),z^\alpha(s))ds$ exploiting the fact that $\mathsf{w}^\alpha+v|_{\partial\mathcal{O}}=0$. Therefore, thanks to the properties of the boundary layer corrector (5.72) and computations already performed we obtain:

$$\begin{aligned}
-\alpha b(\mathsf{w}^\alpha,\Delta\mathsf{u}^\alpha,z^\alpha) &= \alpha\int_{\mathcal{O}}(w_i^\alpha+v_i)\partial_i\mathsf{z}_k^\alpha\partial_{j,j}\mathsf{u}_k^\alpha dx-\alpha\int_{\mathcal{O}}v_i\partial_i\mathsf{z}_k^\alpha\partial_{j,j}\mathsf{u}_k^\alpha dx\\
&= -\alpha\int_{\mathcal{O}}\partial_j(w_i^\alpha+v_i)\partial_i\mathsf{z}_k^\alpha\partial_j\mathsf{u}_k^\alpha dx-\alpha\int_{\mathcal{O}}(w_i^\alpha+v_i)\partial_{i,j}\mathsf{z}_k^\alpha\partial_j\mathsf{u}_j^\alpha dx\\
&+\alpha\|v\|\|\mathsf{z}^\alpha\|_{W^{2,4}}(\|\mathsf{v}^\alpha\|_{H^2}+\|\mathsf{z}^\alpha\|_{H^2})\\
&\leq o(1)+\alpha C(N,u_0)\delta^{1/2}\|\mathsf{z}^\alpha\|_{W^{2,4}}\|\nabla\mathsf{v}^\alpha\|^{1/2}\|\mathsf{v}^\alpha\|_{H^3}^{1/2}\\
&+2\alpha\|\mathsf{z}^\alpha\|_{W^{2,4}}(\|\nabla\mathsf{v}^\alpha\|+\|\nabla v^E\|+\|\nabla v\|)(\|\nabla\mathsf{v}^\alpha\|+\|\nabla\mathsf{z}^\alpha\|)\\
&\leq o(1)+\alpha C\|\mathsf{z}^\alpha\|_{W^{2,4}}\|\nabla\mathsf{v}^\alpha\|^2+\alpha^3\delta\|\mathsf{v}^\alpha\|_{H^3}^2+\delta^{1/2}C(N,u_0)\|\mathsf{z}^\alpha\|_{W^{2,4}}^{3/2}\\
&+C(N,u_0)\alpha\delta^{-1}\|\mathsf{z}^\alpha\|_{W^{2,4}}^4. \qquad (5.90)
\end{aligned}$$

In conclusion, combining (5.88),(5.89),(5.90) we obtain

$$\begin{aligned}
I_6(t) &\leq o(1)+\alpha C\left(\|\mathsf{z}^\alpha\|_{L^\infty(0,T;W^{2,4})}+\|v^E\|_{L^\infty(0,T;W^{2,4})}\right)\int_0^t\|\nabla\mathsf{v}^\alpha(s)\|^2ds\\
&+\alpha^3\delta\sup_{t\in[0,T]}\|\mathsf{v}^\alpha(t)\|_{H^3}^2+\delta^{1/2}C(N,u_0)\|\mathsf{z}^\alpha\|_{L^\infty(0,T;W^{2,4})}^{3/2}\\
&+C(N,u_0)\alpha\delta^{-1}\|\mathsf{z}^\alpha\|_{L^\infty(0,T;W^{2,4})}^4. \qquad (5.91)
\end{aligned}$$

Combining the various estimates on the $I_i(t),\ i\in\{1,\dots,6\}$ we get

$$\begin{aligned}
\|\mathsf{w}^\alpha(t)\|^2+\frac{4}{5}\alpha\|\nabla\mathsf{v}^\alpha(t)\|^2 &\leq o(1)+C\alpha(1+\|v^E\|_{L^\infty(0,T;W^{2,4})}+\|\mathsf{z}^\alpha\|_{L^\infty(0,T;W^{2,4})})\int_0^t\|\nabla\mathsf{v}^\alpha(s)\|^2ds\\
&+C(T)\alpha^3\delta\sup_{t\in[0,T]}\|\mathsf{v}^\alpha(t)\|_{H^3}^2+C\sqrt{\alpha}\sup_{t\in[0,T]}\left|\int_0^t\langle\mathsf{w}^\alpha(s),dW^\alpha(s)\rangle\right|\\
&+C(1+\|\mathsf{z}^\alpha\|_{L^\infty(0,T;W^{1,\infty})}+\|v^E\|_{L^\infty(0,T;W^{1,\infty})})\int_0^t\|\mathsf{w}^\alpha(s)\|^2ds\\
&+C(N,u_0)\|F-\mathsf{z}^\alpha\|_{C([0,T];H^1)}\left(1+\|F-\mathsf{z}^\alpha\|_{C([0,T];H^1)}\right)\\
&+\int_0^t\|\mathsf{w}^\alpha(s)\|\|F(s)-\mathsf{z}^\alpha(s)\|\|\mathsf{z}^\alpha(s)\|_{W^{1,\infty}}ds. \qquad (5.92)
\end{aligned}$$

Applying Grönwall's Lemma to (5.92) we obtain

$$\begin{aligned}
\sup_{t\in[0,T]}\|\mathsf{w}^\alpha(t)\|^2+\alpha\sup_{t\in[0,T]}\|\nabla\mathsf{v}^\alpha(t)\|^2 \leq e^{C(T)(1+\|v^E\|_{L^\infty(0,T;W^{2,4})}+\|\mathsf{z}^\alpha\|_{L^\infty(0,T;W^{2,4})}}\times\\
\Big(o(1)+T\alpha^3\delta\sup_{t\in[0,T]}\|\mathsf{v}^\alpha(t)\|_{H^3}^2
\end{aligned}$$

$$
\begin{aligned}
&+ C(N,u_0)\|F-\mathsf{z}^\alpha\|_{C([0,T];H^1)}\\
&+ C(N,u_0)\|F-\mathsf{z}^\alpha\|^2_{C([0,T];H^1)}\\
&+ \int_0^T \|\mathsf{w}^\alpha(s)\|\|F(s)-\mathsf{z}^\alpha(s)\|\|\mathsf{z}^\alpha(s)\|_{W^{1,\infty}}ds\\
&+ C\sqrt{\alpha}\sup_{t\in[0,T]}\left|\int_0^t \langle \mathsf{w}^\alpha(s), dW^\alpha(s)\rangle\right|\Bigg). \qquad (5.93)
\end{aligned}
$$

Under our assumptions we have $e^{CT(1+\|v^E\|_{L^\infty(0,T;W^{2,4})}+\|\mathsf{z}^\alpha\|_{L^\infty(0,T;W^{2,4})})}\leq C(N,u_0) \quad \mathbb{P}-a.s.$, see (5.12), (5.69). This means that, in order to show that Lemma 5.36 holds, it is enough to prove that

$$
\begin{aligned}
&T\alpha^3\delta\sup_{t\in[0,T]}\|\mathsf{v}^\alpha(t)\|^2_{H^3} + C(N,u_0)\|F-\mathsf{z}^\alpha\|_{C([0,T];H^1)}\left(1+\|F-\mathsf{z}^\alpha\|_{C([0,T];H^1)}\right)\\
&+\int_0^T \|\mathsf{w}^\alpha(s)\|\|F(s)-\mathsf{z}^\alpha(s)\|\|\mathsf{z}^\alpha(s)\|_{W^{1,\infty}}ds + C\sqrt{\alpha}\sup_{t\in[0,T]}\left|\int_0^t \langle \mathsf{w}^\alpha(s), dW^\alpha(s)\rangle\right|\\
&\to 0 \quad \text{in Probability.}
\end{aligned}
$$

Thanks to (5.68), we have $T\alpha^3\delta\sup_{t\in[0,T]}\|\mathsf{v}^\alpha(t)\|^2_{H^3}\to 0$ in probability. $C(N,u_0)\|F-\mathsf{z}^\alpha\|^2_{C([0,T];H^1)}\to 0$ in probability by Lemma 5.34. Lastly, by Lemma 5.34 we have also

$$
\begin{aligned}
&\mathbb{E}\left[\sqrt{\alpha}\sup_{t\in[0,T]}\left|\int_0^t \langle \mathsf{w}^\alpha(s), dW^\alpha(s)\rangle\right| + \int_0^T \|\mathsf{w}^\alpha(s)\|\|F(s)-\mathsf{z}^\alpha(s)\|\|\mathsf{z}^\alpha(s)\|_{W^{1,\infty}}ds\right]\\
&\leq C\mathbb{E}\left[\int_0^T \|\mathsf{w}^\alpha(s)\|^2\right]^{1/2}\left(\sqrt{\alpha}+\mathbb{E}\left[\int_0^T \|F(s)-\mathsf{z}^\alpha(s)\|^2ds\right]^{1/2}\right)\to 0.
\end{aligned}
$$

Now the proof is complete. □

Combining Lemma 5.34 and Lemma 5.36 the second condition in Hypothesis 5.4 holds. Therefore we can apply Theorem 5.5 and complete the proof of Theorem 5.19.

5.5 Some Remarks on the Kato Condition

We end this chapter with a discussion on the Kato-type condition that we assumed in order to prove one of our main results, Theorem 5.17. Recall that the condition Hypothesis 5.14 was the following

Hypothesis 5.37 (Strong Kato Hypothesis)**.** For each $N\in\mathbb{N}$, $u_0^\nu,\ u_0\in\mathcal{E}_0^{NS}$ and $f^\nu,\ f\in\mathcal{P}_2^N$ such that $u_0^\nu\to u_0$ in $\mathcal{E}_0^{NS}$ and $f^\nu\to_{\mathcal{L}} f$ in S^N, if $(\Omega,\mathcal{F},\mathcal{F}_t,\mathbb{P})$ is a filtered probability space where all f^ν, f are defined together and $f^\nu \to f$ $\mathbb{P}-a.s.$ in S^N, then, it exists $c>0$ such that for every $\delta>0$

$$\mathbb{P}\left(\nu\int_0^T \left\|\nabla\mathcal{G}^{NS,\nu}\left(u_0^\nu, \sqrt{\nu}W_\cdot + \int_0^\cdot f^\nu(s)ds\right)\right\|^2_{L^2(\Gamma_{c\nu})} ds > \delta\right) \to 0.$$

Loosely speaking, this condition requires a control on the behaviour in the boundary layer of the solutions of the stochastic Navier-Stokes system with respect to all kind of forcings and initial data. In the course of the proofs we have assumed this condition to verify the Condition 2 in Hypothesis 5.4. As pointed out in Remark 5.15, the uniformity in the initial data is crucial only for the set of initial condition for which we wish to establish a (uniform) LDP, as we can restrict the definition of the set $\mathcal{E}_0^{NS}$ without changing the strength (topology) of the LDP. Thus we can weaken this assumption just by redefining the objects on which we apply the weak convergence approach scheme. On the contrary, if one wishes to use the weak convergence approach, condition 2 in Hypothesis 5.4 (and the definition of the space S^N) does not allow to restrict the space of forcings, without increasing the regularity of the noise W, and thus severely limiting the strength of our result. For this reason, the request on the uniformity with respect to all possible forcing can not be weaken a priori, as for the initial data. Observe that the Strong Kato Hypotesis (SKH) is much stronger than what we asked in the previous chapter to ensure the validity of the inviscid limit, namely

$$\mathbb{E}\left[\nu\int_0^T \left\|\nabla\mathcal{G}^{NS,\nu}\left(u_0^\nu, \sqrt{\nu}W_\cdot\right)\right\|^2_{L^2(\Gamma_{c\nu})}\right] \to 0. \tag{5.94}$$

In the following, we will call the property described by equation (5.94) as Weak Kato Condition (WKC). In particular, this condition does not involve any control of the system for non-zero forcing. In order to weaken the SKH, one can ask if the WKC is enough to ensure the validity of the LDP. Let us notice first that what we have proved, under the SKH is not only a large deviation result for the Navier-Stokes system with zero forcing, $u^\nu := \mathcal{G}^{NS,\nu}(u_0, \sqrt{\nu}W_\cdot)$, but actually we got, as a byproduct, a LDP result for solutions with any forcing in $L^2(0,T;H_0)$, as pointed out in Remark 5.22. Indeed if we want to include in the system a forcing $g \in L^2(0,T;H_0)$ we can just redefine the maps $\mathcal{G}^0_g := \mathcal{G}^0(\cdot, \int_0^\cdot(\cdot + g(s))ds)$ and $\mathcal{G}^\nu_g := \mathcal{G}^\nu(\cdot, \int_0^\cdot(\cdot + g(s))ds)$ and the result that we have proved immediately imply a LDP for the solution with forcing, under the same Hypothesis 5.14. In this sense our condition is optimal for our setting: we ask controls for every forcing and we get a LDP for every forcing term.

Therefore, one way of improving our result could be to prove that the SKH can be deduced from the WKC. In some sense this would requires to be able to pass information between systems with different forcings. We shall notice that the forcing that we are working with all live in the reproducing kernel of W, therefore we might switch from one system to another just by a Girsanov transformation; however this correction explodes exponentially fast in the limit $\nu \to 0$. A posteriori, if one is able to prove that the LDP holds for some forcing, one expects that the explosion of the Girsanov correction gets compensated by the exponential decay of the law of the solutions. A different approach would be to prove that one does not in fact need to ask the strong Kato condition in order to prove only a LDP for equation (1.32) (that is, only for the system with zero forcing). We formulate then the following:

Problem 5.38. (LDP under Weak Kato Hypotesys) Prove that the statement of Theorem 5.17 holds true if we replace Hypothesis 5.14 with the Kato Condition (5.94).

If the answer to this problem was positive, then we would expect to retrieve also the 'full' LDP, that is, a family of LDP for the system with any forcing $g \in L^2(0,T;\mathcal{H}_0)$. This requires to be able to pass a LDP between systems with different forcings. To see why this seems so natural, observe that every time one is able to write a family of solution X_g^ν to some S(P)DE depending on some forcing g as a continuous transformation of a Brownian motion $J(\sqrt{\nu}W_\cdot)$, then by an application of the contraction principle one immediately obtains a LDP for every other forcing. In our setting however, we are not able of proving such a property of the LDP without requiring the strong Kato Condition, that is without having information about the convergence of the systems with different forcings when $\nu \to 0$. Note that, by the uniqueness of the solution to the Euler system in our setting, this convergence is also a necessary condition for the 'full' LDP.
In the end, we believe that the following should be true:

Conjecture. *The Large Deviations of our system hold independently of the choice of the forcing $g \in L^2(0,T;H_0)$, that is, if a LDP holds for at least one such forcing, then it holds for every other.*

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