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In Three Movements: Beliefs, Logic, Music

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Candidato
dr. Matteo Bizzarri

Relatori

Prof. Mario Piazza

Prof. Satoshi Tojo

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"*Die Grammophonplatte, der musikalische Gedanke, die Notenschrift, die Schallwellen, stehen alle in jener abbildenden internen Beziehung zu einander, die zwischen Sprache und Welt besteht.*
Ihnen allen ist der logische Bau gemeinsam."

— Ludwig Wittgenstein, *Tractatus Logico-Philosophicus*, 4.014

Abstract

This dissertation investigates the interplay between logic, belief, and music through a threefold structure, conceived as three "movements."

The first movement explores the philosophical roots of non-classical reasoning, beginning with Hume's notion of vivacity and Wittgenstein's reflections on probability. Their insights are connected to the framework of supraclassical logics, emphasizing how logical systems can be extended beyond the limits of classical closure by incorporating beliefs. The second movement develops Fractional Semantics, a proof-theoretic approach introduced by Piazza and Pulcini, and extends it with the notion of beliefs. By introducing the distinction between Full and Revisable Beliefs—modeled through hyperreal numbers—the system captures the dynamics of belief revision while preserving cut-elimination and decidability. Applications include a formal treatment of the Lottery Paradox, showing how it dissolves within this framework, and an account of belief change operations. The third movement applies proof-theoretic methods to music theory, focusing on the Lambek Calculus and its extensions.

By adapting structural rules and introducing labelled versions of the calculus, the work provides a logical account of harmonic analysis and compositional processes. Case studies on jazz standards demonstrate how depth and complexity of an analysis can be measured proof-theoretically, opening new perspectives on the relation between logic and musical structure. Overall, the dissertation proposes a unified framework where philosophical insights, formal logic, and musical analysis converge. It highlights how logical tools not only address classical problems in epistemology and probability, but also offer novel approaches to understanding the grammar of music.

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Contents

Prelude

The title of this thesis, "*In Three Movements: Beliefs, Logic, Music*", reflects both its thematic scope and its structure. Like a composition unfolding in several movements, the work develops recurring motifs that transform, return, and find resolution. Each movement explores a different domain, as the title states: beliefs, logic, and music.

First Movement: Beliefs. The first movement concerns *beliefs*, a term that is used to indicate an attitude we have to consider something true. From Hume's idea of *vivacity* to Wittgenstein's conception of probability as logical expectation, the notion of belief has oscillated between psychological intensity and logical form. The central question thus arises: how can logic, a language of certainty, represent beliefs and degrees of belief? Particular attention will be devoted to the theory of Belief Revision, which studies how rational agents can coherently update their beliefs when faced with new information.

Second Movement: Logic. The second movement introduces *logic* as the means to address this question. Logic here is not a static calculus of truth values but a dynamic architecture of inference. Within this framework, we develop the *Fractional Semantics*, a semantics that breaks the symmetry between tautology and contradiction. From this we embed beliefs and hyperreal numbers. This addition allows beliefs to be represented as infinitesimal gradations of confidence—*Gradient Beliefs*. Reasoning becomes a process of transformation rather than substitution: a continuous negotiation between certainty and doubt. In the closing part of this movement, we connect Fractional Semantics with Belief Revision, providing a unified formal framework for graded beliefs.

Third Movement: Music. The third movement turns to *music*. Though apparently distant from logic, music is itself a logic of sound—a grammar of expectation and resolution. By extending Lambek Calculus, originally conceived as a categorial grammar for language, harmonic relations can be expressed in proof-theoretic terms. The depth of a proof becomes the depth of a harmonic progression, and the closure of a logical derivation mirrors the cadence that concludes a phrase. The aim, however, is not to reduce music to logic, but to show how proof-theoretic ideas can shed light on tonal architecture and even suggest generative procedures for composition.

The three movements are not isolated, but neither are they mere variations of one another. The first two, beliefs and logic, intertwine throughout the work, alternating between philosophical reflection and formal construction. The final movement, devoted to music, stands as an autonomous exploration: it adopts some of the tools of logic but applies them to a distinct expressive field, where inference gives way to harmony and sound acquires structural meaning.

Structure of the Thesis

The thesis is organised in three movements, corresponding to the three main domains explored: *beliefs*, *logic*, and *music*. Although presented sequentially, these movements are interconnected: the first two alternate and overlap, while the third offers an independent transposition of formal ideas into the musical domain. For this reason, the chapters will not be presented in strict chronological order, but rather according to their thematic role within each movement. This arrangement is intended to make the overall structure of the work clearer as it unfolds, highlighting the conceptual connections that motivate the sequence.

First Movement: Beliefs

Chapter 1, *From Beliefs to Belief Revision*, introduces the philosophical foundations of the formalization of belief. It presents the tradition of Belief Revision theory (AGM) and its central idea: that rational agents must revise their beliefs coherently when confronted with new information.

Chapter 3, *Beliefs, Credences and Probabilities*, explores the transition from qualitative to quantitative models of belief. It analyses the relationship between belief, credence, and probability, and identifies their limitations when reduced to purely numerical forms. Two historical perspectives on belief and probability are examined. The first originates with Hume, who offered one of the earliest attempts to connect belief with probability through the notion of the *vivacity* of an idea. This concept expresses the degree of force with which an agent entertains a belief, that is, the level of commitment or confidence attached to it. A formal interpretation of this notion will be proposed.

The second perspective comes from Wittgenstein's *Tractatus*, in which probability is treated through the framework of truth tables. Although the *Tractatus* has been widely studied, this particular idea has been largely neglected in the literature. It provides, however, a remarkably elegant way of handling probabilities within classical logic, albeit with some non-trivial consequences. Several properties of this system will be demonstrated.

This system is interesting because of the influence it had on some Ramsey. We will also show that Wittgenstein's treatment of probability constitutes, in fact, a form of Supraclassical Logic.

Second Movement: Logic

Chapter 2, *Supraclassical Logic*, extends this discussion to logical frameworks capable of accommodating non-classical reasoning, thus preparing the ground for a semantics that transcends bivalence. The basic notions related to Supraclassical Logic will be introduced here. This chapter, together with the first one, can be read as preparatory material for the developments presented in the rest of the thesis.

Chapter 4, *Fractional Semantics*, introduces the general rules of Fractional Semantics as originally proposed in [80]. This framework provides a proof-theoretic approach to handling contradictions within classical logic. Classical Logic, as it stands, lacks a system capable of going beyond the binary distinction between valid and invalid formulas. Fractional Semantics, by contrast, offers an interpretation that can deal with such intermediate cases. Building upon its foundations, we extend Fractional Semantics to a Supraclassical setting, thereby creating a system capable of treating beliefs as propositions that may be considered as true as tautologies, approaching the status of tautologies.

Chapter 5, *How to Solve the Lottery Paradox in Classical Logic*, applies two different frameworks to the well-known Lottery Paradox, which traditionally poses difficulties when logic is required to handle beliefs or extra-logical assumptions. We show that this paradox can be resolved both within Wittgenstein's conception of probability and within the framework of Fractional Semantics. From a philosophical point of view, the Lottery Paradox represents an important turning point: both systems can be formulated within Classical Logic itself, which implies that the paradox can be addressed without abandoning the classical framework.

Finally, Chapter 6, *Fractional Semantics, Belief Revision and Graded Beliefs*, unifies the results of the previous chapters by integrating fractional reasoning with Belief Revision theory. To achieve this, we introduce a system based on Hyperreal Numbers. Each belief in Fractional Semantics is associated with a hyperreal number that ensures that it will never be regarded as fully equivalent to a tautology. The philosophical motivation, already discussed in Chapter 4, lies in the distinction between beliefs and tautologies: beliefs may change over time, whereas tautologies remain eternally true. The use of Hyperreal Numbers allows us to represent this distinction formally by assigning to a belief a value slightly below one, namely $1 - \delta$.

Thanks to Hyperreal Numbers properties this approach preserves all the properties of the original system while marking the difference between beliefs and tautologies. Indeed, the *standard part* of a hyperreal number—its real counterpart—retains the same logical properties as before. Thus, the introduction of Hyperreal Numbers does not alter the theorems but provides a refined tool for distinguishing between degrees of certainty. The resulting system models belief dynamics as a continuous process of logical transformation.

Third Movement: Music

Chapter 7, *A Proof-Theoretic Perspective on Musical Harmony*, constitutes the final movement, which is presented in a single chapter. This part of the thesis introduces three different ways of relating logic and music, each offering new insights into the inner architecture of harmonic structure.

The first approach is intentionally simple: it considers harmonic relations through tableaux and proof-theoretic reasoning. Although it is the most elementary, it provides a clear and visually accessible representation of harmonic relations, and its structure can easily be inverted to serve as a compositional tool.

The second and third approaches build upon Lambek Calculus. In the first of these, *tagged sequents* are used to represent harmonic regions, namely tonalities. Within this framework, harmonic progressions correspond to proofs and cadences to logical implications. The notion of *proof depth* is introduced as a measure of harmonic complexity, offering a formal account of musical reasoning.

The final framework extends this model by removing explicit labels and retaining only minimal tags for tonalities, allowing for greater flexibility in addressing the problem of modulation. Modulation, indeed, represents one of the most challenging aspects of harmonic analysis; however, through the use of modal logic, it becomes possible to establish *accessibility relations* between tonalities by means of relations analogous to those connecting possible worlds in modal semantics. In this way, a map of tonal transitions can be constructed, inspired by the well-known theory of the *Tonfeld*, which relates chords in a systematic and logically grounded manner.

Across these movements, the same principles recur—coherence, transformation, and resolution. Logic does not stand apart from the other domains, but moves through them, adapting to the needs of each language. Whether in belief, in reasoning, or in music, it expresses a shared aspiration: to make complexity intelligible without reducing it to simplicity. The thesis therefore closes not with a synthesis, but with an open dialogue between logic, belief, and music—between the precision of inference and the complexity of harmony.

Chapter 1

From Beliefs to Belief Revision

1.1 Formalizing beliefs

Is it possible to formalize something that seems as elusive as beliefs? That may seem not fully feasible, but it is possible to explore ways to overcome this difficulty. To face the problem of belief's formalization, in order to pursue this objective, the present section will be devoted to better define the concept of formalizing beliefs.

The first obstacle to overcome is constituted by formal representations. In fact, the choice of a formal framework itself determines which questions are salient to the enquiry and which ones may recede into the background of our research. This would consequently facilitate the formalization of certain items, while making more difficult, if not impeding, the formalization of others. How is it then possible to formalize?

Objects of belief may be defined, broadly speaking, as propositions, and propositions, from their part, as set of possible worlds. Of these latter is possible to provide an intuitive view by stating that: a possible world is a complete description of one reality [30]. We live in one possible world, but there are a lot of worlds where things act differently.

Definition 1.1 (Set of possible worlds)**.** The set $\mathcal{W}$ is the set of all contextually relevant epistemic possibilities.

It is not necessary, in this context, to think of every possible world that represents the total amount of metaphysical possibilities, because that would not be useful for our aim here. Since we could be submerged in obscure, strange and uninteresting details, context usually helps us to select the possible worlds that are interesting to consider.

For example, for the purpose of determining who is the mayor of a city at a certain time in a world, this information could be easily obtained without knowing quantum physics or if said major is left or right-handed.

For our purposes, a possible world is a restriction of what we can think as possible worlds, so it is:

Definition 1.2 (Possible world)**.** A possible world $w_n \in \mathcal{W}$ with $n \in \mathbb{N}$ is a complete specification of all and only those features of the world that are relevant given the context.

The concept of relevant is here used in the most intuitive manner. A set of possible worlds is a proposition and it is defined as follows:

Definition 1.3 (Proposition)**.** A proposition $p \subseteq \mathcal{W}$ is a set of possible worlds. To be certain that p is true is to be certain that the actual world is among the set of worlds $\{\mathcal{W}_n : \mathcal{W}_n \in p\}$ since p is true in a possible world $\mathcal{W}_n$ if and only if $\mathcal{W}_n \in p$.

Given that, beliefs are formalized in classical deductive logic, the consequence operator $\vdash$ must satisfy the following properties for all sets P, Q of formulas.

Reflexivity : $P \vdash p$.

Cumulative Transitivity (CT), *alias* Cut : Whenever $P \vdash q$, for all $q \in Q$ and $P \cup Q \vdash r$ then $P \vdash r$.

Monotony : Whenever $P \vdash r$ and $P \subseteq Q$ then $Q \in r$.

Reflexivity expresses the fact that any sentence p is a deductive consequence of itself. The rule of cut formalize that deductive conclusions are on an equal epistemic footing with their premises: there is no loss of confidence as derivations get longer. Together, these principles may be defined as: consequences of consequences are consequences. Monotony expresses the fact that adding more premises to a deductive argument allows to derive all the same conclusions one could draw with fewer premises.

We define $Cn(p)$ through $\vdash$, the definitions are, in fact, interchangeable. Given the relation $\vdash$, it is possible to define $\vdash$ from Cn by setting $Cn(p) = \{q : p \vdash q\}$. Conversely it is possible to define $\vdash$ from Cn by the rule: $p \vdash q$ iff $q \in Cn(p)$. Both of them are useful, but sometimes one is more convenient the other. Therefore, it is necessary to define the properties defined earlier in Cn-terms:

Reflexivity : $P \subseteq Cn(P)$.

Cumulative Transitivity (CT), *alias* Cut : $P \subseteq Q \subseteq Cn(P)$ implies $Cn(Q) \subseteq Cn(P)$.

Monotony : Whenever $P \subseteq Q$ then $Cn(P) \subseteq Cn(Q)$.

It is possible to find a set-theoretic structure that organizes propositions.

Complementary : $\neg p = \mathcal{W} \setminus p$, is the set of all worlds in which p is false.

Intersection : If p, q are arbitrary propositions, then their intersection $p \cap q$ is the set of all worlds in which p and q are both true.

Disjunction : If p, q are arbitrary propositions, then their disjunction $p \cup q$ is the set of worlds in which at least one of p, q is true.

Material condition : $p \to q$ is the set of worlds where $\neg p \cup q$, in which or p is false or q is true.

Entailment : If $p \subseteq q$ we say that p *entails* q or that p is *logically stronger* than q.

Tautological proposition : is true in every world.

Empty set : is not true in every world.

Even though these types of propositions are valid for any theoretical structure, it is necessary to consider the limits of this kind of formalization.

A doxastic agent can believe a proposition or not, it can also ignore it or believe that said proposition is true and false at the same time. In classical logic there are only two possibilities: either I believe p, either I believe $\neg p$. The aim here is not to discuss the *grey-zone* cases, but only the *all or nothing* cases, will be analyzed i.e. if an agent believes that p, so p is fully true for him.

The present chapter will be focused on all or nothing cases, i.e. cases in which agents consider p either fully true, either completely false. Gray-zone cases in which agents assign various degree of truth to proposition will be put aside for the moment and left to be explored in the last chapter.

Moreover, there is one last peculiarity to discuss with reference to the formalization of beliefs: a set of beliefs must be closed under logical consequence, i.e. every sentence that follows logically from this set is already in the set. This is clearly an unrealistic idealization, since it would mean that the agent is *logically omniscient*: the agent would know that everything can be deduced formally from what it believes. However, it is a useful idealization because it simplifies the logical treatment. Actually, it would not be possible to formalize anything without considering the agent *logically omniscient*, because it would not be possible to formalize the difficulties that may arise from the deduction problems.

1.2 Foundationalism and Coherentism

Gardenfors, in [28], pointed out that when dealing with modeling a state of beliefs it is necessary to consider whether the justifications for the beliefs should be part of the model or not. With respect to this question there are two main approaches: foundational and coherence theories.

The major distinctive feature of the foundational approach is that it relies on a special class of beliefs, which are often referred to as *epistemologically basic beliefs* or *basic beliefs*. Every belief in a foundational system is supposed to be justified in terms of other beliefs which are, in turn, justified by further beliefs and so on, until ultimately basic beliefs which have no need for justification are reached. This set of basic beliefs is called *belief set*.

The most famous frameworks based on this idea are the Truth Maintenance System (TMS) introduced by Jon Doyle[19] and the following called Assumption Based Truth Maintenance System (ATMS). The problems of this approach arise in this system when we it becomes to change the belief base, since ATS is based on a system made by *nodes* and *justifications* (also called *reasons*) representing reason to believe something. Each node may be in one of two states:

in if the node has a valid justification and is considered a current belief;

out if the node does not have a valid justification; it is not believed in this moment.

A justification is a pair of sets of nodes: an *inlist* and a *outlist*. A justification is *valid* if and only if all nodes on its *inlist* are *in* and all those on its *outlist* are out. A proposition is a belief when at least one of its justifications are valid and it is not a belief when none of its justifications is valid.

The TMS-system creates new nodes and adds or retreats justifications. This may become a complex process as other nodes and justifications may be affected. A justification for a proposition c is represented by the propositional clause:

$$a_1 \wedge \cdots \wedge a_n \wedge \neg b_1 \wedge \cdots \wedge \neg b_m \rightarrow c$$

where $a_1 \wedge \cdots \wedge a_n$ represent the *inlist*, while $b_1 \wedge \cdots \wedge b_n$ represent the *outlist*. A contradiction $\perp$ could be represented by the clause:

$$a_1 \wedge \cdots \wedge a_n \rightarrow \perp$$

and it means that $a_1, \ldots, a_n$ cannot be all believed. This is, in brief, a foundationalism approach, based on minimal beliefs that are called *belief base*. A theory is a *foundations* theory if it holds that one should keep track of the justifications for one's belief[28, p. 8]: Propositions that have no justification should not be accepted as beliefs.

On the other hand, there is the *coherence* theory. The coherence theory focuses on the *logical* structure of beliefs - what matters is how a belief coheres with the other beliefs that are accepted in the present state. We don't have anymore a *belief base*, but we have a *belief set* instead. The latter theory will be herein preferred, since it is believed to better fit the purpose of the present work.

The choice of the coherence model lies on the fact that it is the easiest way to update a knowledge. Consider a situation in which a certain belief base (or, respectively, a set of beliefs) has been set, but it becomes necessary to modify it in light of new information, which could severely impact our knowledge about something, let us call this process a *belief revision*. According to the foundations theory, belief revision should consist, first, in giving up all beliefs that no longer have a satisfactory justification and, second, in adding new beliefs that have become justified. On the other hand, according to the coherence theory, the objectives are, first, to maintain consistency in the revised epistemic state and, second, to make minimal changes of the old state that guarantee sufficient overall coherence.

Thus, the two theories of belief revision are based on conflicting ideas of what constitutes rational changes of belief. The most important theory based on the coherentism approach is the AGM framework that owes its name to its creators: Alchourròn, Gärdenfors and Makinson in 1985[1].

1.3 AGM belief revision theory

The AGM first and most important aim is that of retaining as much as possible from the former set of beliefs when it needs to be modified: change needs to be minimal. As it was contended before, in the AGM model beliefs are represented in some formal language. Sentences do not capture all aspects of belief, but they are the best general purpose representation presently available.[40]

The logical omniscient agent, which was briefly presented in the past section, has now to be discussed more in depth. Consider that at time t_1 an agent fully believes that Pisa is north of Rome and that *north of* is a transitive relation. At the time t_2, in addition to what the agent believes at time t_1, the agent comes to believe fully that Milan is north of Pisa. At the time t_3 the agent comes to believe fully that Milan is north of Rome. Isaac Levi[62, p. 6] distinguished clearly two kinds of changing of full belief:

1. Changes in doxastic commitment.

2. Changes in doxastic performance.

As long as the transition from t_1 to t_2 is taken in account, the observed change affects the doxatstic committtment, , whereas the change from t_2 to t_3 is a change in doxastic performance without a change in doxastic commitment.

In *AGM*, as elsewhere, changes in doxastic performance are not considered, because at time t_3 the agent recognized his commitment and fulfills the obligation it entails. Changes in doxastic performance are herein set aside as well, although in a certain sense the new state of belief at t_3 is not completely ignored.

It is, in fact, considered that if the agent knows that Pisa is north of Rome and, at a certain time, he also comes to believe that Milan is north of Pisa, because of the transitive relation he immediately comes to believe that Milan is north of Rome. There is no time between the belief at time t_2 and the belief at time t_3: adding a new belief to the belief set changes what the agent knows and how it changes is the object of enquiry of the *AGM* system. The belief set is the set of the agent's epistemic commitments, and therefore larger than the set of its actually held belief.

In the *AGM* framework, there are three types of belief change:

Contraction : a sentence p is removed, i.e., a belief set B is superseded by another belief set $B \div p$ that is a subset of B not containing p.

Expansion : a sentence p is added to B and nothing is removed, i.e., B is replaced by a set $B + p$ that is the smallest logically closed set that contains both B and p.

Revision : a sentence p is added to B and at the same time other sentences are removed if this is needed to ensure that the resulting belief set $B * p$ is consistent.

Time was discussed here to clarify better what are the analyzed beliefs, but actually in the *AGM* system there isn't an explicit representation of time.

1.3.1 The AGM Postulates for Contraction

Partial Meet Contraction

The contraction of a belief set B by a sentence p aims to obtain a subset of B that no longer implies p, while preserving as much information as possible. In this sense, sentences in B that are not related to p should not be removed, since their elimination would lead to a loss of relevant information.

The first postulate, usually referred to as the *basic Gärdenfors postulate* (or *basic AGM postulate*) [40], requires that the result of any contraction operation must itself be a belief set:

$$\textit{For any sentence } p \textit{ and any belief set } B,\ B \div p \textit{ is a belief set.} \qquad (B \div 1)$$

In set-theoretic terms, contraction seeks the *maximal* subset of B that does not imply p. In other words, if B' is a subset of B such that $B \div p \subset B' \subseteq B$, then B' must also imply p.

Following Gärdenfors [28, p. 11], further postulates can be introduced. The second one, known as *inclusion*, requires that contraction yields a subset of the original set:

$$B \div p \subseteq B \qquad (B \div 2)$$

If the sentence p is not included in the deductive closure of B, then contracting by p should not affect the belief set at all. This is the principle of *vacuity*:

$$\textit{If } p \notin Cn(B) \textit{ then } B \div p = B \qquad (B \div 3)$$

The next postulate, called *success*, states that after contracting B by p, the sentence p should no longer be a consequence of the contracted set, unless p is logically valid:

$$\textit{If } p \notin Cn(\varnothing), \textit{ then } p \notin Cn(B \div p). \qquad (B \div 4)$$

From $(B \div 1)$ to $(B \div 4)$ it follows that:

$$\textit{If } p \notin B, \ \textit{then } (B \div p) + p \subseteq B \tag{1.1}$$

This condition ensures that if p is first retracted and then reintroduced, no new beliefs are accepted that were not already in B. However, a postulate guaranteeing informational economy is still missing. To ensure that no information is lost unnecessarily, the *recovery* postulate requires that expanding $B \div p$ by p restores the original belief set:

$$\textit{If } p \in B, \ \textit{then } B \subseteq (B \div p) + p \qquad (B \div 5)$$

Another principle, *extensionality*, ensures that logically equivalent sentences are treated identically by contraction:

$$\textit{If } p \leftrightarrow q \in Cn(\varnothing), \ \textit{then } B \div p = B \div q \qquad (B \div 6)$$

Postulates $(B \div 1)$ to $(B \div 6)$ are referred to as the *basic postulates* for contraction. Two further principles, known as *Gärdenfors's supplementary postulates*, concern contraction with respect to conjunctions.

The first one, *conjunctive overlap*, requires that contraction by $p \wedge q$ preserves at least what is preserved by both $B \div p$ and $B \div q$:

$$(B \div p) \cap (B \div q) \subseteq B \div (p \wedge q) \qquad (B \div 7)$$

The second, *conjunctive inclusion*, requires that if contracting by $p \wedge q$ results in the loss of p, then all beliefs lost by contracting p alone are also lost by contracting $p \wedge q$:

$$\textit{If } p \notin B \div (p \wedge q), \ \textit{then } B \div (p \wedge q) \subseteq B \div p \qquad (B \div 8)$$

Epistemic entrenchment contraction

When talking about beliefs, even if they are considered as facts (we assign them maximal probability), some of them appear to be considered as more important than others. In particular, certain pieces of our knowledge and beliefs about the world are more important than others when planning future actions, conducting scientific investigations or reasoning in general.

Consider being in need for contracting a set of belief and in front of the necessity to choose what to give up: there are intuitive ways to do it. For example, if it is necessary to give up one of the natural laws or a single factual statement, the choice will likely fall on the latter. That is because the natural law has a greater explanatory power and giving it up would probably affect the validity of some other beliefs, which descended from said natural law. Ceasing to believe a natural law would determine a massive change in the set, contrary to the principle of minimum change expressed earlier.

This is the main idea behind Peter Gärdenfors's proposal that contraction of beliefs should be ruled by a binary relation called *epistemic entrenchment.*[29] Taking into account two elements p and q, to say that q is more entrenched than p, means that q is more useful for some reason and its epistemic value is greater than p's. In belief contraction, the beliefs with the lowest entrenchment should be the ones that we can give up more easily. To define the entrenchment the following symbols are used:

$p \leq q$: p is at most as entrenched as q.

$p < q$: p is less entrenched than q, i.e., $((p \leq q) \wedge \neg(q \leq p))$.

$p \equiv q$: p and q are equally entrenched, i.e., $((p \leq q) \wedge (q \leq p))$

Gärdenfors has proposed the following postulates for epistemic entrenchment.

Transitivity : If $p \leq q$ and $q \leq r$, then $p \leq r$

Dominance : If $p \vdash q$, then $p \leq q$

Conjunctiveness : Either $p \leq (p \wedge q)$ or $q \leq (p \wedge q)$

Minimality : If the belief set B is consistent, then $p \notin B$ iff $p \leq q$ for all q

Maximality : If $q \leq p$ for all q, then $p \in Cn(\varnothing)$

The *dominance* postulate is justified by the fact that, if one has to retract p or q from B, it will be a smaller change to give up p instead of give up q. In fact if q is given up, p should be retracted in order to satisfy the integrity constraint.

The *conjunctiveness* postulate is justified as it follows: if one wants to retract $p \wedge q$ from B, this can only be achieved by giving up either p or q and, consequently, the informational loss incurred by giving up $p \wedge q$ will be the same as the loss incurred by giving up p or that incurred by giving up q.

It is important to remark that the entrenchment $\leq$ relation is defined only in relation to a given set B, different belief sets may be associated with different ordering of epistemic entrenchment for the same belief.

The entrenchment based relation $\leq$ gives rise to an operation $\div$ of entrenchment based contraction according to the following definition:

$$q \in B \div p \text{ iff } q \in B \text{ and either } p < (p \wedge q) \text{ or } q \in Cn(\varnothing)$$

Entrenchment-based contraction has been shown to coincide with transitively relational partial meet contraction by Makinson [65].

1.3.2 Facing the recovery problem

Recovery, one of the postulates originally introduced by Gärdenfors, was largely debated: is it possible to recover a belief that was given up? It is possible to identify examples both supporting and countering the use of recovery. If one believes they have lost their wallet, are unable to find it and start thinking that someone could have stolen it, when the wallet is finally found back, in a pocket, they will certainly return to the previous state of belief. But consider another example:

Example 1.1. Suppose these two beliefs: "Oscar is a criminal" (p) and "Oscar is a serial killer" (q). These information are built in such a way that if p is dropped, q has to be dropped as well, because $\vdash q \rightarrow p$.

Then, a further piece of information proves true the belief "Oscar is a burglar" (r). The resulting new belief set is the expansion of $B \div p$ by r, $(B \div p) + r$. Since p follows from r, $(B \div p) + p$ is a subset of $(B \div p) + r$. By recovery, $(B \div p) + p$ includes q, from which follows that $(B \div p) + r$ includes q.

Thus, since Oscar was previously believed to be a serial killer, it follows from recovery that, after that, he cannot be believed to be a burglar without him being believed to be a serial killer as well.

1.3.3 The AGM Postulates for Revision

The goal of this section is to formulate postulates for rational revision. The underlying motivation for these postulates is the same as before: when changing our beliefs, the goal is to retain as much as possible from our old beliefs, in order to make a *minimal change*. As it was pointed out talking about epistemic entrenchment, sentences with a lower degree of entrenchment will not be addressed.

It is assumed that, for every belief set B and every sentence p in $\mathcal{L}$, there is a unique belief set $B * p$ representing the revision of B with respect to p. In other words $*$ is a function fed with a belief set and a sentence as argument, returning a belief set as result. The first postulate requires that the output of revision is a set of belief.

$$\textit{For any sentence } p \textit{ and any belief set } B,\ B * p \textit{ is a belief set.} \qquad (B * 1)$$

The second postulate, called *success*, guarantees that the input sentence p is accepted in $B * p$:

$$p \in B * p \qquad (B * 2)$$

It could be the case, when revising our belief set, that what is already in B is totally consistent with p. Generally, it is necessary to give up those beliefs that contradict $\neg p \in B$, while if $\neg p \notin B$ revision is identified with expansion. This identification is split in two parts called *inclusion* and *vacuity*:

$$B * p \subseteq B + p \qquad (B * 3)$$

$$\textit{If } \neg p \notin B \textit{ then } B * p = B + p \tag{$B*4$}$$

The purpose of a revision is to produce a new belief set that has to be *consistent.* This means that $B * p$ should be consistent, unless p is logically impossible. From this follows the *consistency postulate*:

$$B * p \text{ is consistent if } p \text{ is consistent.} \tag{$B*5$}$$

Logically equivalent sentences should lead to identical revisions, because the content of the input sentence p is more important than its linguistic formulation. In other words, belief revisions should be analyzed on the knowledge level, not on the syntactic level. This leads to *extensionality*:

$$\textit{If } (p \leftrightarrow q) \in Cn(\varnothing) \textit{ then } B * p = B * q \tag{$B*6$}$$

The postulates $(B * 1)$-$(B * 6)$ presented in this section are grouped under the label of *basic Gärdenfors postulates for revision.* In addition, two supplementary postulates are proposed, concerning composite belief revision. The idea is that, if $B * p$ has to be changed by another sentence q, such a change should be made by expansions of $B * p$ whenever possible. More generally, the minimal change of B to include both p and q, i.e., $B * (p \wedge q)$, ought to be the same as the expansion of $B * p$ by q, but only if q does not contradict the beliefs in $B * p$. This is guaranteed by two postulates called *superexpansion* and *subexpansion.*

$$B * (p \wedge q) \subseteq (B * p) + q \tag{$B*7$}$$

$$\textit{If } \neg q \notin Cn(B * p) \textit{ then } (B * p) + q \subseteq B * (p \wedge q) \tag{$B*8$}$$

A further addition to these postulates is represented by the *Levi identity* and the *Harper identity*[26]. They relate expansion, revision and contraction:

$$B * p = (B \div \neg p) + p \tag{Levi identity}$$

$$B \div p = B \cap (B * \neg p) \qquad \text{(Harper identity)}$$

In fact, postulates $(B*7)$ and $(B*8)$ are related to $(B \div 7)$ and $(B \div 8)$ through Levi identity. $*$ satisfies superexpansion if and only if $\div$ satisfies conjunctive overlap. Furthermore, $*$ satisfies subexpansion if and only if $\div$ satisfies conjunctive inclusion.

1.3.4 Possible world modeling

A different model of belief states can be constructed out of sets of possible worlds. This model was introduced by Grove in 1988[37], in his paper he developed a system only for revision, but easily extensible to contraction and expansion. The idea presented by Grove was inspired by Lewis, who provided a semantic for certain factual logics using spheres. Grove, in fact, uses the spheres system as well. In this section Grove's method will be recalled together with its application to revision, contraction and expansion. A set of images will be provided to clarify better the models.

First, a sphere has to be defined. In order to do that let, for any set B of sentences, $[B]$ denote the set of possible worlds that contain B as a subset and let, for any sentence p, $[p]$ be the set of possible worlds that contains p as an element. If B is a belief set, then $\bigcap[B] = B$.

Definition 1.4 (Sphere)**.** A sphere is defined to be a set of possible worlds.

Definition 1.5 (System of spheres)**.** A system of spheres centered on B is an ordering over sets of possible worlds where $[B]$ is the innermost sphere and $\mathcal{W}$ is the outermost sphere. Where $\mathcal{W}$ is the set of all the possible worlds.

It is now possible to define the system of spheres formally as follows.

Definition 1.6 (Properties of a system of sphere)**.** Let S be any collection of subset of $\mathcal{W}$ and let S be a system of sphere, centered on B for some subset $B \subseteq \mathcal{W}$, if it satisfies the following conditions:

S_1 S is ordered by $\subseteq$, i.e., if $P, Q \in S$ then $P \subseteq Q$ or $Q \subseteq P$.

S_2 B is the $\subseteq$-minimum of S, i.e., $B \in S$ and if $P \in S$, then $B \subseteq P$.

S_3 $\mathcal{W}$ is in S and it is the largest element of S.

S_4 If $p \in \mathcal{L}$ and there is a sphere in S intersecting $[p]$, then there is a smallest sphere in S intersecting $[p]$. There is a sphere $P \in S$ such that $P \cap [p] \neq \varnothing$, and $Q \cap [p] \neq \varnothing$ implies $P \subseteq Q$ for all $q \in Q$.

A pictorial representation of a system of spheres centered on $[B]$ is given in figure 1.1.

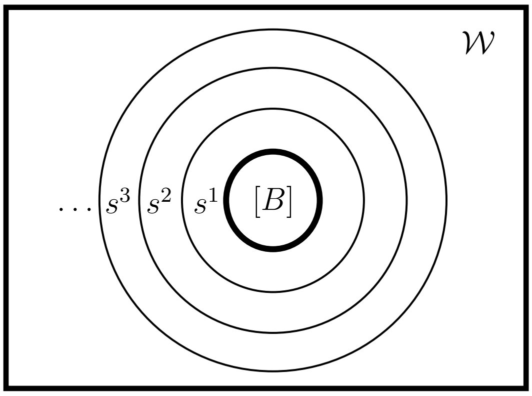


Figure 1.1: Sphere based approach

The system of spheres was inspired by Lewis's idea of a visual semantic, where the concept of spheres is used to provide semantics for certain counter factual logic. There are many differences between Lewis and Grove's spheres. The most significant departure from Lewis's use of spheres is that Grove's ones are centered on arbitrary subset of $\mathcal{W}$, i.e., arbitrary theories over $\mathcal{L}$ [37, p. 159].

Lewis's use of spheres for counter factual logics, in fact, involves one system of spheres for each individual world in $\mathcal{W}$ and his principal interest was in systems of spheres centered on single worlds. Grove's interest instead was not on single worlds, but on a set of worlds.

The condition S_4 guarantees that if any formula p has worlds intersecting $\mathcal{W}$ then there is a smallest sphere (in the sense of set inclusion) or innermost sphere in S intersecting $[p]$. We shall denote such a sphere by $c_S(p)$. We can now define a function called $f_S : \mathcal{L} \to 2^{\mathcal{W}}$ defined as follows for any $p \in \mathcal{L}$.

$$f_S(p) = [p] \cap c_S(p) \qquad \text{(Def } f_S\text{)}$$

Intuitively, f_S is a function that selects those p-worlds, i.e., those worlds $w \in \mathcal{W}$ in which p holds ($[p]$ is the set of all the p-worlds), that are the closest to $[B]$. In other words, it selects the innermost of the p-worlds. Now that the function f_S is defined, it is possible to understand why revision is the easiest operation to perform:

$$[B * p] = f_s(p)$$

The worlds corresponding to a revision of B by p are exactly those worlds closest to $[B]$, because of the principle of minimal change considered above. The minimality can be interpreted as the proximity to $[B]$. In figure 1.2 it is possible to see the graphic illustration of the sphere semantics for belief revision, showing $[B * p] = f_S(p) = c_S(p) \cap [p]$ colored with black.

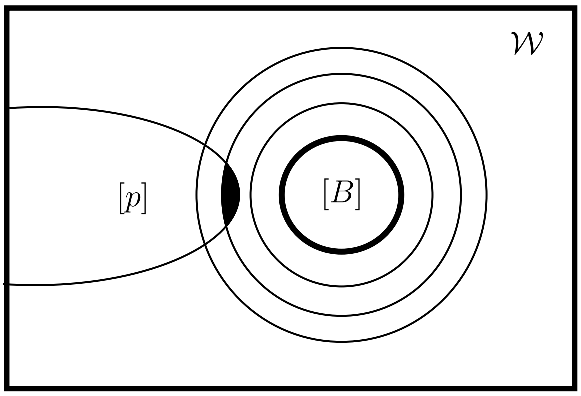


Figure 1.2: Sphere based revision.

It is needed to justify that this semantics is appropriate and to do that Grove provided two representation theorems, formulated as follows:

Theorem 1.1. *Let S be any system of spheres in $\mathcal{W}$ centered on $[B]$ for some belief set B. If one defines, for any $p \in \mathcal{L}, B * p$ to be $t(f_S(p))$, then the postulates from $B * q$ to $B * 8$ are satisfied.*

Theorem 1.2. *Let $* : \mathcal{B} \times \mathcal{L} \to \mathcal{B}$[1] be any function satisfying postulates $B * 1$-$B * 8$. Then for any fixed belief set B there is a system of spheres on $\mathcal{W}$, S say, centered on $[B]$ and satisfying $B * p = t(f_S(p))$, for all $p \in \mathcal{L}$.*

[1]Where $\mathcal{B}$ is the set of all set of beliefs B.

Now, thanks to these representation theorems, it is easy to determine the semantics for belief expansion of an epistemic state B by epistemic input p. In the principal case where $\neg p \notin B$ p is consistent with B and therefore $[B] \cap [p] \neq \varnothing$. That is, the closest p-worlds reside within the innermost sphere $[B]$ and the worlds consistent with the expanded epistemic state are thus given by:

$$[B + p] = [B] \cap [p]$$

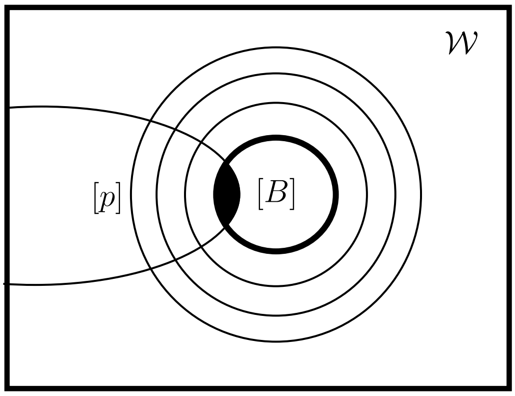


Figure 1.3: Sphere based expansion.

This situation is illustrated in figure 1.3. In the case that $\neg p \in B$, $B + p = B_{\perp}$, i.e., is not consistent. However, in this case $[B] \cap [p] = \varnothing$ and so again, $[B + p] = [B] \cap [p]$.

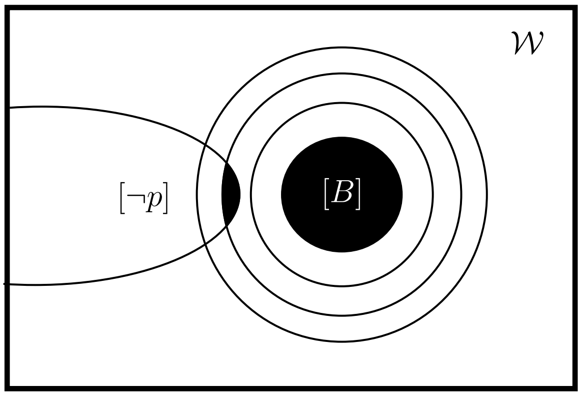


Figure 1.4: Sphere based contraction.

The contraction can be obtained from the revision through Harper identity. In the case of contraction it is necessary to give up something and this means that the possible worlds where B is consistent are increased. Specifically, the system incorporates some $\neg p$-worlds, otherwise p would still be accepted in the contracted epistemic state and therefore violate the postulate of success for belief contraction.

It is necessary to take care of *minimal change*, so the co-sets $\neg p$-worlds are added. Therefore, the worlds consistent with the new epistemic state may be obtained by:

$$[B \div p] = [B] \cup f_S(\neg p)$$

This is illustrated by figure 1.4, where the black color is the surface representing the worlds where $B \div p$ is true, i.e., $[B \div p]$. The surface colored between the sphere and $\neg p$ is $f_S(\neg p)$.

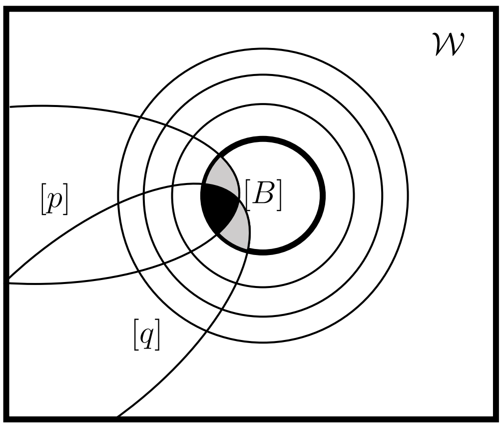


Figure 1.5: Sphere based expansion with conjunction.

To illustrate the conjunction for expansion, revision and contraction it is possible to iterate the operation for the two sentences. The expansion with conjunction is represented in figure 1.5.

$$[B + (p \wedge q)] = [B + p] \cap [B + q]$$

Figure 1.6 represents the conjunction for contraction.

$$[B \div (p \wedge q)] = [B \div p] \cup [B \div q]$$

Finally the conjunction for revision is represented in figure 1.7

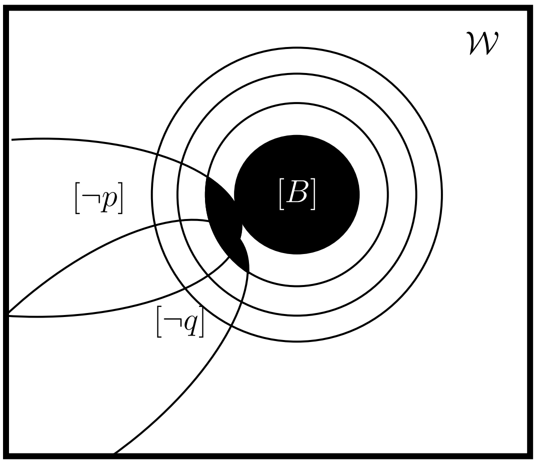


Figure 1.6: Sphere based contraction with conjunction.

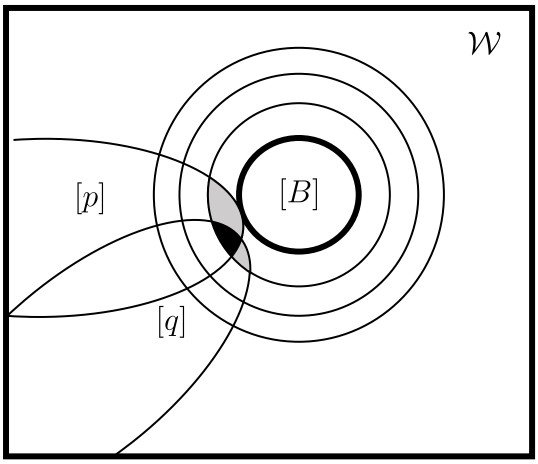


Figure 1.7: Sphere based revision with conjunction.

$$[B * (p \wedge q)] = [B * p] \cap [B * q]$$

1.4 Why is Belief Revision important?

This brief exposition of the main themes in belief revision will play an important role later in the thesis. In the final part, we shall introduce a system, denoted $GS4_B$, which incorporates both a set of beliefs and a sphere-based representation of possible worlds. The discussion here serves as a conceptual foundation for that development.

It is worth concluding this chapter with a few reflections. Belief revision was originally developed as a tool for artificial intelligence, providing a formal method for updating knowledge bases. Yet it also raises deeper philosophical questions about the nature of belief itself: can beliefs really be reduced to sentences?

Consider, for example, the case of partial conviction. One may believe that a proposition is true, but not with full certainty: they might assign it a plausibility of 75% rather than 100%. At the same time, this does not imply that they believe with 100% conviction in its negation. Such intermediate states of belief are difficult to capture in traditional frameworks, since human agents are rarely able to quantify their degree of confidence with precision. Vague expressions such as "I am barely sure" or "I am not so sure" suggest gradations of belief that resist reduction to simple binary values.

One possible way forward is to introduce numerical estimates only in contexts where beliefs are already computable, that is, where probabilistic operations can be carried out effectively. Belief revision, as a formal system, is computable; this means that if uncertain beliefs are to be integrated, they must be represented in terms of probabilities. The challenge of connecting beliefs, vagueness, and probability will be addressed in greater detail in Chapter 3.

This closes the first chapter, which has provided the initial framework for understanding how beliefs can be formally modeled, revised, and potentially extended to account for degrees of uncertainty.

Chapter 2

Supraclassical Logic

The purpose of this chapter is to highlight how supraclassical logics function as an intermediate stage between the rigidity of classical logic and the flexibility of non-monotonic systems. By relaxing certain classical constraints, such as substitution, and by allowing the introduction of background assumptions, supraclassical logics offer a conceptual bridge that prepares the ground for fractional semantics, discussed in Chapter 4.

2.1 We Are All Non-Monotonic: The Intuitive Basis

"*We are all non-monotonic*" is the title of the opening section of *Bridges between Classical and Non-Monotonic Logic* by David Makinson [65]. The claim is rooted in a philosophical observation: our everyday reasoning often fails to behave monotonically, that is, we sometimes retract conclusions even when none of the premises we initially relied on have been discarded.

To see what non-monotonic reasoning means in practice, consider an example. In the novel that introduces Sherlock Holmes and Dr. Watson, Inspector Lestrade finds the word "Rache" written in red on a wall and assumes it to be the unfinished name "Rachel." Holmes, however, interprets it differently: drawing on the victim's background, he concludes that it is the German word for "revenge." This hypothesis is later confirmed when it emerges that the woman had been poisoned.

The moral of this story is clear: one can rationally hold a belief at one point in time, and then, upon receiving new information, justifiably abandon or revise it. Non-monotonic logics succeed where classical logic fails, precisely because classical logic is ill-equipped to handle such dynamic changes in belief.

Formally, to *reason non-monotonically* means that a conclusion, though validly drawn from a set of premises, may need to be retracted once additional information becomes available — even if none of the original premises are discarded.

This idea is not entirely new: traces of non-monotonic reasoning can already be found in Hume's account of belief and Locke's discussion of probability. What is new, however, is its systematic formalization, which has been developed only in recent decades.

Classical logic, by contrast, is fully monotonic: if a proposition is a consequence of a set of premises, it remains a consequence of any larger set formed by adding new premises, provided none of the original premises are abandoned. Formally:

$$p \vdash q \text{ implies } p, r \vdash q \qquad \text{(Monotonicity)}$$

Supraclassical logics have been introduced as a natural bridge between classical and non-monotonic reasoning, and this chapter recalls their most salient features, following Makinson's formulation.

2.2 Eliminating Substitution: Post-Completeness and its Limits

A logic is said to be supraclassical if it allows us to infer more conclusions from a set of premises than classical logic authorizes. According to Makinson [65], there are three main strategies to achieve this:

1. adding background assumptions;
2. restricting the set of valuations;
3. introducing additional rules.

This section focuses on the first of these strategies.

2.3 Adding Background Assumptions: The Pivotal Assumption Framework

The first constructive procedure proposed by Makinson in order to introduce a supraclassical framework is to enrich reasoning with background assumptions. In everyday life, when one engages in a deduction, it is common to rely not only on explicit premises, but also on a set of implicit assumptions. This phenomenon was already recognized in ancient Greece under the name of *enthymeme*, an argument with one or more unstated premises.

Working with propositional language as in classical logic, let $\mathcal{L}$ denote the set of all formulas and let $K \subseteq \mathcal{L}$ be a fixed set of formulas. Intuitively, K plays the role of a set of background assumptions or *expectations*, closely resembling the set of beliefs discussed in Chapter 1. The notion of pivotal-assumption consequence can then be defined as follows:

Definition 2.1 (Pivotal-assumption consequence)**.** If p is a consequence of A modulo the assumption set K, then $A \vdash_K p$ alias $p \in Cn_K(A)$ if and only if there is no valuation v such that $v(K \cup A) = 1$ while $v(p) = 0$, i.e., if $K \cup A \vdash p$. A relation or operation is a pivotal-assumption consequence if and only if it is identical with $\vdash_K$ (Cn_K) for some set K of formulas.

This means that there is not only one pivotal-assumption consequence, but many, one for each value of K. Since classical consequence is monotonic, pivotal-assumption consequence relations and operations are supraclassical. This means that for every K $\vdash \subseteq \vdash_K$, i.e., $Cn \leq Cn_K$. Substitution cannot be accepted here not only because of the reasons explained earlier, but also because a substitution, when new premises are added, could not make any sense. Let the set K be a set of beliefs, doing a substitution in classical sense would be counter intuitive other than something that could bring on weird streets. In fact, if one believes that $\vdash_B p$ is true, where p is 'Rome is the capital of Italy" they cannot do a substitution with a generic q, because they do not believe a generic q that could be, for example, Pisa is the capital of

France" or another assumption. If one believes $\vdash_B p$ true, they would not believe $\vdash_B q$. A very important feature of pivotal-assumption consequence is that *representation theorem* for pivotal-assumption consequence operations is given and it is formulated as follows:

Definition 2.2 (Pivotal-assumption consequence)**.** If p is a consequence of A modulo the assumption set K, then $A \vdash_K p$ (equivalently $p \in Cn_K(A)$) if and only if there is no valuation v such that $v(K \cup A) = 1$ while $v(p) = 0$, i.e., if and only if $K \cup A \vdash p$. A relation or operation is a pivotal-assumption consequence if and only if it is identical with $\vdash_K$ (Cn_K) for some set K of formulas.

This shows that there is not just one pivotal-assumption consequence, but rather a family of them, one for each possible choice of K. Since classical consequence is monotonic, pivotal-assumption consequence relations are supraclassical: for every K, $\vdash \subseteq \vdash_K$, i.e., $Cn \leq Cn_K$.

Substitution cannot be retained in this setting, not only for the reasons discussed earlier, but also because it would clash with the very idea of background assumptions. If K represents a set of beliefs, then applying an arbitrary substitution would be counterintuitive, and might even produce nonsensical results. For example, if one accepts $\vdash_B p$ where p is "Rome is the capital of Italy," they cannot simply replace p with a generic q such as "Pisa is the capital of France." Beliefs are not schema variables; they are contextually grounded assumptions.

A central feature of pivotal-assumption consequence is that it admits a *representation theorem*, formulated as follows:

Theorem 2.1 (Representation theorem for $\vdash_K$ (Cn_K))**.** *Let Cn^+ be a supraclassical consequence operation that is compact. Then there exists a set K of formulas such that $Cn^+ = Cn_K$.*

The compactness clause can be set aside for our purposes, since belief sets are finite. Compactness is relevant primarily in contexts such as classical propositional logic, which assumes an infinite supply of propositional letters. From a philosophical perspective, it is implausible to model belief sets as infinite; in real reasoning, belief sets are always finite, or at least finitely representable.

The theorem is called a *representation theorem* because it captures the way supraclassical logic stands between classical and non-monotonic logic. Whereas classical consequence is unique, supraclassical and non-monotonic consequence relations are multiple: different consequence relations arise from different choices of K, i.e., different sets of background assumptions or beliefs.

By contrast, the completeness theorem of classical logic states that whenever p is a classical consequence of A, then p can be derived from A using the rules of inference of the system. A representation theorem is stronger: it shows that every structure satisfying certain syntactic conditions corresponds exactly to some semantically defined relation. Thus Theorem 2.1 demonstrates that every supraclassical consequence relation satisfying the relevant syntactic constraints is identical to a semantically defined pivotal-assumption relation.

Attempts to establish a completeness theorem for supraclassical logic, however, collapse into triviality. For this reason, representation theorems are the appropriate tool for studying the structure of supraclassical consequence.

2.4 Supraclassical Logic and Belief Revision

It now becomes clear why supraclassical logics were introduced in this context. If in the previous section the set K represented a collection of background assumptions, here it can be naturally replaced with the set of beliefs B. In this way, a particular form of supraclassical logic arises, built precisely by adding background assumptions in the form of beliefs.

The logic of belief change, introduced in Chapter 1, must address questions that are closely related to the challenges of non-monotonic and supraclassical reasoning. Among them, three are particularly central:

- Given the indeterminacy of contraction and revision, are there syntactic conditions they should always satisfy, or at least conditions worth retaining as desirable properties?
- Are there systematic ways of generating belief-change operations, i.e., semantic frameworks that justify them?
- Among the many possible contraction and revision operations that satisfy

reasonable conditions, can we identify one pair that deserves to be called the "correct" ones? [65, p. 143]

These questions are deeply interconnected. On one hand, there is no unique contraction or revision operation that is universally correct; the appropriate operation often depends on the context. On the other hand, it is possible to formulate general constraints and guiding attitudes, as demonstrated by the AGM framework, to identify belief-change operations that behave rationally.

Makinson pointed out that the principles governing revision and contraction strongly resemble those that underlie non-monotonic reasoning, though they are not identical. The supraclassical framework, enriched by background assumptions, provides a bridge: for every supraclassical consequence relation, there exists a set of assumptions K such that $Cn^+ = Cn_K$. This makes supraclassical logics a natural intermediary between classical consequence and the richer dynamics of belief change.

Nevertheless, supraclassical logics — like the AGM system itself — have limitations. The AGM model focuses on a single belief set at a time, without offering methods to compare or integrate multiple sets of beliefs. For instance, suppose that for two belief sets K and B, both entail A, i.e., $\vdash_K A$ and $\vdash_B A$. In this case, it is intuitive that their intersection also entails A: $\vdash_{K \cap B} A$. However, complications arise when neither K nor B entails A. In such a situation, classical logic forces us to conclude that $\vdash_{K \cap B} A$ is false, but this black-or-white outcome does not capture the way reasoning typically unfolds in practice.

In everyday reasoning, agents may still assign some credibility to A, even if it is not derivable from their current beliefs. They may act on uncertain hypotheses or weigh beliefs according to their plausibility. This suggests that a richer semantics is required, one that goes beyond the binary values of truth and falsity.

A natural extension is to employ a semantics where truth values range over the real interval $[0, 1]$. Such a framework allows us to compare the plausibility of different belief sets: instead of asking whether $\vdash_K A$ and $\vdash_B A$ are simply true or false, we can inquire whether one supports A more strongly than the other. This move opens the way toward *fractional semantics*, which will be developed in Chapter 4. There, the graded evaluation of beliefs will provide the tools to overcome the limitations of supraclassical logics and to formally capture the dynamics of uncertain reasoning.

The introduction of Supraclassical Logic has shown that the inferential structure of reasoning can be extended beyond the boundaries of bivalence without abandoning the rigour of classical systems. However, a logic—no matter how general—remains empty without an interpretation of what its formulas represent. The next step, therefore, is to reconnect the formal architecture to the epistemic notions that originally motivated it. In particular, we shall focus on beliefs, credences, and probabilities: on how degrees of conviction can be expressed, compared, and revised within a logical framework that allows for gradation. This transition marks the passage from structural generalisation to semantic interpretation.

Chapter 3

Beliefs, credences and probabilities

Having explored the formal landscape of Supraclassical Logic, we can now return to the question that first motivated this investigation: what does such a logical extension mean for our understanding of belief? Logic alone provides structure, but beliefs give that structure content and direction. If supraclassical inference allows us to move beyond classical logic, it should also allow us to capture the subtler gradations of conviction that shape real reasoning. To do so, we must examine how beliefs, credences, and probabilities relate and where they diverge. The next chapter addresses this problem thanks to an historical degression on David Hume and Ludwig Wittgenstein.

3.1 Introduction

The distinction between beliefs and credences has become central in contemporary epistemology and formal logic. A belief is often understood as a binary attitude: an agent either accepts a proposition as true or does not. Within traditional logical frameworks, beliefs have frequently been assimilated to tautologies, i.e., treated as fully true propositions immune to revision. This perspective, however, is philosophically problematic, since not every belief we hold carries the same degree of certainty as a logical truth.

Credences, by contrast, capture the graded dimension of our epistemic attitudes. Instead of a dichotomy, they assign a numerical value—commonly interpreted in probabilistic terms—to represent the degree of confidence an agent places in a proposition. The Lockean Thesis [45], for example, offers a bridge between the two notions: it maintains that once an agent's credence in a proposition surpasses a given threshold, the proposition may be accepted as a belief. This explains how beliefs can be seen as derived from, but distinct in nature from, underlying credences.

In this sense, beliefs mark the point of commitment in reasoning and action, while credences reflect the underlying gradation of epistemic support. The tension between the binary and the probabilistic view is crucial for understanding how logical systems can integrate uncertainty without collapsing the distinction between absolute and defeasible acceptance.

The aim of this chapter is to examine the bridge between beliefs and credences in two authors, namely Hume and Wittgenstein. On the one hand, Hume was among the first philosophers to treat beliefs and credences in a systematic logical and philosophical manner. We will focus on his notion of *vivacity*, which can be seen as a precursor to the modern concept of credence within subjective probability. On the other hand, we will turn to Wittgenstein who, in his *Tractatus*, introduces a connection between probability and belief that has been largely overlooked in the literature. We will argue that Wittgenstein's account also aligns with the modern notion of credence and subjective probability.

3.2 Hume, Probability, and Vivacity

The question of how to connect beliefs with numerical credences has a long history in philosophy. Long before the formal development of probability theory, philosophers sought ways to capture the "strength" with which an agent holds a belief.

One of the earliest and most influential contributions [59] comes from David Hume, who in *A Treatise of Human Nature* [51] laid the groundwork for a proto-probabilistic conception of belief. Rather than defining belief in purely logical terms, Hume introduced the psychological notion of *vivacity*, thereby anticipating later theories that treat belief as a graded phenomenon closely tied to probability.

> All the perceptions of the mind are of two kinds, viz. impres-

> sions and ideas, which differ from each other only in their different degrees of force and vivacity. Our ideas are copy'd from our impressions, and represent them in all their parts. When you wou'd any way vary the idea of a particular object, you can only encrease or diminish its force and vivacity. If you make any other change on it, it represents a different object or impression. The case is the same as in colours. A particular shade of any colour may acquire a new degree of liveliness or brightness without any other variation. But when you produce any other variation, 'tis no longer the same shade or colour. So that as belief does nothing but vary the manner, in which we conceive any object, it can only bestow on our ideas an additional force and vivacity. An opinion, therefore, or belief may be most accurately defin'd, A lively idea related to or associated with a present impression. [51, Book 1, Part III, Section VII]

Vivacity can be seen as an early attempt to describe subjective probability; in fact, it is something very close to it. Unlike mere products of imagination, beliefs are ideas that impress the mind with greater liveliness, thereby guiding our actions and influencing the passions. Hume insists that this vivacity cannot be further analyzed; rather, it is an immediately recognizable feeling in consciousness.

Patrick Maher [64] argues that Hume's notion of vivacity anticipates the modern concept of subjective probability. On this interpretation, degrees of belief correspond to degrees of vivacity: the higher the probability we assign to an idea, the more vividly it is conceived. Maher reconstructs Hume's view as a subtraction principle, whereby the vivacity of a belief results from weighing the probability of an event against that of its contrary. This interpretation shows that Hume's psychology provides not only a descriptive account of how beliefs arise, but also the outline of a systematic theory of graded belief and probable reasoning.

Seen in this light, Hume's theory of vivacity becomes more than a mere psychological curiosity. It represents an early attempt to capture the graded nature of belief, anticipating later developments in probability theory and the philosophy of subjective credence.

3.2.1 Vivacity of a belief

Maher expresses this idea as follows [64]:

> Since vivacity cannot be negative, it is natural to suppose that $V(h) = 0$ whenever $P(h) < \frac{1}{2}$, while $V(h) = 1$ just in case $P(h) = 1$. The most natural relationship between vivacity and probability which satisfies these constraints is:
>
> $$V(h) = \begin{cases} P(h) - P(\neg h), & \text{if } P(h) \geq \frac{1}{2}, \\ 0, & \text{otherwise.} \end{cases}$$
>
> An equivalent and slightly neater formulation is:
>
> $$V(h) = \max\{\, P(h) - P(\neg h),\, 0\,\}.$$

According to this definition if, for example, $P(h) = 0.7$ then $V(h) = 0.4$, while if $P(h) = 0.3$ the vivacity vanishes, $V(h) = 0$. In other words, on Maher's reconstruction, a belief has vivacity only when its probability outweighs that of its negation as it is shown in Figure 3.1. The idea of considering every probability with value less than 0.5 with vivacity 0 is due to the fact that we don't have any *spark* if the probability of an event is only half probable, because it is how usually things work: either p happens or p does not happen.

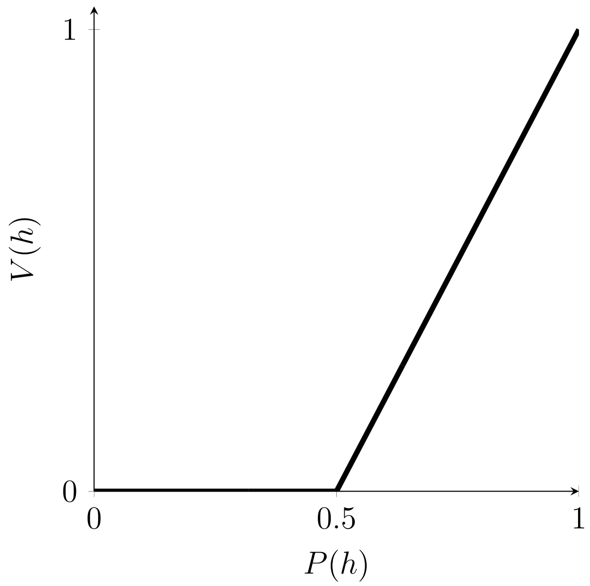


Figure 3.1: Maher's vivacity function.

An alternative formulation On the other hand, for Maher, vivacity remains 0 whenever the probability of a proposition is less than 0.5. His rationale is that if the probability of a statement being true is so low, the "spark" that would normally sustain belief is extinguished, and thus we do not rely on it in our reasoning or actions. In other words, a low probability does not generate any degree of liveliness in the mind.

However, this view overlooks an important point. When we judge that a proposition h is highly improbable, we are not simply suspending belief in h; rather, we are often committing ourselves to the truth of its negation $\neg h$. For example, if we estimate the probability that "it will rain tomorrow" at $P(h) = 0.1$, Maher's definition would assign vivacity $V(h) = 0$, as though no belief were present at all. But in practice we do form a belief: namely, that it will *not* rain tomorrow. The mental vivacity in such cases is not absent but redirected—it attaches to the negation of the proposition.

From the standpoint of classical logic, this is natural. A proposition and its negation are exhaustive: to deny h with confidence is equivalent to affirming $\neg h$ with equal confidence. Therefore, we should not interpret probabilities below 0.5 as signaling the absence of vivacity, but rather as signaling vivacity oriented toward $\neg h$. In this sense, belief is always present, though its polarity may change.

This motivates an alternative formalization of vivacity. Instead of truncating vivacity at 0 whenever $P(h) < 0.5$, we define it symmetrically in terms of the distance between $P(h)$ and $P(\neg h)$:

$$V(h) \;=\; \big|P(h) - P(\neg h)\big| \;=\; \big|2P(h) - 1\big|.$$

This formulation has several advantages. First, it ensures that maximal vivacity occurs not only when a proposition is certain ($P(h) = 1$), but also when it is certainly false ($P(h) = 0$), reflecting the fact that both certainty of truth and certainty of falsity generate strong convictions. Second, it captures the intuitive idea that vivacity should be weakest precisely at the point of maximal uncertainty ($P(h) = 0.5$), where belief in either h or $\neg h$ lacks sufficient force to guide action. Third, it offers a more faithful representation of everyday reasoning, where we often hold strong negative beliefs (e.g., "I firmly believe it will *not* rain") that are just as psychologically vivid as strong positive ones.

In this way, the alternative definition reframes Hume's concept of vivacity as a bidirectional phenomenon: not only the intensity with which we believe a proposition, but also the intensity with which we believe its negation. This symmetry reflects the dual structure of classical logic and aligns better with how human agents actually form and sustain beliefs.

$$V(h) \;=\; \big|P(h) - P(\neg h)\big| \;=\; \big|2P(h) - 1\big|.$$

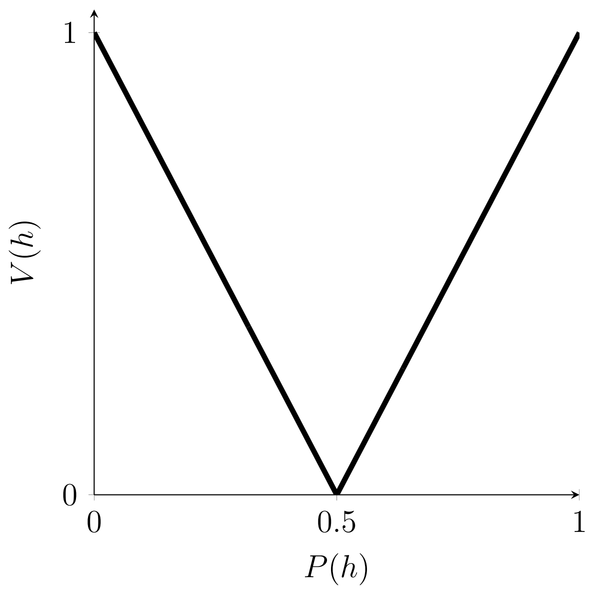


Figure 3.2: Vivacity as absolute difference, $V(h) = |2P(h) - 1|$.

For example, if $P(h) = 0.2$ then Maher's definition yields $V(h) = 0$, while this alternative definition gives $V(h) = 0.6$, reflecting strong confidence in $\neg h$. Both approaches agree that maximal vivacity occurs when $P(h) = 0$ or $P(h) = 1$, and minimal vivacity when $P(h) = \frac{1}{2}$.

To make the contrast clearer, the following table compares the two definitions for some representative values of $P(h)$:

The comparison between Maher's reconstruction and the alternative formulation shows that the notion of vivacity can be interpreted in two distinct ways. Maher's account emphasizes the role of probability as a threshold phenomenon: below 0.5 there is no vivacity, and only when the probability of a proposition exceeds its negation do we obtain a non-zero degree of liveliness. By contrast, the alternative account treats vivacity as a symmetric measure of confidence, equally sensitive to strong beliefs in a proposition and to strong beliefs in its negation.

From a philosophical perspective, this shift is significant. If we adopt Ma-

$P(h)$	Maher's $V(h)$	Alternative $V(h) = \|2P(h) - 1\|$
0	0	1
0.2	0	0.6
0.3	0	0.4
0.5	0	0
0.7	0.4	0.4
0.8	0.6	0.6
1	1	1

Table 3.1: Comparison between Maher's definition and the alternative formulation of vivacity.

her's view, belief appears only as a form of positive support for a proposition, while disbelief remains unaccounted for. The alternative definition, however, reflects the dual character of belief in classical logic: to deny with conviction is not to suspend judgment, but to affirm the opposite. In this sense, vivacity is not extinguished by improbability but redirected toward the negation.

This more balanced treatment of positive and negative belief not only captures Hume's intuition of vivacity as an immediate feeling of conviction, but also aligns with the probabilistic framework in which belief and disbelief are two sides of the same coin. It thereby provides a more comprehensive foundation for understanding graded belief and its role in reasoning under uncertainty.

In summary, Hume's appeal to vivacity provides an early model of belief as a graded attitude, one that varies in strength rather than occurring in an all-or-nothing fashion. Whether interpreted through Maher's reconstruction or through a more symmetric formulation, vivacity highlights the intimate connection between belief and probability. Hume's idea was significant for the development of this thesis from both a historical and philosophical perspective. On the one hand, the novelty of Hume's thoughts on probability has not been sufficiently emphasized; on the other hand, his insight proves meaningful for what will be developed in Chapter 6. In fact, it represents the first historical attempt to achieve what we will later formalize in Chapter 6.

This insight prepares the ground for Wittgenstein's treatment of probability in the *Tractatus*, where probability is no longer understood as a psychological feeling but as a structural feature of logic itself. In the next section, we will see how Wittgenstein transforms the idea of graded belief from a matter of psychology into a formal account of logical possibility and measure.

3.3 Wittgenstein, Probability and Beliefs

The discussion of Hume's notion of vivacity has highlighted how beliefs can be conceived as a graded phenomenon, closely tied to the feeling of conviction and to subjective probability. This perspective already points beyond the boundaries of classical logic, since it suggests that beliefs cannot be reduced to mere tautologies, but rather carry a weight that affects reasoning under uncertainty. It is precisely at this juncture that Wittgenstein's reflections on probability in the *Tractatus* become relevant. While Hume provided a psychological account of how beliefs acquire force, Wittgenstein aimed to capture, within a logical framework, how probability operates when beliefs are taken into account. In this sense, his notion of probability represents one of the earliest attempts to integrate beliefs with logic, anticipating what later developments would call supraclassical logics [8].

The aim of this section is to introduce Wittgenstein's notion of probability as presented in the *Tractatus Logico-Philosophicus* and to connect it with the framework of supraclassical logics. While von Wright [98] famously interpreted Wittgenstein's account as a mere generalization of Laplace's principle of indifference, we argue that it offers a richer and more subtle perspective [46, 72]. Far from being a simplistic account, Wittgenstein's reflections represent one of the earliest systematic attempts to integrate beliefs into logic through a probabilistic lens.

According to Wittgenstein, probability is defined by the relationship between the "belief's truth-possibilities" (*Wahrheitsmöglichkeiten*) and the truth possibilities of the proposition under consideration [15, 20, 46, 72]. Throughout his Tractatus, Wittgenstein posits that probability is *a priori* and maintains this viewpoint in his later writings, wherein he firmly rejects *frequentism* as the correct interpretation of probability (TBT 104e):

> Let's assume that someone playing dice every day were to throw, say, nothing but ones for a whole week, and that he does this with dice that turn out to be good when subjected to all other methods of testing, and that also produce the normal results when someone else throws them. Does he now have reason to assume a natural law here, according to which he always has to throw ones? Does he have reason to believe that things will continue in this way – or

> (rather) to assume that this regularity won't last much longer? So does he have reason to quit the game since it has turned out that he can throw only ones; or to continue playing, because now it is just all the more likely that on the next try he'll throw a higher number? – In actual fact he'll refuse to acknowledge the regularity as a law of nature; at least it will have to last for a long time before he'll consider this view of regularity. But why? – I think it's because so much of his previous experience in life refutes such a law, experience that has to be, so to speak – vanquished before we accept a totally new way of looking at things.

One of the objectives of this section is to prove that Wittgenstein's probability is a supraclassical logic, i.e., a logic that is able to derive more than classical logic usually permits [65], that is able to consider beliefs as axioms. In this sense it is mandatory to lose substitution due to the Post completeness proved by Emil Post in [85]. The idea lies on the fact that for Wittgenstein probability is a sort of extension of classical logic (TLP 5.156):

> [5.156] It is in this way that probability is a generalization.
> It involves a general description of a propositional form.
> We use probability only in default of certainty—if our knowledge of a fact is not indeed complete, but we do know something about its form.
> (A proposition may well be an incomplete picture of a certain situation, but it is always a complete picture of something.)
> A probability proposition is a sort of excerpt from other propositions.

In Wittgenstein's concept of probability, truthfulness is not solely determined by logic but also by knowledge, namely, beliefs. As a result, propositions that do not hold as true in classical logic due to not being tautologies can still be assigned values greater than 0 in probability framework. The objective of this research is to establish a connection between Wittgenstein's idea of probability as a supraclassical logic and the development of his views on probability.

To accomplish this, we draw upon Makinson's foundational work, 'Bridges between classical and non-monotonic logic' (*2005*) that we mentioned in chapter 2, with a specific focus on the section dedicated to probability and beliefs.

This examination seeks to illustrate that Wittgenstein's probability, albeit unconventional, aligns with supraclassical logics and with Kolmogorov's axioms, signifying its compatibility with fundamental principles of probability theory. The peculiar aspect of Wittgenstein's probability should be noted, as it extends beyond the confines of classical logic by incorporating knowledge-based evaluations of truthfulness for propositions.

By delving into these connections, we aim to shed light on the unique characteristics and implications of Wittgenstein's probabilistic approach.

3.3.1 Probability and possibility in Wittgenstein's *Tractatus*

In the first chapter of 'La logica dell'incerto' (*the logic on uncertainty*), de Finetti (*1931*) distinguishes between possibilities, which are objective, and probabilities, which are subjective. Wittgenstein's approach lies somewhere in between these two concepts. On one hand, Wittgenstein analyzes each possibility of falsity and truthfulness, akin to possibilities in de Finetti's framework. However, on the other hand, the agent's ability to choose the initial set of propositions introduces a subjective element, linked to the agent's personal knowledge of a given argument.

Notably, Wittgenstein's primary reflections on the nature of probability can be found in (TLP 5.1):

> [5.1] Truth-functions can be arranged in series.
> That is the foundation of the theory of probability.[1]

Wittgenstein's probability can be seen as an attempt to establish a connection between beliefs and logic. The beliefs of an agent, in this sense, will be considered as a set of propositions considered true from her. Despite its relative lack of extensive treatment, Wittgenstein's reflections on probability in the Tractatus offer valuable insights into this complex subject. Preliminar reflections on these can be found in the Notebooks 1914-1916 and in the reflections made in the *Vienna Circle*. We find other reflections on the theme in *Philosophical Grammar* and in the *Big Typescript*.

[1]Surprisingly, von Wright (*1969*), one of the first and most important authors that worked on Wittgenstein and in particular on the topic of probability in his work, does not include this proposition in the list of meaningful propositions about probability.

How probability works in the Tractatus

The notion of probability presented in Wittgenstein's *Tractatus Logico Philosophicus* may initially seem unusual, particularly when compared to the conventional modern perspective on probability. Wittgenstein's singular definition of probability is articulated in proposition 5.15:

> [5.15] If T_r is the number of the truth-grounds of a proposition r, and if T_{rs} is the number of the truth-grounds of a proposition s that are at the same time truth-grounds of r, then we call the ratio T_{rs}/T_r the degree of probability that the proposition r gives to the proposition s.

In practical terms, this approach entails examining instances where a believed proposition holds true and quantifying how many of these instances align with the true instances of the proposition under analysis, based on the original set of beliefs.

Example 3.1. Let us now examine an example drawn from everyday life: the act of tossing a coin. The main proposition under consideration will be denoted as $x \underline{\vee} y$, where the symbol $\underline{\vee}$ represents the mutually exclusive disjunction. In this context, the two possible outcomes, i.e., 'head' and 'tail', are mutually exclusive. The truth table for the proposition $x \underline{\vee} y$ is as follows:

	$x \underline{\vee} y$	x	y
1	F	T	T
2	**T**	**T**	F
3	**T**	F	**T**
4	F	F	F

Considering only the cases where $x \underline{\vee} y$ is true, we find that only the second and the third rows satisfy this condition. Now, let's determine the probabilities of x and y given the proposition $x \underline{\vee} y$. For the proposition x, out of the two instances where $x \underline{\vee} y$ is true, only the second instance has x as true while the third instance has x as false. Therefore, the probability of x given $x \underline{\vee} y$ is $1/2$. Similarly, for the proposition y, out of the two instances where $x \underline{\vee} y$ is true, only the third instance has y as true while the second instance has y as false. Thus, the probability of y given $x \underline{\vee} y$ is also $1/2$.

	$(x\underline{\vee}y) \wedge (r\underline{\vee}s)$	x	y	r	s	$(x \wedge r)$	$(y \wedge s)$
1	F	T	T	T	T	T	T
2	F	T	T	T	F	T	F
3	F	T	T	F	T	F	T
4	F	T	T	F	F	F	F
5	F	T	F	T	T	T	F
6	**T**	**T**	F	**T**	F	**T**	F
7	**T**	**T**	F	F	**T**	F	F
8	F	T	F	F	F	F	F
9	F	F	T	T	T	F	T
10	**T**	F	**T**	**T**	F	F	F
11	**T**	F	**T**	F	**T**	F	**T**
12	F	F	T	F	F	F	F
13	F	F	F	T	T	F	F
14	F	F	F	T	F	F	F
15	F	F	F	F	T	F	F
16	F	F	F	F	F	F	F

Figure 3.3: The table of truth representing the toss of a coin.

To summarize, when we toss a coin and consider the mutually exclusive disjunction proposition $x\underline{\vee}y$, the probability of x and y given this proposition is 1/2 for both cases, as expected.

Beliefs are perceived by an agent as unequivocally true, and this aligns with Wittgenstein's approach in the Tractatus. To provide further clarity, we shall introduce a novel formalization that was not presented by Wittgenstein, but that is very similar to the one used for conditionalization: this is due to the fact that Wittgenstein's method can be addressed as a formalization of conditionalization. In this formalization, we define the subscript as the set of beliefs under consideration for probability calculus. To illustrate this, let us revisit example 3.1, where the belief set considered was $x\underline{\vee}y$. Accordingly, the probability that x occurs, given $x\underline{\vee}y$, is denoted as $p_{x\underline{\vee}y}(x) = 1/2$.

Example 3.2. Now consider the case of the two coins problem. Imagine we need to toss two coins, and we want to calculate the probabilities using Wittgenstein's method. In this scenario, the proposition we'll analyze is $x \wedge r$, which, represents the probability of obtaining two heads from tossing two coins. To proceed, we need to consider each coin's results separately: let's designate the first coin's outcomes as x or y, and the second coin's outcomes as r or s. The truth table for the proposition $(x\underline{\vee}y) \wedge (r\underline{\vee}s)$ is shown in Figure 3.3.

Let us consider that $A = (x \underline{\vee} y) \wedge (r \underline{\vee} s)$. As per Wittgenstein's method, we find that $p_A(x) = p_A(y) = p_A(r) = p_A(s) = 0.5$, and $p_A(x \wedge r) = p_A(y \wedge s) = 0.25$. This confirms that the probability of getting two consecutive heads is 25%, which is the same for getting the first toss as head and the second toss as tail. Meanwhile, the probability of the first toss resulting in heads is 50%.

Let us examine now the likelihood of achieving a scenario with two heads, denoted as $x \wedge r$. Upon closer inspection, we observe that the condition $x \wedge r$ holds true solely in the sixth line, among a total of four instances where truth is affirmed. Similarly, this pattern emerges with $y \wedge s$, representing the chance of obtaining one head and one tail. This leads us to conclude that despite the method's seemingly unconventional nature, it harmoniously adheres to the principles governing probabilities.

Two elementary propositions

In one of the most significant statements concerning probability, one particular proposition stands out (TLP 5.152):

> [5.152] Two elementary propositions give one another the probability 1/2.

Von Wright (*1969*, p. 262) argues that the interpretation of this statement depends on one's understanding of *elementary propositions*. However, in my view, now that the system is clarified, Wittgenstein's intended meaning becomes quite apparent. Let us consider a single elementary proposition, denoted as x, and an unrelated proposition, denoted as y. The corresponding truth table is as follows:

	x	y
1	**T**	**T**
2	**T**	F
3	F	T
4	F	F

As we observe from the truth table, y is true only once out of the two total instances. This arises from the fact that we have no information about the relationship between x and y; the only knowledge we possess is that they are not mutually related.

It is essential to emphasize that, in 5.152, Wittgenstein stated that 'Two elementary propositions give one another the probability 1/2.' This statement, however, presents a particular problem due to the definition of independence provided in the same proposition in the Prototractatus and in the first version of the Tractatus as stated in *Cuffaro 2010*, *von Wright 1969*: *When propositions have no truth-arguments in common with one another, we call them independent of one another.* This implies that propositions like $x \vee \neg x$ and $y \vee \neg y$ are independent, but their probability is not 1/2; rather, it is 1 because both of them are tautologies. The same holds true for two contradictions, which cannot be calculated because on the left we have an undetermined value. It is plausible that Wittgenstein was aware of this issue and thus modified the text in the second edition of Tractatus (*1933*).

3.3.2 Wittgenstein's supraclassical logic

Makinson's contribution that will be analysed here lies in his introduction of the concept of supraclassical logic, as documented in *Makinson 2005*. Supraclassical logics, a realm of formal reasoning that transcends the limitations of classical logic, have garnered substantial interest due to their capacity to deduce conclusions beyond what classical logic traditionally allows. Makinson employs a diverse array of methodologies to achieve this expansion, with one prominent approach involving the incorporation of sets of beliefs into the logical framework. Through this technique, propositions that typically remain undecidable within classical logic can be validated as true owing to the presence of the supplementary belief sets. Notably, this approach exhibits intriguing parallels with the philosophical underpinnings of Wittgenstein's work.

It is worth noting that while Wittgenstein's approach also explores the notion of probability, Makinson's focus in the initial section of the book centers on propositional logic, however, as the book progresses, Makinson delves into the realm of probability theory. It is from this exploration that the seeds of inspiration for our concept of linking these seemingly distant authors were sown.

The interesting thing of Wittgenstein's method is that it was considered by von Wright in *von Wright 1969* only a generalization of Laplace's principle of indifference: 'if there are not evidences that one outcome is more preferable than another, then the agent must distribute her credences equally among the total number of outcomes'. Actually it is not just that; it is something deeper: Wittgenstein in fact proposed a method that considers one proposition as true and he compares that proposition with what we want to know. The number that he obtains is on one hand the generalization of Laplace principle, but on the other hand it is proposing a different kind of logic, that is not only analyzing probability, but also comparing a proposition with a given set of beliefs.

Let us consider example 3.2, depicting a coin toss. Notably, deriving $\vdash_B x \underline{\vee} y$ is elusive; its truth is not immediately evident. However, Wittgenstein's view offers a fresh angle. It prompts us to assess not only binary truth but also the proposition's frequency amid all potential outcomes.

Kolmogorov's axioms and Wittgenstein truth tables

Another very interesting point to note is that Wittgenstein's truth tables satisfy the Kolmogorov's axioms, i.e., the four Kolmogorov's axioms are proved by the truth tables as intended in the considered part of the *Tractatus*. Informally they were firstly proved as provable in Wittgenstein's system in (Ongaro 2021). The axioms are as follows:

(K1) $0 \leq p(x) \leq 1$

(K2) $p(x) = 1$ for some formula x

(K3) $p(x) \leq p(y)$ whenever $x \vdash y$

(K4) $p(x \vee y) = p(x) + p(y)$ whenever $x \vdash \neg y$

(K1) and (K2) follow from construction: the final value must be a number between 0 and 1. It cannot be less than 0 because the worst that can happen is that, as a belief, we have a contradiction, i.e., all instances are false. On the other hand, if the proposition we are analysing is equivalent to our belief or is a tautology, we will obtain the value of 1, but not more. This last consideration let the proof of (K2) obvious. (K3) can be proved thanks to the following truth table, where instead of $x \vdash y$, we consider $x \rightarrow y$ as true, that is a classical translation:

$K3$	$x \rightarrow y$	x	y
1	**T**	**T**	**T**
2	F	T	F
3	**T**	F	**T**
4	**T**	F	F

where $p_{x \rightarrow y}(x) = 1/3$ and $p_{x \rightarrow y}(y) = 2/3$, so $p_{x \rightarrow y}(x) \leq p_{x \rightarrow y}(y)$ and (K4) can be proved by the following:

$K4$	$x \rightarrow \neg y$	x	y	$x \vee y$
1	F	T	T	T
2	**T**	**T**	F	**T**
3	**T**	F	**T**	**T**
4	**T**	F	F	F

where $p_{x \rightarrow \neg y}(x) = 1/3$, $p_{x \rightarrow \neg y}(y) = 1/3$ and $p_{x \rightarrow \neg y}(x \vee y) = p_{x \rightarrow \neg y}(x) + p_{x \rightarrow \neg y}(y) = 1/3 + 1/3 = 2/3$ as wanted.

It is not easy to prove something generic, because we have to check every case, for example if we want to prove $(K5)$ $p(\neg x) = 1 - p(x)$ we must distinguish between the four combination of truthfulness and falsehood.

$K5$	$Formula$	x	$\neg x$
1	**T**	**T**	F
2	F	F	T

$p_{formula}(x) = 1$, $p_{formula}(\neg x) = 0$ and $p_{formula}(\neg x) = 1 - p_{formula}(x)$.

$K5$	$Formula$	x	$\neg x$
1	**T**	**T**	F
2	**T**	F	**T**

$p_{formula}(x) = 0.5$, $p_{formula}(\neg x) = 0.5$ and $p_{formula}(\neg x) = 1 - p_{formula}(x)$.

$K5$	$Formula$	x	$\neg x$
1	F	T	F
2	**T**	F	**T**

$p_{formula}(x) = 0$, $p_{formula}(\neg x) = 1$ and $p_{formula}(\neg x) = 1 - p_{formula}(x)$.

$K5$	$Formula$	x	$\neg x$
1	F	T	F
2	F	F	T

This last case is obviously special because we are giving a contradiction formula as a belief, so it's always false. Despite this, it was not really useful proving K5 from a formal point of view, because once K1-K4 were proved, than also K5 is provable from the first four axioms without using the truth tables. Proving the Kolmogorov's axioms proves that Wittgenstein's idea of probability is an actual probabilistic logic.

Why is Wittgenstein's probability a supraclassical logic?

We have now shown that Wittgenstein's probability satisfies Kolmogorov's axioms. Now we can address the main problem: is Wittgenstein's probability a supraclassical logic? Until now, we have only described a probabilistic logic, not a supraclassical one.

Definition 3.1 (Supraclassical logic)**.** A supraclassical logic is a logic that can derive more than classical logic usually permits, i.e., if $\vdash_S$ is the symbol for the supraclassical logic derivation, than it can be the case that $p \vdash_S q$ also if $p \nvdash q$.

Makinson in *Makinson 2005* elucidates the process of constructing a supraclassical logic thanks to three different techniques. The first of them, and the one that we will consider here, is to add a new set of axioms, namely beliefs, to let the logic prove more than classical logic usually permits. This method let the logic gain the ability to derive more than usual, also if it loses some property like the substitution, i.e., it loses Post Completeness.

Following Wittgenstein's method it is clear why this is a supraclassical logic: it considers beliefs that are added into the system. Beliefs are exactly the left part of the derivation that we have made in the previous sections: beliefs for Wittgenstein, also if he does not call them this way, are the starting point to derive a certain number greater than 0, i.e., the value that classical logic would assign to the examples made earlier.

Someone can argue that Makinson, later in the book, integrates this with the probability theory, in particular he focuses on the non-monotonic version of the probabilistic supraclassical logic, so why cannot we concentrate on them?

The case that Makinson considers is useful if we want to create a non monotonic supraclassical logic, but this is not possible in Wittgenstein's framework.

Makinson's approach to supraclassical probabilistic logics involves the imposition of constraints on valuations. The crux of this approach lies in the selection of a specific subset, denoted as Q, extracted from the larger set P. Intriguingly, this subset Q possesses the unique property of assigning a probability value of 1 to a designated formula, even in scenarios where the encompassing set P fails to do so. To illustrate this, let us consider an example involving inconsistent sets of beliefs. Typically, an inconsistent set would attribute a probability value of 0 to every proposition. However, through the strategic confinement of the set to a consistent subset, the assignment of a probability value becomes viable.

This principle can be extended to various contexts. For example, imagine the set P representing the logical conjunction $x \wedge y$, and we seek to ascertain the probability value of x. In the absence of constraints, the resultant probability would be 0.5. Nevertheless, by confining the analysis to solely the proposition x, we can derive a probability value of $p(x) = 1$.

The reason why we cannot concentrate on this is that Wittgenstein's framework inherently lacks the capacity to accommodate non-monotonic reasoning. This limitation stems from the framework's heavy reliance on conditionalizations, a foundational aspect that precludes the integration of probabilistic non-monotonic logic. Makinson's development of a probabilistic non-monotonic logic necessitates the abandonment of the very concept of conditionalization, which, as the framework is originally constructed, proves unfeasible within this context.

In conclusion, Makinson's exploration of supraclassical logic creation through probability manipulation can be effectively applied to Wittgenstein's methodology, in fact this way to treat probability is in fact a supraclassical logic. This methodology finds resonance even within Wittgenstein's theoretical paradigm. However, it is crucial to recognize that while Wittgenstein's framework is intrinsically tied to conditionalization, this feature inhibits the emergence of probabilistic non-monotonic logic, a frontier that Makinson's approach admirably advances by relinquishing the constraints of conditionalization.

Conclusions

The analyses of Hume and Wittgenstein offer two complementary perspectives on the relationship between probability, belief, and logic. Hume's account of belief, rooted in the notion of *vivacity*, anticipates the modern conception of subjective probability and provides an early framework for understanding graded belief. In Hume's reconstruction, belief is not merely a binary state but a matter of degree, governed by the comparative liveliness with which ideas present themselves to the mind. Maher's formalization, and the alternative formulation we have proposed, make explicit how this vivacity can be linked to probability measures, highlighting the continuity between philosophical psychology and probabilistic reasoning.

Wittgenstein, by contrast, situates probability directly within the logical structure of propositions. His treatment in the *Tractatus* demonstrates how truth-functions can ground a theory of probability, where the measure of one proposition depends on its alignment with a set of others. In doing so, Wittgenstein not only generalizes Laplace's principle of indifference but also introduces a framework that, when viewed through the lens of Makinson's theory, can be regarded as a form of supraclassical logic. Beliefs, in Wittgenstein's system, serve as truth-grounds that expand the inferential capacity of logic beyond the classical framework, while still preserving compatibility with Kolmogorov's axioms.

Taken together, Hume and Wittgenstein illustrate two paths toward the formalization of belief: one psychological and probabilistic, the other logical and structural. Hume emphasizes the subjective intensity with which we hold propositions, whereas Wittgenstein emphasizes the logical interplay of truth-grounds within a belief system. Their contributions converge on a common point: belief is neither reducible to pure logic nor to mere psychological impression, but instead occupies a space where probability and conviction interact.

This dual insight sets the stage for the introduction of *Fractional Semantics*, a framework that will be used to refine the relationship between beliefs and probability by quantifying the weight of axioms within a proposition. In the next chapter, we will explore how Fractional Semantics builds upon these philosophical foundations to provide a rigorous formal tool for analyzing beliefs and graded belief.

Chapter 4

Fractional Semantics

The motivation for introducing *Fractional Semantics* lies in the attempt to provide a proof-theoretic framework that can capture more than the traditional dichotomy between truth and falsity. Classical semantics evaluates formulas as either valid or invalid, while probability theory interprets them numerically, but both perspectives have their limits when we deal with uncertainty, partial justification, or graded beliefs. Fractional Semantics is designed precisely to fill this gap: it assigns to formulas rational values in the interval $[0, 1]$, representing the ratio of tautological components within a proof structure. In this way, it offers a semantic account that is at once rigorous, proof-theoretically grounded, and sensitive to degrees of truth.

The origins of this approach can be traced back to Piazza and Pulcini's introduction of fractional semantics [80], where the idea was to measure the proximity of a formula to being a tautology or a contradiction by inspecting its proof structure. This framework was subsequently refined for modal logic [81], showing that fractional semantics can accommodate not only propositional but also modal reasoning. More recently, the approach has been extended to epistemic contexts, first by incorporating beliefs into the system [5] and then by applying it to paradoxes such as the Lottery Paradox [3], where fractional values provide a principled way to distinguish between highly probable but not certain events.

The present chapter builds on this tradition, but it also aims to push the framework further. The first goal is to give a refined presentation of $GS4_B$, the system that extends fractional semantics with beliefs. This system allows us to treat beliefs as axioms while still preserving the multi-valued perspective, and it introduces new challenges such as how to account for deductive closure and the role of cut elimination. A second goal is to distinguish between different types of beliefs, using tools from non-standard analysis. In particular, hyperreal numbers allow us to differentiate *Full Beliefs*, which are treated as categorical truths with value 1, from *Revisable Beliefs*, which are assigned an infinitesimally lower value, $1 - \delta$. This refinement reflects the fact that, philosophically, not all beliefs we hold are equivalent to tautologies, and yet they can still be considered true within a broader logical framework.

The choice of hyperreals is inspired by Hansson's work [41, 42], where hyperreal numbers are employed to capture subtle distinctions between beliefs. Our aim, however, is slightly different: rather than focusing on belief dynamics alone, we want to preserve the full proof-theoretic apparatus of fractional semantics, so that we can calculate the value of complex formulas involving tautologies, Full Beliefs, and Revisable Beliefs in a uniform way. This allows us to track not only the logical validity of a derivation, but also its "fractional strength," i.e., how close it remains to a tautology once beliefs are incorporated.

Another key motivation concerns the treatment of cut. In classical proof theory, cut elimination is fundamental both for consistency and for the subformula property. In the context of fractional semantics, the presence or absence of cut affects the fractional value of formulas, making the study of cut elimination crucial for a robust system. This chapter therefore develops a strong cut elimination result for $GS4_B$, showing that the system can be kept cut-free while retaining its fractional interpretation. This result also entails the uniqueness of axiomatization for any cluster of beliefs, connecting our work to recent results on the analyticity of classical logic extensions [79].

Finally, the chapter serves as a basis for further developments. Once the refined system is presented and the distinction between different types of beliefs is established, we will extend the approach to *gradient beliefs* and *belief revision*, showing how the fractional perspective can model more realistic scenarios of epistemic reasoning. In this sense, Fractional Semantics is not only a technical innovation but also a philosophical tool: it offers a bridge between categorical logic and the fluid, graded nature of human belief, with potential applications ranging from formal epistemology to artificial intelligence.

The chapter is structured as follows. Section 1 provides an overview of Fractional Semantics in its original formulation. Section 2 introduces $GS4_B$ and presents proofs of some of its fundamental theorems. Section 3 discusses the incorporation of beliefs and the differentiation between Full and Revisable

Beliefs through hyperreal numbers. Section 4 develops the results on strong cut elimination and uniqueness of axiomatization. In the final sections, we turn to new material, focusing on gradient beliefs and belief revision, and sketching possible future applications of the framework.

4.1 Introduction and Motivation

Fractional Semantics, initially introduced in [80], serves as a powerful tool for discerning the number of proper axioms within a proposition relative to the total number of axioms. This method underwent refinement for modal logic [81] and expanded into the domain of beliefs in [5] and applied to the Lottery Paradox in [3]. The study demonstrated the instrumental role of Fractional Semantics in resolving the Lottery Paradox.

This work has two main objectives: firstly, to present in a refined way $GS4_B$, firstly presented in [5]—the Fractional Semantics System that incorporates beliefs; secondly, to introduce a nuanced categorization of beliefs. In [5], all beliefs are treated as true, akin to tautologies. However, this poses a philosophical challenge, as not every proposition we believe aligns with the certainty of a tautology. To address this, we utilize Hyperreal numbers, signifying that a belief holds a value not of 1, but infinitesimally lower—specifically, $1 - \delta$, where δ represents an infinitesimal value smaller than every real number.

This approach draws inspiration from Hansson [41, 42], who used hyperreal numbers to differentiate between Full Beliefs (assigned a value of 1) and beliefs open to revision in the presence of evidence, termed Revisable Beliefs. However, our aim is different: we seek a system capable of tracking not only the count of Full Beliefs but also beliefs considered true even if subject to revision, differenciating between them thanks to hyperreal numbers. Fractional Semantics enables us to perform derivations and determine the composition of the combination between tautologies, Full Beliefs, and Revisable Beliefs.

The chapter is structured as follows: in the first section, we briefly present Fractional Semantics, referring to [3, 5, 79–81] for more examples and proofs; in the second section, we present proofs for theorems from [5]; and in the last section, we introduce a distinction between Full Beliefs and Revisable Beliefs within the framework of fractional semantics.

4.2 Fractional Semantics

Fractional semantics is a multi-valued approach governed by pure proof-theoretic considerations firstly introduced in [80], assigning truth values as rational numbers in the closed interval [0,1] breaking the symmetry between tautologies and contradictions, allowing values other than 0 for non-logical axioms, i.e., contingent. It measures the proposition's proximity to being a tautology or a contradiction.

To enable fractional interpretation, a decidable logic $\mathcal{L}$ is required, displayed in a sequent system **S** meeting three conditions: bilateralism, invertibility, and stability.

Bilateralism : **S**, as a bilateral system, generates **S**-derivations for any well formed formula A of $\mathcal{L}$: if A is valid, its **S**-derivation will be an actual proof of A; if A is invalid, its **S**-derivation will provide a formal refutation of A, i.e., a proof of its unprovability.

Invertibility : each logical rule of S is invertible, meaning that the provability of its conclusion implies the provability of (each one of) its premise(s). This means that there is an algorithm to decompose uniquely a sequent into an equivalent formula in conjunctive normal form.

Stability : two analytic S-proofs with the same end-sequent share the same multi-set of top-sequents.

Fractional semantics is obtained by focusing on the axiomatic structure of proofs expressed in Kleene's one-side sequent system $GS4$ [54, 93]. The system is as following:

$$\frac{}{\vdash \Gamma, p, \overline{p}}\ (ax.)$$

$$\frac{\vdash \Gamma, p, q}{\vdash \Gamma, p \vee q}\ (\vee) \qquad\qquad \frac{\vdash \Gamma, p \qquad \vdash \Gamma, q}{\vdash \Gamma, p \wedge q}\ (\wedge)$$

GS4 is a one-sided sequent where structural properties are absorbed into the calculus, Γ and Δ are multisets of formulas, and $p, q, \ldots$ are atomic formulas. As usual, $\wedge$ indicates the conjunction and $\vee$ the disjunction. There is not a rule governing negation as it is inductively defined by different atomic formulas

p and $\overline{p}$, where $\overline{p}$ indicates the negation of p. Sequents can be decomposed into initial sequents that are allowed to contain only atomic formulas.

The interpretation of a formula is the result of the ratio between the number of identity top-sequents $(\Delta, p, \overline{p})$ out of the total number of top-sequents occurring in any of its proofs. Weakening and contraction are dropped while cut rule has the form:

$$\frac{\vdash \Gamma, p \qquad \vdash \overline{p}, \Delta}{\vdash \Gamma, \Delta}\ (cut)$$

In order to give a fractional interpretation a counterpart is needed, namely $\overline{\overline{GS4}}$, that is the $GS4$ calculus maximally extended:

Definition 4.1 ($\overline{\overline{GS4}}$)**.** The calculus $\overline{\overline{GS4}}$ is obtained from $GS4$ that is able to prove any sequent and it satisfies cut-elimination à la Gentzen if its axioms introduce only clauses [80], i.e., a sequent which consists solely of atomic formulae [2].

Definition 4.2 (Top-sequents axioms)**.**

$top^1(\pi)$: denotes the multiset of all and only π's *top-sequents* introduced by an identity axiom, i.e., those those sequents directly introduced as instances of the axiom rules.;

$top^0(\pi)$: denotes the multiset of all and only π's *top-sequents* introduced by a complementary axiom, in other words, those axioms that are not tautological.

Any formula A can be interpreted as the ratio between the number of identity top-sequents (sequents introduced by the standard axiom) out of the total number of top-sequents.

$$[\![A]\!] = \frac{top^1(\pi)}{top^1(\pi) + top^0(\pi)}$$

Definition 4.3 (Top-sequents)**.** Top-sequents represent the number of the leaves of the proof as defined in Definition 4.2 and $[\![\Gamma]\!]$ denotes the value of the formula $\vee\Gamma$ where only $\vee-$applications appear.

$top^1(\pi)$: let's call this m

$top^0(\pi)$: let's call this n

$[\![\vee\Gamma]\!]$ is $\frac{m}{n} \in [0,1]$

From this definition it is possible to give general rules with decorated sequents. These decorated sequents are able to keep track of the fractional value along the proof.

$$\frac{}{\mid\!\frac{1}{1}\ \Gamma, p, \overline{p}}\ (ax.) \qquad \frac{}{\mid\!\frac{0}{1}\ \Delta}\ (\overline{ax.})$$

$$\frac{\mid\!\frac{m}{\mathrm{n}}\ \Gamma, A, B}{\mid\!\frac{m}{\mathrm{n}}\ \Gamma, A \vee B}\ (\vee) \qquad \frac{\mid\!\frac{m_1}{\mathrm{n}_1}\ \Gamma, A \qquad \mid\!\frac{m_2}{\mathrm{n}_2}\ \Gamma, B}{\mid\!\frac{m_1+m_2}{\mathrm{n}_1+\mathrm{n}_2}\ \Gamma, A \wedge B}\ (\wedge)$$

Example 4.1. Let's consider an example with the turnstile decorated:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\mid\!\frac{0}{1}\ p, q}\ (\overline{ax.})}{\mid\!\frac{0}{1}\ p \vee q}\ (\vee) \qquad \dfrac{\dfrac{}{\mid\!\frac{1}{1}\ p, \overline{p}}\ (ax.)}{\mid\!\frac{1}{1}\ p \vee \overline{p}}\ (\vee)}{\mid\!\frac{1}{2}\ (p \vee q) \wedge (p \vee \overline{p})}\ (\wedge) \qquad \dfrac{\dfrac{}{\mid\!\frac{0}{1}\ \overline{r}}\ (\overline{ax.}) \qquad \dfrac{}{\mid\!\frac{0}{1}\ \overline{t}}\ (\overline{ax.})}{\mid\!\frac{0}{2}\ (\overline{r} \wedge \overline{t})}\ (\wedge)}{\mid\!\frac{1}{4}\ (p \vee q) \wedge (p \vee \overline{p}) \wedge (\overline{r} \wedge \overline{t})}\ (\wedge)$$

Here, it is possible to observe that for each step of the proof, we can directly read the fractional semantics value on the turnstile.

4.2.1 Framing beliefs into Fractional Semantics for classical logic

From Fractional Semantics we can do a different framework where beliefs are incorporated into fractional semantics for classical logic by introducing a set of axioms denoted as B. These axioms, representing the true beliefs of an agent, are treated as tautologies. The underlying philosophy is that an agent naturally considers their own beliefs to be true.

Beliefs in this context are treated as deductively closed, implying that any deduction made using these true beliefs is also considered true. This reflects the idea of an agent being deductively ideal. Integrating such beliefs into fractional semantics can lead to obtaining values greater than those typically permitted by fractional semantics alone.

The inspiration for this expansion comes from one of Makinson's methods, namely *pivotal-assumption consequence*, used to bridge the gap between classical and non-monotonic logic by adding background assumptions. However, the fractional semantics approach with added beliefs differs from *pivotal-assumption consequence* in two key aspects. Firstly, while Makinson used a classical two-valued semantics, fractional semantics operates within a multi-valued interpretation. Secondly, *pivotal-assumption consequence* assigns the value 0 if any axiom is not a proper axiom or belief, whereas fractional semantics can assign values greater than 0 when a top sequent is a tautology or a belief.

To incorporate beliefs into the system, they must be atomic; otherwise, they need to be decomposed. The definitions of $GS4_B$ and $\vdash_B$ are provided as follows:

Definition 4.4 ($GS4_B$)**.** Let $\mathbb{B} = b_1, \ldots, b_n$ a set of non tautological, non contradictory and of arbitrary complexity formulas; let B be the set of sequents obtained from the decomposition of formulas in $\mathbb{B}$ and closed under cut; let $GS4$ be as defined earlier, then $GS4_B$ is the system where everything that is derived from $\mathbb{B}$ and from $GS4$ is true.

Definition 4.5 ($\vdash_B$)**.** If $\vdash$ is the closure relation of classical logic, then $\vdash_B$ is defined as the closure relation of $GS4_B$.

Now, let's delve deeper into formalizing the system by defining the top sequent incorporating added beliefs.

Definition 4.6. $top^b(\pi)$: represents the multiset of all and only top sequents of π introduced by a belief.

Definition 4.7 ($top^b(\pi)$)**.** represents the multiset of all and only top sequents of π introduced by a belief.

The reason for introducing this new type of top sequent stems from our desire, particularly in this context, to treat beliefs on par with identity axioms. This is because an agent invariably regards her own beliefs as true. The updated method for calculating the value of a sequent is:

$$[\![A]\!]_B = \frac{top^b(\pi) + top^1(\pi)}{top^b(\pi) + top^1(\pi) + top^0(\pi)}$$

It is also possible to see the same tree with multi valued system, adding a new rule:

$$\dfrac{}{\vdash^{1}_{1\,B} B}\ (\overline{b_i})$$

Example 4.2. For example let's consider this example where $B = p, q$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash^{1}_{1\,B} p, q}\ (b_1)}{\vdash^{1}_{1\,B} p \vee q}\ (\vee) \quad \dfrac{\dfrac{}{\vdash^{1}_{1\,B} p, \overline{p}}\ (ax.)}{\vdash^{1}_{1\,B} p \vee \overline{p}}\ (\vee)}{\vdash^{2}_{2\,B} (p \vee q) \wedge (p \vee \overline{p})}\ (\wedge) \qquad \dfrac{\dfrac{}{\vdash^{0}_{1\,B} \overline{r}}\ (\overline{\overline{ax.}}) \quad \dfrac{}{\vdash^{0}_{1\,B} \overline{t}}\ (\overline{\overline{ax.}})}{\vdash^{0}_{2\,B} (\overline{r} \wedge \overline{t})}\ (\wedge)}{\vdash^{2}_{4\,B} (p \vee q) \wedge (p \vee \overline{p}) \wedge (\overline{r} \wedge \overline{t})}\ (\wedge)$$

It is worth noting that if this sequent was considered in classical logic, any valuation would assign either the value 0 or 1. Something similar happens in Makinson pivotal assumption consequence, also if the belief set is the same that we have defined earlier, because a two valued logic is there considered.

4.3 Strong cut elimination

The last section pointed out that the agent is an ideal one and that they are aware of every deduction between beliefs. This means that the belief set is deductively closed: nothing that was not already in the set can be derived. In order to have a deductively closed belief set it is important that every combination of sentences, when it is possible, must be closed under cut and the new sentences obtained in this way will be added to the belief set.

In order to eliminate cut from $GS4_B$ the method is taken from [79], but it is simplified because of the nature of one-sided sequents. The method is the following:

1. let's consider a propositional formula $b_i \in B$ (B being the set of beliefs) and decompose it using the invertible rules;
2. the procedures gives identity and non-logical sequents. Remove the identity ones;
3. let's contract every sequent thus obtained;
4. let's consider two sequents Γ, p and $\Delta, \overline{p}$ and add the sequent Γ, Δ to the set of beliefs and let's contract the set thus obtained;

5. the procedure terminates;

6. finally, take the set closed under weakening.

To emphasize the importance of accounting for the fractional value of a formula incorporating beliefs, it is necessary to consider, as initial sequents, not only those obtained directly but also sequents derived via closure under cut. Let's illustrate this with the following example:

Example 4.3. It is easy to show why the step 4. is so important. Suppose that an agent has a new belief: $A = (\overline{p} \wedge (\overline{t} \vee q)) \vee (t \wedge (\overline{t} \vee q))$. The first thing to do in order to add that belief is to transform A in a conjunctive form: it is easy to show that it is equivalent to $\vdash (\overline{p} \vee t) \wedge (\overline{t} \vee q) \wedge (t \vee \overline{t} \vee q) \wedge (\overline{t} \vee q)$. Let's decompose it in a set of clauses: $\vdash \overline{p}, t$, $\vdash \overline{t}, q$, $\vdash t, \overline{t}, q$, $\vdash \overline{t}, q$ and remove one of the copies of $\vdash \overline{t}, q$ and the axiom $\vdash t, \overline{t}, q$. By the method presented earlier the agent has to add $(\overline{p} \vee t)$ and $(\overline{t} \vee q)$ to the system, but these beliefs are not cut free. To let them be cut free, it is necessary to close them under the cut.

$$\frac{\vdash \overline{p}, t \qquad \vdash \overline{t}, q}{\vdash \overline{p}, q} \,(cut)$$

From the last point of the method presented earlier, it is needed to add not only $\vdash \overline{p}, t$ and $\vdash \overline{t}, q$, but also $\vdash \overline{p}, q$. Let's see why: $[\![(\overline{p} \vee t) \wedge (\overline{t} \vee q)]\!]$ has value 1 if $B = \{(\overline{p}, t); (\overline{t}, q)\}$

$$\frac{\dfrac{\vdash^{1}_{1\,B} \overline{p}, t}{\vdash^{1}_{1\,B} \overline{p} \vee t}\,(\vee) \qquad \dfrac{\vdash^{1}_{1\,B} \overline{t}, q}{\vdash^{1}_{1\,B} \overline{t} \vee q}\,(\vee)}{\vdash^{2}_{2\,B} (\overline{p} \vee t) \wedge (\overline{t} \vee q)}\,(\wedge)$$

As it was showed, the cut is really important for a complete set of beliefs, but it is also necessary to see how the cut can be eliminated from the calculus.

4.3.1 Elimination of cut

The elimination of cut in presence of proper axioms was firstly proposed by Girard [33], as noted by Avron [2], upgrading the Gentzen's standard cut elimination algorithm. The procedure here proposed, i.e., the decomposition of the formula, the add to the system and the cut of the formula to obtain all the derivations, owes a lot to the one presented in [79].

In the article, in fact, is proved that, for any cluster of extra-logical assumptions, there exists exactly one axiomatic extension of classical propositional logic that admits cut elimination. We can prove that Fractional value does not decrease in $GS4_B$ with relation to the addition of formulas:

Theorem 4.1. *For any multiset of atomic formulas* $\vdash_B \Gamma$ *and* $\vdash_B \Delta$, $[\![\bigvee \Gamma \vee \bigvee \Delta]\!]_B \geq [\![\bigvee \Gamma]\!]_B$.

Proof. To prove this is sufficient to consider a transformation of $\vdash_B$. In fact if $B = b_1, \dots b_n$, then $\vdash_B \Gamma$ is equal to $\vdash \Gamma, \overline{b_1}, \dots \overline{b_n}$, changing the kind of turnstile from the one introduced here to the classical one, as pointed out in [65][1]. Intuitively this is due to the fact that the sequent is true iff there is a disjunction between a letter and its negation (for example b_i and $\overline{b_i}$). From this fact it is possible to consider four cases:

- if $[\![\Gamma]\!]_B = [\![\Delta]\!]_B = 1$, than obviously $[\![\Gamma \vee \Delta]\!]_B = 1$ as well;
- if $[\![\Gamma]\!]_B = [\![\Gamma, \overline{b_1}, \dots, \overline{b_n}]\!] = 1$ and $[\![\Delta]\!]_B = 0$, then $[\![\Gamma \vee \Delta]\!]_B = 1$ as well;
- if $[\![\Delta]\!]_B = [\![\Delta, \overline{b_1}, \dots, \overline{b_n}]\!] = 1$ and $[\![\Gamma]\!]_B = 0$, then $[\![\Gamma \vee \Delta]\!]_B \geq [\![\Gamma]\!]_B$, whatever value assumes $[\![\Gamma]\!]_B$;
- if $[\![\Gamma]\!]_B = [\![\Delta]\!]_B = 0$, then $[\![\Gamma \vee \Delta]\!]_B \geq [\![\Gamma]\!]_B$.

□

It is possible to generalize this result for any context:

Theorem 4.2. *For any context* Γ *and a formula* A*, such that* A *is not contradictory with the set* B, $[\![\bigvee \Gamma \vee A]\!]_B \geq [\![\Gamma]\!]_B$.

Proof. Let's prove it by induction on the complexity of the formula A.

Base case: Let's consider A atomic, then we have two cases:

$A \in B$**:** if $A \in B$, then $[\![\bigvee \Gamma, A]\!]_B = 1$ and $[\![\Gamma, A]\!]_B \geq [\![\Gamma]\!]_B$

[1] In the text the two sided version of this transformation was used, so $\vdash_B \Gamma$ becomes $b_1, \dots, b_n \vdash \Gamma$, but here because of the choice to use $GS4$ as main system, it is used the one-sided classically equivalent version $\vdash \Gamma, \overline{b_1}, \dots, \overline{b_n}$.

$A \notin B$**:** if $A \notin B$, then if $[\![\bigvee \Gamma]\!]_B = 1$, there is an atomic formula in Γ that is in the belief set, so also $[\![\bigvee \Gamma, A]\!]_B = 1$. If $[\![\bigvee \Gamma]\!]_B = 0$, $b_1, \ldots, b_n \notin \Gamma$ and then $[\![\bigvee \Gamma \vee A]\!]_B = 0$

Inductive step: Let's consider two cases:

$A \equiv p \wedge q$**:** by inductive hypothesis $[\![\bigvee \Gamma \vee p]\!]_B \geq [\![\Gamma]\!]_B$ and $[\![\bigvee \Gamma \vee q]\!]_B \geq [\![\Gamma]\!]_B$. If at least one between $[\![\bigvee \Gamma \vee p]\!]_B$ and $[\![\bigvee \Gamma \vee q]\!]_B$ is equal to 0, then $[\![\bigvee \Gamma]\!]_B = 0$ for inductive hypothesis and then $[\![\bigvee \Gamma \vee (p \wedge q)]\!]_B \geq [\![\Gamma]\!]_B$. The only remaining case is when $[\![\bigvee \Gamma \vee p]\!]_B = 1$ and $[\![\bigvee \Gamma \vee q]\!]_B = 1$:

$$\dfrac{\dfrac{\dfrac{}{\vdash^{1}_{1\,B} \Gamma, p}}{\vdash^{1}_{1\,B} \bigvee \Gamma \vee p}\,(\vee) \qquad \dfrac{\dfrac{}{\vdash^{1}_{1\,B} \Gamma, q}}{\vdash^{1}_{1\,B} \bigvee \Gamma \vee q}\,(\vee)}{\vdash^{2}_{2\,B} \bigvee \Gamma \vee (p \wedge q)}\,(\wedge)$$

Thus $[\![\bigvee \Gamma \vee (p \wedge q)]\!]_B \geq [\![\Gamma]\!]_B$.

$A \equiv p \vee q$**:** by theorem 4.1.

□

Theorem 4.3 (Strong cut elimination of $GS4_B$)**.** *The cut rule is redundant when added to $GS4_B$.*

Proof. Girard was the first to notice that a different procedure could preserve cut elimination even in the presence of axioms [33, 75]. The proof is as usual with double induction, the alorithm is similar to the one presented in [82].

□

The set of beliefs can be "completed" through cut or without that. This means that $GS4_B$ is a cut-free system, because it is an axiomatic extension of classical logic. By the way, the use of cut can alter the fractional semantics value as shown in [80]. Thanks to theorem 4.3 the algorithm presented in section 4.3 can be transformed in an algorithm without the presence of cut. As a corollary of the strong cut elimination it can be obtained:

Theorem 4.4 (Uniqueness of axiomatization in $GS4_B$)**.** *For any cluster of axioms in the set of beliefs B the axiomatization is unique.*

Proof. See [79]. □

4.3.2 Towards the Lottery Paradox

The introduction of Fractional Semantics has provided a formal framework capable of representing beliefs with graded strength by means of hyperreal-valued evaluations. However, while such a framework allows for a fine-grained account of epistemic attitudes, its conceptual adequacy must be tested against paradigmatic problems in epistemology. Among these, the Lottery Paradox stands out as a crucial benchmark: it exposes the tension between rational belief and probabilistic reasoning. In the next chapter, we shall examine how the Fractional Semantics can offer a consistent and structurally simple solution to this paradox, preserving both logical coherence and the intuitive notion of belief.

Chapter 5

How to solve the Lottery Paradox in Classical Logic

5.1 Introduction

The Lottery Paradox, first formulated by Kyburg [57], poses a challenge for the rational treatment of probabilistic beliefs[1].

> Let us consider a fair 1000-ticket lottery that has only one winning ticket. A perfectly rational agent knows that each ticket has a probability of 999/1000 of not winning. Thus, it is rational for the agent to accept that each ticket will not win because this probability is greater than her Lockean threshold. This reasoning can be extended to every other ticket in the lottery, leading to the conclusion that somehow every ticket will not be the winning ticket. However, the lottery is fair, so the conjunction of all these statements has to be false, rather than true as it appears.

The paradox illustrates the tension between probabilistic reasoning and the logical closure of beliefs under conjunction. Different strategies have been proposed to resolve it. One approach, following Kyburg, is to abandon closure under conjunction and accept that rational agents may hold inconsistent belief sets. Another line of thought insists on maintaining closure, but then seeks alternative semantic frameworks to dissolve the paradox rather than weaken logic itself.

[1]For a more detailed discussion on the Lottery Paradox cf. for example *Hajek 2019*, *Hawthorne 2004*, *Hawthorne 2009* and *Leitgeb 2017*.

In what follows, we will explore two such frameworks. First, Wittgenstein's conception of probability as a conservative extension of classical logic, which shows how the paradox dissolves once beliefs are treated as graded truth-grounds rather than dichotomous judgments. Second, Fractional Semantics, which refines this idea by introducing a fine-grained account of how individual conjuncts contribute to the overall evaluation, offering a transparent explanation of why the paradox does not genuinely arise.

5.2 Dissolving the Lottery Paradox as a lump of sugar in water

Starting from what we introduced in Section 3.3, it is possible to try to solve the Lottery Paradox through Wittgenstein's probability. Wittgenstein's probability is a probabilistic logic and a supraclassical logic. This is useful to prove that, thanks to Wittgenstein's method, it is possible to solve (or better, *dissolve*)[2] a class of belief paradoxes, such as the Lottery Paradox.

Let us also consider Proposition 5.156 from Wittgenstein's Tractatus:

> [5.156] It is in this way that probability is a generalization.
> It involves a general description of a propositional form.
> We use probability only in default of certainty—if our knowledge of a fact is not indeed complete, but we do know something about its form.
> (A proposition may well be an incomplete picture of a certain situation, but it is always a complete picture of something.)
> A probability proposition is a sort of excerpt from other propositions.

Although Wittgenstein uses the term *generalisation*, we believe that in modern logic it is more appropriate to refer to it as a *conservative extension*, while maintaining the essence of his idea. In fact, by following Wittgenstein's perspective, we can resolve the Lottery Paradox within a conservative extension of classical logic. This solution can be given without dropping the principle of conjunction between rational beliefs as the author of the paradox, Kyburg, has originally suggested in [57]

[2]The use of this word and the name of this section are quoted from *TBT*, 38, page 310e.

The Lottery Paradox is not a paradox within Wittgenstein's framework due to the nature of this probabilistic logic. The paradox arises when dichotomous beliefs and probabilistic beliefs are combined, but in Wittgenstein's view, only probabilistic beliefs exist, still within the generalization of classical logic. Our idea is that this paradox dissolves in Wittgenstein's concept of probability 'as a lump of sugar in water,' quoting Philosophical Occasions.

To achieve this, it is necessary to establish that when dealing with a conjunction involving a finite yet arbitrarily large number of elementary propositions, where all but one are negative, only a singular True line emerges. Furthermore, it becomes essential to demonstrate that this true line occupies a specific position within the matrix and maintains its uniqueness as selecting distinct propositions, each with the positive formula in a different position. Using these insights, it is possible to construct a disjunction that encompasses all possible scenarios, resulting in exactly n True lines, where n represents the count of literals within the formula. Firstly we have constructed the following true table but, instead of considering to include both T (True) and F (False) values for each proposition, we have opted for a more readable table format. This is why, in Figure 5.1, $\neg p_1$ is represented as F in its initial entry.
The table represented in Figure 5.1 needs some hint to let it be cleared:

1. To enhance clarity in tracking transitions from T to F, we have explicitly highlighted the most significant changes. For example, 2^{n-1} represents the last row where the truth value of $\neg p_1$ changes, occurring exactly at the midpoint of the entire truth table. Similarly, $2^{n-1} + 2^{n-2}$ marks the last row before the intermediate change of $\neg p_2$.

2. The most interesting row in the table is $2^n - 2^{n-x}$ because it consists entirely of T instances. This is a result of the fact that on the left side of p_x, we only have T instances that continually double in number with each iteration. On the right side, we observe a similar pattern, but with 'T' instances halving until we reach the single T instance for $\neg p_n$.

3. The value of $2^n - 2^{n-x}$ corresponds to the last row before the truth value of p_x changes. It can also be expressed as $\sum_{i=1}^{x} 2^{n-i}$ as it requires summing the halved values successively, reflecting the decreasing number of T instances with each new proposition considered.

	$\neg p_1$ $\wedge$	$\neg p_2$ $\wedge$ $\ldots$ $\wedge$	p_x $\wedge$	$\neg p_{x+1}$ $\wedge$ $\ldots$ $\wedge$	$\neg p_n$
1	F	F	T	F	F
2	F	F	T	F	T
⋮	⋮	⋮	⋮	⋮	⋮
2^{n-1}	F	T	T	T	T
$2^{n-1}+1$	T	F	F	F	F
⋮	⋮	⋮	⋮	⋮	⋮
$2^{n-1}+2^{n-2}$	T	F	T	F	T
$2^{n-1}+2^{n-2}+1$	T	T	F	T	F
⋮	⋮	⋮	⋮	⋮	⋮
$2^n-2^{n-x}-1$	T	T	T	T	F
2^n-2^{n-x}	**T**	**T**	**T**	**T**	**T**
$2^n-2^{n-x}+1$	T	T	F	F	F
⋮	⋮	⋮	⋮	⋮	⋮
2^n	T	T	F	F	T

Figure 5.1: The truth table representing the generalization of Lottery Paradox in Wittgenstein's method.

The following claim has to be proved in order to generalize Wittgenstein's probability:

Claim 5.1. *If a proposition made by an arbitrary number of elementary letters is made by all negated formulas and one positive formula, the only line that is made by true instances is the line marked with the number* $2^n - 2^{n-x}$*, where* x *is the position of the elementary letter starting from the left.*

We will first explain the process by which this truth table was created before providing the proof: intuitively, to find which line is true we have to consider the two extreme cases, i.e., $p_1 \wedge \neg p_2 \wedge \cdots \wedge \neg p_n$ and $\neg p_1 \wedge \cdots \wedge \neg p_{n-1} \wedge p_n$ where in the first case only the first one is positive and in the second case only the last one is not negated. Then we have to prove it for a generic p_x between p_1 and p_n. Let us consider then the following where the positive letter is the first, i.e., p_1:

	p_1	$\wedge$	$\neg p_2$	$\wedge$	$\neg p_3$	$\wedge$	$\dots$	$\wedge$	$\neg p_n$
1	**T**		F		F				F
2	**T**		F		F				T
$\vdots$	$\vdots$		$\vdots$		$\vdots$				$\vdots$
2^{n-2}	**T**		F		T				T
$2^{n-2}+1$	**T**		**T**		F				F
$\vdots$	$\vdots$		$\vdots$		$\vdots$				$\vdots$
$2^{n-2}+2^{n-3}$	**T**		**T**		F				F
$2^{n-2}+2^{n-3}+1$	**T**		**T**		**T**				T
$\vdots$	$\vdots$		$\vdots$		$\vdots$				$\vdots$
2^{n-1}	**T**		**T**		**T**				**T**
$2^{n-1}+1$	F		F		F				F
$\vdots$	$\vdots$		$\vdots$		$\vdots$				$\vdots$
2^n	F		T		T				T

As observed, to the right of a positive propositional letter, we notice a diminishing count of admissible lines. The truth line is in fact 2^{n-1} that is exactly $\sum_{i=1}^{x} 2^{n-i} = 2^n - 2^{n-x}$ where $x = 1$, following that $2^n - 2^{n-1} = 2^{n-1}$. This phenomenon arises because each time the upper half consists solely of false instances and because of the conjunction property, it is possible to consider only the bottom half each time. This pattern persists until we reach the final propositional letter, which renders only one line true among the total of 2^n lines.

On the other hand, if we consider the other limit case, considering that the only positive formula is p_n we obtain:

	$\neg p_1$	$\wedge$	$\neg p_2$	$\wedge$	$\neg p_3$	$\wedge$	$\dots$	$\wedge$	p_n
1	F		F		F				T
2	F		F		F				F
$\vdots$	$\vdots$		$\vdots$		$\vdots$				$\vdots$
2^{n-1}	F		T		T				F
$2^{n-1}+1$	$\mathbf{T}$		F		F				T
$\vdots$	$\vdots$		$\vdots$		$\vdots$				$\vdots$
$2^{n-1}+2^{n-2}$	$\mathbf{T}$		F		T				F
$2^{n-1}+2^{n-2}+1$	$\mathbf{T}$		$\mathbf{T}$		F				T
$\vdots$	$\vdots$		$\vdots$		$\vdots$				$\vdots$
$2^{n-1}+2^{n-3}$	$\mathbf{T}$		$\mathbf{T}$		F				F
$2^{n-1}+2^{n-3}+1$	$\mathbf{T}$		$\mathbf{T}$		$\mathbf{T}$				T
$\vdots$	$\vdots$		$\vdots$		$\vdots$				$\vdots$
2^n-1	$\mathbf{T}$		$\mathbf{T}$		$\mathbf{T}$				$\mathbf{T}$
2^n	T		T		T				F

This implies that to the left of p_n, we witness a diminishing set of potential truth instances, as previously explained, with only the lower half being considered for conjunction. Ultimately, the penultimate line stands as the sole truth-bearing one, i.e. 2^n-1 that satisfies the formula 2^n-2^{n-x}, where $x=n$: $2^n-2^{n-n}=2^n-1$.

For any generic positive value of p_x between p_1 and p_n, we need to consider a range of values between these two extremes. If we examine the first truth table, we can observe that when we start with $\neg p_1$, we only need to focus on the second half of the truth values since the false instances in the first half are not relevant. Moving on to $\neg p_2$, we continue to concentrate on the second half, and this pattern continues until we reach $\neg p_{x-1}$.

When we consider p_x as true, the order of the true values changes. In other words, the first half of the remaining true instances now becomes false, while the second half remains true. As we proceed to $\neg p_{x+1}$, the true instances again halve in the lower part, and this process continues until we arrive at $\neg p_n$, which represents the last line where truth is possible. In conclusion, the exact line where p_x is true in the table corresponds to $2^n - 2^{n-x}$.

Combining these outcomes elucidates why precisely the line $2^n - 2^{n-x}$ is replete with truth instances while the others cannot be true. Furthermore, this unique truth-bearing line varies for each distinct variable x, as evidenced by the changing values of $2^n - 2^{n-x}$. From these considerations it is easy to understand why the line made only by true instances is the last line where p_x has a T-value: the idea of proof lies on this fact.

Proof of Theorem 5.1. We can prove Theorem 5.1 by induction, leveraging the fact that $2^n - 2^{n-x}$ represents the last line where p_x has a truth value of T. The idea is to establish by induction that $2^{n+1} - 2^{n+1-x}$ remains the last line where p_x has a truth value of T and that each other propositional letters have a T value in that line when we have $n + 1$ propositional letters.

Base case: For the base case when $n = 1$, we note that the only true line is the first one. This can be verified by calculating $2^1 - 2^0 = 2 - 1 = 1$, which matches the truth value in the first line.

Inductive step: Now, let us consider the inductive step. Assuming that the line number $2^n - 2^{n-x}$ has only T instances for some value of n, we aim to show that if x remains the same, then the new line should be twice the value of $2^n - 2^{n-x}$.

This is because of the construction of a truth table: if a line for a certain propositional letter, let's say line number i for letter a, was labeled as T (F) in a truth table created for n elementary propositions, then line $2i$ for letter a will also be labeled as T (F) when adding a new elementary letter.

Proving this implies that at line $2(2^n - 2^{n-x})$ for $n+1$ elementary letters, each propositional letter between 1 and n will be true. Moreover, the new elementary letter, $\neg p_{n+1}$, will be true in that line because it alternates between F and T (initially F because $\neg p_{n+1}$ is false in the first line, being a negated formula). This means that the T instances will appear on even lines, and $2(2^n - 2^{n-x})$ is even, completing the correspondence between the two truth tables.

To complete the proof, we need to establish a correspondence between $2(2^n - 2^{n-x})$ and $2^{n+1} - 2^{n+1-x}$. We can easily demonstrate that:

$$2^{n+1} - 2^{n+1-x} = 2(2^n - 2^{n-x})$$

This equation establishes the desired relationship between the new line and the previous one, confirming that it aligns with our expectations. As the inductive hypothesis establishes, the line $2^n - 2^{n-x}$ was true for n. Therefore, the line $2(2^n - 2^{n-x})$ will also be true for $n+1$ because the number of lines doubles, concluding the proof.

□

Going back to the initial problem of the Lottery Paradox: due to the uniqueness of each value of x, the disjunction of the conjunctions, i.e.,

$$\begin{aligned} &(p_1 \land \neg p_2 \land \cdots \land \neg p_n)\lor \\ &\qquad \cdots \lor (\neg p_1 \land \cdots \land p_x \land \cdots \land \neg p_n)\lor \\ &\qquad\qquad \cdots \lor (\neg p_1 \land \neg p_2 \land \cdots \land p_n) \end{aligned}$$

will have exactly n lines. This is due to the fact that we want to formalize a lottery and this means that we want the exclusive disjunction for each ticket.

Remarkably, this composite expression will consistently exhibit precisely n instances of truth lines across all possible configurations of truth values. It is noteworthy that our previous investigation has conclusively established the singularity of the line characterized by a T-value. This uniqueness materializes as $2^n - 2^x$ for values of x ranging from 1 to n, with each individual value of x generating a distinct outcome. We can see it in the following table:

	$(p_1 \land \cdots \land \neg p_n)$	$\lor$	$\ldots$	$\lor$	$(\neg p_1 \land \cdots \land p_n)$	
1	F				F	F
$\ldots$						
2^{n-1}	**T**				F	**T**
$\ldots$						
$2^n - 1$	F				**T**	**T**
2^n	F				F	F

In total we have n true instances and this means that when we consider only one proposition it will be true $1/n$ times.

Example 5.1. Let us see an example: let us consider that the lottery has 1000 tickets, then the proposition will be $A = (p_1 \wedge \cdots \wedge \neg p_{1000}) \vee \cdots \vee (\neg p_1 \wedge \cdots \wedge p_{1000})$ and let's say that the ticket that we have bought is the ticket number 543, then we have to compare A with $\neg p_1 \wedge \cdots \wedge p_{543} \wedge \cdots \wedge \neg p_{1000}$. The only true line for $\neg p_1 \wedge \cdots \wedge p_{543} \wedge \cdots \wedge \neg p_{1000}$ will be $2^{1000} - 2^{1000-543} = 2^{1000} - 2^{457}$. This line will be one of the 1000 true lines of the proposition A for construction and this means that the final probabilistic value of the truthfullness of $\neg p_1 \wedge \cdots \wedge p_{543} \wedge \cdots \wedge \neg p_{1000}$ given A will be $1/1000$.

The analysis of Wittgenstein's notion of probability has shown how the Lottery Paradox can be dissolved once beliefs are treated as graded and tied to truth-grounds, rather than as purely dichotomous judgments. Within this framework, the paradox ceases to arise, since the logical structure guarantees that exactly one winning ticket is singled out without contradiction.

However, Wittgenstein's approach, while illuminating, remains bound to the combinatorial structure of truth-grounds and does not provide a fine-grained measure of how much each axiom or conjunct contributes to the final evaluation. To move beyond this limitation, we now turn back to *Fractional Semantics*, which will be used to address the same paradox. This allows us to capture more precisely the graded nature of beliefs and to explain why the Lottery Paradox, far from being a genuine contradiction, dissolves once the internal structure of the belief set is made explicit.

5.3 Fractional Semantics and Probability: The Lottery Paradox

Wittgenstein's framework allows us to reinterpret probability as a logical relation between propositions, showing how the Lottery Paradox can dissolve once beliefs are treated as graded and logically structured. Yet, while illuminating, this approach remains essentially qualitative: it demonstrates consistency, but it does not provide a fine-grained account of the contribution of each axiom or conjunct to the overall evaluation.

To address this limitation, we now turn to *Fractional Semantics*. An interesting application of this framework for classical logic, as stated in the introduction, can be found in the solution of the same problem: the Lottery Paradox. The Lottery Paradox is closely related to the Lockean Thesis, which defines how classical logic aligns with beliefs.

> But another Man who never took the pains to observe the Demonstration, hearing a Mathematician, a Man of credit, affirm the three Angles of a Triangle, to be equal to two right ones, assents to it: i.e. receives it for true. [63, 4^{th} book, chapter XV, §1]

Beliefs are related to the level of assent that one agent can give to another and so to those beliefs that are not 100% true, but in which the agent still has a high degree of confidence in them.

> Being that which [the probability] makes us presume things to be true, before we know them to be so. [63, 4^{th} book, chapter XV, §3]

Here probability is treated as the level of assent in a certain proposition, in fact the renewed version of the Lockean Thesis is formulated as it follows in [23]:

> It is epistemically rational for us to believe a proposition just in case it is epistemically rational for us to have a sufficiently high degree of confidence in it, sufficiently high to make our attitude towards it one of belief.

The Lockean Thesis has many important positive aspects. For instance, it implies that even a logically ideal agent whose degrees of confidence satisfy the axioms of probability theory can rationally believe each of a large body of propositions that are jointly inconsistent. This can be beneficial in situations where the agent has incomplete information or where there are various sources of uncertainty.

However, as we have seen in Section 3.3, beliefs under the Lockean Thesis are not closed under conjunction because inconsistent beliefs with varying degrees of confidence are possible. This means that an agent can simultaneously hold contradictory beliefs with different levels of confidence. One criticism of the Lockean Thesis is that it does not provide clear guidance on how to select the threshold. Determining which propositions an agent considers to be true or false is crucial, and the threshold can also affect the coherence and consistency of an agent's beliefs. However, the Lockean Thesis does not specify how an agent should select the threshold, and some argue that it can seem arbitrary.

Despite its strengths, the Lockean Thesis faces a significant challenge in the form of the Lottery Paradox. The paradox can appear straightforward at first glance, but it comes into conflict with the Lockean Thesis.

The seemingly simple Lottery Paradox highlights an interesting problem that can be formalized in modal logic through the Barcan formula (BF). The BF is problematic when considered with the Lottery Paradox, whereas the converse Barcan formula (CBF) is not problematic.

$$\forall x \Box F(x) \rightarrow \Box \forall x F(x) \tag{BF}$$

$$\Box \forall x F(x) \rightarrow \forall x \Box F(x) \tag{CBF}$$

The converse Barcan formula is not problematic in this case, as it simply states that if it is necessary for all x to have the property F, then each individual x must have the property F. Therefore, if it is necessary for every ticket to be a losing one, then each individual ticket must be a losing one.

However, the Barcan formula itself presents a problem in the case of the Lottery Paradox. It states that if every x is necessarily F, then it is necessary that every x is F. In the context of the Lottery Paradox, if it is necessary that each individual ticket is a losing one, then it is necessary that every ticket is a losing one. This leads to a contradiction: each individual ticket considered alone is a losing ticket, but the conjunction of all losing tickets couldn't be true, as there must be at least one winning ticket.

One proposed solution to this problem, as discussed in sources such as [45] and [59], is to reject the closure of beliefs under conjunction. This is a strong thesis, as it implies that inconsistent beliefs can be rational. The authors propose that this is due to the fact that beliefs are not completely certain and can change over time, allowing for the possibility of holding multiple inconsistent beliefs simultaneously.

However, it may be desirable to maintain belief closure under conjunction. Through Fractional Semantics, it is possible to achieve this while using classical logic. To apply this approach to the Lottery Paradox, we can represent each ticket as a proposition, denoting whether or not it is the winning ticket. Let p_n represent the proposition that ticket n will win, where $1 \leq n \leq 1000$, and let p_i represent the winning ticket.

To simplify the problem, but without loss of generality, we can assume that the first and last tickets are not winning tickets and enumerate the tickets as follows:

$$p_1, \ldots, p_{i-1}, p_i, p_{i+1}, \ldots, p_{1000}$$

We can then consider the negations of these propositions, $\overline{p_1}, \ldots, \overline{p_i}, \ldots, \overline{p_{1000}}$, and represent them in a tree.

By using the Fractional Semantics expansion presented in section 4.2.1, we can assign a truth value chosen between 0 and 1 to each proposition, indicating whether the proposition is a belief or not. In this case, the value of each non-winning ticket will be 1, and the value of the winning ticket will be 0, because $\overline{p_i}$ is false, i.e., that p_i will win. This system preserves classical logic and allows us to maintain belief closure under conjunction while also resolving the paradox.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\vdash^{1}_{1\,B} \overline{p_1} \qquad \cdots \qquad \vdash^{1}_{1\,B} \overline{p_{i-1}}}{\vdash^{\text{i}-1}_{\text{i}-1\,B} \overline{p_1} \wedge \cdots \wedge \overline{p_{i-1}}} \qquad \vdash^{0}_{1\,B} \overline{p_i}}{\vdash^{\text{i}-1}_{\text{i}\,B} \overline{p_1} \wedge \cdots \wedge \overline{p_{i-1}} \wedge \overline{p_i}} \qquad \vdash^{1}_{1\,B} \overline{p_{i+1}}}{\vdash^{\text{i}}_{\text{i}+1\,B} \overline{p_1} \wedge \cdots \wedge \overline{p_{i-1}} \wedge \overline{p_i} \wedge \overline{p_{i+1}}} \qquad \cdots \qquad \vdash^{1}_{1\,B} \overline{p_{1000}}}{\vdots} }{\vdash^{999}_{1000\,B} \overline{p_1} \wedge \cdots \wedge \overline{p_{i-1}} \wedge \overline{p_i} \wedge \overline{p_{i+1}} \wedge \cdots \wedge \overline{p_{1000}}}$$

Fractional Semantics deals perfectly with this paradox, providing a very simple solution to this problem. In fact it is easy to see that in Fractional Semantics it is not useful to have a threshold, because in every moment is possible to control the value of a proposition and also the history of how that value becomes itself thanks to the proof tree. The final value will be 999/1000 and means that 999 parts of the conjunction are true out of the 1000 joints and this coheres with the fact that one ticket must be the winning one.

Fractional semantics and probability Must be stressed here that the fractional semantics value is not a probabilistic one, in fact the probability measure of the conjunction is made by the conditionalization formula as pointed out in [39]. In this formalism $\mathcal{P}$ indicates the probability, $(\mathcal{P}(\overline{p_1})|\mathcal{P}(\overline{p_2}))$ indicates the probability that the event $\overline{p_2}$ happens once $\overline{p_1}$ happened. This is called conditionalization because it is the probability that a certain event happens if another happened, *conditioning* the final probability and it is calculated by the following formula.

$$\mathcal{P}(\overline{p_1} \wedge \overline{p_2}) = \mathcal{P}(\overline{p_1}) \cdot (\mathcal{P}(\overline{p_1}) | \mathcal{P}(\overline{p_2}))$$

This means that, once $\overline{p_1}$ is realized, also $\overline{p_2}$ realizes, so:

$$\mathcal{P}(\overline{p_1} \wedge \overline{p_2}) = \frac{999}{1000} \cdot \frac{998}{999} = \frac{998}{1000}$$

The meaning of the fractional semantics value is that the conjunction is true for 999 of the joints and false for only one of them. Where the probability or classical logic assigns value 0, fractional semantics helps us to understand that, also if the final conjunction results false in classical logic, actually all but one of the propositions are true and this result is not explicit in classical framework.

5.4 Conclusions

In this chapter we have addressed the Lottery Paradox by combining two complementary perspectives: Wittgenstein's conception of probability as a conservative extension of classical logic, and the refinement introduced by Fractional Semantics. Both share the central intuition that beliefs should not be treated as merely dichotomous, but as graded and logically structured.

Wittgenstein's framework shows that the paradox dissolves once probability is understood as a relation between truth-grounds. Within this setting, the contradiction never arises: each ticket is part of a structured disjunction in which exactly one true line persists, guaranteeing that the probability of a specific ticket winning is $1/n$. The paradox therefore ceases to exist, "dissolving like a lump of sugar in water."

Fractional Semantics pushes this intuition further. By assigning fractional values to conjunctions, it clarifies why the total conjunction of all losing tickets fails: it is false not because every component is unreliable, but because exactly one conjunct must fail while all the others remain true. This explains why an agent is rational in believing that almost all tickets will lose, without falling into the contradiction of believing that no ticket will win. Unlike Lockean accounts, this solution preserves closure under conjunction and avoids reliance on arbitrary thresholds.

The significance of these results is twofold. First, they show that the Lottery Paradox is not an unavoidable feature of rational belief: within richer logical frameworks it simply disappears. Second, they highlight how logic, probability, and belief can be integrated without sacrificing consistency, while still reflecting the graded character of rational conviction.

In conclusion, the Lottery Paradox is not solved by external fixes or pragmatic concessions but dissolved within a unified framework that blends Wittgenstein's insights with the precision of Fractional Semantics. What initially appears as a threat to rationality is revealed instead as a limitation of classical binary reasoning—a limitation that can be overcome by treating belief as both probabilistic and logically structured. This opens the way for extending Fractional Semantics to other paradoxes of belief and to the broader theory of belief revision.

Chapter 6

Fractional Semantics, Belief Revision and Graded Beliefs

6.1 Introduction

The discussion of the Lottery Paradox in the previous chapter highlighted a central tension in the theory of beliefs: on the one hand, agents tend to reason in a classical manner, treating beliefs as if they were categorical; on the other hand, paradoxes arise precisely because not all beliefs can be regarded as tautologies. Fractional Semantics offered a way to preserve classical proof-theoretic structure while introducing a measure of epistemic strength.

In this chapter, we develop this idea further by distinguishing between two kinds of beliefs: those that are held with absolute firmness, and those that remain open to revision in the light of new evidence. To capture this distinction, we introduce infinitesimal values drawn from the hyperreal numbers, thereby extending Fractional Semantics into a framework capable of differentiating between *Full Beliefs* and *Revisable Beliefs*.

This move not only refines the expressive power of our logical system but also aligns it more closely with philosophical accounts of belief as inherently revisable. Building on this, we shall then consider a more general extension: the introduction of *Gradient Beliefs*, which allow for degrees of commitment beyond the dichotomy of full and revisable acceptance. The result is a richer formal apparatus for modelling belief revision, contraction, expansion, and changes of strength, while maintaining continuity with the proof-theoretic foundation established earlier.

6.2 Full Beliefs and Revisable Beliefs

The formal model of beliefs used so far is dichotomous: an all-or-nothing structure in which a belief is either fully accepted or not accepted at all. We previously claimed that, within this framework, beliefs are treated as true in the same way as tautologies. This binary picture can be refined. In this section we distinguish between *Full Beliefs* and *Revisable Beliefs*. A *Full Belief* is a belief that the agent is unwilling to discard in any situation; its value is immutable. A *Revisable Belief* is instead taken as true *for now*, but is explicitly open to retraction in light of new evidence. Following Hansson [41], we model this distinction by means of hyperreal numbers, but we slightly change the terms. *Full Beliefs* in Hansson, in fact, are what now we call Revisable Beliefs, because in his paper the aim was to distinguish between probabilistic and not probabilistic beliefs but we concern only about non probabilistic beliefs here.

6.2.1 Hyperreal Numbers: Definitions and Basic Properties

We briefly recall the basic vocabulary we use for hyperreals; see [53] for a standard introduction and [41] for their use in modeling beliefs.

Definition 6.1 (Hyperreal numbers)**.** Hyperreal numbers are an extension of the real numbers $\mathbb{R}$ in $^*\mathbb{R}$ obtained by adjoining certain classes of infinite and infinitesimal elements. A hyperreal number δ is said to be:

- *finite* iff $|\delta| < n$ for some integer n,
- *infinitesimal* iff $|\delta| < \frac{1}{n}$ for all integers n.

Practically, there exists a hyperreal number that is greater than 0 but smaller than $\frac{1}{n}$ for every $n \in \mathbb{N}$. Hyperreal numbers and real numbers share many properties: in particular, both form fields, and any algebraic expression that can be written in the field of real numbers can also be written in the field of hyperreal numbers. The mathematical difference between the two is more subtle: while $\mathbb{R}$ is unique up to isomorphism, the existence of isomorphisms between different hyperreal fields requires additional assumptions on the underlying mathematical universe. Moreover, unlike the real field, the field of hyperreals is not bounded.

In this paper, however, we will not make use of the algebraic structure of fields. What matters for our purposes is the so-called *transfer principle* between hyperreal and real numbers, which is made precise through the *standard part* mapping. Moreover, we will restrict attention to *finite* hyperreal numbers, while maintaining their essential properties.

Definition 6.2 (Standard Part)**.** Every finite hyperreal x is *infinitely close* to a unique real number $r \in \mathbb{R}$, called its *standard part*, written $\mathbf{st}(x) = r$.

Definition 6.3 (Hyperreal notation and standard part)**.** We use the following relational symbols:

$$\begin{aligned} a \approx b &\quad \text{iff } \mathbf{st}(a) = \mathbf{st}(b), \\ a \not\approx b &\quad \text{iff } \mathbf{st}(a) \neq \mathbf{st}(b), \\ a \prec b &\quad \text{iff } a < b \text{ and } \mathbf{st}(a) = \mathbf{st}(b), \\ a \ll b &\quad \text{iff } \mathbf{st}(a) < \mathbf{st}(b). \end{aligned}$$

For finite hyperreals, the standard part behaves as a real-valued homomorphism on the basic arithmetic operations [41]:

$$\begin{aligned} \mathbf{st}(-s) &= -\mathbf{st}(s), \\ \mathbf{st}(s+t) &= \mathbf{st}(s) + \mathbf{st}(t), \\ \mathbf{st}(s-t) &= \mathbf{st}(s) - \mathbf{st}(t), \\ \mathbf{st}(s \times t) &= \mathbf{st}(s) \times \mathbf{st}(t), \\ \text{if } \mathbf{st}(t) \neq 0 \text{ then } \mathbf{st}(s/t) &= \mathbf{st}(s)/\mathbf{st}(t). \end{aligned}$$

Definition 6.4 (Arithmetic)**.** Infinitesimals are stable under arithmetic: if δ_1, δ_2 are infinitesimal and s is finite and non-infinitesimal, then $\delta_1+\delta_2,\ \delta_1\delta_2,\ s\delta_1,\ \delta_1/s$ are infinitesimal.

Historically, the mathematical legitimacy and fruitfulness of Nonstandard Analysis is well documented [16, p. 474]; in particular, the Transfer Principle guarantees that ${}^*\mathbb{R}$ is a conservative extension of $\mathbb{R}$.

Anyway, we employ hyperreal numbers because they allow us to adjust the values of beliefs, expressing their inherent impossibility of ever reaching the same level of confidence as a tautology. At the same time, the internal properties of hyperreal numbers permit straightforward operations on their standard part, ensuring that all proofs valid in the realm of $\mathbb{R}$ remain valid in this extended setting ${}^*\mathbb{R}$.

6.2.2 Why Hyperreals for Beliefs?

On the Ramsey–de Finetti picture, an agent treats a proposition as true when she is disposed to accept fair bets whose stakes reflect her degree of confidence. Yet even an extremely confident agent may refuse a bet with a tiny gain and a catastrophic loss: practical certainty falls short of absolute certainty. For instance, if I am completely sure that I have put the orange juice in the fridge, but someone offers to bet one million euros against my one euro that it is indeed there, I will likely decline the offer: the potential gain is not sufficient to warrant absolute confidence.

Hyperreal valuations capture this: assigning $1-\delta$ (with δ infinitesimal) expresses "almost sure" acceptance while leaving room for revision under adverse evidence or stakes. Hyperreals bring precise mathematical leverage:

- **Full vs. Revisable Beliefs.** We set $\mathrm{val}(A) = 1$ for *Full Beliefs* and $\mathrm{val}(A) = 1 - \delta_A$ for *Revisable Beliefs*. The infinitesimal δ_A labels dependence on potentially retractable assumptions.

- **Compatibility with $\overline{\overline{GS4}}_{\mathbb{B}}$.** Since $\mathbf{st}(1-\delta) = 1$, all standard-value proofs in the base calculus survive unchanged. Thus the enriched system is a conservative extension at the level of standard parts.

- **Fine-grained weakening.** Infinitesimals compose additively/multiplicatively, providing a cumulative "fragility" measure for derivations (how many and how strongly they rely on revisable steps).

- **Bridge to probabilistic intuition.** Values $1-\delta$ behave like near-certainties without committing to a full probability semantics.

6.2.3 Sequent Decorations and Belief Axioms

To internalize the distinction, we decorate the turnstile with a hyperreal value. We replace the belief axiom

Axioms:

$$\frac{}{\vdash^{0}_{1\,\mathbb{B}} \Gamma}\ (\overline{ax.}) \qquad \frac{}{\vdash^{1}_{1\,\mathbb{B}} \Gamma, p, \neg p}\ (\text{ax.})$$

Beliefs:

$$\frac{}{\vdash^{1-\delta_{\{p_1,\ldots,p_n\}}}_{1\,\mathbb{B}}\ p_1, \ldots, p_n}\ (b_j) \qquad \text{If } \{p_1, \ldots, p_n\} \in \mathbb{B}$$

$$\frac{}{\vdash^{\delta_{\{p_1,\ldots,p_n\}}}_{1\,\mathbb{B}}\ \neg(p_1, \ldots, p_n)}\ (b_j) \qquad \text{If } \{p_1, \ldots, p_n\} \in \mathbb{B}$$

$$\frac{}{\vdash^{0}_{1\,\mathbb{B}}\ \neg(p_1, \ldots, p_n)}\ (b_j) \qquad \text{If } \{p_1, \ldots, p_n\} \notin \mathbb{B} \text{ and } \{\neg(p_1, \ldots, p_n)\} \notin \mathbb{B}$$

Rules:

$$\frac{\vdash^{m}_{n\,\mathbb{B}} \Gamma, A, B}{\vdash^{m}_{n\,\mathbb{B}} \Gamma, A \vee B}\ (R\vee) \qquad \frac{\vdash^{m_1}_{n_1\,\mathbb{B}} \Gamma, A \qquad \vdash^{m_2}_{n_2\,\mathbb{B}} \Gamma, B}{\vdash^{m_1+m_2}_{n_1+n_2\,\mathbb{B}} \Gamma, A \wedge B}\ (R\wedge)$$

Figure 6.1: The $\overline{\overline{GS4}}_{\mathbb{B}}$ sequent calculus with hyperreal decorations.

$$\frac{}{\vdash^{1}_{1\,\mathbb{B}} \Gamma, b_j}\ (b_j)$$

with

$$\frac{}{\vdash^{1-\delta_{b_j}}_{1\,\mathbb{B}} \Gamma, b_j}\ (b_j)$$

where δ_{b_j} is either 0 (if b_j is a Full Belief) or an infinitesimal (if b_j is Revisable). The δ thus functions as a trace of revisability along the proof.

This notation does not affect admissibility results such as Cut and Weakening: since $\mathbf{st}(1-\delta) = 1$, standard-part arguments go through as before [41]. What changes is the expressive power of Fractional Semantics: the infinitesimal residue records the extent to which a derivation depends on revisable premises.

Example 6.1 (Conjunction).
$$\frac{\vdash^{1-\delta_p}_{1\,\mathbb{B}}\ p \qquad \vdash^{1-\delta_q}_{1\,\mathbb{B}}\ q}{\vdash^{2-(\delta_p+\delta_q)}_{2\,\mathbb{B}}\ p \wedge q}\ (\wedge)$$

Here the standard value remains 2 (so, under Fractional Semantics, the truth-value is 1), while the infinitesimal residue $\delta_p + \delta_q$ records that both leaves rely on revisable beliefs. If later evidence forces revision, the fractional value may decrease accordingly. Intuitively, the cumulative δ measures the *fragility* of the derivation.

Remark. The historical legitimacy of using R^* alongside R, and its conservativity via Transfer, is well noted in [16]. This supports our use of hyperreal decorations: proof-theoretically conservative at the level of standard parts, yet semantically more expressive in tracking revisability.

6.2.4 Decomposition of a Revisable Belief

To decompose a belief, the rules remain the same as before; we must decompose it and close under cut. Suppose we aim to incorporate the belief $\vdash q \wedge (r \vee \neg s)$ into the system, but it is not a full belief. To achieve this, we need to decompose it:

$$\dfrac{\vdash q \qquad \dfrac{\vdash r, \neg s}{\vdash r \vee \neg s}\,(\vee)}{\vdash q \wedge (r \vee \neg s)}\,(\wedge)$$

Now, to indicate that the original belief $\vdash q \wedge (r \vee \neg s)$ was neither a Full Belief nor a Tautology, we adjust its value by adding the number $1 - \delta$ instead of 1. This adjustment accounts for the infinitesimal nature of δ, and its division by 2 ensures the preservation of infinitesimal characteristics.

$$\dfrac{\vdash^{\frac{1-\delta_q}{1}}_{\mathbb{B}} q \qquad \dfrac{\vdash^{\frac{1-\delta_{\{r,\neg s\}}}{1}}_{\mathbb{B}} r, \neg s}{\vdash^{\frac{1-\delta_{\{r,\neg s\}}}{1}}_{\mathbb{B}} r \vee \neg s}\,(\vee)}{\vdash^{\frac{1-(\delta_q+\delta_{\{r,\neg s\}})}{2}}_{\mathbb{B}} q \wedge (r \vee \neg s)}\,(\wedge)$$

In the event that either $\vdash q$ or $\vdash r, \neg s$ is employed in a derivation, we explicitly denote this value in the sequent derivation. For instance:

$$\dfrac{\dfrac{\dfrac{\vdash^{\frac{1}{1}}_{\mathbb{B}} p, \neg p}{\vdash^{\frac{1}{1}}_{\mathbb{B}} p \vee \neg p}\,(\vee) \qquad \vdash^{\frac{1-\delta_q}{1}}_{\mathbb{B}} q}{\vdash^{\frac{2-\delta_q}{2}}_{\mathbb{B}} (p \vee \neg p) \wedge q}\,(\wedge) \qquad \dfrac{\vdash^{\frac{0}{1}}_{\mathbb{B}} \neg p, q}{\vdash^{\frac{0}{1}}_{\mathbb{B}} \neg p \vee q}\,(\vee)}{\vdash^{\frac{2-\delta_q}{3}}_{\mathbb{B}} (p \vee \neg p) \wedge q \wedge (\neg p \vee q)}\,(\wedge)$$

This implies that, even without knowing the initial values of the leaves, we can still make observations about the value $(2 - \delta_q)/3$: the standard part of the derivation is the Fractional Semantics value in $GS4_{\mathbb{B}}$ is $2/3$ and we can observe that there is only one infinitesimal number, indicating that only one of the initial beliefs is a Revisable Belief, in particular the proposition q. The portion that is neither a Full Belief nor a Revisable Belief is then $1/3$, representing what remains between $2/3$ and 1.

Example 6.2. Suppose an agent believes that their smartphone is either on the table or in the backpack. We formalize:

$$\dfrac{\left|\dfrac{1-\delta_{\{p,q\}}}{1}\right._{\mathbb{B}}\; p, q}{\left|\dfrac{1-\delta_{\{p,q\}}}{1}\right._{\mathbb{B}}\; p \vee q}\;(\vee)$$

This means the agent believes (almost) that the phone is either on the table or in the backpack. If the agent conjoins a tautology, the fractional value increases, while the normalized truth component stays the same:

$$\dfrac{\dfrac{\left|\dfrac{1-\delta_{\{p,q\}}}{1}\right._{\mathbb{B}}\; p, q}{\left|\dfrac{1-\delta_{\{p,q\}}}{1}\right._{\mathbb{B}}\; p \vee q} \qquad \dfrac{\left|\dfrac{1}{1}\right._{\mathbb{B}}\; r, \neg r}{\left|\dfrac{1}{1}\right._{\mathbb{B}}\; r \vee \neg r}}{\left|\dfrac{2-\delta_{\{p,q\}}}{2}\right._{\mathbb{B}}\; (p \vee q) \wedge (r \vee \neg r)}\;(\wedge)$$

In particular,

$$\frac{2 - \delta_{\{p,q\}}}{2} = 1 - \frac{\delta_{\{p,q\}}}{2} > \frac{1 - \delta_{\{p,q\}}}{1} = 1 - \delta_{p,q} \quad \text{for } \delta_{\{p,q\}} > 0.$$

Thus, adding a tautology raises the value (the hyperreal term halves from $\delta_{\{p,q\}}$ to $\delta_{\{p,q\}}/2$), while the normalized truth component remains 1.

6.3 Belief Revision in Fractional Semantics

Belief revision represents a crucial test case for any logic of credences. In classical AGM theory, the dynamics of beliefs are described by operations of contraction, expansion, and revision, but the revision process acts only on the content of the belief set. The logical status of a belief—whether it is categorical or defeasible—remains external to the formalism.

In the framework presented here, by contrast, belief change is internal to the semantics itself. Each belief carries a fractional decoration that makes explicit whether it is a Full Belief, a Revisable Belief, or a gradient commitment. Revision operations therefore do not simply update the set of sentences an agent accepts, but directly affect the fractional value of sequents. This way, the system provides a transparent account of which beliefs can be revised, which cannot, and how the revision reshapes the outcome of a derivation.

This feature marks an essential difference with respect to AGM: while AGM operates at the level of sets, our approach embeds revision into the semantics, so that the proof-theoretic value itself keeps track of epistemic dynamics.

To make this idea concrete, we now refine the semantics with hyperreal numbers, which allow us to mark revisability without changing the standard part of the valuation.

Hyperreal numbers provide the technical machinery to realize this distinction. Their role is twofold: on one hand, it is easy to distinguish between Revisable and Full Beliefs (i.e. tautologies); on the other hand, hyperreal numbers do not alter the fractional final value, thereby validating all the proofs that we have made for $GS4_{\mathbb{B}}$ also for this settlement. In Figure 6.1 we have stated only one Axiom for beliefs, but from a philosophical point of view is mandatory to distinguish between the following:

- Tautology: $\vdash^{1}_{1} p, \neg p$.
- Belief in p: $p \in \mathbb{B}$, $\vdash^{1-\delta_p}_{1\;\mathbb{B}} p$ and $\vdash^{\delta_p}_{1\;\mathbb{B}} \neg p$.
- Disbelief in p: $\neg p \in \mathbb{B}$, $\vdash^{1-\delta_{\neg p}}_{1\;\mathbb{B}} \neg p$ and $\vdash^{\delta_{\neg p}}_{1\;\mathbb{B}} p$.
- Unsettledness about p if $p \notin \mathbb{B}$ and $\neg p \notin \mathbb{B}$: $\vdash^{0}_{1\;\mathbb{B}} p$.

The unsettledness about p is what in Figure 6.1 was the complementary axiom labelled as ($\overline{ax.}$) The rationale behind the values assigned to p and $\neg p$ is that if a belief p belongs to the set $\mathbb{B}$, then its negation is assigned a hyperreal value, reflecting the possibility that the belief may be revised in the future.

6.3.1 Operations

It is possible to define a relation between p, $\neg p$, δ_p, and $\delta_{\neg p}$ for atomic formulas as follows:

$$\vdash^{1-\delta_p}_{1\;\mathbb{B}} p \quad \longleftrightarrow \quad \vdash^{\delta_{\neg p}}_{1\;\mathbb{B}} \neg p$$

$$\vdash^{1-\delta_{\neg p}}_{1\;\mathbb{B}} \neg p \quad \longleftrightarrow \quad \vdash^{\delta_p}_{1\;\mathbb{B}} p$$

This equivalence follows directly from the hyperreal interpretation, where each infinitesimal δ tracks the revisability of a proposition. However, this neat correspondence becomes more subtle when dealing with compound beliefs.

To generalize this reasoning, we can extend it to sequences such as:

$$\frac{{\vdash^{1-\delta_{\{p_1,p_2,\ldots,p_n\}}}_{1}}_{\mathbb{B}}\; p_1, p_2, \ldots, p_n}{{\vdash^{1-\delta_{\{p_1,p_2,\ldots,p_n\}}}_{1}}_{\mathbb{B}}\; p_1 \vee p_2 \vee \cdots \vee p_n}$$

Yet this introduces a complication. According to our assumptions, a belief has two sides: if an agent believes p to be true, she also believes $\neg p$ to be false. However, the negation of a disjunction becomes a conjunction, which must be decomposed into atomic components. For instance, in $GS4$ we have:

$$\vdash p, q \quad \text{is equivalent to} \quad \vdash p \vee q$$

Therefore, the negation of such a disjunction becomes:

$$\neg(p \vee q) \equiv \neg p \wedge \neg q$$

This entails that the negation of a disjunction cannot be uniformly assigned a single δ value; instead, it must be decomposed into conjunctive branches, each carrying its own infinitesimal.

Example 6.3. To better illustrate the point, consider the following example: the agent knows that she has passed the exam, and that the condition for passing was to answer correctly either question 1 or question 2 — formally, $\vdash^{1-\delta_{\{p_1,p_2\}}}_{1} p_1, p_2$. Given this knowledge, she also knows that she would not have passed the exam had she answered both questions incorrectly. That is, she recognizes the falsity of the proposition $\neg p_1 \wedge \neg p_2$:

$$\vdash^{\delta_{\neg p_1}+\delta_{\neg p_2}}_{2} \neg p_1 \wedge \neg p_2$$

and decomposing it:

$$\frac{\vdash^{\delta_{\neg p_1}}_{1} \neg p_1 \qquad \vdash^{\delta_{\neg p_2}}_{1} \neg p_2}{\vdash^{\delta_{\neg p_1}+\delta_{\neg p_2}}_{2} \neg p_1 \wedge \neg p_2}$$

To preserve a meaningful relationship between a formula and its negation, we proceed as follows. If:

$$\vdash^{\frac{n-(\delta_{p_1}+\cdots+\delta_{p_n})}{n}}_{\mathbb{B}} p_1 \vee \cdots \vee p_n$$

then its negation:

$$\neg(p_1 \vee \cdots \vee p_n) \equiv \neg p_1 \wedge \cdots \wedge \neg p_n$$

is assigned the value:

$$\vdash^{\frac{\delta_{\neg p_1}+\cdots+\delta_{\neg p_n}}{n}}_{\mathbb{B}} \neg p_1 \wedge \cdots \wedge \neg p_n$$

notice that the new assigned values are negative, since the considered atoms are now negative. This can be visualized in the following proof tree:

$$\dfrac{\dfrac{\vdash^{\frac{\delta_{\neg p_1}}{1}}_{\mathbb{B}} \neg p_1 \qquad \vdash^{\frac{\delta_{\neg p_2}}{1}}_{\mathbb{B}} \neg p_2}{\vdash^{\frac{\delta_{\neg p_1}+\delta_{\neg p_2}}{2}}_{\mathbb{B}} \neg p_1 \wedge \neg p_2}(\wedge) \qquad \cdots \qquad \vdash^{\frac{\delta_{\neg p_n}}{1}}_{\mathbb{B}} \neg p_n}{\dfrac{\vdots}{\vdash^{\frac{\delta_{\neg p_1}+\cdots+\delta_{\neg p_n}}{n}}_{\mathbb{B}} \neg p_1 \wedge \cdots \wedge \neg p_n}}(\wedge)$$

This construction ensures that the reversal of polarity (i.e., switching from a disjunction to the conjunction of its negation) preserves the total weight of the infinitesimal components, thus maintaining a consistent semantic interpretation within the fractional framework.

On the other hand, when working only on the positive side, it is often sufficient to write:

$$\vdash^{\frac{1-\delta_{\Delta}}{1}}_{\mathbb{B}} \Delta$$

since all the δ_i are infinitesimal. There is no appreciable difference in the standard (real) part between:

$$\frac{n-(\delta_{p_1}+\cdots+\delta_{p_n})}{n} \quad \text{and} \quad \frac{1-\delta_{\Delta}}{1}$$

because they have the same standard value:

$$\mathbf{st}\left(\frac{n-(\delta_{p_1}+\cdots+\delta_{p_n})}{n}\right) \quad = \quad \mathbf{st}\left(\frac{1-\delta_{\Delta}}{1}\right)$$

Definition 6.5. If an agent holds the primitive belief $\Delta = p_1, p_2, \ldots, p_n$, i.e., $\vdash \Delta$, its semantic value is $1 - \delta_\Delta$, where δ_Δ is an infinitesimal representing the revisability of the entire belief. This is represented as:

$$\left|\frac{1-\delta_\Delta}{1}\right._{\mathbb{B}} \Delta$$

By the rules of $GS4_{\mathbb{B}}$, only disjunctions may appear on the right-hand side of the sequent, otherwise, it can be decomposed. The agent is also committed to the falsity of $\neg\Delta$, i.e., $\vdash \neg\Delta$. Since the negation of a disjunction is a conjunction, the value of the negated belief is computed as:

$$\left|\frac{n-(\delta_{p_1}+\delta_{p_2}+\cdots+\delta_{p_n})}{n}\right._{\mathbb{B}} \neg p_1 \wedge \neg p_2 \wedge \cdots \wedge \neg p_n$$

This relationship can be formalized as:

$$\left|\frac{1-\delta_\Delta}{1}\right._{\mathbb{B}} \Delta \quad\longleftrightarrow\quad \cfrac{\cfrac{\cfrac{\left|\frac{\delta_{\neg p_1}}{1}\right._{\mathbb{B}} \neg p_1 \qquad \left|\frac{\delta_{\neg p_2}}{1}\right._{\mathbb{B}} \neg p_2}{\left|\frac{\delta_{\neg p_1}+\delta_{\neg p_2}}{2}\right._{\mathbb{B}} \neg p_1 \wedge \neg p_2}\,(\wedge) \qquad \cdots \qquad \left|\frac{\delta_{\neg p_n}}{1}\right._{\mathbb{B}} \neg p_n}{\vdots}\,(\wedge)}{\left|\frac{\delta_{\neg p_1}+\cdots+\delta_{\neg p_n}}{n}\right._{\mathbb{B}} \neg p_1 \wedge \cdots \wedge \neg p_n}$$

Intuitively, this means that if an agent believes the disjunctive proposition $\Delta = p_1, p_2, \ldots, p_n$ to be true, she is also committed to the falsity of its negation $\neg\Delta$. Moreover, she considers each individual component $\neg p_i$ to be likely false.

As additional propositions are added to Δ, the likelihood that some part of $\neg\Delta$ fails increases. This is reflected in the additive behavior of the infinitesimal values:

$$\frac{\delta_{p_1}}{1} \leq \frac{\delta_{p_1} + \delta_{p_2}}{2} \leq \cdots \leq \frac{\delta_{p_1} + \cdots + \delta_{p_n}}{n}$$

This cumulative pattern captures the idea that as a belief becomes more complex, its negation becomes epistemically weaker: the more conjuncts in $\neg\Delta$, the more plausible it is that at least one of them is false, thereby undermining the reliability of the entire conjunction.

6.3.2 Belief Change Operations

To formally describe belief revision within the framework of Fractional Semantics, we introduce the following notion:

Definition 6.6 (Generalisation of a Proposition)**.** The generalisation of a proposition in Fractional Semantics with Hyperreal Numbers is expressed as:

$$\left|\frac{k-x\delta}{n}\right._{\mathbb{B}} \Delta$$

where:

Δ a multiset of formulas containing Revisable Beliefs, Tautologies, etc.;

k the real part of the Fractional Semantics value;

x the number of true Revisable Beliefs in Δ;

δ a hyperreal number associated with one proposition in Δ;

n the total number of propositions in Δ.

This definition provides a generalized form of a sequent that can encompass different cases. Intuitively, the formula represents an ordinary sequent equipped with a fractional semantics value composed of three components: k, which denotes the standard fractional semantics value; $x\delta$, which stands for the cumulative contribution of revisable beliefs (obtained by simplifying multiple applications of δ); and n, the number of leaves in a given proof π. Since, by Definition 6.3, $x\delta$ is still a hyperreal number, the overall fractional semantics value remains unchanged, i.e.,

$$\mathbf{st}\left(\frac{k-x\delta}{n}\right) = \frac{k}{n}.$$

This kind of generalisation could be seen as imprecise, but it is more convenient than the following, which—although syntactically more accurate in our system—results in:

$$\left|\frac{k-(\delta_{\Delta_1}+\cdots+\delta_{\Delta_n})}{n}\right._{\mathbb{B}} \Delta_1 \wedge \cdots \wedge \Delta_n$$

where each Δ_i is a proposition composed solely of applications of $\vee$. $\left|\frac{k-x\delta}{n}\right._{\mathbb{B}} \Delta$ is unprecise because of the relationship between conjunctions and disjunctions introduced in paragraph 6.3.1. This follows from the construction of $GS4_{\mathbb{B}}$.

Definition 6.7 (Contraction)**.** Let $\mathbb{B} - p$ denote the *contraction* of $\mathbb{B}$ by p. If p is a Revisable Belief, i.e., $\left|\frac{1-\delta_p}{1}\right._{\mathbb{B}} p$, then p must become an unsettled belief:

$$\left|\frac{0}{1}\right._{\mathbb{B}-p} p.$$

More generally, assume $\left|\frac{k-x\delta}{n}\right._{\mathbb{B}} \Delta$ with $p \in \Delta$, $1 \leq k \leq n$, $x \leq k$. Then:

$$\left|\frac{k-x\delta-(1-\delta_p)}{n}\right._{\mathbb{B}-p} \Delta - \{p\}.$$

The reason why k must be greater than 1 is that at least one p is a Revisable Belief in Δ, without that, it is not possible to make any contraction to Δ because it would have been only a multiset made by tautologies.

Example 6.4. Suppose that the belief set $\mathbb{B} = \{p, q\}$ and we want to evaluate the proposition $\vdash p \wedge q$. Firstly we have to decompose it and assign the Fractional Semantics values:

$$\frac{\left|\frac{1-\delta_p}{1}\right._{\mathbb{B}} p \qquad \left|\frac{1-\delta_q}{1}\right._{\mathbb{B}} q}{\left|\frac{2-\delta_p-\delta_q}{2}\right._{\mathbb{B}} p \wedge q} (\wedge)$$

By the generalized Definition in 6.17, contracting $\mathbb{B}$ with p we have:

$$\left|\frac{2-\delta_p-\delta_q-(1-\delta_p)}{2}\right._{\mathbb{B}-p} p \wedge q \; = \left|\frac{1-\delta_q}{2}\right._{\mathbb{B}-p} p \wedge q$$

But we can obtain the same result changing the values at the level of the leaves of the tree:

$$\frac{\left|\frac{0}{1}\right._{\mathbb{B}-p} p \qquad \left|\frac{1-\delta_q}{1}\right._{\mathbb{B}-p} q}{\left|\frac{1-\delta_q}{2}\right._{\mathbb{B}-p} p \wedge q} (\wedge)$$

Obtaining the same result.

Definition 6.8 (Expansion)**.** Let $\mathbb{B} + p$ denote the *expansion* of $\mathbb{B}$ and p unsettled in $\mathbb{B}$, i.e., $\left|\frac{0}{1}\right._{\mathbb{B}} \neg p$. Then:

$$\left|\frac{1-\delta_p}{1}\right._{\mathbb{B}+p} p.$$

More generally, if $\left|\frac{k-x\delta}{n}\right._{\mathbb{B}} \Delta$ and p is unsettled in Δ, then:

$$\left|\frac{k-x\delta+(1-\delta_p)}{n}\right._{\mathbb{B}+p} \Delta + \{p\}.$$

Example 6.5. Suppose that my girlfriend knows it is raining outside because of the skylight, so the belief set is $\mathbb{B} = q$. She wants to evaluate the proposition "It is raining outside and the umbrella is in the car", but she does not know where the umbrella actually is. In this case, we need to evaluate $\vdash p \wedge q$. First, we decompose it and assign the Fractional Semantics values:

$$\dfrac{\left|\frac{0}{1}\right._{\mathbb{B}} p \qquad \left|\frac{1-\delta_q}{1}\right._{\mathbb{B}} q}{\left|\frac{1-\delta_q}{2}\right._{\mathbb{B}} p \wedge q}\ (\wedge)$$

Now suppose I call her and tell her that the umbrella is in the car. By the generalized Definition in 6.18, we expect that, after expanding $\mathbb{B}$ with p, we obtain.

$$\left|\frac{1-\delta_q+(1-\delta_p)}{2}\right._{\mathbb{B}+p} p \wedge q \;= \left|\frac{2-\delta_q}{2}\right._{\mathbb{B}+p} p \wedge q$$

However, we can obtain the same result by changing the values at the level of the leaves of the tree:

$$\dfrac{\left|\frac{1-\delta_p}{1}\right._{\mathbb{B}+p} p \qquad \left|\frac{1-\delta_q}{1}\right._{\mathbb{B}+p} q}{\left|\frac{2-\delta_p-\delta_q}{2}\right._{\mathbb{B}+p} p \wedge q}\ (\wedge)$$

Obtaining the same result.

Definition 6.9 (Revision)**.** Let $\mathbb{B} * p$ denote the *revision* of $\mathbb{B}$ by p, where $\neg p \in \mathbb{B}$ and $p \notin B$, i.e., $\left|\frac{1-\delta_{\neg p}}{1}\right._{\mathbb{B}} \neg p$. Then:

$$\left|\frac{1-\delta_p}{1}\right._{\mathbb{B}*p} p \quad \text{and} \quad \left|\frac{0}{1}\right._{\mathbb{B}*p} \neg p.$$

More generally, if $\left|\frac{k-x\delta-\delta_{\neg p}}{n}\right._{\mathbb{B}} \Delta, \neg p$ then:

$$\left|\frac{k-x\delta-\delta_p}{n}\right._{\mathbb{B}*p} \Delta * \{p\}.$$

Remark 6.1. *It is important to notice that in Definition 6.19 the value of $\delta_{\neg p}$ is changed into δ_p because of the fact that the new considered proposition, i.e. the revisable one, is p and not $\neg p$ anymore.*

We recall here the notion of Levi Identity introduced in Section 1.3.1:t

Definition 6.10 (Levi Identity)**.** Revision satisfies the Levi identity:

$$\mathbb{B} * p = (\mathbb{B} - \neg p) + p.$$

Proposition 6.1. *Fractional Semantics is consistent with Levi Identity:*

$$\left|\frac{k-x\delta}{n}\right._{\mathbb{B}*p} \Delta * \{p\}.$$

this is the same as the operations of contraction and expansion. Starting from contraction we have that:

$$\vdash^{\frac{k-x\delta-(1-\delta_p)}{n}}_{\mathbb{B}-\neg p} \Delta - \{\neg p\}$$

By definition 6.18, since $\neg p \notin \Delta$ by contraction, the expansion is as follows:

$$\vdash^{\frac{k-x\delta-(1-\delta_p)+(1-\delta_p)}{n}}_{\mathbb{B}-\neg p+p} (\Delta - \{\neg p\}) + \{p\} \;= \vdash^{\frac{k-x\delta}{n}}_{\mathbb{B}-\neg p+p} (\Delta - \{\neg p\}) + \{p\}$$

Then the Fractional Semantics value is $k - x\delta$ as expected and by Definition 6.10 $\mathbb{B} * p = (\mathbb{B} - \neg p) + p$, so:

$$\vdash^{\frac{k-x\delta}{n}}_{\mathbb{B}*p} \Delta * \{p\}$$

Example 6.6. Suppose that the belief set $\mathbb{B} = \{p, q\}$ and we want to evaluate the proposition $\vdash p \wedge q$. Firstly we have to decompose it and assign the Fractional Semantics values:

$$\frac{\vdash^{\frac{1-\delta_p}{1}}_{\mathbb{B}} p \qquad \vdash^{\frac{1-\delta_q}{1}}_{\mathbb{B}} q}{\vdash^{\frac{2-\delta_p-\delta_q}{2}}_{\mathbb{B}} p \wedge q} (\wedge)$$

By the generalized Definition in 6.19, revise $\mathbb{B}$ by p will be the same as the contraction:

$$\vdash^{\frac{2-\delta_p-\delta_q-(1-\delta_p)}{2}}_{B*\neg p} p \wedge q \;= \vdash^{\frac{1-\delta_q}{2}}_{B*\neg p} p \wedge q$$

Or, at the level of the leaves:

$$\frac{\vdash^{\frac{\delta p}{1}}_{\mathbb{B}*\neg p} p \qquad \vdash^{\frac{1-\delta_q}{1}}_{\mathbb{B}*\neg p} q}{\vdash^{\frac{1-\delta_q+\delta_p}{2}}_{\mathbb{B}*\neg p} p \wedge q} (\wedge)$$

On the other hand, since this is a Revision we can consider another example: the evaluation of the proposition $\vdash \neg p \wedge q$.

$$\frac{\vdash^{\frac{0}{1}}_{\mathbb{B}} \neg p \qquad \vdash^{\frac{1-\delta_q}{1}}_{\mathbb{B}} q}{\vdash^{\frac{1-\delta_q}{2}}_{\mathbb{B}} \neg p \wedge q} (\wedge)$$

Revising $\mathbb{B}$ by p we obtain:

$$\frac{\left|\frac{1-\delta_{\neg p}}{1}\right._{B*\neg p} \neg p \qquad \left|\frac{1-\delta_{q}}{1}\right._{B*\neg p} q}{\left|\frac{2-\delta_{\neg p}-\delta_{q}}{2}\right._{B*\neg p} \neg p \wedge q} (\wedge)$$

This is the same with the double operation of Contraction and Expansion. The Contraction doesn't change the Fractional Semantics value, since p doesn't appear between the atomic formulas of $\vdash p \wedge q$.

$$\left|\frac{1-\delta_{q}}{2}\right._{\mathbb{B}} \neg p \wedge q \; = \left|\frac{1-\delta_{q}}{2}\right._{\mathbb{B}-p} \neg p \wedge q \; = \left|\frac{1-\delta_{q}+(1-\delta_{\neg p})}{2}\right._{\mathbb{B}} \neg p \wedge q$$
$$= \left|\frac{2-\delta_{\neg p}-\delta_{q}}{2}\right._{\mathbb{B}*\neg p} \neg p \wedge q$$

It is interesting to notice that in a conjunction, if we change one conjunctive with its opposite, the Fractional Semantics Value changes accordingly to the rules presented before. In fact:

$$\left|\frac{2-\delta_{\neg p}-\delta_{q}}{2}\right._{\mathbb{B}*\neg p} \neg p \wedge q \longleftrightarrow \left|\frac{1-\delta_{q}+\delta_{p}}{2}\right._{\mathbb{B}*\neg p} p \wedge q$$

Generalizing:

$$\left|\frac{k-x\delta-\delta_{p_i}}{n}\right._{\mathbb{B}} p_1 \wedge \cdots \wedge p_i \wedge \cdots \wedge p_n \longleftrightarrow \left|\frac{k-x\delta-(1-\delta_{\neg p_i})}{n}\right. p_1 \wedge \cdots \wedge \neg p_i \wedge \cdots \wedge p_n$$

The operations presented so far describe how belief sets can evolve through contraction, expansion, and revision. However, these mechanisms still rely on a binary distinction between beliefs that are fully accepted and those that are rejected. In reality, reasoning often involves intermediate degrees of conviction. To capture this more nuanced epistemic landscape, we now introduce the notion of *Gradient Beliefs*.

6.4 Gradient Beliefs

Which are the limits of starting with assumptions that must be either true or false? Let's consider the following example:

Example 6.7. Suppose that my girlfriend believes that outside is raining because she is in her bedroom and she has a skylight, so that the sound of the rain is evident. She has to go to the grocery store and she starts to wonder where her umbrella is. She supposes that it is in the car, but she is not sure that the umbrella is in the car. In $GS4_{\mathbb{B}}$ we have to formalize such an assumption as following: $\mathbb{B} = p$ where p is "outside is raining", but we can't have q in the set of beliefs, where q is "the umbrella is in the car" because my girlfriend is not sure about it. This means that the tree will be as following:

$$\frac{\vdash^{1-\delta_p}_{1} p \qquad \vdash^{0}_{1} q}{\vdash^{1-\delta_p}_{2} p \wedge q}\,(\wedge)$$

The result means that the conjunction "outside is raining and the umbrella is in the car" will be sure only $1 - \delta_p/2$, even if my girlfriend had some clues about the umbrella being in the car. To face this problem we need to extend $GS4_{\mathbb{B}}$ to $GS4_{\mathbb{B},\mathbb{G}}$.

In extending $GS4_{\mathbb{B}}$ to better handle nuanced epistemic states, we now introduce the notion of *gradient beliefs*. These allow an agent to commit to a proposition with a degree of confidence, rather than fully accepting or rejecting it. This provides a continuum between belief and disbelief, offering a richer framework for modeling uncertainty.

Definition 6.11 (Gradient beliefs)**.** A *gradient belief* is a doxastic commitment to a proposition A with a degree of confidence in the closed interval $[0, 1]$. This degree is interpreted as the agent's epistemic strength or confidence in the truth of A. Let $\mathbb{G} = \{g_1, g_2, \ldots, g_n\}$ be the set of such gradient beliefs, where each g_i is a formula paired with a real-valued degree $v^{g_i} \in [0, 1]$ expressing the strength of belief in g_i.

Gradient beliefs thus serve to generalize full beliefs ($v^{g_i} = 1$) and revisable beliefs (which we may now treat as those for which $v^{g_i} = 1 - \delta$ for infinitesimal δ). This prepares the system to accommodate a more fine-grained spectrum of attitudes toward propositions.

Definition 6.12 (Gradient belief values v^{g_i}). Each gradient belief $g_i \in \mathbb{G}$ is associated with a rational value $v^{g_i} \in [0,1]$, representing its strength. This value may be interpreted semantically as the degree to which the agent accepts g_i as true.

The system we now define, $GS4_{\mathbb{B},\mathbb{G}}$, extends the base system $GS4_{\mathbb{B}}$ by adding support for these gradient judgments.

Definition 6.13 (System $GS4_{\mathbb{B},\mathbb{G}}$). The system $\mathbf{GS4}_{\mathbb{B},\mathbb{G}}$ extends $GS4_{\mathbb{B}}$ by including gradient belief axioms $g_1, g_2, \ldots, g_n \in G$, each associated with a strength value v^{g_i}. The logic thus has:

- full and revisable beliefs (as in $GS4_{\mathbb{B}}$),
- graded beliefs represented by rational values in $[0, 1]$.

To account for the presence of gradient beliefs at the top of a proof, we extend the notion of top sequents.

Definition 6.14 (Top gradient sequents). Let $\#top^{\mathbb{G}}(\pi)$ denote the number of top-level sequents in a derivation π that originate from gradient belief axioms, i.e., *cardinalities.* These are added to the other categories of top sequents. The following is the complete list:

- $\#top^{\mathbb{G}}(\pi)$: gradient beliefs;
- $\#top^{b}(\pi)$: revisable beliefs;
- $\#top^{1}(\pi)$: full beliefs;
- $\#top^{0}(\pi)$: neutral or non-committed premises.

We now update the semantic value of a formula in this expanded context to reflect the contribution of gradient beliefs.

Definition 6.15 (Fractional semantics in $GS4_{\mathbb{B},\mathbb{G}}$). The semantic value of a formula A under $GS4_{\mathbb{B},\mathbb{G}}$ is given by:

$$[\![A]\!]_{\mathbb{B},\mathbb{G}} = \frac{v^{g_1}(\pi) + v^{g_2}(\pi) + \cdots + v^{g_n}(\pi) + \#top^{b}(\pi) + \#top^{1}(\pi)}{\#top^{\mathbb{G}}(\pi) + \#top^{b}(\pi) + \#top^{1}(\pi) + \#top^{0}(\pi)}$$

where:

- $v^{g_i}(\pi)$ is the weight contributed by each gradient belief at the top level of π;
- $\#top^b(\pi)$, $\#top^1(\pi)$, $\#top^0(\pi)$ are cardinalities of top-level sequents from revisable, full, and neutral beliefs respectively.

This value reflects the overall epistemic support for A as derived from both graded and categorical beliefs.

To incorporate gradient beliefs into proofs, we extend the axiom system with the following rule:

Definition 6.16 (Gradient belief axiom rule)**.** Axioms for gradient beliefs are introduced in $GS4_{\mathbb{B},\mathbb{G}}$ via:

$$\frac{}{\vdash^{v^{g_i}-\gamma_{g_i}}_{1}{}_{\mathbb{B},\mathbb{G}}\ g_i}\ (g_i)$$

Here:

- γ_{g_i} is a (possibly negative) hyperreal number that indicates that g_i is a belief, i.e., g_i can be revised. γ has the same meaning of δ in the previous sections, but to distinguish them we decided to differantiate them.
- $v^{g_i} - \gamma_{g_i}$ represents the effective belief strength under current epistemic conditions;
- the fractional semantics remains well-defined, analogously to how we treated infinitesimal variation in $GS4_{\mathbb{B}}$.

Remark 6.2. *The definition 6.15 remains unchanged when interpreted over hyperreal numbers. This is due to the fact that infinitesimals, by construction, do not affect the standard part of a hyperreal number. That is, for any expression of the form $z - \gamma$, with $z \in \mathcal{Q}$, where δ is infinitesimal, we have:*

$$\mathbf{st}(z - \gamma) = z$$

and therefore all reasoning and derivations based on Definition 6.15 continue to hold.

Example 6.8. Let's consider an example: an agent believes that her belief g_1 has value $v^{g_1} = 0.8$ and her belief g_2 has value $v^{g_2} = 0.7$. If we consider the proposition $\vdash g_1 \wedge g_2$ we obtain the following derivation:

$$\dfrac{\overset{0.8-\gamma_{g_1}}{\underset{1}{\vdash}}\, g_1 \qquad \overset{0.7-\gamma_{g_2}}{\underset{1}{\vdash}}\, g_2}{\overset{0.8+0.7-(\gamma_{g_1}+\gamma_{g_2})}{\underset{2}{\vdash}}\, g_1 \wedge g_2}\ (\wedge_R)$$

According to the conjunction rule, the numerator is the sum of the values of the premises, while the denominator is the sum of the number of premises. Hence the final Fractional Value is

$$[\![A]\!]_{\mathbb{B},\mathbb{G}} = \frac{1.5 - (\gamma_{g_1} + \gamma_{g_2})}{2}.$$

If we ignore the infinitesimal parts $\gamma_{g_1}, \gamma_{g_2}$, the real part is $[\![A]\!]_{\mathbb{B},\mathbb{G}} = \frac{1.5}{2}$. This could be simplified arithmetically as $\frac{3/2}{2} = \frac{3}{4}$. However, such a simplification would hide one of the main features of Fractional Semantics: the denominator explicitly keeps track of the number of propositions involved in the derivation. Thus, writing $\frac{1.5}{2}$ is more informative than $\frac{3}{4}$, since it preserves the structural information carried by the proof tree.

Example 6.9 (Combination of Gradient Beliefs)**.** Let's consider again Example 6.7. In $GS4_{\mathbb{B},\mathbb{G}}$ we can formalize as follows: $\mathbb{B} = p$, where p is "outside is raining", and the proposition "the umbrella is in the car" has a confidence value of 0.7, i.e. $q = 0.7$. This yields the following derivation:

$$\dfrac{\overset{1-\delta_p}{\underset{1}{\vdash}}{}_{\mathbb{B},\mathbb{G}}\, p \qquad \overset{0.7-\gamma_q}{\underset{1}{\vdash}}{}_{\mathbb{B},\mathbb{G}}\, q}{\overset{1.7-(\delta_p+\gamma_q)}{\underset{2}{\vdash}}{}_{\mathbb{B},\mathbb{G}}\, p \wedge q}\ (\wedge)$$

The result indicates that the conjunction "outside is raining and the umbrella is in the car" has a total confidence of

$$\mathbf{st}\frac{1.7 - (\delta_p + \gamma_q)}{2} \approx 0.85,$$

a value higher than 0.5 and closer to truth, thereby providing a more faithful approximation of the agent's epistemic state.

6.5 Belief Change for Gradient Beliefs

The same Belief Change operations can be done in the new setting as following:

Definition 6.17 (Contraction)**.** Let $G - p$ denote the *contraction* of G by p. If p is a Gradient Belief, i.e.,

$$\overset{v^p-\delta_p}{\underset{1}{\vdash}}{}_G\, p,$$

Axioms:

$$\frac{}{\mid\!\frac{0}{1}{}_{\mathbb{B},\mathbb{G}}\ \Gamma}\ (\overline{ax.}) \qquad \frac{}{\mid\!\frac{1}{1}{}_{\mathbb{B},\mathbb{G}}\ \Gamma, p, \neg p}\ (\text{ax.})$$

Beliefs:

$$\frac{}{\mid\!\frac{1-\delta_{\{p_1,\ldots,p_n\}}}{1}{}_{\mathbb{B},\mathbb{G}}\ p_1, \ldots, p_n}\ (b_j) \qquad \{p_1, \ldots, p_n\} \in B$$

$$\frac{}{\mid\!\frac{\delta_{\{p_1,\ldots,p_n\}}}{1}{}_{\mathbb{B},\mathbb{G}}\ \neg(p_1, \ldots, p_n)}\ (b_j) \qquad \{p_1, \ldots, p_n\} \in B$$

$$\frac{}{\mid\!\frac{0}{1}{}_{\mathbb{B},\mathbb{G}}\ \neg(p_1, \ldots, p_n)}\ (b_j) \qquad \{p_1, \ldots, p_n\} \notin B \text{ and } \{\neg(p_1, \ldots, p_n)\} \notin B$$

Gradient Beliefs:

$$\frac{}{\mid\!\frac{v^{g_i}-\delta g_i}{1}{}_{\mathbb{B},\mathbb{G}}\ g_i}\ (g_j) \qquad g_i \in G$$

Rules:

$$\frac{\mid\!\frac{m}{n}\ \Gamma, A, B}{\mid\!\frac{m}{n}\ \Gamma, A \vee B}\ (R\vee) \qquad \frac{\mid\!\frac{m_1}{n_1}\ \Gamma, A \qquad \mid\!\frac{m_2}{n_2}\ \Gamma, B}{\mid\!\frac{m_1+m_2}{n_1+n_2}\ \Gamma, A \wedge B}\ (R\wedge)$$

Figure 6.2: The $\overline{\overline{GS4}}_{\mathbb{B},\mathbb{G}}$ sequent calculus.

then p must become an unsettled belief:

$$\Big|\frac{0}{1}{}_{G-p}\; p.$$

More generally, assume

$$\Big|\frac{v^{g_1}+\cdots+v^{g_n}+v^{p}-(\delta_{g_1}+\cdots+\delta_{g_n}+\delta_p)}{n+1}{}_{G}\; \Delta.$$

Then:

$$\Big|\frac{v^{g_1}+\cdots+v^{g_n}-(\delta_{g_1}+\cdots+\delta_{g_n})}{n+1}{}_{G-p}\; \Delta.$$

Definition 6.18 (Expansion)**.** Let $\mathbb{G}+p$ denote the *expansion* of $\mathbb{G}$ by p. If p is unsettled in G, i.e.,

$$\Big|\frac{0}{1}{}_{G}\; p,$$

then p must become a Gradient Belief:

$$\Big|\frac{v^{p}-\delta_p}{1}{}_{G+p}\; p.$$

More generally, assume

$$\Big|\frac{v^{g_1}+\cdots+v^{g_n}-(\delta_{g_1}+\cdots+\delta_{g_n})}{n}{}_{\mathbb{G}}\; \Delta.$$

Then:

$$\Big|\frac{v^{g_1}+\cdots+v^{g_n}+v^{p}-(\delta_{g_1}+\cdots+\delta_{g_n}+\delta_p)}{n+1}{}_{\mathbb{G}+p}\; \Delta.$$

Definition 6.19 (Revision)**.** Let $G*p$ denote the *revision* of $\mathbb{G}$ by p, where $\neg p\in\mathbb{G}$ and $p\notin G$. If $\neg p$ is a Gradient Belief, i.e.,

$$\Big|\frac{v^{\neg p}-\delta_{\neg p}}{1}{}_{G}\; \neg p,$$

then the revision requires that:

$$\Big|\frac{v^{p}-\delta_p}{1}{}_{\mathbb{G}*p}\; p \qquad \text{and} \qquad \Big|\frac{0}{1}{}_{\mathbb{G}*p}\; \neg p.$$

More generally, assume

$$\Big|\frac{v^{g_1}+\cdots+v^{g_n}+v^{\neg p}-(\delta_{g_1}+\cdots+\delta_{g_n}+\delta_{\neg p})}{n}{}_{\mathbb{G}}\; \Delta, \neg p.$$

Then:

$$\mid\!\frac{v^{g_1}+\cdots+v^{g_n}+v^p-(\delta_{g_1}+\cdots+\delta_{g_n}+\delta_p)}{n}_{\mathbb{G}*p}\ \Delta, p.$$

Definition 6.20 (Change of value)**.** Let us consider a Gradient Belief p with value v_1^p at time t_1:

$$\mid\!\frac{v_1^p-\delta_p}{1}\ p.$$

If, at a later time t_2, the agent acquires new information but still maintains p as true with a different degree of confidence v_2^p, then the belief is updated as:

$$\mid\!\frac{v_2^p-\delta_p}{1}\ p.$$

More generally, a change of value corresponds to a transformation of the fractional semantics decoration of p from v_1^p to v_2^p, while preserving its status as a Gradient Belief.

Remark 6.3. *The operation of* Change of value *must be distinguished from the operation of* Revision*. In fact, while Revision replaces a belief p, Change of value preserves the same belief p, but updates the strength of commitment associated with it.*
Formally, Revision transforms:

$$\mid\!\frac{v^{\neg p}-\delta_{\neg p}}{1}\ \neg p \quad \textit{into} \quad \mid\!\frac{v^p-\delta_p}{1}\ p,$$

whereas Change of value transforms:

$$\mid\!\frac{v_1^p-\delta_p}{1}\ p \quad \textit{into} \quad \mid\!\frac{v_2^p-\delta_p}{1}\ p.$$

Thus, Revision alters the content *of the belief set, while Change of value alters only the* degree *with which a belief is held.*

Example 6.10. Suppose that an agent holds the belief p with value $v_1^p = 0.6$ at time t_1, i.e.,

$$\mid\!\frac{0.6-\delta_p}{1}\ p.$$

At a later time t_2, she acquires new information that increases her confidence in p to $v_2^p = 0.9$. The updated belief is then:

$$\mid\!\frac{0.9-\delta_p}{1}\ p.$$

This illustrates that the belief remains p, but its associated strength changes.

6.6 Concluding Remarks

The development of Fractional Semantics leads us to a conception of reasoning in which uncertainty is not a limitation but a constitutive element of thought. Through the use of hyperreal numbers, beliefs can be expressed as gradients rather than isolated points of certainty, revealing the continuous nature of our epistemic states. Each belief is no longer fixed once and for all, but subject to infinitesimal adjustments that mirror the way human reasoning actually evolves — by approximation, revision, and negotiation between conviction and doubt.

This perspective allows us to describe the dynamics of belief not as a sequence of abrupt replacements, but as a process of transformation within a structured space of possibilities. A belief may lose or gain infinitesimal strength, shift its relation to others, or merge into more complex configurations. What matters is not the static content of what is believed, but the continuous trajectory by which a system of beliefs maintains coherence through change.

The idea of Gradient Beliefs thus captures a deeper intuition: that reasoning, like perception and emotion, unfolds in degrees. The formalism of Fractional Semantics provides a language for representing this graduality, showing that logical structures can account for smooth transitions as well as sharp distinctions. In this sense, the logic developed here is less a calculus of certainty than a geometry of belief.

In the next chapter, this vision will encounter another domain where continuity and transformation coexist: music. The movement from logic to harmony will not simply juxtapose two different languages, but will explore how a system of inference can describe motion, tension, and resolution — the same principles that govern both reasoning and sound.

Chapter 7

A Proof-Theoretic Perspective on Musical Harmony

"Why is some music so much deeper and more beautiful than other music? It is because form, in music, is expressive–expressive to some strange subconscious regions of our minds. The sounds of music do not refer to serfs or city-states, but they do trigger clouds of emotion in our innermost selves; in that sense musical meaning IS *dependent on intangible links from symbols to things in the world–those 'things', in this case, being secret software structures in our minds."*

— Douglas R. Hofstadter, *Gödel, Escher, Bach: an Eternal Golden Braid*

7.1 Introduction

This chapter applies the formal framework of Proof Theory and the Lambek Calculus to tonal harmony, interpreting harmonic derivations as proof-theoretic processes. The aim is to show how logical inference and harmonic progression share a common structural foundation: both are systems of transformation governed by rules.

The analysis presented here builds upon my previous works [4, 7] and on a series of papers developed in collaboration with Professor Satoshi Tojo [9–12]. While parts of those studies are incorporated, the present formulation extends and refines them within a unified proof-theoretic perspective.

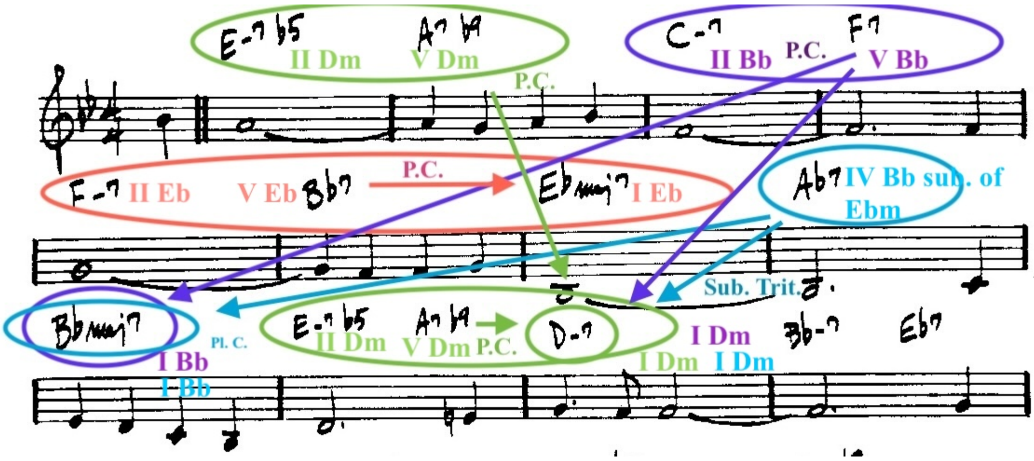


Figure 7.1: First bars of "Stella by Starlight". Example of traditional analysis, with harmonic relations annotated directly on the score. The degree is written before the tonality: for instance, the Em7(^{b}5) is indicated as the second degree of D minor (II Dm). **P.C.** stands for "Perfect Cadence", and the arrows show where cadences resolve.

The idea of describing music through logic is not entirely new, yet most previous approaches have relied on set-theoretical or algebraic tools. My proposal is to bring proof theory—and in particular the Lambek Calculus—into music analysis. In this framework, harmonic progressions can be viewed as derivations: rules determine the admissible transitions between chords, and the structure of a proof itself provides a measure of harmonic complexity. Proof theory thus offers a rigorous language for musical reasoning while preserving a flexibility that remains faithful to musical intuition.

The motivation for this work is both theoretical and practical. During my studies in jazz piano, I realized that despite the abundance of harmonic analyses, there was no shared formal system comparable to logical calculi. Traditional analyses often appeared ad hoc or inconsistent, and existing systematic approaches such as Neo-Riemannian theory or Tonfeld analysis, while insightful, lack the formal precision of proof theory. As shown in Figure 7.1, the conventional way of annotating progressions directly on the score may capture local relations but tends to obscure the overall structure. This experience motivated the search for a more systematic and logically grounded method.

The general idea parallels familiar proof-theoretic techniques. Just as a complex proposition can be decomposed into simpler components, a chord progression can be reduced step by step until a *core set* of chords remains. This process highlights the essential harmonic relations underlying a passage.

Because the rules are invertible, the same system can also be used in a generative direction, enabling the mechanical composition of progressions that satisfy logical constraints.

The chapter is structured to reflect this twofold ambition: to motivate and to formalize. We begin by situating our work within existing traditions such as Neo-Riemannian theory and Tonfeld analysis, and by recalling the role of the Circle of Fifths in organizing tonal relations. We then introduce the first set of inference rules for chord progressions, with examples of their analytic and generative use. Building on this, we refine the system into the *Labelled Lambek Calculus*, which incorporates a notion of *harmonic depth*. This allows us to measure the complexity of derivations and distinguish between superficially similar but structurally different harmonic paths. We then introduce the Labelled-free Lambek Calculus for music, to point out the differences between the systems. Applications to case studies such as *Stella by Starlight*, *In Your Own Sweet Way*, and *All the Things You Are* illustrate the potential of the method across diverse harmonic environments.

The chapter concludes with reflections on the advantages and limitations of this approach. The overarching aim is not merely to propose a novel calculus for harmony, but to show how logic and music can enrich one another: logic offering structure, transparency, and a standard of rigor, and music providing a challenging domain where logical ideas can acquire new depth and resonance. Moreover, logic can be used as an important tool in Music Composition, allowing the creation of consistent Harmonic Structure with the only use of Logical Rules, that can be varied as the composer whishes.

Notation. Throughout this chapter, we adopt the *Berklee style* chord notation, e.g. $\mathrm{C}\textsc{ma}^7$, Dm7, G7, $\mathrm{E}\flat\mathrm{m}7^{(\flat 5)}$, instead of, for example, C°, d7, $C^{\oslash}7$, etc... This choice ensures clarity and avoids ambiguities that can arise with alternative systems of chord symbols. All harmonic progressions and formal rules will therefore be presented in this standardized notation.

Definition 7.1 (Berklee Chord Names)**.** The input will be as following:

$\mathbf{X}\textsc{ma}^7$: major seventh;

Xm7 : minor seventh;

$\mathbf{Xm}\textsc{ma}^7$: minor chord with major seventh;

Xm7$^{\flat 5}$: semi diminished or minor ;

Xdim7 : is the diminished chord;

Xsus4 : is the suspended fourth;

X7 : dominant seventh;

X$^{\sharp 5}$: augmented fifth

7.2 Tonfeld and Circle of Fiths

The Tonfeld Theory [84] provides a geometric and highly systematic way to represent tonal relations. It can be thought of as an infinite plane graph in which each node corresponds to a pitch class (i.e., an equivalence class of pitches modulo octaves), and the edges encode harmonic relations between them. From the perspective of logic, one may see the Tonfeld as the analogue of a Kripke frame: a structured domain of points (here, pitches) together with accessibility relations (here, harmonic connections). In this setting, chords are no longer seen as isolated symbols but as structured configurations of nodes in the graph.

The theory identifies three fundamental cyclic structures that generate the harmonic space:

- **Octatonic sets:** collections of eight pitch classes that form a symmetric structure. They are invariant under certain transpositions and can be described by Messiaen's "modes of limited transposition" [66]. Their cyclic nature makes them analogous to regular subgraphs or automorphic structures in logic.

- **Hexatonic sets:** six-element sets, also describable as limited transposition modes, which capture important tonal symmetries. These subsets act like smaller invariants within the global harmonic graph, organizing many common triadic relations.

- **Stacks of fifths:** unlike the octatonic or hexatonic sets, these are not modes of limited transposition and therefore require explicit enumeration. A stack of fifths $Fif_{C,n}$ begins at a given pitch (say C) and adds successive perfect fifths (C, G, D, A, ...). This process can be iterated

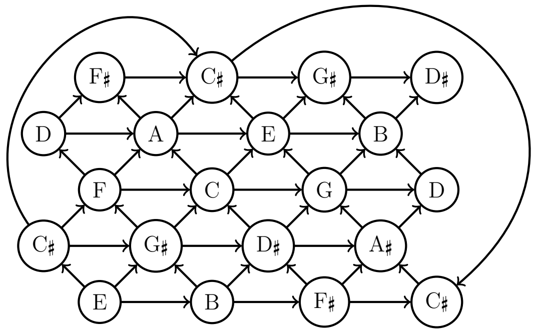


Figure 7.2: A part of the Tonfeld. The lines outside the figure outline the relation between the same chords throughout the plane.

without bound, producing the familiar *circle of fifths* when considered modulo the octave.

The octatonic and hexatonic sets admit a compact representation: instead of defining all possible sets for every root, it suffices to specify only a few, because the others are obtained by transposition. For example:

$$Oct_0 = \{C, D\flat, E\flat, E, G\flat, G, A, B\flat\} \tag{7.1}$$

$$Oct_1 = \{D\flat, D, E, F, G, A\flat, B\flat, C\flat\} \tag{7.2}$$

$$Oct_2 = \{C, D, E\flat, F, G\flat, A\flat, A, B\} \tag{7.3}$$

and similarly for the four hexatonic collections:

$$Hex_0 = \{C, E\flat, E, G, A\flat, B\}, \quad Hex_1 = \{C\sharp, E, F, G\sharp, A, C\} \tag{7.4}$$

$$Hex_2 = \{D, F, F\sharp, A, B\flat, C\sharp\}, \quad Hex_3 = \{E\flat, F\sharp, G, B\flat, B, D\} \tag{7.5}$$

By contrast, stacks of fifths must be explicitly enumerated, since no mode-of-limited-transposition symmetry is available. For example:

$$Fif_{C,1} = \{C, G\}, \quad Fif_{C,2} = \{C, G, D\}, \quad Fif_{C,3} = \{C, G, D, A\}, \quad \ldots \tag{7.6}$$

The structure underlying stacks of fifths is nothing but the well-known *circle of fifths* (Figure 7.3). This circle plays a central role in traditional harmonic

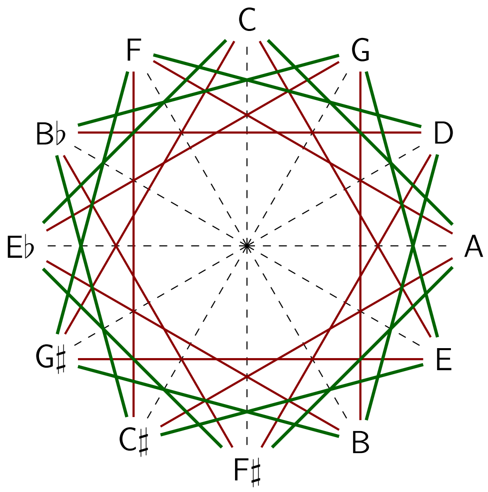


Figure 7.3: The circle of fifths. Major thirds are highlighted in red, minor thirds in green. This structure will serve as the "frame" for the inference rules developed later.

theory, and in our framework it will serve as the primary reference structure for encoding rules. In proof-theoretic terms, the circle of fifths provides the *background frame* in which inference rules operate, much as the accessibility relation does in modal logic.

In summary, the Tonfeld, the octatonic and hexatonic sets, and the circle of fifths constitute the raw material on which our proof-theoretic analysis will operate. They play a role analogous to the *language* and *semantic structures* in logic: before rules can be formulated, one must first establish the domain of objects and the relations connecting them.

Remark 7.1. *The tableaux-like derivations presented in this early section represent a preliminary attempt to visualise harmonic inference through a tree structure. While the idea provides an intuitive parallel with logical tableaux, it lacks the full formal development required for a rigorous proof-theoretic account. For this reason, it should be regarded as an exploratory step rather than a definitive formulation.*

7.3 Why Structural Proof Theory?

The central questions, in Proof Theory, address the nature of proof, the relationship between syntax and semantics, and the role of proofs in the development of mathematics. Proof theory not only provides techniques for assessing consistency and completeness of formal systems, but also offers fine-grained methods for analyzing how complex structures can be systematically decomposed into simpler ones. Among its various strands, the one most relevant for our purposes is *structural proof theory.*

The abstract structure of a proof can be represented as a tree: at the top, the *leaves* correspond to initial sequents such as $p \vdash p$, while at the bottom we find the sequent being proved. As a simple illustration, the following derivation shows how to prove the commutativity of conjunction:

$$\dfrac{\dfrac{\dfrac{\dfrac{q \vdash q \qquad p \vdash p}{q, p \vdash q \wedge p}\,(\wedge_{R,I})}{p, q \vdash q \wedge p}\,(ex.)}{p \wedge q \vdash q \wedge p}\,(\wedge_{L,I})}{\vdash (p \wedge q) \to (q \wedge p)}\,(\to_{R,I})$$

The dual representation is also possible: instead of constructing a proof from the axioms upwards, one may start from the goal sequent and expand it downward into simpler subgoals, producing a *tableau.* For the same commutativity property, a tableau representation looks as follows:

$$\begin{array}{c}
\vdash (p \wedge q) \to (q \wedge p) \\
| \\
p \wedge q \vdash q \wedge p \\
| \\
p,\ q \vdash q \wedge p \\
\diagup \qquad \diagdown \\
p,\ q \vdash q \qquad p,\ q \vdash p
\end{array}$$

This duality—deriving from the axioms upward or decomposing a complex sequent downward—will be essential for our musical application. Just as in logic one may view a proof both as a verification (analysis) and as a construction (synthesis), so too in harmony we may view a sequence of chords both as something to be reduced to its underlying structure and as something to be generated step by step from a minimal core.

In short, structural proof theory offers not only a rigorous formal language, but also a methodological stance: it clarifies how rules decompose structures, how complexity is measured by proof depth, and how derivations can be inverted. These features make it an especially suitable framework for the logical treatment of chord progressions, as will be shown in the next sections.

7.4 Rule based Presentation

The purpose of this section is to present a more systematic account of how certain harmonic rules can be formalized within a proof-theoretic framework, combining principles from traditional harmony and Tonfeld theory. The system proposed here should be regarded as a preliminary attempt to understand music through logic. Nonetheless, it is of particular interest because, so far, it is the only model we have developed that is fully invertible.

While music is not a proof, a set of fundamental rules can still be outlined and adapted by adding or removing rules. This article focuses on jazz tonal harmony and presents a construction using a limited set of rules, but it can be hopefully expanded. Our attempt is to find a way to combine the generative theory of tonal music (GTTM) [87] with the ability of proof theory to explicitly indicate when and where a certain rule must or can be applied. This aims to provide a more systematic and logical approach to understanding and analyzing harmony in tonal music. It must be stressed that this attempt will not adhere to all of the structural rules typically found in proof theory. In fact, the only structural rule that we will use is the Contraction Rule, which will play a crucial role.

$$\frac{\vdash p, p}{\vdash p}\,(Contraction)$$

but there is no place for weakening, because we don't want that new chords can appear spontaneously:

$$\frac{\vdash p}{\vdash p, q}\,(Weakening)$$

and exchange, because we don't want that chords change position:

$$\frac{\vdash p, q}{\vdash q, p}\,(Exchange)$$

7.4.1 The First Rules

In a Gentzen-style presentation, a rule has the following form:

$$\frac{\vdash p \qquad \vdash q}{\vdash p \wedge q}\,(\wedge_I)$$

where the letter I indicates the *introduction* rule. The musical analogue of this procedure is the elimination of non-essential chords in order to highlight the fundamental components of a harmonic sequence. The goal is to isolate the *core set* of harmonies, which represents the essential harmonic skeleton of a passage.

Cadences In tonal harmony, a *cadence* is a standard pattern of chords that provides closure or resolution to a musical phrase. Cadences play a role similar to logical rules: they establish how one harmony follows from another in a way that is widely recognized and structurally necessary. Among the different types, the *perfect cadence* (or authentic cadence) is the most conclusive one. It consists of a dominant seventh chord built on the fifth degree (V7) followed by the tonic major chord ($\text{I}\textsc{ma}^7$). This progression is perceived as the strongest form of resolution, and thus serves as a natural candidate for a formal inference rule in our system.

Perfect Cadence:

$$\frac{\text{V7} \qquad \text{I}\textsc{ma}^7}{\text{I}\textsc{ma}^7}\ (Fif_{I,1})$$

In this representation, the dominant (V7) and the tonic (Imaj7) are combined by the rule $Fif_{I,1}$, which encodes the relation between the root of the tonic and its dominant a fifth above. The effect is analogous to an introduction rule in logic: once the dominant has resolved, the tonic remains as the stable conclusion.

This kind of cadence can be expanded with the stack of fifths:

$$\frac{\text{IIm7} \qquad \text{V7} \qquad \text{I}\textsc{ma}^7}{\text{I}\textsc{ma}^7}\ (Fif_{I,2})$$

$$\frac{\text{VIm7} \qquad \text{IIm7} \qquad \text{V7} \qquad \text{I}\textsc{ma}^7}{\text{I}\textsc{ma}^7}\ (Fif_{I,3})$$

$$\vdots$$

Another rule is the *Plagal Cadence*, which is extensively used in ancient as well as modern pop music. *Plagal Cadence* is a type of cadence that goes from the fourth scale degree to the first one. It can be schematized in three main ways, to also explicitly show the movement from the fourth minor scale degree to the first one. It can be interpreted as a particular case of the circle of fifths: from the last instances to the first one.

Plagal cadence :

$$\frac{\mathrm{IVMA}^7 \quad \mathrm{IMA}^7}{\mathrm{IMA}^7}\ \text{P.C.} \qquad \frac{\mathrm{IVm7} \quad \mathrm{IMA}^7}{\mathrm{IMA}^7}\ \text{P.C.} \qquad \frac{\mathrm{IVMA}^7 \quad \mathrm{IVm7} \quad \mathrm{IMA}^7}{\mathrm{IMA}^7}\ \text{P.C.}$$

To explicitly explain the other types of cadences, like the *Deceptive Cadence*, it must be noted that the tonic and the submediant have a lot of notes in common, which allows for their mutual substitution. For example, an authentic cadence can be transformed into a deceptive one by substituting the tonic with the submediant. Something similar occurs between the tonic and the mediant.

Inversions:

$$\frac{\mathrm{VI}}{\mathrm{I}}\ (i.) \qquad \frac{\mathrm{III}}{\mathrm{I}}\ (i.) \qquad \frac{\mathrm{I}}{\mathrm{III}}\ (i.) \qquad \frac{\mathrm{I}}{\mathrm{VI}}\ (i.)$$

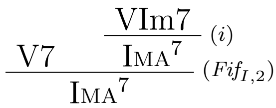
In jazz and classical music, another rule is deduced: the tritone substitution. This rule allows for the substitution of a dominant chord with its relative tritone. This is particularly useful in the context of jazz improvisation and the creation of complex harmonic progressions in classical music. The tritone substitution adds more dissonance to the progression, and is one of the most important feature of jazz harmony. It can be used to create tension and dissonance and it is an essential tool to understand the harmonic language of jazz.

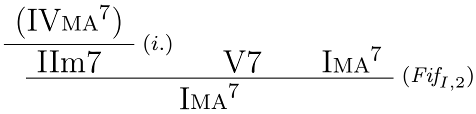
$$\frac{\mathrm{V}}{\mathrm{I}\sharp}\ (tr)$$

From the Authentic Cadence and its inversions, it is possible to also obtain the Deceptive Cadence and the Authentic Cadence with the subdominant scale degree instead of the supertonic:

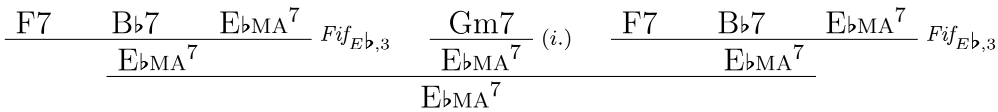
- Deceptive Cadence: V-VI;
- Authentic Cadence with subdominant: IV-I;

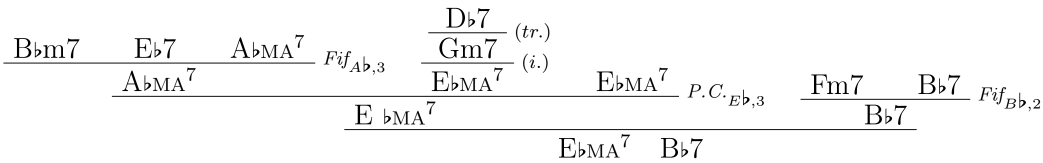
These variations help to create a more rich and complex harmonic language and can be used to create a different emotional or stylistic effect in the music.

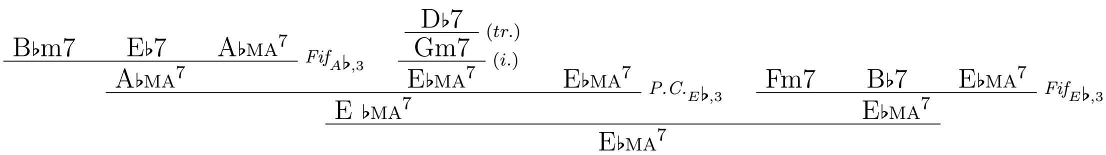




Example 7.1. The following example is taken from "But not for me" (Figure 7.4) by George Gershwin. The structure is divided into sections, because the tree was too long.







As we can see the main chord is E♭MA^7, that is always the main chord.

Figure 7.4: The jazz standard "But not for me" by George and Ira Gershwin.

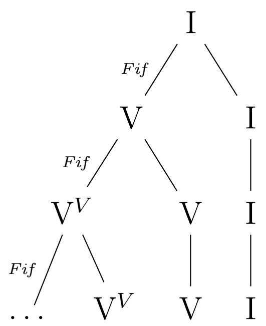
7.5 Composition of New Musical Structures through a Rule based Presentation

This method uses invertible rules to create novel harmonic tonal structures. The process is straightforward and involves selecting a fundamental node, determining the desired length of the structure, and applying the rules to expand the system until the required number of chords is reached. By using this systematic and logical approach based on the invertibility of the rules, it's possible to compose original harmonies.

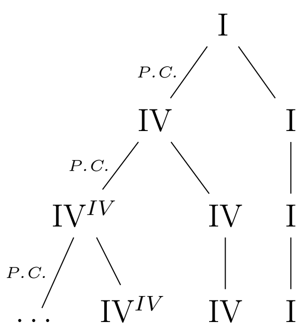
7.5.1 Tableux

The inversion of the rules made in section 7.4.1 can be inverted thanks to the dual of proof theory: tableux.

Generation of the Authentic Cadence:



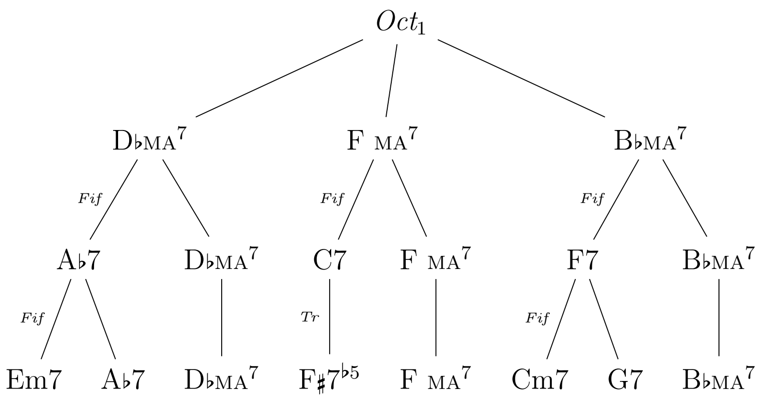
Generation of the Plagal Cadence:



Inversions:

Triton substitution:

Example 7.2. Firstly, we'll form a basic arrangement of three notes taken from the initial Octatonic set. Then, we'll establish a preferred duration for the arrangement, say 8 measures. We'll utilize various techniques, such as stack of fifths, plagal cadence, and tritone substitution, to extend the pattern following certain guidelines until we attain the desired number of chords. Consequently, we'll obtain 11 distinctive chords, resulting in an 8-measure arrangement with increased intricacy and musical appeal.



This way of composition can be automatized to create new and different harmonic structures, always remaining into a tonal configuration, but what if we want to create a non-tonal structure?

7.5.2 New rules

One of the interesting thing about our system is that it is possible to add new rules giving flexibility to the system. Music and harmony, in fact, can change between ages and the rule-based system can be expanded with new rules if they are considered useful for a certain kind of analysis or a certain kind of composition. Suppose that the analyzed piece is taken from the baroque period and so it is important to explicit the *picardy third.* Then it is easy to add a rule that could be something like:

$$\frac{\text{Im} \quad \text{V} \quad \text{I}}{\text{Im} \qquad \text{I}}\ P.T.$$

This could seem redundant in respect of the rule of the stack of fifths, but the attempt here is to create something general that could be useful also in specific cases. In the baroque chorals, for example, understand when there is an authentic minor cadence or a picardy third could be useful, because a picardy third indicates the end of a phrase or of the piece.

Example 7.3. Let's consider, for example, J.S. Bach's Jesu, meine Freude (figure 7.5), this is an example of picardy third.

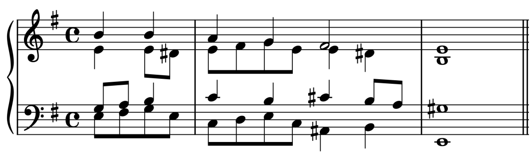
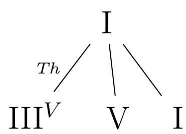


Figure 7.5: J.S. Bach's Jesu, meine Freude; mm. 11-13

To analyze these bars it is possible to use the new rule:

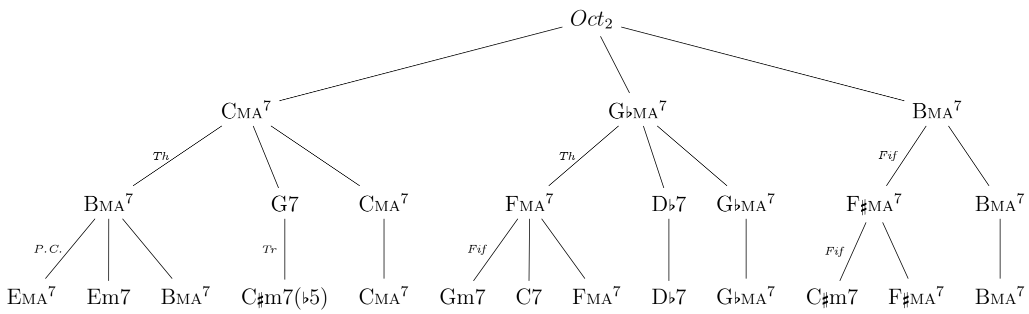
Em Am Em *P.C.* F♯7 B7 E *P.T.*
Em E
Em E

New rules in composition In addition to their traditional use in tonality, add rules can also be employed in composition to generate unconventional results. For instance, a new rule could be introduced to mandate the inclusion of the third degree of a chord in any dominant chord that appears. This rule might read as follows: "Whenever a dominant chord is encountered, it must contain also the third degree" and let's call that *Th*..



Incorporating this additional layer would introduce an added level of intricacy and diversity to the harmonic arrangements produced by the system, leading to a greater potential for novel and unforeseen outcomes. It is crucial to acknowledge that the regulations you integrate will shape the final composition to align with your specific requirements.

Example 7.4. For example try to write a new harmonic form using this rule with also some other rule:



7.5.3 Proof theory method and CCG

The presented method shares similarities with the one presented in [36] that uses Combinatorial Category Grammar (CCG), but there are some notable differences. On one hand, the reduction method we propose is more flexible than the one presented in [36]. On the other hand, our method is not currently linked to machine learning or automatic analysis, which are areas that we plan to explore in the future. It's worth noting that the two methods are not in conflict and can ideally be combined in the future.

The contraction method we propose is particularly useful because it's malleable: we can add new rules to analyze and stress different musical aspects, and we can even invert the method to create new harmonic structures. In contrast, the method presented in [36] relies on a statistical machine learning technique that may not be as straightforward to implement using our proof theoretic presentation. Moreover, our proof theoretic presentation can be inverted to create new harmonic structures, which CCG can only achieve by working on the rules. However, this inversion process is not as straightforward as it is with our method. A common point between CCG and our method is the philosophical and musicological idea that chord progressions are driven by the listener's *expectations* of progression, based on the same harmonic Riemannian concept. However, our method can be expanded due to its ability to incorporate new rules.

Remarks

The method of composition based on invertible rules demonstrates how proof-theoretic ideas can be used not only for analysis but also for the generation of new harmonic structures. By applying the same rules used in harmonic reduction in reverse, it becomes possible to construct progressions of varying length and complexity, always within a consistent tonal framework.

The flexibility of the system also allows for the introduction of additional rules to capture stylistic features such as the Picardy third or tritone substitutions, thereby adapting the method to different historical or aesthetic contexts. In this way, the approach highlights both the systematic and creative potential of proof-theoretic reasoning when applied to musical composition.

In the next section we will present an evolution of this methodology by introducing the *Labelled Lambek Calculus*. Building on the framework developed so far, this new system allows us to organize chords in a more precise and systematic way. On the other hand, it also increases the complexity of the model: while the rule-based approach presented above makes it relatively straightforward to design algorithms for composition, the labelled version is more intricate and poses additional challenges for generative applications.

7.6 LLCMA: Labelled Lambek Calculus for Music Analysis

The methodology presented so far provides a first logical framework for chord analysis, but it still lacks a rigorous way to manage tonal contexts and modulations. In particular, when two different interpretations of the same chord are possible, or when a sequence involves distant tonal regions, the system tends to become ambiguous. To address these limitations, we introduce here the *Labelled Lambek Calculus for Music Analysis* (LLCMA).

The idea of LLCMA is to enrich the traditional Lambek Calculus with labels that explicitly encode the tonal centre and the degree of each chord. This additional layer allows us to keep track of modulations, cadential structures, and accessibility relations between tonalities. In this way, the calculus is able not only to reduce chord sequences to their essential core, but also to provide a consistent interpretation of their harmonic function.

The introduction of labels also makes it possible to measure the depth of a harmonic analysis, following the analogy with proof theory where the complexity of a derivation is related to the number of rules applied. At the same time, the system becomes more sophisticated: while the rule-based method of the previous section was easy to invert and use for composition, the labelled framework is more demanding but also more expressive, opening new perspectives for both analysis and generative approaches.

7.6.1 Definition of the Calculus

Definition 7.2 (Tonalities)**.** The key, i.e. the tonality, will be indicated in general with a low case greek letter, such as $\alpha, \beta, \dots$ and in the analysis with a low case letter for minor tonalities and upper case for major tonalities, for example C stands for C major, a stands for A minor, etc…

Definition 7.3 (Degree)**.** The degree will generally be indicated with $x, y, \dots,$ and in analysis, it will be represented by a Roman numeral associated with the function, such as IIIm7, IIMA7… , unless the function is clear from the tonal context. In such cases, we will simply use the Roman numeral for simplicity.

Definition 7.4 (Interpretation of a chord)**.** The interpretation of a chord is based on definition will be divided by two dots, i.e., $\alpha : x$.

We provide a *Lexicon*, where a Berklee chord is looked up and it can be *interpreted* in multiple ways, as follows.

F	$\Rightarrow$	$C\colon$IV, $F\colon$I, $B\flat\colon$V, $\cdots$
G	$\Rightarrow$	$G\colon$I, $D\colon$IV, $a\colon$VII, $\cdots$
B$\flat$	$\Rightarrow$	$F\colon$IV, $\cdots$
C7	$\Rightarrow$	$F\colon$V7, $\cdots$
$\vdots$		$\vdots$

Definition 7.5 (Initial Sequent)**.** The initial sequent of an analysis will be the interpretation of a chord written similarly to a tautology in logic:

$$\frac{\textit{Chord}}{\alpha\colon x \vdash \alpha\colon x}$$

We can have multiple interpretation of the same chord, based on the given lexicon, for example:

$$\frac{\mathrm{C}_{\mathrm{MA}}{}^{7}}{C\text{:I} \vdash C\text{:I}}$$

$$\frac{\mathrm{C}_{\mathrm{MA}}{}^{7}}{G\text{:IV} \vdash G\text{:IV}}$$

$$\frac{\mathrm{C}_{\mathrm{MA}}{}^{7}}{a\text{:III} \vdash a\text{:III}}$$

$$\vdots$$

Definition 7.6 (Multiset of formulas)**.** The multiset of formulas are, as usual, indicated by upper case greek letters, i.e., $\Gamma, \Gamma'\Delta, \Pi, \Phi, \ldots$

Definition 7.7 (Accessibility Relation R)**.** The accessibility relation $\alpha R \alpha^{\star}$ is a relation between the two different tonalities α and $\alpha^{\star}$ that permits to go between the two tonalities, changing accordingly the degree. We will use the simbol R when there is not a modulation, but only a changing of function.

The idea of the accessibility relationship is due to the fact that sometimes a chord has a function if considered as the result of a cadence that appears before that, but maybe it has another function if it is considered as the first of a new chain.

Example 7.5 (Use of the rule $\alpha R \alpha^{\star}$)**.** Let's consider a classical secondary dominant, i.e., D7, G7, $\mathrm{C{\small MA}}^7$.

$$\dfrac{\dfrac{\dfrac{\dfrac{\mathrm{D7}}{G{:}\mathrm{V} \vdash G{:}\mathrm{V}} \qquad \dfrac{\mathrm{G7}}{G{:}\,\mathrm{I} \vdash G{:}\,\mathrm{I}}}{G{:}\mathrm{V}\ ,\ G{:}\,(\mathrm{V} \rhd \mathrm{I}\ ) \vdash G{:}\,\mathrm{I}}\,(\rhd)}{G{:}\mathrm{V}\ ,\ G{:}\,(\mathrm{V} \rhd \mathrm{I}\ ),\ GRC \vdash C{:}\,V}\,(R) \qquad \dfrac{\mathrm{C{\small MA}}^7}{C{:}\,\mathrm{I} \vdash C{:}\mathrm{I}}}{G{:}\mathrm{V}\ ,\ G{:}\,(\mathrm{V} \rhd \mathrm{I}\ ),\ GRC,\ C{:}\,(\mathrm{V} \rhd \mathrm{I}\ ) \vdash C{:}\,\mathrm{I}}\,(\rhd)$$

Reading the final sequent we can say that the cadence begins on the fifth degree of G major. Then we have a cadence on the tonic (first degree) of G major, which is related to the key of C major, changing its harmonic function. Afterward, there is a cadence from the fifth to the tonic (first degree) of C major, with the entire passage interpreted in the key of C major, which is the true tonality of this cadence.

Example 7.6 (The use of $\wedge$)**.** Let's consider two cadences belonging to different tonalities that don't have any relationship between them, for example B♭7, $\mathrm{E\flat{\small MA}}^7$, G7, $\mathrm{C{\small MA}}^7$.

$$\dfrac{\dfrac{\dfrac{B\flat 7}{E\flat{:}\,V \vdash E\flat{:}\,V} \qquad \dfrac{\mathrm{E\flat{\small MA}}^7}{E\flat{:}\mathrm{I} \vdash E\flat{:}\mathrm{I}}}{E\flat{:}\,V,\ E\flat{:}\,(V \rhd I) \vdash E\flat{:}\mathrm{I}}\,(\rhd) \qquad \dfrac{\dfrac{G7}{C{:}\,V \vdash C{:}\,V} \qquad \dfrac{\mathrm{C{\small MA}}^7}{C{:}\mathrm{I} \vdash C{:}\mathrm{I}}}{C{:}\,V,\ C{:}\,(V \rhd I) \vdash C{:}\mathrm{I}}\,(\rhd)}{E\flat{:}\,V,\ E\flat{:}\,(V \rhd I),\ E\flat RC,\ C{:}\,V,\ C{:}\,(V \rhd I) \vdash E\flat{:}\mathrm{I} \wedge C{:}\mathrm{I}}\,(\wedge_R)$$

Reading this harmonic sequence we can say that we have a cadence in E♭ suddenly solved, as well as the one in C. They are connected by a relation R that doesn't change the harmonic function of the chords.

Example 7.7 (The use of the Melting Rule)**.** The melting rule is based on the observation that sometimes a chord progression ends in a specific tonality, followed by another progression that also concludes in the same tonality. In this case, there has been no change in tonality, and the system is designed to remain unchanged when the tonality is consistent.

$$\dfrac{\dfrac{\dfrac{G7}{C{:}\,V \vdash C{:}\,V} \qquad \dfrac{\mathrm{C{\small MA}}^7}{C{:}\,\mathrm{I} \vdash C{:}\,\mathrm{I}}\,(\rhd)}{C{:}\,\mathrm{V},\ C{:}\,(\mathrm{V} \rhd \mathrm{I}) \vdash C{:}\,\mathrm{I}}\,(\rhd) \qquad \dfrac{\dfrac{D\flat m7^{(\flat 5)}}{C{:}\,\mathrm{II}\flat \vdash C{:}\,\mathrm{II}\flat} \qquad \dfrac{\mathrm{C{\small MA}}^7}{C{:}\,\mathrm{I} \vdash C{:}\,\mathrm{I}}\,(\rhd)}{C{:}\,\mathrm{II}\flat,\ C{:}\,(\mathrm{II}\flat \rhd \mathrm{I}) \vdash C{:}\,\mathrm{I}}\,(\rhd)}{C{:}\,\mathrm{V},\ C{:}\,(\mathrm{V} \rhd \mathrm{I}),\ C{:}\,\mathrm{II}\flat,\ C{:}\,(\mathrm{II}\flat \rhd \mathrm{I}) \vdash C{:}\,\mathrm{I}}\,(melt.)$$

Example 7.8 (Long distant dependency)**.** Let's consider the same example we discussed earlier, but now in an entrenched form: B♭7, G7, E♭MA^7, CMA^7. This is just a miniature version of what extensively occurs in songs like "Stella by Starlight", and addressing exactly this type of problem was one of the main goals of the method presented here.

$$\dfrac{\dfrac{\dfrac{\dfrac{B\flat 7}{E\flat : V \vdash E\flat : V} \quad \dfrac{G7}{C : V \vdash C : V}}{E\flat : V,\ E\flat RC,\ C : V \vdash E\flat : V \wedge C : V}\,(\wedge_R) \qquad \dfrac{\mathrm{E\flat MA}^7}{E\flat :\mathrm{I} \vdash E\flat :\mathrm{I}}}{E\flat : V,\ E\flat RC,\ C : V,\ CRE\flat,\ E\flat : (V \triangleright I) \vdash C : V \wedge E\flat :\mathrm{I}}\,(\triangleright) \qquad \dfrac{\mathrm{CMA}^7}{C :\mathrm{I} \vdash C :\mathrm{I}}}{E\flat : V,\ E\flat RC,\ C : V,\ CRE\flat,\ E\flat : (V \triangleright I),\ E\flat RC,\ C : (V \triangleright I) \vdash E\flat :\mathrm{I} \wedge C :\mathrm{I}}\,(\triangleright)$$

The final result can be interpreted as follows: from the fifth degree of the key of E♭, we modulate to the key of C (denoted as $CRE\flat$), where we again find the fifth degree. We then return to the key of E♭, resolving from the fifth degree to the tonic. After that, we modulate back to C, where we conclude with a cadence from the fifth degree to the tonic. Everything is interpreted in the two tonalities on the right, i.e., E♭major and C major, that are the two tonalities of the harmonic progression.

Initial Sequent:

$$\frac{\textit{Chord}}{\alpha\colon x,\ \alpha_1\colon x_1, \cdots \vdash \alpha\colon x}$$

Rules:

$$\frac{\Gamma,\ \alpha\colon x,\ \Gamma' \vdash \Delta \wedge \alpha\colon x \wedge \Delta' \qquad \Pi,\ \alpha\colon y,\ \Pi' \vdash \Phi \wedge \alpha\colon y \wedge \Phi'}{\Gamma,\ \alpha\colon x,\ \Gamma',\ \Pi,\ \alpha\colon (x \triangleright y),\ \alpha R \alpha_{\Pi'},\ \Pi' \vdash \Delta \wedge \Delta' \wedge \Phi \wedge \alpha\colon y \wedge \Phi'}\ (\triangleright)$$

$$\frac{\Gamma,\ \beta\colon w,\ \Gamma' \vdash \Delta \wedge \alpha\colon x \wedge \Delta' \qquad \Pi,\ \alpha\colon y,\ \Pi' \vdash \Phi \wedge \gamma\colon z \wedge \Phi'}{\Gamma,\ \beta\colon w,\ \Gamma',\ \Pi,\ \alpha\colon (x \triangleright y),\ \alpha R \alpha_{\Pi'},\ \Pi' \vdash \Delta \wedge \Delta' \wedge \Phi \wedge \gamma\colon z \wedge \Phi'}\ (\triangleright)$$

Where $\alpha_{\Pi'}$ *is the first tonality encountered in* Π' *and* $\alpha \neq \alpha_{\Pi'}$.

$$\frac{\Gamma,\ \alpha\colon x,\ \Gamma' \vdash \Delta \wedge \alpha\colon y \wedge \Delta'}{\Gamma,\ \alpha\colon x,\ \Gamma',\ \alpha\colon (x \triangleright y) \vdash \Delta \wedge \Delta'}\ (\triangleright_I)$$

$$\frac{\Gamma,\ \alpha\colon x,\ \Gamma' \vdash \Delta \wedge \alpha\colon x \wedge \Delta'}{\Gamma,\ \alpha\colon x,\ \alpha R \alpha^\star,\ \Gamma' \vdash \Delta \wedge \alpha^\star\colon x^\star \wedge \Delta'}\ (R)$$

$$\frac{\Gamma,\ \alpha\colon x \vdash \Delta \wedge \alpha\colon x \qquad \beta\colon y,\ \Pi' \vdash \beta\colon y \wedge \Phi}{\Gamma,\ \alpha\colon x,\ \alpha R \beta,\ \beta\colon y,\ \Pi' \vdash \Delta \wedge (\alpha\colon x) \wedge (\beta\colon y) \wedge \Phi}\ (\wedge_R)$$

$$\frac{\Gamma,\ \alpha\colon x \vdash \Delta \wedge \alpha\colon x \qquad \beta\colon y,\ \Pi' \vdash \alpha\colon x \wedge \Phi}{\Gamma,\ \alpha\colon x,\ \alpha R \beta,\ \beta\colon y,\ \Pi' \vdash \Delta \wedge (\alpha\colon x) \wedge \Phi}\ (\textit{melt.})$$

$$\frac{\Gamma,\ \alpha\colon x,\ \Gamma' \vdash \Delta \wedge \alpha\colon x \wedge \Delta' \qquad \Gamma,\ \alpha\colon x,\ \Gamma' \vdash \Delta \wedge \alpha\colon x \wedge \Delta'}{\Gamma,\ \alpha\colon x,\ \Gamma' \vdash \Delta \wedge \alpha\colon x \wedge \Delta'}\ (\textit{contr.})$$

With $\alpha \neq \beta$ and $x \neq y$.

Figure 7.6: Rules of LLCMA. Here all the multisets can be empty. Moreover, only $\wedge_R$-applications appear in Δ, Δ', Φ, and Φ', in fact on the right side, only one formula can appear, which means that we will only find conjunctions.

Example 7.9 (The use of contraction)**.** The contraction is useful for harmonic analysis of structures that contain repetitions, such as the A section of *I Got Rhythm* or any typical Anatole progression: Am7, Dm7, G7, $\mathrm{C}\textsc{ma}^7$; Am7, Dm7, G7, $\mathrm{C}\textsc{ma}^7$.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\mathrm{Am7}}{C\!:VI \vdash C\!:VI} \quad \dfrac{\mathrm{Dm7}}{C\!:II \vdash C\!:II}}{C\!:VI,\ C\!:(VI \rhd II) \vdash C\!:II}\,(\rhd) \quad \dfrac{\mathrm{G7}}{C\!:V \vdash C\!:V}}{C\!:VI,\ C\!:(VI \rhd II),\ (II \rhd V) \vdash C\!:V}\,(\rhd) \quad \dfrac{\mathrm{CMA}^7}{C\!:\mathrm{I} \vdash C\!:\mathrm{I}}}{C\!:VI,\ C\!:(VI \rhd II),\ (II \rhd V),\ (V \rhd I) \vdash C\!:\mathrm{I}}\,(\rhd) \quad \dfrac{\dfrac{\dfrac{\dfrac{\mathrm{Am7}}{C\!:VI \vdash C\!:VI} \quad \dfrac{\mathrm{Dm7}}{C\!:II \vdash C\!:II}}{C\!:VI,\ C\!:(VI \rhd II) \vdash C\!:II}\,(\rhd) \quad \dfrac{\mathrm{G7}}{C\!:V \vdash C\!:V}}{C\!:VI,\ C\!:(VI \rhd II),\ (II \rhd V) \vdash C\!:V}\,(\rhd) \quad \dfrac{\mathrm{CMA}^7}{C\!:\mathrm{I} \vdash C\!:\mathrm{I}}}{C\!:VI,\ C\!:(VI \rhd II),\ (II \rhd V),\ (V \rhd I) \vdash C\!:\mathrm{I}}\,(\rhd)}{C\!:VI,\ C\!:(VI \rhd II),\ (II \rhd V),\ (V \rhd I) \vdash C\!:\mathrm{I}}\,(contr.)$$

Example 7.10 (Multiple Relations)**.** Let's consider a more complicated chord structure like the following one: $A7^{alt}, Dm7, G7, B\flat, E\flat\mathrm{MA}^7, C\mathrm{MA}^7$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A7^{alt}}{d\!:V \vdash d\!:V} \quad \dfrac{Dm7}{d\!:\mathrm{I} \vdash d\!:\mathrm{I}}}{d\!:V,\ d\!:(V \rhd I) \vdash d\!:\mathrm{I}}\,(\rhd)}{d\!:V,\ d\!:(V \rhd I),\ dRC \vdash C\!:II}\,(R) \quad \dfrac{G7}{C\!:V \vdash C\!:V}}{d\!:V,\ d\!:(V \rhd I),\ dRC,\ C\!:(II \rhd V) \vdash C\!:V}\,(\rhd) \quad \dfrac{\dfrac{B\flat 7}{E\flat\!:V \vdash E\flat\!:V} \quad \dfrac{E\flat\mathrm{MA}^7}{E\flat\!:\mathrm{I} \vdash E\flat\!:\mathrm{I}}}{E\flat\!:V,\ E\flat\!:(V \rhd I) \vdash E\flat\!:\mathrm{I}}\,(\rhd)}{d\!:V,\ d\!:(V \rhd I),\ dRC,\ C\!:(II \rhd V),\ CRE\flat,\ E\flat\!:V,\ E\flat\!:(V \rhd I) \vdash C\!:V \wedge E\flat\!:\mathrm{I}}\,(\wedge_R) \quad \dfrac{C\mathrm{MA}^7}{C\!:\mathrm{I} \vdash C\!:\mathrm{I}}}{d\!:V,\ d\!:(V \rhd I),\ dRC,\ C\!:(II \rhd V),\ CRE\flat,\ E\flat\!:V,\ E\flat\!:(V \rhd I),\ C\!:(V \rhd I) \vdash E\flat\!:\mathrm{I} \wedge C\!:\mathrm{I}}\,(\rhd)$$

This example represents a synthesis of all previous examples. We begin with the dominant of D minor, which resolves. However, we also observe a relation to the tonality of C major, as we encounter the dominant G7. Thus, we establish an accessibility relation between D minor and C major, transforming the function of D minor from a tonic into a subdominant (labelled with a second degree) within a cadence, which remains for now unresolved.

Following the G7 chord, a different cadence appears in E♭, moving from the dominant to the tonic. This creates a new accessibility relation, R, which does not alter the functional role of the preceding chords. The resolution of the original cadence is deferred until after the cadence in E♭. The tonalities of this cadence will be E♭major and C major.

7.6.2 The depth of an Harmonic Analysis

It is insightful to use the logical notion of the depth of a proof to track the complexity of a Harmonic Analysis. The idea is that if a large number of rules are applied during the analysis of a harmonic structure, the complexity of that structure increases. This suggests that it is possible to assign a numerical value to a Harmonic Structure, representing its level of complexity. As we will demonstrate later, the Harmonic Analyses introduced earlier are associated with a fixed number—indicating a consistent level of complexity—which allows us to easily compare different sequences of chords and determine which is more complex. This measure of complexity is objective and not based on aesthetics; rather, it provides a systematic way of quantifying harmonic intricacy. Composers and musicians can then interpret this information in their own way, drawing conclusions that align with their artistic and interpretive preferences.

Definition 7.8. Let $\mathfrak{a}, \mathfrak{b}, \ldots$ be analyzed sequent at the end of the harmonic analysis.

Definition 7.9 (Depth of an Harmonic Analysis)**.** The Depth of an harmonic set of chords $\mathfrak{a}$, $gr(\mathfrak{a})$, is defined as follows:

$$dp(\alpha\colon x \vdash \alpha\colon x) \;=0$$

$$dp(\Gamma,\ \alpha\colon x,\ \Delta,\ \alpha\colon (x\rhd y) \vdash \alpha\colon y) \;= dp(\Gamma,\ \alpha : x \vdash \alpha : x) + dp(\Delta,\ \alpha : y \vdash \alpha : y) + 1$$

$$dp(\Gamma,\ \alpha\colon x,\ \alpha R\beta,\ \Delta,\ \beta\colon y \vdash (\alpha\colon x) \wedge (\beta\colon y)) \;= dp(\Gamma,\ \alpha\colon x \vdash \alpha\colon x) + + dp(\Delta,\ \beta\colon y \vdash \beta\colon y) + 1$$

$$dp(\Gamma,\ \alpha\colon x,\ \alpha R\beta,\ \Delta,\ \beta\colon y \vdash (\alpha\colon x) \;= dp(\Gamma,\ \alpha\colon x \vdash \alpha\colon x) + + dp(\Delta,\ \beta\colon y \vdash \beta\colon y) + 1$$

$$dp(\Gamma,\ \alpha\colon x,\ \Gamma' \vdash \Delta \wedge \alpha\colon x \wedge \Delta') = gr(\Gamma,\ \alpha\colon (x\rhd x),\ \Gamma' \vdash \Delta \wedge \alpha\colon x \wedge \Delta') - 1$$

The depth of a Harmonic Analysis in LLCMA is increased only by the rules ($\rhd$), ($\lhd$) and ($\wedge_R$), but not by the R rule or the contraction rule. This is due to the fact that when there is a progression from a chord to the same chord, we do not want to increase the complexity of the harmonic structure, because the harmony didn't changed. Furthermore, when the harmonic analysis is contracted, the depth of the harmonic set of chords will correspond to that of only one of the two branches of the analysis.

Example 7.11. Let's consider the analysis in Example 7.6, the depth of the Harmonic Analysis will be 3, because we have applied only 3 rules to the derivation.

To track the evolution of the depth in the Harmonic Analysis, we can decorate the turnstiles with a number that increases according to the rules in Figure 7.6. The rules can then be rewritten with decorated sequents indicating the depth of the Harmonic Analysis, as shown in Figure 7.7.

Example 7.12. Let's take the same example as before and decorate the sequents:

$$\dfrac{\dfrac{\dfrac{B\flat 7}{E\flat : V \vdash_0 E\flat : V} \qquad \dfrac{\mathrm{E\flat MA}^7}{E\flat :\mathrm{I} \vdash_0 E\flat :\mathrm{I}}}{E\flat : V,\ E\flat : (V \rhd I) \vdash_1 E\flat :\mathrm{I}}\,(\rhd) \qquad \dfrac{\dfrac{G7}{C : V \vdash_0 C : V} \qquad \dfrac{\mathrm{CMA}^7}{C :\mathrm{I} \vdash_0 C :\mathrm{I}}}{C : V,\ C : (V \rhd I) \vdash_1 C :\mathrm{I}}\,(\rhd)}{E\flat : V,\ E\flat : (V \rhd I),\ E\flat RC,\ C : V,\ C : (V \rhd I) \vdash_3 E\flat :\mathrm{I} \wedge C :\mathrm{I}}\,(\wedge_R)$$

Theorem 7.1 (Uniqueness of Harmonic Analysis)**.** *Given a sequence of chords the number of rules applied to obtain a terminating derivation is the same for every possible derivation.*

Proof. Prove this by induction on the depth of the analysis. The base case, where only one rule is applied, is straightforward by definition. Now, consider a formula $\Gamma \vdash \Delta$ that involves n applications of rules. We need to prove the statement for the case when there are $n+1$ rules. By definition, the depth $n+1$ can only be reached through the application of the rules $(\rhd)$, $(\lhd)$ and $(\wedge_R)$. Therefore, it follows that the depth $n+1$ must necessarily be reached by an application of one of these rules, and no other rules. This means that the final depth of the analysis will remain fixed, even if the analysis is carried out in a different order. □

Example 7.13 (Different choices, different trees, same depth)**.** To understand better the stability of our system let's consider another example. This example introduces an intriguing ambiguity in harmonic analysis, specifically regarding the dual interpretation of the D7 chord in Stella by Starlight (the full example is in the following section). On one hand, D7 can be understood as the dominant of G, adhering to the conventional dominant-to-tonic progression. On the other hand, D7 may be seen as the tonic in a cadence that begins with A7, where A7 functions as its dominant. Despite these distinct interpretive paths, both analyses apply the same number of rules and arrive at the same final sequent, demonstrating the system's ability to accommodate varying harmonic perspectives while maintaining structural consistency.

This ambiguity underscores a broader analytical challenge—determining when chords signify distinct tonalities and when they form part of a larger harmonic structure. The example illustrates the flexibility of the system in handling such ambiguities, raising important considerations for refining the rules, particularly those concerning conjunction (when tonalities diverge) and implication (when harmonic progression is maintained). The stability of the system can be seen both on the final sequent and on the final value of the depth of the Harmonic Analysis.

In the first example, we initially consider D7 as the dominant of $G7^{\sharp 5}$. Before this, we take it into account with $Am7^{(\flat 5)}$, and after applying the $\alpha R \alpha^*$-rule, we observe how the interpretation changes on the right, allowing the application of the ($\triangleright$)-rule. Then, a new application of the $\alpha R \alpha^*$ rule is required to complete the analysis.

$$
\dfrac{
\dfrac{
\dfrac{
\dfrac{A7}{D\!:\!\mathrm{V} \vdash_0 D\!:\!\mathrm{V}}
\qquad
\dfrac{
\dfrac{
\dfrac{Am7^{(\flat 5)}}{G\!:\!\mathrm{II} \vdash_0 G\!:\!\mathrm{II}}
\quad
\dfrac{D7}{G\!:\!\mathrm{V} \vdash_0 G\!:\!\mathrm{V}}
}{G\!:\!\mathrm{II},\ G\!:(\mathrm{II} \triangleright \mathrm{V}) \vdash_1 G\!:\!\mathrm{V}}\,(\triangleright)
}{G\!:\!\mathrm{II},\ G\!:(\mathrm{II} \triangleright \mathrm{V}),\ GRD \vdash_1 D\!:\mathrm{I}}\,(R)
}{D\!:\!\mathrm{V},\ DRG,\ G\!:\!\mathrm{II},\ G\!:(\mathrm{II} \triangleright \mathrm{V}),\ GRD,\ D\!:(\mathrm{V} \triangleright \mathrm{I}) \vdash_2 D\!:\!\mathrm{I}}\,(\triangleright)
}{D\!:\!\mathrm{V},\ DRG,\ G\!:\!\mathrm{II},\ G\!:(\mathrm{II} \triangleright \mathrm{V}),\ GRD,\ D\!:(\mathrm{V} \triangleright \mathrm{I}),\ DRG \vdash_2 G\!:\!\mathrm{V}}\,(R)
\qquad
\dfrac{G7^{\sharp 5}}{G\!:\mathrm{I} \vdash_0 G\!:\mathrm{I}}
}{D\!:\!\mathrm{V},\ DRG,\ G\!:\!\mathrm{II},\ G\!:(\mathrm{II} \triangleright \mathrm{V}),\ GRD,\ D\!:(\mathrm{V} \triangleright \mathrm{I}),\ DRG,\ G\!:(\mathrm{V} \triangleright I) \vdash_3 G\!:\!\mathrm{I}}\,(\triangleright)
$$

In the following example, the D7 is once again interpreted as the dominant of G, but this time we evaluate the first two chords separately, followed by the rest of the harmonic structure. The application of the (R)-rule is less frequent than before, but the final result remains the same.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A7}{D{:}\mathrm{V} \vdash_0 D{:}\mathrm{V}} \quad \dfrac{Am7^{(\flat 5)}}{G{:}\mathrm{II} \vdash_0 G{:}\mathrm{II}}}{D{:}\mathrm{V}\ ,\ G{:}\mathrm{II} \vdash_1 D{:}\mathrm{V} \wedge G{:}\mathrm{II}}(\wedge)_R \quad \dfrac{D7}{G{:}\mathrm{V} \vdash_0 G{:}\mathrm{V}}}{D{:}\mathrm{V},\ DRG,\ G{:}\mathrm{II},\ G{:}\,(\mathrm{II} \rhd \mathrm{V}) \vdash_2 D{:}\mathrm{V} \wedge G{:}\mathrm{V}}(\rhd)}{D{:}\mathrm{V},\ DRG,\ G{:}\mathrm{II},\ G{:}\,(\mathrm{II} \rhd \mathrm{V}),\ GRD \vdash_2 D{:}\mathrm{V} \wedge D{:}\mathrm{I}}(R)}{D{:}\mathrm{V},\ DRG,\ G{:}\mathrm{II},\ G{:}\,(\mathrm{II} \rhd \mathrm{V}),\ GRD,\ D{:}\,(\mathrm{V} \rhd \mathrm{I}) \vdash_2 D{:}\mathrm{V}}(\rhd) \quad \dfrac{G7^{\sharp 5}}{G{:}\ \mathrm{I} \vdash_0 G{:}\ \mathrm{I}}}{D{:}\mathrm{V},\ DRG,\ G{:}\mathrm{II},\ G{:}\,(\mathrm{II} \rhd \mathrm{V}),\ GRD,\ D{:}\,(\mathrm{V} \rhd \mathrm{I}),\ DRG,\ G{:}\,(\mathrm{V} \rhd I) \vdash_3 G{:}\mathrm{I}}(\rhd)$$

In the following example, we initially interpret D7 as the tonic of a cadence on D. We then use the $\alpha R \alpha^*$ rule to shift it into the tonality of G. Thanks to the $(\rhd)$ rule, it becomes straightforward to establish a second accessibility relation within the rule, allowing us to easily achieve the same result.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A7}{D{:}\mathrm{V} \vdash_0 D{:}\mathrm{V}} \quad \dfrac{Am7^{(\flat 5)}}{G{:}\mathrm{II} \vdash_0 G{:}\mathrm{II}}}{D{:}\mathrm{V},\ DRG,\ G{:}\mathrm{II} \vdash_1 D{:}\mathrm{V} \wedge G{:}\mathrm{II}}(\wedge_R)}{D{:}\mathrm{V},\ DRG,\ G{:}\mathrm{II},\ G{:}\,(\mathrm{II} \rhd \mathrm{V}) \vdash_1 D{:}\mathrm{V}}(\rhd)_I \quad \dfrac{D7}{D{:}\mathrm{I} \vdash_0 D{:}\mathrm{I}}}{D{:}\mathrm{V},\ DRG,\ G{:}\mathrm{II},\ G{:}\,(\mathrm{II} \rhd \mathrm{V}),\ GRD,\ D{:}\,(\mathrm{V} \rhd \mathrm{I}) \vdash_2 D{:}\mathrm{I}}(\rhd)}{D{:}\mathrm{V},\ DRG,\ G{:}\mathrm{II},\ G{:}\,(\mathrm{II} \rhd \mathrm{V}),\ GRD,\ D{:}\,(\mathrm{V} \rhd \mathrm{I}),\ DRG \vdash_2 G{:}\mathrm{V}}(R) \quad \dfrac{G7^{\sharp 5}}{G{:}\ \mathrm{I} \vdash_0 G{:}\ \mathrm{I}}}{D{:}\mathrm{V},\ DRG,\ G{:}\mathrm{II},\ G{:}\,(\mathrm{II} \rhd \mathrm{V}),\ GRD,\ D{:}\,(\mathrm{V} \rhd \mathrm{I}),\ DRG,\ G{:}\,(\mathrm{V} \rhd I) \vdash_3 G{:}\mathrm{I}}(\rhd)$$

7.6.3 Stella By Starlight

To exemplify the power of LLCMA, we will illustrate a famous song, namely "Stella By Starlight" (Figure 7.8. We will analyze the structure of the song and calculate its complexity using the notion of depth introduced in the previous section.

As demonstrated earlier, we can start our analysis from any set of chords. Therefore, we will begin our analysis from the third chord and continue through measure 9, before examining measures 10-12. This approach allows us to highlight the significance of the first two chords, specifically the perfect cadence in D minor found at measure 13.

Initial Sequent:

$$\frac{\text{Chord}}{\alpha\colon x \vdash_0 \alpha\colon x}$$

Depth of the trees:

$$dp(\mathfrak{a}) = dp(\Gamma,\ \alpha\colon x,\ \Gamma' \vdash \Delta \wedge \alpha\colon x \wedge \Delta')$$

$$dp(\mathfrak{b}) = dp(\Pi,\ \alpha\colon y,\ \Pi' \vdash \Phi \wedge \alpha\colon y \wedge \Phi')$$

$$dp(\mathfrak{c}) = dp(\Gamma,\ \alpha\colon x,\ \Gamma' \vdash \Delta \wedge \alpha\colon y \wedge \Delta')$$

$$dp(\mathfrak{d}) = dp(\beta\colon y,\ \Pi' \vdash \alpha\colon x \wedge \Phi)$$

Rules:

$$\frac{\Gamma,\ \alpha\colon x,\ \Gamma' \vdash_{\mathrm{dp}(\mathfrak{a})} \Delta \wedge \alpha\colon x \wedge \Delta' \qquad \Pi,\ \alpha\colon y,\ \Pi' \vdash_{\mathrm{dp}(\mathfrak{b})} \Phi \wedge \alpha\colon y \wedge \Phi'}{\Gamma, \alpha\colon x,\ \Gamma',\ \Pi, \alpha\colon (x \triangleright y),\ \alpha R \alpha_{\Pi'}, \Pi' \vdash_{\mathrm{dp}(\mathfrak{a})+\mathrm{dp}(\mathfrak{b})+1} \Delta \wedge \Delta' \wedge \Phi \wedge \alpha\colon y \wedge \Phi'}\ (\triangleright)$$

$$\frac{\Gamma,\ \alpha\colon x,\ \Gamma' \vdash_{\mathrm{dp}(\mathfrak{c})} \Delta \wedge \alpha\colon y \wedge \Delta'}{\Gamma,\ \alpha\colon (x \triangleright y),\ \Gamma' \vdash_{\mathrm{dp}(\mathfrak{c})} \Delta \wedge \Delta'}\ (\triangleright_I)$$

$$\frac{\Gamma,\ \alpha\colon x,\ \Gamma' \vdash_{\mathrm{dp}(\mathfrak{a})} \Delta \wedge \alpha\colon x \wedge \Delta'}{\Gamma,\ \alpha\colon x,\ \alpha R \alpha^{\star},\ \Gamma' \vdash_{\mathrm{dp}(\mathfrak{a})} \Delta \wedge \alpha^{\star}\colon x^{\star} \wedge \Delta'}\ (R)$$

$$\frac{\Gamma,\ \alpha\colon x \vdash_{\mathrm{dp}(\mathfrak{a})} \Delta \wedge \alpha\colon x \qquad \beta\colon y,\ \Pi' \vdash_{\mathrm{dp}(\mathfrak{b})} \beta\colon y \wedge \Phi}{\Gamma,\ \alpha\colon x,\ \alpha R \beta,\ \beta\colon y,\ \Pi' \vdash_{\mathrm{dp}(\mathfrak{a})+\mathrm{dp}(\mathfrak{b})+1} \Delta \wedge (\alpha\colon x) \wedge (\beta\colon y) \wedge \Phi}\ (\wedge_R)$$

$$\frac{\Gamma,\ \alpha\colon x \vdash_{\mathrm{dp}(\mathfrak{a})} \Delta \wedge \alpha\colon x \qquad \beta\colon y,\ \Pi' \vdash_{\mathrm{dp}(\mathfrak{d})} \alpha\colon x \wedge \Phi}{\Gamma,\ \alpha\colon x,\ \alpha R \beta,\ \beta\colon y,\ \Pi' \vdash_{\mathrm{dp}(\mathfrak{a})+\mathrm{dp}(\mathfrak{d})} \Delta \wedge (\alpha\colon x) \wedge \Phi}\ (melt.)$$

$$\frac{\Gamma,\ \alpha\colon x,\ \Gamma' \vdash_{\mathrm{dp}(\mathfrak{a})} \Delta \wedge \alpha\colon x \wedge \Delta' \qquad \Gamma,\ \alpha\colon x,\ \Gamma' \vdash_{\mathrm{dp}(\mathfrak{a})} \Delta \wedge \alpha\colon x \wedge \Delta'}{\Gamma,\ \alpha\colon x,\ \Gamma' \vdash_{\mathrm{dp}(\mathfrak{a})} \Delta \wedge \alpha\colon x \wedge \Delta'}\ (ctr.)$$

With $\alpha \neq \beta$ and $x \neq y$.

Figure 7.7: Rules of LLCMA, with the decorated sequents that track the depth of the Harmonic Analysis.

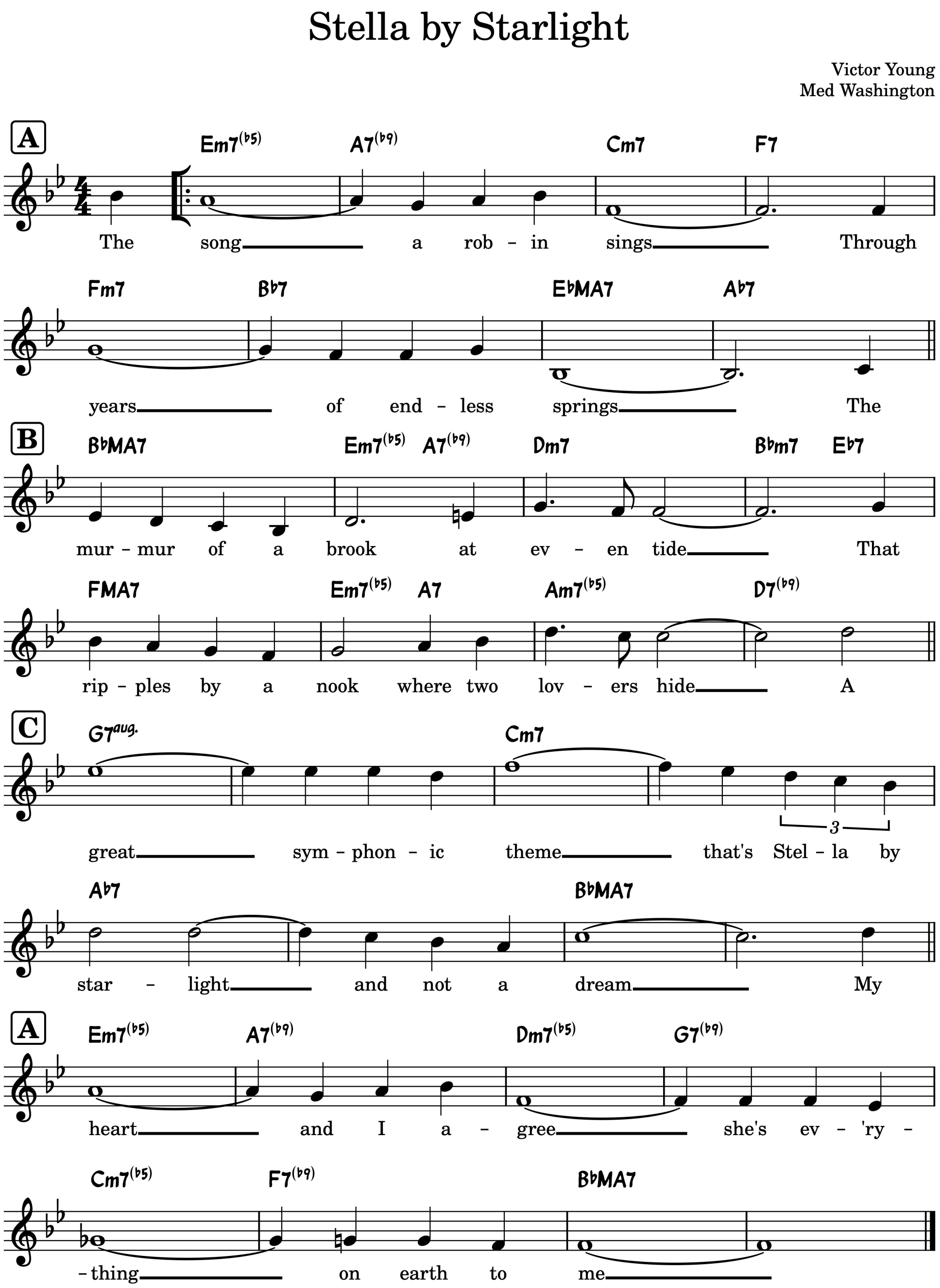


Figure 7.8: The jazz standard “Stella by Starlight” written by Victor Young and Med Washington.

A :

$$\frac{\text{Fm7}}{E\flat{:}\mathrm{II} \vdash_0 E\flat{:}\mathrm{II}} \qquad \frac{B\flat 7}{E\flat{:}\mathrm{V} \vdash_0 E\flat{:}\mathrm{V}}$$

$$\frac{Cm7}{B\flat{:}\mathrm{II} \vdash_0 B\flat{:}\mathrm{II}} \quad \frac{F7}{B\flat{:}\mathrm{V} \vdash_0 B\flat{:}\mathrm{V}} \;(\rhd) \qquad E\flat{:}\mathrm{II},\ E\flat{:}\,(\mathrm{II} \rhd \mathrm{V}) \vdash_1 E\flat{:}\mathrm{V} \;(\rhd) \quad \frac{E\flat\ {\scriptstyle \mathrm{MA}}^7}{E\flat{:}\mathrm{I} \vdash_0 E\flat{:}\mathrm{I}} \;(\rhd) \qquad \frac{A\flat 7}{B\flat{:}\ \mathrm{VII}\flat \vdash_0 B\flat{:}\ \mathrm{VII}\flat} \quad \frac{B\flat \mathrm{MA}^7}{B\flat{:}\mathrm{I} \vdash_0 B\flat{:}\mathrm{I}}$$

$$B\flat{:}\mathrm{II},\ B\flat{:}\,(\mathrm{II} \rhd \mathrm{V}) \vdash_1 B\flat{:}\mathrm{V} \qquad E\flat{:}\mathrm{II},\ E\flat{:}\,(\mathrm{II} \rhd \mathrm{V}),\ E\flat{:}\,(\mathrm{V} \rhd \mathrm{I}) \vdash_2 E\flat{:}\ \mathrm{I} \;(\wedge_R) \qquad B\flat{:}\ \mathrm{VII}\flat,\ B\flat{:}\,(\ \mathrm{VII}\flat \rhd I) \vdash_1 B\flat{:}\mathrm{I} \;(\rhd)$$

$$B\flat{:}\mathrm{II},\ B\flat{:}\,(\mathrm{II} \rhd \mathrm{V}),\ B\flat R E\flat,\ E\flat{:}\mathrm{II},\ E\flat{:}\,(\mathrm{II} \rhd \mathrm{V}),\ E\flat{:}\,(\mathrm{V} \rhd \mathrm{I}) \vdash_4 B\flat{:}\mathrm{V} \wedge E\flat{:}\ \mathrm{I} \;(\rhd)$$

$$B\flat{:}\mathrm{II},\ B\flat{:}\,(\mathrm{II} \rhd \mathrm{V}),\ B\flat R E\flat,\ E\flat{:}\mathrm{II},\ E\flat{:}\,(\mathrm{II} \rhd \mathrm{V}),\ E\flat{:}\,(\mathrm{V} \rhd \mathrm{I}),\ B\flat{:}\ \mathrm{VII}\flat,\ B\flat{:}\,(\ \mathrm{VII}\flat \rhd I),\ B\flat{:}\,(\mathrm{V} \rhd I) \vdash_6 E\flat{:}\mathrm{I} \wedge B\flat{:}\mathrm{I}$$

B :

$$\frac{Em7^{(\flat 5)}}{d{:}\mathrm{II} \vdash_0 d{:}\mathrm{II}} \quad \frac{A7}{d{:}\mathrm{V} \vdash_0 d{:}\mathrm{V}} \;(\rhd) \qquad \boxed{\text{A}}\text{-analysis} \qquad \frac{Em7^{(\flat 5)}}{d{:}\mathrm{II} \vdash_0 d{:}\mathrm{II}} \quad \frac{A7}{d{:}\mathrm{V} \vdash_0 d{:}\mathrm{V}} \;(\rhd) \qquad \frac{Dm}{d{:}\mathrm{I} \vdash_0 d{:}\mathrm{I}}$$

$$d{:}\mathrm{II},\ d{:}\,(\mathrm{II} \rhd \mathrm{V}) \vdash_1 d{:}\mathrm{V} \qquad B\flat{:}\mathrm{II},\ \ldots,\ B\flat{:}\,(\mathrm{V} \rhd \mathrm{I}) \vdash_6 E\flat{:}\mathrm{I} \wedge B\flat{:}\mathrm{I} \;(\wedge_R) \qquad d{:}\mathrm{II},\ d{:}\,(\mathrm{II} \rhd \mathrm{V}) \vdash_1 d{:}\mathrm{V} \;(\rhd)$$

$$d{:}\mathrm{II},\ d{:}\,(\mathrm{II} \rhd \mathrm{V}),\ d R B\flat,\ B\flat{:}\mathrm{II},\ \ldots,\ B\flat{:}\,(\mathrm{V} \rhd I) \vdash_8 d{:}\mathrm{V} \wedge E\flat{:}\mathrm{I} \wedge B\flat{:}\mathrm{I} \qquad d{:}\mathrm{II},\ d{:}\,(\mathrm{II} \rhd \mathrm{V}),\ d{:}\,(\mathrm{V} \rhd I) \vdash_2 d{:}\mathrm{I} \;(\rhd)$$

$$^{(*2)}d{:}\mathrm{II},\ d{:}\,(\mathrm{II} \rhd \mathrm{V}),\ d R B\flat,\ B\flat{:}\mathrm{II},\ \ldots,\ B\flat{:}\,(\mathrm{V} \rhd I),\ d{:}\,(\mathrm{II} \rhd \mathrm{V}),\ d{:}\,(\mathrm{V} \rhd I) \vdash_{11} E\flat{:}\mathrm{I} \wedge B\flat{:}\mathrm{I} \wedge d{:}\mathrm{I}$$

C :

$$\dfrac{\dfrac{\dfrac{B\flat m7}{F:\mathrm{iv}\vdash_0 F:\mathrm{iv}} \qquad \dfrac{E\flat 7 \Rightarrow F:\mathrm{VII}\flat}{F:\mathrm{VII}\flat \vdash_0 F:\mathrm{VII}\flat}}{F:\mathrm{iv},\ F:(\mathrm{iv}\triangleright\mathrm{VII}\flat)\vdash_1 F:\mathrm{VII}\flat}(\triangleright) \qquad \dfrac{F_{\mathrm{MA}}{}^{7}}{F:\mathrm{I}\vdash_0 F:\mathrm{I}}}{F:\mathrm{iv},\ F:(\mathrm{iv}\triangleright\mathrm{VII}\flat),\ F:(\mathrm{VII}\flat\triangleright\mathrm{I})\vdash_2 F:\mathrm{I}}(\triangleright)$$

D :

$$\dfrac{\dfrac{\dfrac{Em7^{(\flat 5)}}{d:\mathrm{II}\vdash_0 d:\mathrm{II}}}{d:\mathrm{II},\ dRD\vdash_0 D:\mathrm{II}}(R) \qquad \dfrac{A7}{D:\mathrm{V}\vdash_0 D:\mathrm{V}}}{d:\mathrm{II},\ dRD,\ D:(\mathrm{II}\triangleright\mathrm{V})\vdash_1 D:\mathrm{V}}$$

$$\dfrac{\dfrac{\dfrac{Am7^{(\flat 5)}}{G:\mathrm{II}\vdash_0 G:\mathrm{II}} \qquad \dfrac{D7}{G:\mathrm{V}\vdash_0 G:\mathrm{V}}}{G:\mathrm{II},\ G:(\mathrm{II}\triangleright\mathrm{V})\vdash_1 G:\mathrm{V}}(\triangleright)}{G:\mathrm{II},\ G:(\mathrm{II}\triangleright\mathrm{V}),\ GRD\vdash_1 D:\mathrm{I}}(R)$$

$$\dfrac{d:\mathrm{II},\ dRD,\ D:(\mathrm{II}\triangleright\mathrm{V})\vdash_1 D:\mathrm{V} \qquad G:\mathrm{II},\ G:(\mathrm{II}\triangleright\mathrm{V}),\ GRD\vdash_1 D:\mathrm{I}}{\dfrac{D:\mathrm{V},\ G:\mathrm{II},\ G:(\mathrm{II}\triangleright\mathrm{V}),\ GRD,\ D:(\mathrm{V}\triangleright\mathrm{I})\vdash_3 D:\mathrm{I}}{D:\mathrm{V},\ G:\mathrm{II},\ G:(\mathrm{II}\triangleright\mathrm{V}),\ GRD,\ D:(\mathrm{V}\triangleright\mathrm{I}),\ DRG\vdash_3 G:\mathrm{V}}(R)}(\triangleright)$$

$$\dfrac{D:\mathrm{V},\ G:\mathrm{II},\ G:(\mathrm{II}\triangleright\mathrm{V}),\ GRD,\ D:(\mathrm{V}\triangleright\mathrm{I}),\ DRG\vdash_3 G:\mathrm{V} \qquad \dfrac{G7^{\sharp 5}}{G:\mathrm{I}\vdash_0 G:\mathrm{I}}}{\dfrac{D:\mathrm{V},\ G:\mathrm{II},\ G:(\mathrm{II}\triangleright\mathrm{V}),\ GRD,\ D:(\mathrm{V}\triangleright\mathrm{I}),\ DRG,\ G:(\mathrm{V}\triangleright I)\vdash_4 G:\mathrm{I}}{d:\mathrm{II},\ dRD,\ D:(\mathrm{II}\triangleright\mathrm{V}),\ G:\mathrm{II},\ G:(\mathrm{II}\triangleright\mathrm{V}),\ GRD,\ D:(\mathrm{V}\triangleright I),\ G:(\mathrm{V}\triangleright I)GRc\vdash_4 c:\mathrm{V}}(R)}(\triangleright)$$

$$\dfrac{d:\mathrm{II},\ dRD,\ D:(\mathrm{II}\triangleright\mathrm{V}),\ G:\mathrm{II},\ G:(\mathrm{II}\triangleright\mathrm{V}),\ GRD,\ D:(\mathrm{V}\triangleright I),\ G:(\mathrm{V}\triangleright I)GRc\vdash_4 c:\mathrm{V} \qquad \dfrac{Cm7}{c:\mathrm{I}\vdash_0 c:\mathrm{I}}}{\dfrac{d:\mathrm{II},\ dRD,\ D:(\mathrm{II}\triangleright\mathrm{V}),\ G:\mathrm{II},\ G:(\mathrm{II}\triangleright\mathrm{V}),\ GRD,\ D:(\mathrm{V}\triangleright I),\ G:(\mathrm{V}\triangleright\mathrm{I}),\ GRc,\ c:(\mathrm{V}\triangleright\mathrm{I})\vdash_5 c:\mathrm{I}}{d:\mathrm{II},\ dRD,\ D:(\mathrm{II}\triangleright\mathrm{V}),\ G:\mathrm{II},\ G:(\mathrm{II}\triangleright\mathrm{V}),\ GRD,\ D:(\mathrm{V}\triangleright\mathrm{I}),\ G:(\mathrm{V}\triangleright\mathrm{I}),\ GRc,\ c:(\mathrm{V}\triangleright\mathrm{I}),\ cRB\flat\vdash_5 B\flat:\mathrm{II}}(R)}$$

$$\dfrac{\dfrac{A\flat 7}{B\flat:\mathrm{VII}\flat\vdash_0 B\flat:\mathrm{VII}\flat} \qquad \dfrac{B\flat_{\mathrm{MA}}{}^{7}}{B\flat:\mathrm{I}\vdash_0 B\flat:\mathrm{I}}}{{}^{(*2)}B\flat:\mathrm{VII}\flat,\ B\flat:(\ \mathrm{VII}\flat\triangleright I)\vdash_1 B\flat:\mathrm{I}}(\triangleright)$$

$$\dfrac{d:\mathrm{II},\ dRD,\ \ldots,\ cRB\flat\vdash_5 B\flat:\mathrm{II} \qquad {}^{(*2)}B\flat:\mathrm{VII}\flat,\ B\flat:(\ \mathrm{VII}\flat\triangleright I)\vdash_1 B\flat:\mathrm{I}}{d:\mathrm{II},\ dRD,\ D:(\mathrm{II}\triangleright\mathrm{V}),\ G:\mathrm{II},\ G:(\mathrm{II}\triangleright\mathrm{V}),\ GRD,\ D:(\mathrm{V}\triangleright I),\ G:(\mathrm{V}\triangleright I),\ GRc,\ c:(\mathrm{V}\triangleright I),\ cRB\flat,\ B\flat:(\mathrm{II}\triangleright\mathrm{VII}\flat),\ B\flat:(\ \mathrm{VII}\flat\triangleright I)\vdash_7 B\flat:\mathrm{I}}$$

$\boxed{\mathrm{E}}$:

$$\dfrac{\dfrac{\dfrac{Em7^{(\flat 5)}}{d{:}\mathrm{II} \vdash_0 d{:}\mathrm{II}} \quad \dfrac{A7}{d{:}\mathrm{V} \vdash_0 d{:}\mathrm{V}}}{d{:}\mathrm{II},\ d{:}\,(\mathrm{II} \triangleright \mathrm{V}) \vdash_1 d{:}\mathrm{V}}\,(\triangleright) \quad \dfrac{Dm^{\flat 5}}{d{:}\mathrm{I} \vdash_0 d{:}\mathrm{I}}}{d{:}\mathrm{II},\ d{:}\,(\mathrm{II} \triangleright \mathrm{V}),\ d{:}\,(\mathrm{V} \triangleright \mathrm{I}) \vdash_2 d{:}\mathrm{I}}\,(\triangleright)$$

$$\dfrac{d{:}\mathrm{II},\ d{:}\,(\mathrm{II} \triangleright \mathrm{V}),\ d{:}\,(\mathrm{V} \triangleright \mathrm{I}) \vdash_2 d{:}\mathrm{I}}{d{:}\mathrm{II},\ d{:}\,(\mathrm{II} \triangleright \mathrm{V}),\ d{:}\,(\mathrm{V} \triangleright \mathrm{I}),\ dRC \vdash_2 C{:}\mathrm{II}}\,(R) \quad \dfrac{G7}{G{:}\,\mathrm{I} \vdash_0 G{:}\,\mathrm{I}}$$

$$\dfrac{d{:}\mathrm{II},\ d{:}\,(\mathrm{II} \triangleright \mathrm{V}),\ d{:}\,(\mathrm{V} \triangleright \mathrm{I}),\ dRC \vdash_2 C{:}\mathrm{II} \quad G{:}\,\mathrm{I} \vdash_0 G{:}\,\mathrm{I}}{d{:}\mathrm{II},\ d{:}\,(\mathrm{II} \triangleright \mathrm{V}),\ d{:}\,(\mathrm{V} \triangleright \mathrm{I}),\ dRC,\ G{:}\,(\mathrm{II} \triangleright \mathrm{V}) \vdash_3 G{:}\mathrm{I}}\,(\triangleright)$$

$$\dfrac{d{:}\mathrm{II},\ d{:}\,(\mathrm{II} \triangleright \mathrm{V}),\ d{:}\,(\mathrm{V} \triangleright \mathrm{I}),\ dRC,\ G{:}\,(\mathrm{II} \triangleright \mathrm{V}) \vdash_3 G{:}\mathrm{I}}{d{:}\mathrm{II},\ d{:}\,(\mathrm{II} \triangleright \mathrm{V}),\ d{:}\,(\mathrm{V} \triangleright \mathrm{I}),\ dRC,\ G{:}\,(\mathrm{II} \triangleright \mathrm{V}),\ GRc \vdash_3 c{:}\mathrm{V}}\,(R) \quad \dfrac{Cm7^{\flat 5}}{c{:}\,I \vdash_0 c{:}\,I}$$

$$\dfrac{d{:}\mathrm{II},\ d{:}\,(\mathrm{II} \triangleright \mathrm{V}),\ d{:}\,(\mathrm{V} \triangleright \mathrm{I}),\ dRC,\ G{:}\,(\mathrm{II} \triangleright \mathrm{V}),\ GRc \vdash_3 c{:}\mathrm{V} \quad c{:}\,I \vdash_0 c{:}\,I}{d{:}\mathrm{II},\ d{:}\,(\mathrm{II} \triangleright \mathrm{V}),\ d{:}\,(\mathrm{V} \triangleright \mathrm{I}),\ dRC,\ c{:}\,(\mathrm{II} \triangleright \mathrm{V}),\ GRc,\ c{:}\,(\mathrm{V} \triangleright \mathrm{I}) \vdash_4 c{:}\mathrm{I}}\,(\triangleright)$$

$$\dfrac{d{:}\mathrm{II},\ d{:}\,(\mathrm{II} \triangleright \mathrm{V}),\ d{:}\,(\mathrm{V} \triangleright \mathrm{I}),\ dRC,\ c{:}\,(\mathrm{II} \triangleright \mathrm{V}),\ GRc,\ c{:}\,(\mathrm{V} \triangleright \mathrm{I}) \vdash_4 c{:}\mathrm{I}}{d{:}\mathrm{II},\ d{:}\,(\mathrm{II} \triangleright \mathrm{V}),\ d{:}\,(\mathrm{V} \triangleright \mathrm{I}),\ dRC,\ c{:}\,(\mathrm{II} \triangleright \mathrm{V}),\ GRc,\ c{:}\,(\mathrm{V} \triangleright \mathrm{I}),\ cRB\flat \vdash_4 B\flat{:}\mathrm{II}}\,(R) \quad \dfrac{F7}{B\flat{:}\,V \vdash_0 B\flat{:}\,V}$$

$$\dfrac{d{:}\mathrm{II},\ d{:}\,(\mathrm{II} \triangleright \mathrm{V}),\ d{:}\,(\mathrm{V} \triangleright \mathrm{I}),\ dRC,\ c{:}\,(\mathrm{II} \triangleright \mathrm{V}),\ GRc,\ c{:}\,(\mathrm{V} \triangleright \mathrm{I}),\ cRB\flat \vdash_4 B\flat{:}\mathrm{II} \quad B\flat{:}\,V \vdash_0 B\flat{:}\,V}{d{:}\mathrm{II},\ d{:}\,(\mathrm{II} \triangleright \mathrm{V}),\ d{:}\,(\mathrm{V} \triangleright \mathrm{I}),\ dRC,\ c{:}\,(\mathrm{II} \triangleright \mathrm{V}),\ GRc,\ c{:}\,(\mathrm{V} \triangleright \mathrm{I}),\ cRB\flat,\ B\flat{:}\,(\mathrm{V} \triangleright \mathrm{I}) \vdash_5 B\flat{:}\mathrm{II}}\,(\triangleright) \quad \dfrac{B\flat}{B\flat{:}\,I \vdash_0 B\flat{:}\,I}$$

$$\dfrac{d{:}\mathrm{II},\ d{:}\,(\mathrm{II} \triangleright \mathrm{V}),\ d{:}\,(\mathrm{V} \triangleright \mathrm{I}),\ dRC,\ c{:}\,(\mathrm{II} \triangleright \mathrm{V}),\ GRc,\ c{:}\,(\mathrm{V} \triangleright \mathrm{I}),\ cRB\flat,\ B\flat{:}\,(\mathrm{V} \triangleright \mathrm{I}) \vdash_5 B\flat{:}\mathrm{II} \quad B\flat{:}\,I \vdash_0 B\flat{:}\,I}{d{:}\mathrm{II},\ d{:}\,(\mathrm{II} \triangleright \mathrm{V}),\ d{:}\,(\mathrm{V} \triangleright \mathrm{I}),\ dRC,\ c{:}\,(\mathrm{II} \triangleright \mathrm{V}),\ GRc,\ c{:}\,(\mathrm{V} \triangleright \mathrm{I}),\ cRB\flat,\ B\flat{:}\,(\mathrm{II} \triangleright \mathrm{V}),\ B\flat{:}\,(\mathrm{V} \triangleright \mathrm{I}) \vdash_6 B\flat{:}\mathrm{I}}\,(\triangleright)$$

$$\boxed{\mathrm{B}}\quad d{:}\mathrm{II},\ldots,\ d{:}(\mathrm{V}\triangleright I)\vdash_{11} E\flat{:}\mathrm{I}\wedge B\flat{:}\mathrm{I}\wedge d{:}\mathrm{I}$$

$$\boxed{\mathrm{C}}\quad F{:}\mathrm{iv},\ F{:}(\mathrm{iv}\triangleright \mathrm{VII}\flat),\ F{:}(\mathrm{VII}\flat\triangleright\mathrm{I})\vdash_{2} F{:}\mathrm{I}$$

$$\frac{\boxed{\mathrm{B}}\qquad \boxed{\mathrm{C}}}{d{:}\mathrm{II},\ldots,\ d{:}(\mathrm{V}\triangleright I),\ F{:}\mathrm{iv},\ldots,\ F{:}(\mathrm{VII}\flat\triangleright\mathrm{I}),\ FRE\flat\vdash_{14} E\flat{:}\mathrm{I}\wedge B\flat{:}\mathrm{I}\wedge d{:}\mathrm{I}\wedge F{:}\mathrm{I}}\ (\wedge_R)$$

$$\boxed{\mathrm{D}}\quad d{:}\mathrm{II}\ ,\ldots,\ B\flat{:}(\ \mathrm{VII}\flat\flat\triangleright I)\vdash_{7} B\flat{:}\mathrm{I}$$

$$\frac{\cdots\qquad \boxed{\mathrm{D}}}{d{:}\mathrm{II},\ldots,\ FRE\flat,\ d{:}\mathrm{II}\ ,\ldots,\ B\flat{:}(\ \mathrm{VII}\flat\flat\triangleright I)\vdash_{22} E\flat{:}\mathrm{I}\wedge B\flat{:}\mathrm{I}\wedge d{:}\mathrm{I}\wedge F{:}\mathrm{I}\wedge B\flat{:}\mathrm{I}}\ (\wedge_R)$$

$$\boxed{\mathrm{E}}\quad d{:}\mathrm{II}\ ,\ldots,\ B\flat{:}(\mathrm{V}\triangleright\mathrm{I})\vdash_{6} B\flat{:}\mathrm{I}$$

$$\frac{\cdots\qquad \boxed{\mathrm{E}}}{d{:}\mathrm{II},\ldots,\ B\flat{:}(\ \mathrm{VII}\flat\flat\triangleright I),\ d{:}\mathrm{II}\ ,\ldots,\ B\flat{:}(\mathrm{V}\triangleright\mathrm{I})\vdash_{29} E\flat{:}\mathrm{I}\wedge B\flat{:}\mathrm{I}\wedge d{:}\mathrm{I}\wedge F{:}\mathrm{I}\wedge B\flat{:}\mathrm{I}}\ (melt.)$$

Figure 7.9: This tree illustrates the section in its entirety, obtained by combining the various derivations presented in the previous parts.

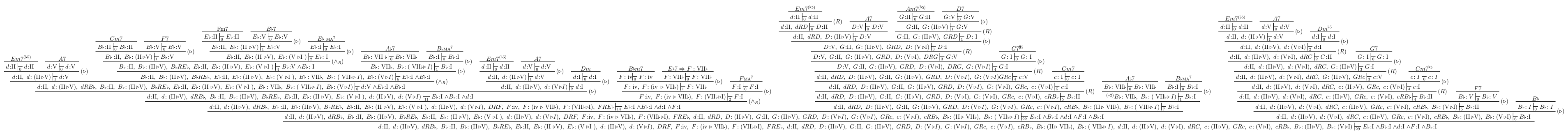


Figure 7.10: This tree combines all the derivations of “Stella by Starlight” into a single structure. Although in practice it is almost impossible to read in a printed version, it represents the cumulative result of all the steps we have introduced so far. It highlights the general structure of the song.

7.7 Labelled-Free Lambek Calculus

While the labelled system introduced in the previous section provides a precise way to track tonalities, degrees, and accessibility relations, it can also become rather heavy in practice. Each step of the analysis requires carrying explicit information about the tonal centre, which is useful for clarity but not always necessary. For many purposes, a lighter system that abstracts away from labels is sufficient, provided it still captures the structural relations between chords.

The initial system was conceived in a labelled style, inspired by modal sequent calculi. However, we soon realised that the operators $\Box$ and $\Diamond$ could not be introduced in a structurally consistent way within our sequent framework. Consequently, we opted for a lighter, unlabelled version, which preserves the harmonic interpretation while avoiding the modal inconsistencies inherent in the labelled approach.

This transition marks a shift from a representational model of harmonic space to a purely structural one: the calculus now focuses on the inferential relations between chords rather than on the explicit encoding of tonal contexts.

The *Labelled-Free Lambek Calculus* aims at this simplification. By removing explicit labels, the calculus focuses on the pure logical structure of harmonic progressions, highlighting the hierarchical relations and the syntactic properties of chord sequences without being tied to a specific tonal framework. In this way, it connects more directly to the tradition of categorial grammar and to the original formulation of Lambek Calculus, while still preserving the essential insights obtained in the labelled version.

The advantage of this approach is twofold: on the one hand, it provides a cleaner and more elegant formalism, closer to the methods of proof theory and linguistic analysis; on the other hand, it offers a flexible tool that can be applied beyond strictly tonal contexts, opening the possibility of extending the method to modal or non-tonal music.

7.7.1 Grammar and Calculus

The goal of this subsection is to outline the basic principles of categorial grammar and of the Labelled-Free Lambek Calculus, emphasizing their proof-theoretic foundations. This background will serve as a bridge between the labelled-free version introduced above and the application to harmonic progressions. By recalling the fundamental rules of grammar and calculus, we will be able to show how linguistic methods can be adapted to music, treating chord sequences as syntactic objects that can be parsed and analyzed through logical inference.

Categorial Grammar

Categorial grammar (CG) [70] consists of a set of categories, assigned to each lexical word. A category is constructed recursively from a given basic category, with the following (/) and (\); let X and Y be categories, then

$$Y/X: \text{biting } X \text{ from the right-hand side, produces } Y,$$
$$Y\backslash X: \text{biting } Y \text{ from the left-hand side, produces } X.$$

We can argue if a given sequence of categories is *grammatical*, dependent on if we can finally obtain sentence category S, e.g.,

$$Y,\ Y\backslash(S/X),\ X \Longrightarrow S/X,\ X \Longrightarrow S.$$

A human language is said to belong approximately to the class of context-free grammar (CFG) in the Chomsky hierarchy [48], though several exceptions are known. The generative power of CG is equivalent to CFG, shown as follows. A set of production rules of CFG can be rewritten in Chomsky normal form [55], each of which produces two non-terminal symbols (Y, Z below) or one terminal symbol (w) from a non-terminal symbol (X) as below

$$X \to Y\ Z, \quad X \to w,$$

and the branching in a syntactic tree becomes always binary except for terminal symbols at the leaves. The nodes in the tree are non-terminal symbols of CFG, which correspond to categories in CG. We can replace each CFG rule $X \to Y\ Z$ either for $Z = Y\backslash X$ or for $Y = X/Z$, and the resulting set of categories becomes equivalent to CFG.

For example, a verb phrase (VP) bites a subject noun phrase (NP) from left to be S, VP = NP\S. Also, a determiner (Det) bites a noun from right to be NP, Det = NP/N. Thus, we can compose a sentence as follows.

$$\cfrac{\cfrac{\overset{\text{A}}{\text{NP/N}} \quad \overset{\text{bird}}{\text{N}}}{\text{NP}} \quad \overset{\text{flies}}{\text{NP}\backslash\text{S}}}{\text{S}}$$

Lambek Calculus

We now translate the categorial constructions into a Gentzen-style sequent calculus. The turnstile symbol ($\vdash$) is read as "derives," and commas on the left-hand side denote logical conjunctions ($\wedge$). In accordance with intuitionistic logic, the right-hand side of the sequent is restricted to a single formula [68]. This system is known as the Lambek Calculus [58].

In this formalism, lowercase Latin letters $(x, y, z, \dots)$ denote individual formulas, while uppercase Greek letters (e.g., Γ, Δ, Σ) represent sequences (or multisets) of formulas on the left-hand side of a sequent.

For example, consider the sentence "A bird flies." In categorial grammar, this can be expressed as a derivation using NP/N, N, and $NP\backslash S$. This construction is translated into the sequent calculus as follows:

$$\cfrac{\cfrac{\text{NP} \vdash \text{NP} \qquad \text{N} \vdash \text{N}}{\text{NP/N, N} \vdash \text{NP}}\,(/_L) \qquad \text{S} \vdash \text{S}}{\text{NP/N, N, NP}\backslash\text{S} \vdash \text{S}}\,(\backslash_L)$$

In this example, each category is introduced by an axiom and combined through inference rules, forming a syntactic proof that corresponds to the grammatical derivation.

Modal Logic

Modality is variously interpreted in music [92], and especially modal extension of Lambek Calculus [67] is useful to capture modulation in music, so we introduce a modal operator '$\Box$' by modal logic [50].

Modal logic is classically interpreted via Kripke semantics [13], which assumes a set of possible worlds connected by an accessibility relation R. A proposition P holds in a world w is written as $w \Vdash P$, and $\Box P$ is interpreted as:

$$w \Vdash \Box P \iff \forall w' \, ({}_wR_{w'}) \, w' \Vdash P$$

where ${}_wR_{w'}$ states that w' is accessible from w. Namely, $\Box P$ holds in w if P holds in all the accessible worlds w' from w. Usual modal logic provides also

$\Diamond$-operator as the dual of '$\Box$' as $\Diamond = \neg\Box\neg$, however, since Lambek Calculus does not possess negation ($\neg$) we do not include '$\Diamond$' in our system.

In linguistics, $\Box$-operator can represent hypothetical contexts (e.g., conditionals or beliefs). A clause that has type S may be embedded as '$\Box S$' in another sentence, meaning it is evaluated in a different context.

In our framework, we treat musical keys as possible worlds. For example, given a set of diatonic notes under the chord name C, we may interpret it as the IV-th degree in the key of G major, written as G $\Vdash$ IV(C). In music, we employ '$\Box$' to express the shift to another tonality. We formalize modulations as transitions between possible tonal worlds, using modal operators to encode harmonic motion across key regions, that will be discussed in Section 7.7.2.

7.7.2 Multiple Accessibility in Tonal Pitch Space

To formally model modulations in tonal music, we introduce a family of modal operators $\Box_X$, each corresponding to a specific tonal relationship. These operators allow us to capture structured transitions between keys, inspired by accessibility relations in Kripke semantics.

In our system, the tonal pitch space is structured according to traditional music theory. For instance:

- $\Box_D$ denotes modulation to the dominant key,
- $\Box_S$ denotes modulation to the subdominant key,
- $\Box_R$ denotes modulation to the relative major or minor,
- $\Box_P$ denotes modulation to the parallel key (same tonic, opposite mode).

Each modal operator corresponds to an accessibility rule in a directed graph of tonal centers. Applying $\Box_X$ to a chord function F means interpreting F in the new tonal center reached via modulation type X. For example, the chord G7 in the key of C can be interpreted as $\Box_D$(I), since G is the dominant of C and functions as I in the key of G.

These modal annotations enrich the expressive power of Lambek derivations. Rather than flattening all harmonic functions into a single key, we enable layered reasoning across multiple tonal centers. This is especially useful in jazz, where temporary tonicizations and frequent modulations are structurally significant.

We also adopt a simplification principle: when multiple derivations exist, the one with the fewest modal transitions (or the shallowest modal nesting) is preferred. This principle of minimality complements the proof depth metric introduced in the next section.

Key Space

Figure 7.11 illustrates the space of accessible keys, based on the Tonal Pitch Space model [60]. Similar ideas are also found in [14, 34, 56].

We use $R_{\mathbb{D}}$, $R_{\mathbb{S}}$, $R_{\mathbb{P}}$, and $R_{\mathbb{R}}$ to represent modulations to the dominant, subdominant, parallel, and relative keys, respectively. It is important to distinguish that $\mathbb{D}$, $\mathbb{S}$, $\mathbb{P}$, and $\mathbb{R}$ refer to scale degrees, while $R_{\mathbb{D}}$ etc. refer to modulations between keys.

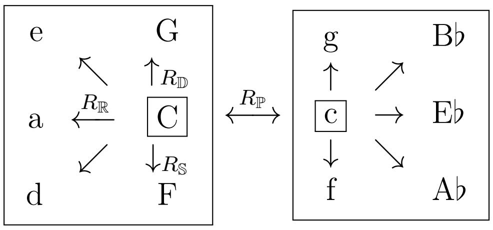


Figure 7.11: Accessible keys in the Tonal Pitch Space

We summarize the relations among modalities in Table 7.1, where R represents key-to-key accessibility and φ denotes a key.

$$R_{\mathbb{D}}R_{\mathbb{S}}\varphi = R_{\mathbb{S}}R_{\mathbb{D}}\varphi = \varphi \tag{7.7}$$

$$R_{\mathbb{D}}^{12}\varphi = R_{\mathbb{S}}^{12}\varphi = \varphi \tag{7.8}$$

$$R_{\mathbb{R}}^{2}\varphi = \varphi,\ R_{\mathbb{P}}^{2}\varphi = \varphi \tag{7.9}$$

$$\begin{cases} R_{\mathbb{D}}R_{\mathbb{P}}\varphi = R_{\mathbb{P}}R_{\mathbb{D}}\varphi \\ R_{\mathbb{S}}R_{\mathbb{P}}\varphi = R_{\mathbb{P}}R_{\mathbb{S}}\varphi \\ R_{\mathbb{D}}R_{\mathbb{R}}\varphi = R_{\mathbb{R}}R_{\mathbb{D}}\varphi \\ R_{\mathbb{R}}R_{\mathbb{S}}\varphi = R_{\mathbb{S}}R_{\mathbb{R}}\varphi \end{cases} \tag{7.10}$$

Table 7.1: Relations in accessibility

In Table 7.1, (7.7) expresses that the dominant of the subdominant (or vice versa) returns to the original key. Equation (7.8) shows that applying the dominant or subdominant modulation 12 times brings us back to the starting key, reflecting the circle of fifths in equal temperament. The relations in (7.9) show that parallel and relative modulations are involutive. Finally, the commutativity relations in (7.10) describe symmetric transitions between modulations.

In fact, these operators are not all strictly necessary: for example, we could use just one modality to track motion clockwise or counter-clockwise on the circle of fifths. Since $R_{\mathbb{D}}^{-1}\varphi = R_{\mathbb{S}}\varphi$, we can write:

$$R_{\mathbb{D}}^{m}\varphi = R_{\mathbb{S}}^{n}\varphi \quad \text{where } m + n \equiv 0 \mod 12.$$

Similarly, one can express a bridge between the major and minor circles with:

$$R_{\mathbb{P}}\varphi = R_{\mathbb{S}}^{3}R_{\mathbb{R}}\varphi.$$

Nevertheless, for the sake of musical clarity and interpretability, we retain all four modal operators explicitly in our system.

Degree Calculi

The key modulations are applied here to degree calculation. We write the degrees of chords, as well as those headed by the modal operators in $\{\mathbb{D}, \mathbb{S}, \mathbb{P}, \mathbb{R}\}$, as propositions in a key (or possible world).

For example, a Berklee chord name C is I in C major, as well as IV in G major, as

$$\mathrm{C} \Vdash \mathrm{I}(C) \iff \mathrm{G} \Vdash \mathrm{IV}(C).$$

In general, the x-th degree in key φ is equivalent to the $(x+3)$-rd degree in $R_{\mathbb{D}}\varphi$ (φ represents a key, x a Roman numeral, and c a chord name or a set of diatonic notes):

$$\varphi \Vdash x(c) \leftrightarrow R_{\mathbb{D}}\varphi \Vdash \mathbb{D}x(c) \quad (\mathbb{D}x = x + 3 \mod 7)$$

where we need to distinguish upper-case degree numerals (major chords) from lower-case ones (minor chords). The full calculations are shown in Table 7.2. The first three can be considered the main ones, whereas the fourth is derived, since the Subdominant function can be derived from the Dominant function, and the Relative function can be derived from the Parallel one. Here, M represents a major key, and m a minor key.

$\mathbb{P}$	$key,\ deg_M$	$=$	$key,\ deg_m$
$\mathbb{D}$	$key,\ deg_M$	$=$	$\mathbb{D}\{key\},\ \{deg+3 \quad mod7\}_M$
$\mathbb{D}$	$key,\ deg_m$	$=$	$\mathbb{D}\{key\},\ \{deg+3 \quad mod7\}_m$
$\mathbb{S}$	$key,\ deg_M$	$=$	$\mathbb{S}\{key\},\ \{deg+4 \quad mod7\}_M$
$\mathbb{S}$	$key,\ deg_m$	$=$	$\mathbb{S}\{key\},\ \{deg+4 \quad mod7\}_m$
$\mathbb{R}$	$key,\ deg_M$	$=$	$\mathbb{R}\{key\},\ \{deg+5 \quad mod7\}_m$
$\mathbb{R}$	$key,\ deg_m$	$=$	$\mathbb{R}\{key\},\ \{deg+2 \quad mod7\}_M$

Table 7.2: Summarizing table for functions.

The rules of Lambek Calculus for chord analysis

We have now introduced the essential components of our formal system. Figure 7.12 summarizes the inference rules used to construct derivations. The expression y/x denotes a chord y that expects chord x to its right—i.e., y precedes x in the sequence. Conversely, $x\backslash y$ has the same interpretation but with reversed directionality: x expects y to its left. These rules can be applied either on the left side of the sequent ($\backslash_L$, $/_L$) or on the right ($\backslash_R$, $/_R$).

$$\frac{\Delta, y, \Sigma \vdash z \qquad \Gamma \vdash x}{\Delta, y/x, \Gamma, \Sigma \vdash z}\ (/_L)$$

$$\frac{\Gamma \vdash x \qquad \Delta, y, \Sigma \vdash z}{\Delta, \Gamma, x\backslash y, \Sigma \vdash z}\ (\backslash_L)$$

$$\frac{\Gamma, x \vdash y}{\Gamma \vdash y/x}\ (/_R) \qquad \frac{x, \Gamma \vdash y}{\Gamma \vdash x\backslash y}\ (\backslash_R)$$

$$\frac{\Gamma \vdash x \qquad x \vdash y}{\Gamma \vdash y}\ (\text{Cut}) \qquad \frac{}{x \vdash x}\ (\text{Init})$$

Figure 7.12: Set of rules of Lambek Calclus for chord analysis; we exclude $(\cdot)$ rules for they do not concern music.

In the notation, Δ, Γ, and Σ represent sequences of formulas (i.e., harmonic contexts), while z stands for another chord formula. The *Cut* rule is a fundamental principle in proof theory, enabling compositional reasoning across intermediate formulas. However, in this paper, it is included for completeness but never invoked in actual derivations.

The *(Init)* rule represents the initial axiom or base case of a derivation, typically corresponding to the harmonic interpretation of a chord within a given tonality.

Finally, we add the following axiom (K) to the system where $\Box$ collectively represents either one of $\Box_D, \Box_S, \Box_R$ or $\Box_P$.

$$\frac{\Gamma \vdash x}{\Box\Gamma \vdash \Box x}\ (\text{K})$$

In the following section, we introduce *Tagged Sequents*, which allow us to formally represent tonal centers and their role in harmonic derivations.

Tagged sequents

In front of the sequents we proceed to use a key with a colon (:) for mnemonic annotations, i.e., to indicate the key in which the analysis is carried out. To avoid confusion, throughout this chapter we will always specify the tonal function of each key ($R_{\mathbb{D}}$, $R_{\mathbb{S}}$, $R_{\mathbb{P}}$, and $R_{\mathbb{R}}$) alongside the tonal centres, even when they could be interpreted as belonging to a new region. For instance, $R_{\mathbb{D}}$G corresponds to D, while $R_{\mathbb{D}}R_{\mathbb{R}}$d corresponds to C major. The key annotation is valid for the entire line, not only for the subsequent sequent, and it will not be repeated when identical to the previous line.

An *Init* sequent is introduced from the '$\Vdash$' interpretation, e.g., G $\Vdash$ IV(C), as:

$$\overset{C}{\text{G: IV} \vdash \text{IV}}$$

without a horizontal line. The horizontal line, in fact, is used only for derivations.

Remark 7.2. *In the final version of our system, presented in [12], we introduced the notion of* tagged sequents. *Although the tags proved useful for tracing tonal relations across derivations, they were not formally integrated into the syntax of the calculus. From the standpoint of proof theory, this remains an open issue. Future work should aim to provide a rigorous syntactic treatment of tags, possibly by extending the structure of sequents themselves.*

Example 7.14. Consider the example of *D*7–*G*7–*C*MA^7. When we interpret the sequence in C major, we obtain:

$$\dfrac{\dfrac{\overset{D7}{\text{C: II} \vdash \text{II}} \qquad \overset{G7}{\text{V} \vdash \text{V}}}{\text{II}, \text{II}\backslash\text{V} \vdash \text{V}}\,(\backslash_L) \qquad \overset{C\text{MA}^7}{\text{I} \vdash \text{I}}}{\text{II}, \text{II}\backslash\text{V}, \text{V}\backslash\text{I} \vdash \text{I}}\,(\backslash_L)$$

The final sequent i.e., $\text{II}, \text{II}\backslash\text{V}, \text{V}\backslash\text{I} \vdash \text{I}$, can be interpreted as follows: in the tonality of C, we move from the second degree to the fifth, then to the first, which is also the last chord of the sequent, as can be seen on the right side.

Example 7.15. The flexibility of the system also enables the analyst to describe this type of tree in a different way, emphasizing the function of the *Doppel-dominant* as follows.

$$\dfrac{\dfrac{\dfrac{\dfrac{\overset{D7}{\text{G: V} \vdash \text{V}} \qquad \overset{G7}{\text{G: I} \vdash \text{I}}}{\text{V}, \text{V}\backslash\text{I} \vdash \text{I}}\,(\backslash_L)}{R_{\mathbb{S}}\text{G: } \mathbb{S}\{\text{V}, \text{V}\backslash\text{I}\} \vdash \mathbb{S}\text{I}}\,(\text{K}_{\mathbb{S}})}{\mathbb{S}\{\text{V}, \text{V}\backslash\text{I}\} \vdash \text{V}} \qquad \overset{C\text{MA}^7}{\text{I} \vdash \text{I}}}{R_{\mathbb{S}}\text{G: } \mathbb{S}\{\text{V}, \text{V}\backslash\text{I}\}, \text{V}\backslash\text{I} \vdash \text{I}}\,(\backslash_L)$$

In this case, the tagged sequents are rewritten in different tonalities because they change.

In some cases, modulations cannot be restricted to those with single operators; complex operators are also admissible.

Example 7.16. For example, *A*7^{alt}–*Dm*7– *G*7 is analyzed as follows.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\overset{A7^{alt}}{\text{d: V} \vdash \text{V}} \qquad \overset{Dm7}{\text{i} \vdash \text{i}}}{\text{V}, \text{V}\backslash\text{i} \vdash \text{i}}\,(\backslash_L)}{R_{\mathbb{R}}\text{d: } \mathbb{R}\{\text{V}, \text{V}\backslash\text{i}\} \vdash \mathbb{R}\text{i}}\,(\text{K}_{\mathbb{R}})}{R_{\mathbb{D}}R_{\mathbb{R}}\text{d: } \mathbb{D}\mathbb{R}\{\text{V}, \text{V}\backslash\text{i}\} \vdash \mathbb{D}\mathbb{R}\text{i}}\,(\text{K}_{\mathbb{D}})}{\text{VI}, \text{VI}\backslash\text{ii} \vdash \text{ii}} \qquad \overset{G7}{\text{V} \vdash \text{V}}}{\text{VI}, (\text{VI}\backslash\text{ii})/\text{V}, \text{V} \vdash \text{ii}}\,(/_L)$$

The sequent $\text{VI}, (\text{VI}\backslash\text{ii})/\text{V}, \text{V} \vdash \text{ii}$ expresses that, in the key of C (i.e., the tonal center reached via the tagged sequent $R_{\mathbb{D}}R_{\mathbb{R}}d$), we first interpret a chord as the sixth degree, then move through a cadential progression that brings us to the second minor degree (ii), and finally to the fifth degree (V). This sequence reflects a typical jazz turnaround embedded in a modulated tonal context.

7.7.3 Analysis 1 – *In Your Own Sweet Way*

Now, we present an example analysis of the first eight bars of *In Your Own Sweet Way* by Dave Brubeck shown in Figure 7.13, and in Figure 7.16. To provide a clearer understanding, the main points are as follows:

- In the first tree, we track key shifts using shift modalities, namely, $\mathbb{R}$ (relative), $\mathbb{D}$ (dominant), and $\mathbb{S}$ (subdominant).
- Moreover, during the analysis, it is necessary to apply (7.7) to simplify $\mathbb{SDR}$ into $\mathbb{R}$. Indeed, it is easy to verify that g (G minor) is the relative key of B♭.
- The second tree presents an alternative interpretation of the same analysis. The final result remains unchanged, but a notable difference is that the cadence $Cm7$–$F7$–$B\flat^6$ is analyzed separately. Then, using the rule ($\backslash_L$), we obtain the same result as in the first tree.
- The third tree (bottom left in Figure 7.16) is separate from the first one because it features a brief but significant key shift, which is better represented independently.
- The last tree (bottom right in Figure 7.16) illustrates a *ii–V–I* progression concluding the A-section of the piece. It is presented as a separate tree because it belongs to a different tonal region.

The final result will be a sequence of the three distinct regions of B♭ – G♭ – B♭.

7.7.4 Depth of a proof

Now that the basic system is set, it is possible to introduce the notion of the depth of a proof in a way similar to how it is defined in Logic:

Definition 7.10 (Depth of a proof)**.** The depth of a proof is the maximum level of nesting in a proof, determined according to the rules in Figure 7.14.

The concept of depth allows us to more precisely determine how directly a proof represents the harmonic analysis of a given sequence of chords. A lower depth indicates a more immediate derivation, while a higher depth may

Figure 7.13: The beginning of *In your own sweet way* by Dave Brubeck.

result from additional structural elements, mainly modal operators, that clarify underlying harmonic relationships. Furthermore, the same set of chords can be analyzed in multiple ways, leading to different depth values, as demonstrated in Example 7.17. This highlights both the flexibility of the system in capturing different levels of harmonic interpretation and the shift in perspective that the analyst can choose to emphasize. In Figure 7.14 all the rules are described. The idea is that the depth increases whenever it is necessary to do an operation on the proposition, e.g., to write a cadence or to change the key.

Is is also possible to define what is *Minimality* in this context:

Definition 7.11 (Minimality of a proof)**.** A proof is *minimal* when its depth is lowest among all possible analyses of a sequence of chords.

Even though we cannot ensure that a given proof is minimal or not, it is possible to define the minimal proof as such that does not employ the axiom (K):

Definition 7.12. The minimality of a proof that does not require modal operators is the number of chords minus 1.

In fact, it is not difficult to see that, based on the rules in Figure 7.14, if we do not use the modal rules the depth can only increase every when we concatenate two different chords.

Example 7.17. This example presents the analysis of the chords (transposed to the key of F for practicality, so that the last chord is C7) that make up

$$\frac{\text{Chord}}{x \vdash_0 x} \text{(Init)}$$

$$\frac{\Delta, y, \Sigma \vdash_\delta z \qquad \Gamma \vdash_\gamma x}{\Delta, y/x, \Gamma, \Sigma \vdash_{\delta+\gamma+1} z} (/_L)$$

$$\frac{\Gamma \vdash_\delta x \qquad \Delta, y, \Sigma \vdash_\gamma z}{\Delta, \Gamma, x\backslash y, \Sigma \vdash_{\delta+\gamma+1} z} (\backslash_L)$$

$$\frac{\Gamma, x \vdash_\delta y}{\Gamma \vdash_\delta y/x} (/_R) \qquad \frac{x, \Gamma \vdash_\delta y}{\Gamma \vdash_\delta x\backslash y} (\backslash_R)$$

$$\frac{\alpha : \Gamma \vdash_\delta \Delta}{R_\Box \alpha : \Box\Gamma \vdash_{\delta+1} \Box\Delta} (K)$$

Where:

- $\delta, \gamma \in \mathbb{N}$;
- $K_\mathbb{F}$, $R_\mathbb{F}$ are respectively one of the modal functions applied and one of the modal relations;
- $\Box$ represents again each modality.

Figure 7.14: Rules of the Lambek Calculus for Chord Analysis with the calculus of the depths.

the B-part of *I Got Rhythm* (A7, D7, G7, C7). As is well known, this section consists of nothing more than a chain of dominants. Using Lambek Calculus, we provide two different analyses. The first one is the most direct: we simply write the degree of the chords in the key of C, resulting in the minimal possible depth of 3. The second analysis, instead, changes the key twice using modal operators ($K_{\mathbb{S}}$). Although this increases the depth to 5, it makes the harmonic structure clearer, explicitly showing that the chords form a chain of dominants.

$$\dfrac{\dfrac{\dfrac{\overset{A7}{\text{C:VI} \vdash_0 \text{VI}} \quad \overset{D7}{\text{II} \vdash_0 \text{II}}}{\text{VI}, \text{VI}\backslash\text{II} \vdash_1 \text{II}}(\backslash_L) \quad \overset{G7}{\text{V} \vdash_0 \text{V}}}{\text{VI}, \text{VI}\backslash\text{II}, \text{II}\backslash\text{V} \vdash_2 \text{II}}(\backslash_L) \quad \overset{C7}{\text{I} \vdash_0 \text{I}}}{\text{VI}, \text{VI}\backslash\text{II}, \text{II}\backslash\text{V}, \text{V}\backslash\text{I} \vdash_3 \text{I}}(\backslash_L)$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\overset{A7}{\text{D:V} \vdash_0 \text{V}} \quad \overset{D7}{\text{I} \vdash_0 \text{I}}}{\text{V}, \text{V}\backslash\text{I} \vdash_1 \text{I}}(\backslash_L)}{R_{\mathbb{S}}\text{D: } \mathbb{S}\{\text{V}, \text{V}\backslash\text{I}\} \vdash_2 \mathbb{S}\text{I}}(K_{\mathbb{S}})}{\mathbb{S}\{\text{V}, \text{V}\backslash\text{I}\} \vdash_2 \text{V}} \quad \overset{G7}{\text{I} \vdash_0 \text{I}}}{\mathbb{S}\{\text{V}, \text{V}\backslash\text{I}\}, \text{V}\backslash\text{I} \vdash_3 \text{I}}(\backslash_L)}{R_{\mathbb{S}}R_{\mathbb{S}}\text{D: } \mathbb{S}\{\mathbb{S}\{\text{V}, \text{V}\backslash\text{I}\}, \text{V}\backslash\text{I}\} \vdash_4 \mathbb{S}\text{I}}(K_{\mathbb{S}})}{\mathbb{S}\{\mathbb{S}\{\text{V}, \text{V}\backslash\text{I}\}, \text{V}\backslash\text{I}\} \vdash_4 \text{V}} \quad \overset{C_{\text{MA}}{}^{7}}{\text{I} \vdash_0 \text{I}}}{\mathbb{S}\{\mathbb{S}\{\text{V}, \text{V}\backslash\text{I}\}, \text{V}\backslash\text{I}\}, \text{V}\backslash\text{I} \vdash_5 \text{I}}(\backslash_L)$$

Finally we can see another tree with no modal operators that, maintains the same depth although the derivation is a little bit different from the first one that we have seen:

$$\dfrac{\dfrac{\overset{A7}{\text{C:VI} \vdash_0 \text{VI}} \quad \overset{D7}{\text{II} \vdash_0 \text{II}}}{\text{VI}, \text{VI}\backslash\text{II} \vdash_1 \text{II}}(\backslash_L) \quad \dfrac{\overset{G7}{\text{V} \vdash_0 \text{V}} \quad \overset{C7}{\text{I} \vdash_0 \text{I}}}{\text{V}, \text{V}\backslash\text{I} \vdash_1 \text{I}}(\backslash_L)}{\text{VI}, \text{VI}\backslash\text{II}, \text{II}\backslash\text{V}, \text{V}\backslash\text{I} \vdash_3 \text{I}}(\backslash_L)$$

The idea of using depth arises from the fact that this number allows for comparing two different proofs and finding the one that is both the most explanatory for the analyst and the easiest to compute, thanks to the principle of minimality.

7.7.5 Analysis 2 – *All the Things You Are*

Figure 7.15: A part of *All the things you are.*

All the things you are is a well-known song in the jazz repertoire, as was recorded by many celebrated artists, like John Coltrane, Keith Jarrett or Chet Baker (Figure 7.15). The interesting part of this song is the rapid succession of different keys and we can acquire various analyses thanks to Lambek Calculus in Figure 7.17. We should pay attention to the following interesting points about these analyses:

- The first point is that the initial tree analyzes a classic vii-ii-V-I progression that resolves to IV, after which there is a change in tonality. We have chosen to emphasize the fact that G7 functions as the VII major degree of $A\flat$, requiring multiple applications of the modal axiom (K) to reconnect the parts to the new key. However, we would also be able to divide the tree into two parts, providing a more specific analysis of the sudden modulation:

$$\dfrac{\dfrac{\begin{array}{c}\vdots\\ \mathrm{vi}, \mathrm{vi}\backslash\mathrm{ii} \vdash_1 \mathrm{ii}\end{array} \quad \begin{array}{c}\vdots\\ \mathrm{V}\backslash\mathrm{I} \vdash_1 \mathrm{I}\end{array}}{\mathrm{vi}, \mathrm{vi}\backslash\mathrm{ii}, \mathrm{ii}\backslash\mathrm{V}, \mathrm{V}\backslash\mathrm{I} \vdash_3 \mathrm{I}}\,(\backslash_L) \qquad \begin{array}{c} D\flat\mathrm{MA}^7 \\ \mathrm{IV} \vdash_0 \mathrm{IV}\end{array}}{\mathrm{vi}, \mathrm{vi}\backslash\mathrm{ii}, \mathrm{ii}\backslash\mathrm{V}, \mathrm{V}\backslash\mathrm{I}/\mathrm{IV}, \mathrm{IV} \vdash_4 \mathrm{I}}\,(/_L)$$

$$\dfrac{\begin{array}{c} G7 \\ \mathrm{C{:}V} \vdash_0 \mathrm{V}\end{array} \qquad \begin{array}{c} C\mathrm{MA}^7 \\ \mathrm{I} \vdash_0 \mathrm{I}\end{array}}{\mathrm{V}\backslash\mathrm{I} \vdash_1 \mathrm{I}}\,(\backslash_L)$$

In contrast, in the tree in Figure 7.17, we have grouped everything together for two reasons. First, D♭MA^{7} is closely related to $G7$, as they are separated by only a tritone. Second, the same pattern appears in the second tree. Ultimately, these differences reflect the main purpose of Lambek Calculus: to serve as a tool for the analyst, who can choose the approach that best fits their analyses.

- Another interesting point to note is that the results of the first tree in Figure 7.17 and the second tree are exactly the same in a different tonality, where they reveal the same harmonic structure transposed.

- Finally, in the last tree, it is possible to see that a similar mechanism is applied to the modulations, with the difference that from GMA^{7}, the modulation is to EMA^{7}—- only three steps on the circle of fifths – whereas in the rest of the cases, four steps were needed to connect the parts. This is also evident from the fact that the depth after the modulation increases by 3 instead of 4.

7.8 Concluding Remarks

In the course of this work, we have gradually introduced several formal devices for music analysis. Starting from the treatment of cadences as logical implications, we extended the framework to handle unrelated or intermediate tonalities through conjunction-like rules. The notion of derivational depth then provided a way to compare different analyses, yielding a measure of harmonic complexity. Finally, by combining these components within Lambek Calculus, we obtained a system capable of unifying tonal, modal, and even non-tonal contexts under a single proof-theoretic perspective.

What emerges from this progression is a coherent toolkit: each rule captures a specific kind of harmonic motion, while the depth hierarchy gives a global criterion for evaluating proofs. Taken together, these systems illustrate how logic can model the internal grammar of harmony and provide a bridge between local rules (such as cadential patterns) and global structures (such as modulation chains).

This integrative view also clarifies why Lambek Calculus is particularly suitable for music theory: it can flexibly incorporate new rules without losing its proof-theoretic structure, thus adapting to different harmonic idioms.

$$\dfrac{\overset{Am7^{\flat 5}}{\text{g:}\ \mathrm{ii} \vdash \mathrm{ii}} \qquad \overset{D7}{\mathrm{V} \vdash \mathrm{V}}}{\mathrm{ii}, \mathrm{ii}\backslash \mathrm{V} \vdash \mathrm{V}}\ (\backslash_L)$$

$$\dfrac{\mathrm{ii}, \mathrm{ii}\backslash \mathrm{V} \vdash \mathrm{V} \qquad \overset{Gm7}{\mathrm{I} \vdash \mathrm{I}}}{\mathrm{ii}, \mathrm{ii}\backslash \mathrm{V}, \mathrm{V}\backslash \mathrm{I} \vdash \mathrm{I}}\ (\backslash_L)$$

$$\dfrac{\mathrm{ii}, \mathrm{ii}\backslash \mathrm{V}, \mathrm{V}\backslash \mathrm{I} \vdash \mathrm{I}}{\text{F:}\ \mathbb{DR}\{\mathrm{ii}, \mathrm{ii}\backslash \mathrm{V}, \mathrm{V}\backslash \mathrm{I}\} \vdash \mathbb{DR}\mathrm{I}}\ (\mathrm{K})$$

$$\dfrac{\text{F:}\ \mathbb{DR}\{\mathrm{ii}, \mathrm{ii}\backslash \mathrm{V}, \mathrm{V}\backslash \mathrm{I}\} \vdash \mathbb{DR}\mathrm{I}}{\mathbb{DR}\{\mathrm{ii}, \mathrm{ii}\backslash \mathrm{V}, \mathrm{V}\backslash \mathrm{I}\} \vdash \mathrm{ii}}$$

$$\dfrac{\mathbb{DR}\{\mathrm{ii}, \mathrm{ii}\backslash \mathrm{V}, \mathrm{V}\backslash \mathrm{I}\} \vdash \mathrm{ii} \qquad \overset{C7}{\mathrm{V} \vdash \mathrm{V}}}{\mathbb{DR}\{\mathrm{ii}, \mathrm{ii}\backslash \mathrm{V}, \mathrm{V}\backslash \mathrm{I}\}, \mathrm{ii}\backslash \mathrm{V} \vdash \mathrm{V}}\ (\backslash_L)$$

$$\dfrac{\mathbb{DR}\{\mathrm{ii}, \mathrm{ii}\backslash \mathrm{V}, \mathrm{V}\backslash \mathrm{I}\}, \mathrm{ii}\backslash \mathrm{V} \vdash \mathrm{V}}{\text{B}\flat\text{:}\ \mathbb{S}\{\mathbb{DR}\{\mathrm{ii}, \mathrm{ii}\backslash \mathrm{V}, \mathrm{V}\backslash \mathrm{I}\}\}, \mathbb{S}\{\mathrm{ii}\backslash \mathrm{V}\} \vdash \mathbb{S}\mathrm{V}}\ (\mathrm{K})$$

$$\dfrac{\dfrac{\text{B}\flat\text{:}\ \mathbb{S}\{\mathbb{DR}\{\mathrm{ii}, \mathrm{ii}\backslash \mathrm{V}, \mathrm{V}\backslash \mathrm{I}\}\}, \mathbb{S}\{\mathrm{ii}\backslash \mathrm{V}\} \vdash \mathbb{S}\mathrm{V}}{\mathbb{R}\{\mathrm{ii}, \mathrm{ii}\backslash \mathrm{V}, \mathrm{V}\backslash \mathrm{I}\}, \mathbb{S}\{\mathrm{ii}\backslash \mathrm{V}\} \vdash \mathbb{S}\mathrm{V}}}{\mathbb{R}\{\mathrm{ii}, \mathrm{ii}\backslash \mathrm{V}, \mathrm{V}\backslash \mathrm{I}\}, \mathbb{S}\{\mathrm{ii}\backslash \mathrm{V}\} \vdash \mathrm{II}}$$

$$\dfrac{\mathbb{R}\{\mathrm{ii}, \mathrm{ii}\backslash \mathrm{V}, \mathrm{V}\backslash \mathrm{I}\}, \mathbb{S}\{\mathrm{ii}\backslash \mathrm{V}\} \vdash \mathrm{II} \qquad \overset{Cm7}{\mathrm{ii} \vdash \mathrm{ii}}}{\mathbb{R}\{\mathrm{ii}, \mathrm{ii}\backslash \mathrm{V}, \mathrm{V}\backslash \mathrm{I}\}, \mathbb{S}\{\mathrm{ii}\backslash \mathrm{V}\}, \mathrm{II}\backslash \mathrm{ii} \vdash \mathrm{ii}}\ (\backslash_L)$$

$$\dfrac{\mathbb{R}\{\mathrm{ii}, \mathrm{ii}\backslash \mathrm{V}, \mathrm{V}\backslash \mathrm{I}\}, \mathbb{S}\{\mathrm{ii}\backslash \mathrm{V}\}, \mathrm{II}\backslash \mathrm{ii} \vdash \mathrm{ii} \qquad \overset{F7}{\mathrm{V} \vdash \mathrm{V}}}{\mathbb{R}\{\mathrm{ii}, \mathrm{ii}\backslash \mathrm{V}, \mathrm{V}\backslash \mathrm{I}\}, \mathbb{S}\{\mathrm{ii}\backslash \mathrm{V}\}, \mathrm{II}\backslash \mathrm{ii}, \mathrm{ii}\backslash \mathrm{V} \vdash \mathrm{V}}\ (\backslash_L)$$

$$\dfrac{\mathbb{R}\{\mathrm{ii}, \mathrm{ii}\backslash \mathrm{V}, \mathrm{V}\backslash \mathrm{I}\}, \mathbb{S}\{\mathrm{ii}\backslash \mathrm{V}\}, \mathrm{II}\backslash \mathrm{ii}, \mathrm{ii}\backslash \mathrm{V} \vdash \mathrm{V} \qquad \overset{B\flat^{6}}{\mathrm{I} \vdash \mathrm{I}}}{\mathbb{R}\{\mathrm{ii}, \mathrm{ii}\backslash \mathrm{V}, \mathrm{V}\backslash \mathrm{I}\}, \mathbb{S}\{\mathrm{ii}\backslash \mathrm{V}\}, \mathrm{II}\backslash \mathrm{ii}, \mathrm{ii}\backslash \mathrm{V}, \mathrm{V}\backslash \mathrm{I} \vdash \mathrm{I}}\ (\backslash_L)$$

$$\dfrac{\mathbb{R}\{\mathrm{ii}, \mathrm{ii}\backslash \mathrm{V}, \mathrm{V}\backslash \mathrm{I}\}, \mathbb{S}\{\mathrm{ii}\backslash \mathrm{V}\}, \mathrm{II}\backslash \mathrm{ii}, \mathrm{ii}\backslash \mathrm{V}, \mathrm{V}\backslash \mathrm{I} \vdash \mathrm{I} \qquad \overset{E\flat\mathrm{MA}^{7}}{\mathrm{IV} \vdash \mathrm{IV}}}{\mathbb{R}\{\mathrm{ii}, \mathrm{ii}\backslash \mathrm{V}, \mathrm{V}\backslash \mathrm{I}\}, \mathbb{S}\{\mathrm{ii}\backslash \mathrm{V}\}, \mathrm{II}\backslash \mathrm{ii}, \mathrm{ii}\backslash \mathrm{V}, (\mathrm{V}\backslash \mathrm{I})/\mathrm{IV}, \mathrm{IV} \vdash \mathrm{I}}\ (/_L)$$

$$\dfrac{\overset{Am7^{\flat 5}}{\text{g:}\ \mathrm{ii} \vdash \mathrm{ii}} \qquad \overset{D7}{\mathrm{V} \vdash \mathrm{V}}}{\mathrm{ii}, \mathrm{ii}\backslash \mathrm{V} \vdash \mathrm{V}}\ (\backslash_L)$$

$$\dfrac{\mathrm{ii}, \mathrm{ii}\backslash \mathrm{V} \vdash \mathrm{V} \qquad \overset{Gm7}{\mathrm{I} \vdash \mathrm{I}}}{\mathrm{ii}, \mathrm{ii}\backslash \mathrm{V}, \mathrm{V}\backslash \mathrm{I} \vdash \mathrm{I}}\ (\backslash_L)$$

$$\dfrac{\mathrm{ii}, \mathrm{ii}\backslash \mathrm{V}, \mathrm{V}\backslash \mathrm{I} \vdash \mathrm{I}}{\text{F:}\ \mathbb{DR}\{\mathrm{ii}, \mathrm{ii}\backslash \mathrm{V}, \mathrm{V}\backslash \mathrm{I}\} \vdash \mathbb{DR}\mathrm{I}}\ (\mathrm{K})$$

$$\dfrac{\text{F:}\ \mathbb{DR}\{\mathrm{ii}, \mathrm{ii}\backslash \mathrm{V}, \mathrm{V}\backslash \mathrm{I}\} \vdash \mathbb{DR}\mathrm{I}}{\mathbb{DR}\{\mathrm{ii}, \mathrm{ii}\backslash \mathrm{V}, \mathrm{V}\backslash \mathrm{I}\} \vdash \mathrm{ii}}$$

$$\dfrac{\mathbb{DR}\{\mathrm{ii}, \mathrm{ii}\backslash \mathrm{V}, \mathrm{V}\backslash \mathrm{I}\} \vdash \mathrm{ii} \qquad \overset{C7}{\mathrm{V} \vdash \mathrm{V}}}{\mathbb{DR}\{\mathrm{ii}, \mathrm{ii}\backslash \mathrm{V}, \mathrm{V}\backslash \mathrm{I}\}, \mathrm{ii}\backslash \mathrm{V} \vdash \mathrm{V}}\ (\backslash_L)$$

$$\dfrac{\mathbb{DR}\{\mathrm{ii}, \mathrm{ii}\backslash \mathrm{V}, \mathrm{V}\backslash \mathrm{I}\}, \mathrm{ii}\backslash \mathrm{V} \vdash \mathrm{V}}{\text{B}\flat\text{:}\ \mathbb{S}\{\mathbb{DR}\{\mathrm{ii}, \mathrm{ii}\backslash \mathrm{V}, \mathrm{V}\backslash \mathrm{I}\}\}, \mathbb{S}\{\mathrm{ii}\backslash \mathrm{V}\} \vdash \mathbb{S}\mathrm{V}}\ (\mathrm{K})$$

$$\dfrac{\dfrac{\text{B}\flat\text{:}\ \mathbb{S}\{\mathbb{DR}\{\mathrm{ii}, \mathrm{ii}\backslash \mathrm{V}, \mathrm{V}\backslash \mathrm{I}\}\}, \mathbb{S}\{\mathrm{ii}\backslash \mathrm{V}\} \vdash \mathbb{S}\mathrm{V}}{\mathbb{R}\{\mathrm{ii}, \mathrm{ii}\backslash \mathrm{V}, \mathrm{V}\backslash \mathrm{I}\}, \mathbb{S}\{\mathrm{ii}\backslash \mathrm{V}\} \vdash \mathbb{S}\mathrm{V}}}{\mathbb{R}\{\mathrm{ii}, \mathrm{ii}\backslash \mathrm{V}, \mathrm{V}\backslash \mathrm{I}\}, \mathbb{S}\{\mathrm{ii}\backslash \mathrm{V}\} \vdash \mathrm{II}}$$

$$\dfrac{\overset{Cm7}{\text{B}\flat\text{:}\ \mathrm{ii} \vdash \mathrm{ii}} \qquad \overset{F7}{\mathrm{V} \vdash \mathrm{V}}}{\mathrm{ii}, \mathrm{ii}\backslash \mathrm{V} \vdash \mathrm{V}}\ (\backslash_L)$$

$$\dfrac{\mathrm{ii}, \mathrm{ii}\backslash \mathrm{V} \vdash \mathrm{V} \qquad \overset{B\flat^{6}}{\mathrm{I} \vdash \mathrm{I}}}{\mathrm{ii}, \mathrm{ii}\backslash \mathrm{V}, \mathrm{V}\backslash \mathrm{I} \vdash \mathrm{I}}\ (\backslash_L)$$

$$\dfrac{\mathbb{R}\{\mathrm{ii}, \mathrm{ii}\backslash \mathrm{V}, \mathrm{V}\backslash \mathrm{I}\}, \mathbb{S}\{\mathrm{ii}\backslash \mathrm{V}\} \vdash \mathrm{II} \qquad \mathrm{ii}, \mathrm{ii}\backslash \mathrm{V}, \mathrm{V}\backslash \mathrm{I} \vdash \mathrm{I}}{\mathbb{R}\{\mathrm{ii}, \mathrm{ii}\backslash \mathrm{V}, \mathrm{V}\backslash \mathrm{I}\}, \mathbb{S}\{\mathrm{ii}\backslash \mathrm{V}\}, \mathrm{II}\backslash \mathrm{ii}, \mathrm{ii}\backslash \mathrm{V}, \mathrm{V}\backslash \mathrm{I} \vdash \mathrm{I}}\ (\backslash_L)$$

$$\dfrac{\mathbb{R}\{\mathrm{ii}, \mathrm{ii}\backslash \mathrm{V}, \mathrm{V}\backslash \mathrm{I}\}, \mathbb{S}\{\mathrm{ii}\backslash \mathrm{V}\}, \mathrm{II}\backslash \mathrm{ii}, \mathrm{ii}\backslash \mathrm{V}, \mathrm{V}\backslash \mathrm{I} \vdash \mathrm{I} \qquad \overset{E\flat\mathrm{MA}^{7}}{\mathrm{IV} \vdash \mathrm{IV}}}{\mathbb{R}\{\mathrm{ii}, \mathrm{ii}\backslash \mathrm{V}, \mathrm{V}\backslash \mathrm{I}\}, \mathbb{S}\{\mathrm{ii}\backslash \mathrm{V}\}, \mathrm{II}\backslash \mathrm{ii}, \mathrm{ii}\backslash \mathrm{V}, (\mathrm{V}\backslash \mathrm{I})/\mathrm{IV}, \mathrm{IV} \vdash \mathrm{I}}\ (/_L)$$

$$\dfrac{\dfrac{\dfrac{\overset{A\flat m7^{\flat 5}}{\mathrm{G}\flat\!:\ \mathrm{ii} \vdash \mathrm{ii}} \quad \overset{D\flat 7}{\mathrm{V} \vdash \mathrm{V}}}{\mathrm{ii}, \mathrm{ii}\backslash\mathrm{V} \vdash \mathrm{V}}\,(\backslash_L) \quad \overset{G\flat\textsc{ma}^7}{\mathrm{I} \vdash \mathrm{I}}}{\mathrm{ii}, \mathrm{ii}\backslash\mathrm{V}, \mathrm{V}\backslash\mathrm{I} \vdash \mathrm{I}}\,(\backslash_L) \quad \overset{C\flat\textsc{ma}^7}{\mathrm{IV} \vdash \mathrm{IV}}}{\mathrm{ii}, \mathrm{ii}\backslash\mathrm{V}, (\mathrm{V}\backslash\mathrm{I})/\mathrm{IV}, \mathrm{IV} \vdash \mathrm{I}}\,(/_L)
\qquad
\dfrac{\dfrac{\overset{Cm7^{\flat 5}}{\mathrm{B}\flat\!:\ \mathrm{ii} \vdash \mathrm{ii}} \quad \overset{F7}{\mathrm{V} \vdash \mathrm{V}}}{\mathrm{ii}, \mathrm{ii}\backslash\mathrm{V} \vdash \mathrm{V}}\,(\backslash_L) \quad \overset{B\flat 6}{\mathrm{I} \vdash \mathrm{I}}}{\mathrm{ii}, \mathrm{ii}\backslash\mathrm{V}, \mathrm{V}\backslash\mathrm{I} \vdash \mathrm{I}}\,(\backslash_L)$$

Figure 7.16: Analysis of the first 8 bars of *In your own sweet way*

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\overset{Fm7}{A\flat\text{: vi} \vdash_0 \text{vi}} \quad \overset{B\flat\text{m}7}{\text{ii} \vdash_0 \text{ii}}}{\text{vi}, \text{vi}\backslash\text{ii} \vdash_1 \text{ii}}(\backslash_L) \quad \dfrac{\overset{E\flat 7}{\text{V} \vdash_0 \text{V}} \quad \overset{A\flat\text{MA}^7}{\text{I} \vdash_0 \text{I}}}{\text{V}\backslash\text{I} \vdash_1 \text{I}}(\backslash_L)}{\text{vi}, \text{vi}\backslash\text{ii}, \text{ii}\backslash\text{V}, \text{V}\backslash\text{I} \vdash_3 \text{I}}(\backslash_L) \quad \overset{D\flat\text{MA}^7}{\text{IV} \vdash_0 \text{IV}}}{\text{vi}, \text{vi}\backslash\text{ii}, \text{ii}\backslash\text{V}, \text{V}\backslash\text{I}/\text{IV}, \text{IV} \vdash_4 \text{I}}(/_L) \quad \overset{G7}{\text{VII} \vdash_0 \text{VII}}}{\text{vi}, \text{vi}\backslash\text{ii}, \text{ii}\backslash\text{V}, \text{V}\backslash\text{I}/\text{IV}, \text{IV}, \text{I}\backslash\text{VII} \vdash_5 \text{VII}}(\backslash_L)}{R^4_{\mathbb{D}}\text{A}\flat : \mathbb{D}^4\{\text{vi}, \text{vi}\backslash\text{ii}, \text{ii}\backslash\text{V}, \text{V}\backslash\text{I}/\text{IV}, \text{IV}, \text{I}\backslash\text{VII}\} \vdash_9 \mathbb{D}^4\text{VII}}(K^4_{\mathbb{D}})}{\mathbb{D}^4\{\text{vi}, \text{vi}\backslash\text{ii}, \text{ii}\backslash\text{V}, \text{V}\backslash\text{I}/\text{IV}, \text{IV}, \text{I}\backslash\text{VII}\} \vdash_9 \text{V}} \quad \overset{C\text{MA}^7}{\text{I} \vdash_0 \text{I}}}{R^4_{\mathbb{D}}\text{A}\flat : \mathbb{D}^4\{\text{vi}, \text{vi}\backslash\text{ii}, \text{ii}\backslash\text{V}, \text{V}\backslash\text{I}/\text{IV}, \text{IV}, \text{I}\backslash\text{VII}\}, \text{V}\backslash\text{I} \vdash_{10} \text{I}}(\backslash_L)$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\overset{Cm7}{\text{E}\flat\text{: vi} \vdash_0 \text{vi}} \quad \overset{Fm7}{\text{ii} \vdash_0 \text{ii}}}{\text{vi}, \text{vi}\backslash\text{ii} \vdash_1 \text{ii}}(\backslash_L) \quad \dfrac{\overset{B\flat 7}{\text{V} \vdash_0 \text{V}} \quad \overset{E\flat\text{MA}^7}{\text{I} \vdash_0 \text{I}}}{\text{V}\backslash\text{I} \vdash_1 \text{I}}(\backslash_L)}{\text{vi}, \text{vi}\backslash\text{ii}, \text{ii}\backslash\text{V}, \text{V}\backslash\text{I} \vdash_3 \text{I}}(\backslash_L) \quad \overset{A\flat\text{MA}^7}{\text{IV} \vdash_0 \text{IV}}}{\text{vi}, \text{vi}\backslash\text{ii}, \text{ii}\backslash\text{V}, \text{V}\backslash\text{I}/\text{IV}, \text{IV} \vdash_4 \text{I}}(/_L) \quad \overset{D7}{\text{VII} \vdash_0 \text{VII}}}{\text{vi}, \text{vi}\backslash\text{ii}, \text{ii}\backslash\text{V}, \text{V}\backslash\text{I}/\text{IV}, \text{IV}, \text{I}\backslash\text{VII} \vdash_5 \text{VII}}(\backslash_L)}{R^4_{\mathbb{D}}\text{E}\flat : \mathbb{D}^4\{\text{vi}, \text{vi}\backslash\text{ii}, \text{ii}\backslash\text{V}, \text{V}\backslash\text{I}/\text{IV}, \text{IV}, \text{I}\backslash\text{VII}\} \vdash_9 \mathbb{D}^4\text{VII}}(K^4_{\mathbb{D}})}{\mathbb{D}^4\{\text{vi}, \text{vi}\backslash\text{ii}, \text{ii}\backslash\text{V}, \text{V}\backslash\text{I}/\text{IV}, \text{IV}, \text{I}\backslash\text{VII}\} \vdash_9 \text{V}} \quad \overset{G\text{MA}^7}{\text{I} \vdash_0 \text{I}}}{R^4_{\mathbb{D}}\text{E}\flat : \mathbb{D}^4\{\text{vi}, \text{vi}\backslash\text{ii}, \text{ii}\backslash\text{V}, \text{V}\backslash\text{I}/\text{IV}, \text{IV}, \text{I}\backslash\text{VII}\}, \text{V}\backslash\text{I} \vdash_{10} \text{I}}(\backslash_L)$$

$$\dfrac{\overset{Am7}{\text{G: ii} \vdash_0 \text{ii}} \qquad \overset{D7}{\text{C: V} \vdash_0 \text{V}}}{\text{ii}, \text{ii}\backslash\text{V} \vdash_1 \text{V}}\,(\backslash_L)$$

$$\dfrac{\text{ii}, \text{ii}\backslash\text{V} \vdash_1 \text{V} \qquad \overset{G\text{MA}^7}{\text{I} \vdash \text{I}}}{\text{ii}, \text{ii}\backslash\text{V}, \text{V}\backslash\text{I} \vdash_2 \text{I}}\,(\backslash_L)$$

$$\dfrac{\text{ii}, \text{ii}\backslash\text{V}, \text{V}\backslash\text{I} \vdash_2 \text{I} \qquad \dfrac{F\sharp\text{m7}}{\text{vii} \vdash \text{vii}}}{\text{ii}, \text{ii}\backslash\text{V}, \text{V}\backslash\text{I}, \text{I}\backslash\text{vii} \vdash_3 \text{VII}}\,(\backslash_L)$$

$$\dfrac{\text{ii}, \text{ii}\backslash\text{V}, \text{V}\backslash\text{I}, \text{I}\backslash\text{vii} \vdash_3 \text{VII}}{\dfrac{R^3_{\mathbb{D}}\{\text{ii}, \text{ii}\backslash\text{V}, \text{V}\backslash\text{I}, \text{I}\backslash\text{vii}\} \vdash_6 R^3_{\mathbb{D}}\text{vii}}{R^3_{\mathbb{D}}\{\text{ii}, \text{ii}\backslash\text{V}, \text{V}\backslash\text{I}, \text{I}\backslash\text{vii}\} \vdash_6 \text{ii}}}\,(K^3_{\mathbb{D}})$$

$$\dfrac{R^3_{\mathbb{D}}\{\text{ii}, \text{ii}\backslash\text{V}, \text{V}\backslash\text{I}, \text{I}\backslash\text{vii}\} \vdash_6 \text{ii} \qquad \dfrac{B7}{\text{V} \vdash \text{V}}}{R^3_{\mathbb{D}}\{\text{ii}, \text{ii}\backslash\text{V}, \text{V}\backslash\text{I}, \text{I}\backslash\text{vii}\}, \text{ii}\backslash\text{V} \vdash_7 \text{V}}\,(\backslash_L)$$

$$\dfrac{R^3_{\mathbb{D}}\{\text{ii}, \text{ii}\backslash\text{V}, \text{V}\backslash\text{I}, \text{I}\backslash\text{vii}\}, \text{ii}\backslash\text{V} \vdash_7 \text{V} \qquad \dfrac{E\text{MA}^7}{\text{I} \vdash \text{I}}}{R^3_{\mathbb{D}}\{\text{ii}, \text{ii}\backslash\text{V}, \text{V}\backslash\text{I}, \text{I}\backslash\text{vii}\}, \text{ii}\backslash\text{V}, \text{V}\backslash\text{I} \vdash_8 \text{I}}\,(\backslash_L)$$

Figure 7.17: Analysis of the first 24 bars of *All the Things You Are* by Jerome Kern. Notice how the first two trees are exactly the same but in different tonalities. However, thanks to the $R^4_{\mathbb{D}}$ rule, it is also possible to highlight that the relationship between the two tonalities in both sections is identical. Without using this rule and instead attaching the new tonality directly—C in the first case and G in the second—the similarity between the two might be harder to perceive. The double lines indicate instances of the same rule.

Reprise and Finale

This dissertation has explored a wide spectrum of topics lying at the intersection of logic, probability, belief theory, and music analysis. Although each chapter pursued a specific research question, the overall structure reveals a coherent trajectory: starting from philosophical and proof-theoretic motivations, advancing through the technical development of Fractional Semantics and belief revision, and culminating in an original application of proof-theoretic methods to music analysis via the Lambek Calculus. The present chapter aims to bring together these lines of inquiry into a unified perspective.

One of the central threads of this work has been the exploration of supraclassical logics. Building on Makinson's intuition that reasoning in everyday life is essentially non-monotonic, we showed how classical logic can be enriched by additional structures without sacrificing rigor. Wittgenstein's treatment of probability in the *Tractatus* was reinterpreted as an early attempt at supraclassical reasoning, anticipating the role of beliefs as axioms. This reinterpretation provided a natural foundation for the subsequent development of belief-based systems and for the idea that probability can be embedded within logical structures.

A major technical achievement of this dissertation lies in the refinement of Fractional Semantics. Initially introduced by Piazza and Pulcini, this semantics measures the ratio of tautological axioms in a proof, thus quantifying the degree of validity of a formula. We extended this approach in two crucial directions. First, by integrating beliefs as proper axioms, leading to the definition of $GS4_B$, a system capable of representing both tautologies and beliefs. Second, by introducing hyperreal numbers to distinguish between full beliefs and revisable beliefs, thereby enriching the expressive power of the semantics while preserving cut elimination. These extensions demonstrated how Fractional Semantics can serve as a bridge between logical rigor and the graded nature of human belief, maintaining a precise proof-theoretic foundation while accounting for philosophical intuitions about uncertainty.

The dissertation has also provided two complementary approaches to the Lottery Paradox. From the perspective of Wittgenstein's probability, the paradox dissolves: since all propositions are probabilistic rather than categorical, the contradiction never arises. From the perspective of Fractional Semantics, the paradox is resolved by quantifying the truth value of beliefs in a way that prevents inconsistency. Both approaches highlight how classical paradoxes can be revisited and reinterpreted through the lens of supraclassical reasoning, suggesting that what appear to be insurmountable problems in a categorical framework can often be alleviated once graded notions of belief are admitted.

A further development was the treatment of belief revision. By employing hyperreal numbers, the system could account for beliefs that are maintained under normal circumstances but revisable in light of new information. This idea of revisable beliefs reflects the dynamic aspect of human reasoning, capturing how we are able to adapt our convictions without collapsing into inconsistency. The formalism thereby provides a faithful model of belief dynamics, grounded both in logical structure and in philosophical insight.

Finally, the application of proof-theoretic methods to music analysis represents one of the most innovative aspects of this dissertation. By adapting the Lambek Calculus to harmonic progressions and introducing rules that capture cadences, conjunctions of unrelated tonalities, and the notion of depth, we opened a novel pathway for reasoning about music within a logical framework. This approach not only enriches the analytical toolbox of music theory but also demonstrates the versatility of proof theory itself, extending its reach to domains traditionally seen as distant from logic. The case study on "Stella By Starlight" illustrated both the challenges and the potential of this method, showing how logical formalisms can illuminate the structure and complexity of musical works.

Taken together, the contributions of this dissertation show that the gap between logic, probability, belief, and music is narrower than it might first appear. Logic, far from being a purely abstract discipline, can engage with the uncertainty of human reasoning and the expressive depth of musical structures. By bridging classical and supraclassical perspectives, by refining Fractional Semantics, and by venturing into the semiotics of music, this work demonstrates that logical methods remain both adaptable and profoundly interdisciplinary.

Looking forward, several directions of future research remain open. On the logical side, the integration of further modalities into Fractional Semantics could provide even richer models of belief and knowledge. On the philosophical side, extending the analysis to other belief paradoxes could further test the robustness of the proposed frameworks. On the musical side, the formal system developed here could be applied to a wider corpus of pieces, as well as refined to account for rhythmic and contrapuntal dimensions beyond harmony. Finally, the possibility of computational implementations suggests that these ideas might not remain purely theoretical but could find practical application in automated reasoning systems and music analysis software.

In conclusion, this thesis has sought to unify diverse domains through the lens of logic. The journey from Hume's vivacity and Wittgenstein's probability to hyperreal belief revision and Lambek-style harmonic analysis exemplifies the vitality of logical thought when it is allowed to cross disciplinary boundaries. The hope is that this work not only contributes to technical discussions in proof theory and semantics but also opens new possibilities for dialogue between logic, philosophy, and the arts.

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