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Calculus of Variations

Quantitative spectral stability for operators with compact resolvent

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Abstract

This paper deals with quantitative spectral stability for operators with compact resolvent acting on $L^2(X, m)$, where (X, m) is a measure space. Under fairly general assumptions, we provide a characterization of the dominant term of the asymptotic expansion of the eigenvalue variation in this abstract setting. Many of the results about quantitative spectral stability available in the literature can be recovered by our analysis. Furthermore, we illustrate our result with several applications, e.g. quantitative spectral stability for a Neumann limit of a Robin problem, conformal transformations of Riemannian metrics, Dirichlet forms under the removal of sets of small capacity, and for families of Fourier-multipliers.

Mathematics Subject Classification 35P15 · 47A10 · 47A55

1 Introduction

Let us consider the Hilbert space $L^2(X, m)$, where (X, m) is a measure space endowed with a positive measure m. Given $\{(Y_\varepsilon, m_\varepsilon)\}_{\varepsilon\in[0,1]}$ a family of measure spaces, where $Y_\varepsilon \subseteq X$ and m_ε are positive measures over Y_ε, let $\{H_\varepsilon\}_{\varepsilon\in[0,1]}$ be a family of non-negative, self-adjoint operators

$$H_\varepsilon : D[H_\varepsilon] \subseteq L^2(Y_\varepsilon, m_\varepsilon) \to L^2(Y_\varepsilon, m_\varepsilon),$$

so that m_ε is absolutely continuous with respect to m and $(Y_0, m_0) = (X, m)$. Moreover, we suppose that the spectrum $\sigma(H_\varepsilon)$ of H_ε is discrete for any $\varepsilon \in [0, 1]$ and we denote it by $\sigma(H_\varepsilon) = \{\lambda_{n,\varepsilon}\}_{n\in\mathbb{N}}$, where we are repeating each eigenvalues according to its multiplicity.

The present paper is aimed to the study of the asymptotic of the eigenvalue's variation $\lambda_{n,\varepsilon} - \lambda_{n,0}$ as $\varepsilon \to 0^+$ for simple eigenvalues, under suitable assumptions on the family $\{H_\varepsilon\}_{\varepsilon\in[0,1]}$. In particular we are interested in a *quantitative* result, that is, in *how* the eigenvalues $\lambda_{n,\varepsilon}$ converge to $\lambda_{n,0}$ as $\varepsilon \to 0^+$.

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There is a vast literature about quantitative spectral stability for compact operators, with many examples in the field of elliptic PDEs on the Euclidean space or on Riemannian manifolds, in particular under smooth and nonsmooth deformation of the domain, see for example the classical paper [35]. We also refer to [25] for an overview.

A very classic problem is quantitative spectral stability for elliptic operators under some geometrical perturbation on the domain, for example, removing a small set and imposing Dirichlet boundary condition on the removed set.

In the seminal paper [34], the variation of the eigenvalues of the Laplace operator acting on an Euclidean domain D, and perturbed by removing small balls, is characterized explicitly by the means of Green functions. For generic holes concentrating in a compact set K, the eigenvalue's variation is characterized in terms of weighted capacity of the removed sets in [17]. In dimension 2, a precise first order asymptotic expansion of the eigenvalue's variation for similar problems is obtained in [1] and in [3]. In particular, in this latter work the authors provided an expansion in series of the weighted capacity of a scaling hole εK. In an arbitrary dimension N, for scaling holes εK, it is possible to explicitly estimate the vanishing order, see [17]. In the general setting of Riemannian manifolds we refer to [12] for a precise description of the eigenvalue's variation in terms of the capacity of the removed sets and for a more detailed overview of the literature for the Dirichlet Laplacian on manifolds.

For the fractional Laplacian operator in a bounded domain of $\mathbb{R}^N$, a similar characterization of the eigenvalue's variation in terms of a weighted fractional capacity was proved in [2]. Furthermore, in the same paper, the authors managed to compute the eigenvalue's variation in the case of a sequence of shrinking holes of the form εK, where K is a compact subset. Such computation is sharp in the case of set of positive N-dimension Lebesgue measure.

For the ground state of a generic Dirichlet form $(\mathcal{E}, \mathcal{F})$ in a measure space (X, m), under the removal of sets of small capacity, the eigenvalue's variation is expanded at the second order in [32], under suitable assumption on the resolvent and semigroup associated to $\mathcal{E}$.

For the polyharmonic operator $(-\Delta)^m$ acting on a bounded Euclidean domain, the eigenvalue's variation is computed in [22] in terms of a suitable notion of weighted capacity. If the removed sets are given by εK for some compact set K, a more precise description of the eigenvalue's variation, with explicit estimates on its vanishing order, is also provided in the aforementioned work.

The Neumann Laplacian operator in a bounded domain $\Omega \subset \mathbb{R}^N$, where Neumann conditions are imposed on the boundary of the removed set, was studied for example in [16, 29]. In [29] removing a set of size ε, the author expresses the eigenvalue of the perturbed problem as an analytic function in two variables composed with the perturbing parameter ε and $\delta_N \varepsilon \log(\varepsilon)$, where $\delta_N = 1$ if N is odd and 0 if N is even. In [16] the eigenvalue's variation is characterized by the *weighted torsion* of the removed set. Furthermore, for holes of the form $x_0 + \varepsilon K$ for some $x_0 \in \Omega$ and compact $K \subset \Omega$, the eigenvalue's variation can be computed more explicitly. It is worth noticing that its vanishing order strongly depends on the point x_0. A completely explicit expression of the eigenvalue's variation is obtained for spherical holes. We also mention that the study of the convergence in Hausdorff distance of the spectrum for generics holes in $\mathbb{R}^2$ was studied in [5].

A different case of geometric perturbation was considered in [21], for the Euclidean Dirichlet Laplacian. More precisely the authors attached a small cylindrical tube to the unperturbed domain Ω and let the section of the cylinder shrinks. The vanishing order of the eigenvalue's variation is computed with monotonicity formula's techniques.

The case where mixed Dirichlet and Neumann boundary condition are imposed on the boundary of a bounded domain $\Omega \subset \mathbb{R}^N$ for the Laplacian operator were considered in [17, 18]. More precisely, if the Dirichlet region vanishes, then the eigenvalue's variation can be quantified in terms of a weighted capacity of a disappearing set. With techniques based on a combination min-max estimates and a monotonicity formula, the case of a vanishing Neumann region was studied in [18].

Finally, we mention that, for Aharonov-Bohm operators with an arbitrary numbers of moving poles, the eigenvalue's variation was recently studied in [19] and [20]. In this case, the quantity that characterizes the eigenvalue's variation may be seen as an intermediate notion between a weighted torsion and a weighted capacity. There is a vast literature about quantitative spectral stability for Aharonov-Bohm operators, see [19] for a detailed overview.

In the present paper, we focus on simple eigenvalues providing an abstract viewpoint that unifies some of the aforementioned results in a general setting. In particular we are referring to [19, Theorem 5.2], [20, Theorem 5.3], [16, Theorem 2.3] and, see Subsection 4.3 for details, to [17, Theorem 2.5], [2, Theorem 1.5] and [22, Thoerem 1.2]. Furthermore Theorem 3.1 generalizes [32, Theorem 3.6] to the case of simple eigenvalues and removes several assumptions, see Subsection 4.3. The case of multiple eigenvalues is object of current investigation, see [7].

Our main result, see Theorem 3.1, characterizes the eigenvalue's variation as

$$\lambda_{n,\varepsilon} - \lambda_{n,0} = \frac{\lambda_0 \int_{Y_\varepsilon} \phi_{n,0} V_{n,\varepsilon} \, \mathrm{d}m_\varepsilon + O\left(\left\| V_{n,\varepsilon} \right\|^2_{L^2(Y_\varepsilon, m_\varepsilon)} \right)}{\int_{Y_\varepsilon} |\phi_{n,0}|^2 \, \mathrm{d}m_\varepsilon + O(\left\| V_{n,\varepsilon} \right\|_{L^2(Y_\varepsilon, m_\varepsilon)})} \quad \text{as } \varepsilon \to 0,$$

where $\phi_{n,0}$ is a normalized eigenfunction of the operator H_0 associated to the simple eigenvalue $\lambda_{n,0}$ and $V_{\varepsilon,n}$ is the unique minimizer of an energy functional $J_{\varepsilon,n}$ which depends on $\phi_{n,0}$, see (2.8), and we also obtain the convergence of the associated eigenfunctions in a suitable sense. We also presents several applications of Theorem 3.1.

Some different and more classical methods to study eigenvalues variations are available in the literature, as perturbation theory of families of analytic forms in the sense of Kato, see [28], and the min-max characterization of eigenvalues. Our approach, however, allows us to deal with more singular type of perturbations than Kato's theory and to characterize the eigenvalues variation as a leading term with a reminder, while the min-max approach often yields only estimates. Finally, we mention that the function $\psi_\varepsilon := \phi_{0|Y_\varepsilon} - V_\varepsilon$ can be interpreted as a quasimod (see for example [11, Chapter 20]) for the eigenfunction ϕ_0, see the proof of Theorem 3.1.

The paper is organized as follows. In Section 2 we provide some abstract preliminaries results and present the assumptions needed to prove Theorem 3.1. In Section 3 we prove Theorem 3.1 and discuss several results usefull to ensure the validity of its assumption, focusing on qualitative spectral stability in Subsection 3.3. Finally, in Section 4, as applications of Theorem 3.1, we study quantitative spectral stability for a Neumann limit of a Robin problem, conformal transformations of Riemannian metrics, Dirichlet forms under the removal of sets of small capacity, and for families of Fourier-multipliers.

2 Preliminaries and assumptions

Throughout the paper, for any measure space (X, m) we will denote with $(\cdot, \cdot)_{L^2(X,m)}$ the scalar product in $L^2(X, m)$. Furthermore we denote the resolvent and the spectrum of an operator H respectively by $\rho(H)$ and $\sigma(H)$, while we will use $D[H]$ for its domain. We also recall that a symmetric, non-negative form $(\mathcal{E}, \mathcal{F})$ on $L^2(X, m)$ is *closed* if its domain $\mathcal{F}$ is complete with respect to the norm induced by the scalar product

$$\mathcal{E}_1 := \mathcal{E} + (\cdot,\cdot)_{L^2(X,m)}. \tag{2.1}$$

To any non-negative, self-adjoint and densely defined operator

$$H : D[H] \subseteq L^2(X,m) \to L^2(X,m),$$

it is possible to associate a unique symmetric bilinear form $(\mathcal{E},\mathcal{F})$ given by

$$\mathcal{E}(u,v) := (\sqrt{H}u, \sqrt{H}v)_{L^2(X,m)} \quad \text{for any } u, v \in \mathcal{F}, \tag{2.2}$$

where $\mathcal{F} := D[\sqrt{H}]$. Moreover, the form $(\mathcal{E},\mathcal{F})$ is closed. On the other hand, any non-negative densely defined, closed symmetric form admits a unique operator H such that (2.2) holds; see for example [24, Theorem 1.3.1].

2.1 Preliminaries

In this subsection we present some abstract results that will be used throughout the paper. For the sake of completeness we also provide a short proof of each result.

Proposition 2.1 *Let $(\mathcal{E},\mathcal{F})$ be a symmetric, non-negative, densely defined and closed form on $L^2(X,m)$ and let H be the associated densely defined, non-negative and self-adjoint operator. Suppose $\lambda := \min\sigma(H) > 0$ is an eigenvalue of H and that*

$$H^{-1} : L^2(X,m) \to L^2(X,m) \quad \textit{is a compact operator.}$$

Then the embedding $i : \mathcal{F} \hookrightarrow L^2(X,m)$ is compact.

Proof Since H^{-1} is compact, by [36, Theorem 12.30] and the Spectral Theorem for bounded, self-adjoint operators, $H^{-\frac{1}{2}}$ is a compact operator. Let $v_n \rightharpoonup v$ weakly in $\mathcal{F}$ as $n \to \infty$. Then for any $w \in \mathcal{F}$

$$\lim_{n\to\infty} \left(\sqrt{H}v_n, \sqrt{H}w\right)_{L^2(X,m)} = \lim_{n\to\infty} \mathcal{E}(v_n, w) = \mathcal{E}(v,w) = \left(\sqrt{H}v, \sqrt{H}w\right)_{L^2(X,m)}.$$

Hence $\sqrt{H}v_n \rightharpoonup \sqrt{H}v$ weakly in $L^2(X,m)$ as $n \to \infty$ and so, by the compactness of $H^{-\frac{1}{2}}$, $v_n \to v$ strongly in $L^2(X,m)$ as $n \to \infty$. □

In the following we will make use of the next (abstract) Poincare's inequality.

Proposition 2.2 *Let $(\mathcal{E},\mathcal{F})$ be a symmetric, densely defined, non-negative and closed form on $L^2(X,m)$ and let H be the associated densely defined, non-negative and self-adjoint operator. Suppose that $\lambda := \min\sigma(H) > 0$ and that H^{-1} is a compact operator. Then*

$$\|w\|^2_{L^2(X,m)} \le \lambda^{-1}\mathcal{E}(w,w) \quad \textit{for any } w \in \mathcal{F}. \tag{2.3}$$

Proof Let us consider the minimization problem

$$\inf\left\{\frac{\mathcal{E}(w,w)}{\|w\|^2_{L^2(X,m)}} : w \in \mathcal{F}\setminus\{0\}\right\}. \tag{2.4}$$

Let us show that there exists a minimizer of (2.4). Let $\{u_n\}_{n\in\mathbb{N}}$ be a minimizing sequence and let

$$v_n := \frac{u_n}{\|u_n\|_{L^2(X,m)}}.$$

Then also $\{v_n\}_{n\in\mathbb{N}}$ is a minimizing sequence and $\|v_n\|_{L^2(X,m)} = 1$. Since $\mathcal{E}$ is closed (hence $(\mathcal{E}_1, \mathcal{F})$ is Hilbert) and $\{v_n\}_n$ is bounded, up to a subsequence, we can assume that $v_n \rightharpoonup v$ weakly in the norm $\mathcal{E}_1 = \mathcal{E} + (\cdot,\cdot)_{L^2(X,m)}$ as $n \to \infty$ to some $v \in \mathcal{F}$.

In particular $\|v\|_{L^2(X,m)} = 1$ thanks to Proposition 2.1. By the lower semicontinuity of the norm $\mathcal{E}_1$ we have

$$\mathcal{E}_1(v, v) \le \liminf_{n\to\infty} \mathcal{E}_1(v_n, v_n)$$

and so

$$\mathcal{E}(v, v) \le \liminf_{n\to\infty} \mathcal{E}(v_n, v_n).$$

We conclude that v minimizes (2.4). Hence, v is a solution of the equation

$$\mathcal{E}(v, w) = \frac{\mathcal{E}(v, v)}{\|v\|^2_{L^2(X,m)}} (v, w)_{L^2(X,m)} \quad \text{for any } w \in \mathcal{F}.$$

It follows that $\frac{\mathcal{E}(v,v)}{\|v\|^2_{L^2(X,m)}}$ is an eigenvalue of H and that v an associated eigenfunction. Hence, $\lambda \le \frac{\mathcal{E}(v,v)}{\|v\|^2_{L^2(X,m)}}$. Since H^{-1} is compact it follows that there exists an eigenfunction associated to λ, see for example [36, Theorem 12.29]. Evaluating the ratio in (2.4) with such an eigenfunction we obtain the opposite inequality thus concluding that

$$\inf\left\{\frac{\mathcal{E}(w, w)}{\|w\|^2_{L^2(X,m)}} : w \in \mathcal{F}\setminus\{0\}\right\} = \lambda,$$

which proves (2.3). □

We will use the following regularity result in Subsection 4.4. It is a rather simple and standard fact in spectral theory, we provide a detailed proof for the sake of completeness.

Proposition 2.3 *Let $(\mathcal{E}, \mathcal{F})$ be a non-negative, densely defined, symmetric and closed bilinear form and H the corresponding densely defined, non-negative and self-adjoint operator. Let $f \in L^2(X, m)$ and suppose that $u \in \mathcal{F}$ solves the equation*

$$\mathcal{E}(u, v) = (f, v)_{L^2(X,m)} \quad \text{for any } v \in \mathcal{F}. \tag{2.5}$$

Then $u \in D[H]$ and $Hu = f$.

Proof It is not restrictive to suppose that $\inf \sigma(H) > 0$. Indeed $\inf \sigma(H + \mathrm{Id}) \ge 1$, $D[H] = D[H + \mathrm{Id}]$ and if u is a solution to (2.5) and only if u solves

$$\mathcal{E}_1(u, v) = (f + u, v)_{L^2(X,m)} \quad \text{for any } v \in \mathcal{F}.$$

Since $\inf \sigma(H) > 0$, the operator H^{-1} is well-defined and bounded on $L^2(X, m)$. Let us set $\tilde{u} := H^{-1} f$. Clearly $\tilde{u} \in D[H]$, $H\tilde{u} = f$ and, by definition of the operator H, $\tilde{u}$ solves (2.5). It follows that

$$\mathcal{E}(u - \tilde{u}, u - \tilde{u}) = 0$$

and so, by Proposition 2.2, $u = \tilde{u}$. In conclusion $u \in D[H]$ and $Hu = f$. □

2.2 Assumptions

Let us consider the Hilbert space $L^2(X, m)$, where (X, m) is a measure space endowed with a positive measure m. Let $\{(Y_\varepsilon, m_\varepsilon)\}_{\varepsilon\in[0,1]}$ be a family of measure spaces, where $Y_\varepsilon \subseteq X$ and m_ε are positive measures over Y_ε, so that m_ε is absolutely continuous with respect to m and $(Y_0, m_0) = (X, m)$. Let $\{(\mathcal{E}^\varepsilon, \mathcal{F}_\varepsilon)\}_{\varepsilon\in[0,1]}$ be a family of non-negative, symmetric forms defined on their domain $\mathcal{F}_\varepsilon \subseteq L^2(Y_\varepsilon, m_\varepsilon)$.

For any $\varepsilon \in [0, 1]$ let Z_ε be a linear subspace of $\mathcal{F}_\varepsilon$ that is closed with respect to the norm $(\mathcal{E}_1^{(\varepsilon)})^{\frac{1}{2}}$, where $\mathcal{E}_1^{(\varepsilon)}$ is as in (2.1). We are going to consider the eigenvalues of the form $\mathcal{E}^{(\varepsilon)}$ restricted to the subspace Z_ε. We need to consider such a restriction to encompass several problems that have been studied in the literature. For example, this is crucial in [19] and Subsection 4.3.

More precisely, let us define

$$\overline{\mathcal{E}^{(\varepsilon)}} := \mathcal{E}^{(\varepsilon)}_{|Z_\varepsilon\times Z_\varepsilon}.$$

It is clear that $\overline{\mathcal{E}^{(\varepsilon)}}$ is a closed, non-negative defined and symmetric form with domain Z_ε. If we define

$$\mathcal{Z}_\varepsilon := \overline{Z_\varepsilon}^{\|\cdot\|_{L^2(Y_\varepsilon, m_\varepsilon)}}$$

that is, $\mathcal{Z}_\varepsilon$ is the closure of Z_ε with respect to the norm of $L^2(Y_\varepsilon, m_\varepsilon)$ and we endow it with the norm induced by $L^2(Y_\varepsilon, m)$, then clearly $\overline{\mathcal{E}^{(\varepsilon)}}$ is densely defined in $\mathcal{Z}_\varepsilon$. Let $\overline{H_\varepsilon}$ be the associated non-negative, densely defined and self-adjoint operator, see [24, Theorem 1.3.1]. In what follows we suppose that for any $\varepsilon \in [0, 1]$

$$\text{the spectrum of } \overline{H_\varepsilon} \text{ is discrete and it consists of a non-decreasing diverging sequence of non-negative eigenvalues } \{\lambda_{\varepsilon,n}\}_{n\in\mathbb{N}\setminus\{0\}}, \tag{A1}$$

where each eigenvalue is repeated according to its multiplicity.

Under the assumption (A1), let $\{\phi_{\varepsilon,n}\}_{n\in\mathbb{N}\setminus\{0\}}$ be an associated basis of $\mathcal{Z}_\varepsilon$ given by eigenfunctions of $\overline{H_\varepsilon}$. In particular,

$$\overline{\mathcal{E}^{(\varepsilon)}}(\phi_{n,\varepsilon}, w) = \lambda_{n,\varepsilon}\left(\phi_{n,\varepsilon}, w\right)_{L^2(Y_\varepsilon, m_\varepsilon)} \quad \text{for any } w \in Z_\varepsilon. \tag{2.6}$$

Since we are interested in the quantitative spectral stability, it is natural to suppose that

$$\lim_{\varepsilon\to 0} \lambda_{\varepsilon,n} = \lambda_{0,n}, \quad \text{for any } n \in \mathbb{N}\setminus\{0\} \tag{A2}$$

In addition, it is not restrictive to suppose that

$$\lambda_{\varepsilon,1} > 0 \quad \text{for any } \varepsilon \in [0, 1], \tag{A3}$$

as detailed in the following remark.

Remark 2.4 If $\lambda_{\varepsilon,1} = 0$ then we can perform our analysis for the family of operators $\{\overline{H_\varepsilon} + \mathrm{Id}\}_{\varepsilon\in[0,1]}$. Indeed, $\{\overline{H_\varepsilon} + \mathrm{Id}\}_{\varepsilon\in[0,1]}$ is a family of densely defined, positive, self-adjoint operators that possesses the same properties of $\{\overline{H_\varepsilon}\}_{\varepsilon\in[0,1]}$. The spectrum of $\overline{H_\varepsilon} + \mathrm{Id}$ is given by $\{\lambda_{\varepsilon,n} + 1\}_{n\in\mathbb{N}\setminus\{0\}}$.

More generally, we may perform our analysis for a family of forms $\{\overline{(\mathcal{E}^{(\varepsilon)}}, Z_\varepsilon)\}_{\varepsilon\in[0,1]}$ that are uniformly lower semi-bounded by a constant $c \leq 0$. Indeed, we can simply shift the associated operators $\{\overline{H_\varepsilon}\}_{\varepsilon\in[0,1]}$ by $-c\,\mathrm{Id}$.

It is a standard fact that, under the above assumptions, the operator $(\overline{H_\varepsilon})^{-1}$ is continuous, non-negative and self-adjoint. Moreover, in view of [36, Theorem 12.30], by (A1) the operator $(\overline{H_\varepsilon})^{-1}$ also turns out to be compact.

Fix a simple eigenvalue $\lambda_{0,n}$ and a corresponding eigenfunction $\phi_{0,n}$ For the sake of simplicity we write λ_0 instead of $\lambda_{0,n}$ and ϕ_0 instead of $\phi_{0,n}$. We assume that

$$\|\phi_0\|_{L^2(X,m)} = 1 \quad \text{and} \quad \lim_{\varepsilon\to 0} \int_{Y_\varepsilon} |\phi_0|^2 \, \mathrm{d}m_\varepsilon = 1. \tag{A4}$$

We suppose that the space $\mathcal{F}_0$ is included in any $\mathcal{F}_\varepsilon$ in the following sense

$$u \in \mathcal{F}_0 \quad \Rightarrow \quad u_{|Y_\varepsilon} \in \mathcal{F}_\varepsilon. \tag{A5}$$

Such condition is well-posed since m_ε is absolutely continuous with respect to m_0. Furthermore, for any $\varepsilon \in (0, 1]$

$$L_\varepsilon(u) := \mathcal{E}^{(\varepsilon)}(\phi_0, u) - \lambda_0(\phi_0, u)_{L^2(Y_\varepsilon, m_\varepsilon)} \quad \text{for any } u \in \mathcal{F}_\varepsilon, \tag{2.7}$$

defines a continuous linear functional on Z_ε in view of (A5). For the sake of simplicity, we will still denote with $\mathcal{E}^{(\varepsilon)}$ the quadratic form

$$\mathcal{E}^{(\varepsilon)}(w) := \mathcal{E}^{(\varepsilon)}(w, w) \quad \text{for any } \varepsilon \in [0, 1] \text{ and } w \in \mathcal{F}_\varepsilon.$$

Let us define for any $\varepsilon \in (0, 1]$ the functional

$$J_\varepsilon(u) := \frac{1}{2}\mathcal{E}^{(\varepsilon)}(u) - L_\varepsilon(u) \quad \text{for any } u \in \mathcal{F}_\varepsilon. \tag{2.8}$$

Proposition 2.5 *For any $\varepsilon \in [0, 1]$ there exists a unique $V_\varepsilon \in \mathcal{F}_\varepsilon$ solving the minimization problem*

$$\inf\{J_\varepsilon(u) : u \in Z_\varepsilon + \phi_{0|Y_\varepsilon}\}. \tag{2.9}$$

Furthermore, V_ε is the unique solution to the equation

$$\mathcal{E}^{(\varepsilon)}(V_\varepsilon, u) = L_\varepsilon(u) \quad \textit{for any } u \in Z_\varepsilon \tag{2.10}$$

such that $V_\varepsilon - \phi_{0|Y_\varepsilon} \in Z_\varepsilon$. If

$$L_\varepsilon \not\equiv 0 \textit{ in } Z_\varepsilon \quad \textit{or} \quad \phi_{0|Y_\varepsilon} \notin Z_\varepsilon.$$

then $V_\varepsilon \neq 0$.

Proof Thanks to (2.3), denoting by $||\cdot||_{\mathcal{F}_\varepsilon}$ the operator norm over $\mathcal{F}_\varepsilon$, it is enough to notice that

$$\begin{aligned} J_\varepsilon(u) &\geq \frac{1}{2}\mathcal{E}^{(\varepsilon)}(u) - (\mathcal{E}_1^{(\varepsilon)}(u))^{\frac{1}{2}} \, \|L_\varepsilon\|_{(\mathcal{F}_\varepsilon)^*} \\ &\geq \frac{1}{2}\mathcal{E}^{(\varepsilon)}(u) - (1 + \lambda_{\varepsilon,1}^{\frac{1}{2}})(\mathcal{E}^{(\varepsilon)}(u))^{\frac{1}{2}} \, \|L_\varepsilon\|_{(\mathcal{F}_\varepsilon)^*} \end{aligned}$$

to see that J_ε is coercive on $\mathcal{F}_\varepsilon$. Since J_ε is also convex and continuous on $\mathcal{F}_\varepsilon$ we obtain the existence of a minimizer $V_\varepsilon \in \mathcal{F}_\varepsilon$ of (2.9) which solves (2.10). Indeed, the affine subspace $Z_\varepsilon + \phi_{0|Y_\varepsilon}$ is closed with respect to the norm $||\cdot||_{\mathcal{F}_\varepsilon}$ in view of Proposition 2.2.

Furthermore, if v_1 and v_2 solve (2.10) and $v_i - \phi_{0|Y_\varepsilon} \in Z_\varepsilon$ for $i = 1, 2$, then $v := v_1 - v_2$ solves

$$\mathcal{E}^{(\varepsilon)}(v, u) = 0 \quad \text{for any } u \in Z_\varepsilon.$$

Since $v \in Z_\varepsilon$, testing with v we conclude that $v = 0$ by (2.3).

Finally, if $\phi_{0|Y_\varepsilon} \notin Z_\varepsilon$ then clearly V_ε is not trivial. On the other hand, suppose that $\phi_{0|Y_\varepsilon} \in Z_\varepsilon$. Then we are minimizing the functional J_ε on the linear space Z_ε. Since $L_\varepsilon \not\equiv 0$, there exists a function $w \in Z_\varepsilon$ such that $L_\varepsilon(w) > 0$. Then, choosing $t > 0$ small enough, $J_\varepsilon(tw) = t^2\mathcal{E}^{(\varepsilon)}(w) - tL_\varepsilon(w) < 0$. We conclude that $J_\varepsilon(V_\varepsilon) < 0$ and so $V_\varepsilon \neq 0$. □

Remark 2.6 We underline that while the potential V_ε strongly depends on the eventual shifting we made for the form $\mathcal{E}^{(\varepsilon)}$, see Remark 2.4, this has no influence on the eigenvalues variation.

The last assumption of the present section is that

$$\lim_{\varepsilon\to 0} \|V_\varepsilon\|_{L^2(Y_\varepsilon, m_\varepsilon)} = 0. \tag{A6}$$

In view of (2.7) and (2.10), this is a natural requirement, since we are interested in stability results for the spectrum of H_0.

3 Quantitative spectral stability

In this section we prove our main theorem and some additional results to quantify the asymptotic of the eigenvalues variation more explicitly. Furthermore, we discuss the validity of its assumption presenting a criterion for qualitative spectral stability.

3.1 Asymptotics of the eigenvalues variation

Let λ_0 and ϕ_0 be as in Section 2 and for the sake of simplicity let $\lambda_\varepsilon := \lambda_{n,\varepsilon}$. In view of (A2), it is not restrictive to suppose that λ_ε is simple for any $\varepsilon \in [0, 1]$. Let $\phi_\varepsilon := \phi_{n,\varepsilon}$ be an eigenfunction corresponding to the eigenvalue λ_ε such that

$$\|\phi_\varepsilon\|^2_{L^2(Y_\varepsilon, m_\varepsilon)} = 1.$$

Let us define the orthogonal projection

$$\Pi_\varepsilon : L^2(Y_\varepsilon, m_\varepsilon) \to Z_\varepsilon, \quad \Pi_\varepsilon(w) = (w, \phi_\varepsilon)_{L^2(Y_\varepsilon, m_\varepsilon)}\, \phi_\varepsilon.$$

Since $(\overline{\mathcal{E}^{(\varepsilon)}}, \mathcal{Z}_\varepsilon)$ is closed, in view of Proposition 2.2, it follows that $\mathcal{Z}_\varepsilon$ is a Hilbert space with respect to the scalar product $\overline{\mathcal{E}^{(\varepsilon)}}$. Let

$$R_\varepsilon := (\overline{H_\varepsilon})^{-1} : \mathcal{Z}_\varepsilon \to \mathcal{Z}_\varepsilon. \tag{3.1}$$

By the very definition of $\overline{\mathcal{E}^{(\varepsilon)}}$, it holds

$$\overline{\mathcal{E}^{(\varepsilon)}}(R_\varepsilon u, v) = (u, v)_{L^2(Y_\varepsilon, m_\varepsilon)} \quad \text{for any } u, v \in \mathcal{Z}_\varepsilon \tag{3.2}$$

and so if we endow $\mathcal{Z}_\varepsilon$ with the norm $\overline{\mathcal{E}^{(\varepsilon)}}^{\frac{1}{2}}$ then R_ε is linear and continuous. In particular, the restriction of R_ε to Z_ε, which we still denote with R_ε, is linear and continuous as well. By the Spectral Theorem for bounded, self-adjoint operators it follows that

$$(\mathrm{dist}(\mu, \sigma(R_\varepsilon)))^2 \le \frac{\overline{\mathcal{E}^{(\varepsilon)}}(R_\varepsilon w - \mu w)}{\overline{\mathcal{E}^{(\varepsilon)}}(w)} \quad \text{for any } \mu \in \rho(R_\varepsilon) \text{ and } w \in \mathcal{Z}_\varepsilon \setminus \{0\}, \tag{3.3}$$

see for example [27, Proposition 8.20]. Clearly, the eigenvalues of R_ε satisfy $\mu_{\varepsilon,n} := \frac{1}{\lambda_{\varepsilon n}}$ for any $n \in \mathbb{N} \setminus \{0\}$.

The proof of the following theorem is inspired by [1, Appendix A], [17, Theorem 2.5], [2, Theorem 1.5] and [19, Theorem 5.2].

Theorem 3.1 *Let $\{H_\varepsilon\}_{\varepsilon\in[0,1]}$ be a family of non-negative, densely defined and self-adjoint operators*

$$H_\varepsilon : D[H_\varepsilon] \subseteq L^1(Y_\varepsilon, m_\varepsilon) \to L^2(Y_\varepsilon, m_\varepsilon)$$

with associated bilinear forms $(\mathcal{E}^{(\varepsilon)}, \mathcal{F}_\varepsilon)$. Fix a simple eigenvalue $\lambda_0 := \lambda_{0,n}$ of H_0 with associated normalized eigenfunction ϕ, let $\lambda_\varepsilon := \lambda_{\varepsilon,n}$. Suppose that all the assumptions (A1) to (A6) hold.

Then,

$$\lambda_\varepsilon - \lambda_0 = \frac{\lambda_0 \int_{Y_\varepsilon} \phi_0 V_\varepsilon \, \mathrm{d}m_\varepsilon + O\left(\|V_\varepsilon\|^2_{L^2(Y_\varepsilon,m_\varepsilon)}\right)}{\int_{Y_\varepsilon} |\phi_0|^2 \, \mathrm{d}m_\varepsilon + O(\|V_\varepsilon\|_{L^2(Y_\varepsilon,m_\varepsilon)})} \quad \text{as } \varepsilon \to 0, \tag{3.4}$$

where V_ε is the function provided by Proposition 2.5. Furthermore,

$$\mathcal{E}^{(\varepsilon)}(\phi_0 - V_\varepsilon - \Pi_\varepsilon(\phi_0 - V_\varepsilon)) = O\left(\|V_\varepsilon\|^2_{L^2(Y_\varepsilon,m_\varepsilon)}\right) \quad \text{as } \varepsilon \to 0, \tag{3.5}$$

and

$$\|\phi_0 - \Pi_\varepsilon(\phi_0 - V_\varepsilon)\|_{L^2(Y_\varepsilon,m_\varepsilon)} = O\left(\|V_\varepsilon\|_{L^2(Y_\varepsilon,m_\varepsilon)}\right), \tag{3.6}$$

$$\begin{aligned} \mathcal{E}^{(\varepsilon)}(\phi_0 - \Pi_\varepsilon(\phi_0 - V_\varepsilon)) &= \mathcal{E}^{(\varepsilon)}(V_\varepsilon) + O\left(\|V_\varepsilon\|_{L^2(Y_\varepsilon,m_\varepsilon)} (\mathcal{E}^{(\varepsilon)}(V_\varepsilon))^{\frac{1}{2}}\right) \\ &= O\left(\mathcal{E}^{(\varepsilon)}(V_\varepsilon)\right), \end{aligned} \tag{3.7}$$

as $\varepsilon \to 0$.

Proof Suppose that $V_\varepsilon = 0$. Then $L_\varepsilon = 0$ and $\phi_{0|Y_\varepsilon} \in Z_\varepsilon$. It follows that $\phi_{0|Y_\varepsilon}$ is an eigenfunction of $\mathcal{E}_\varepsilon$. Hence by (A2), for ε small enough, $\lambda_\varepsilon = \lambda_0$. In conclusion (3.4) holds trivially. Therefore it is not restrictive to suppose that $V_\varepsilon \neq 0$ for any $\varepsilon \in (0, 1]$.

Let $\psi_\varepsilon := \phi_{0|Y_\varepsilon} - V_\varepsilon$. Then $\psi_\varepsilon \in Z_\varepsilon$ by Proposition 2.5 and

$$\overline{\mathcal{E}^{(\varepsilon)}}(\psi_\varepsilon, w) = \lambda_0 \, (\phi_0, w)_{L^2(Y_\varepsilon,m_\varepsilon)} \quad \text{for all } w \in Z_\varepsilon, \tag{3.8}$$

in view of (2.7) and (2.10). From (3.8) we deduce that

$$\overline{\mathcal{E}^{(\varepsilon)}}(\psi_\varepsilon, w) - \lambda_0 \, (\psi_\varepsilon, w)_{L^2(Y_\varepsilon,m_\varepsilon)} = \lambda_0 \, (V_\varepsilon, w)_{L^2(Y_\varepsilon,m_\varepsilon)} \quad \text{for all } w \in Z_\varepsilon. \tag{3.9}$$

Since $\phi_\varepsilon \in Z_\varepsilon$, we may choose

$$w := \Pi_\varepsilon \psi_\varepsilon = (\psi_\varepsilon, \phi_\varepsilon)_{L^2(Y_\varepsilon,m_\varepsilon)} \phi_\varepsilon$$

in (3.9), thus obtaining

$$\begin{aligned} &(\lambda_\varepsilon - \lambda_0) \, (\psi_\varepsilon, \Pi_\varepsilon \psi_\varepsilon)_{L^2(Y_\varepsilon,m_\varepsilon)} \\ &= \lambda_0 \, (V_\varepsilon, \phi_0)_{L^2(Y_\varepsilon,m_\varepsilon)} + \lambda_0 \, (V_\varepsilon, \Pi_\varepsilon \psi_\varepsilon - \phi_0)_{L^2(Y_\varepsilon,m_\varepsilon)} \, . \end{aligned} \tag{3.10}$$

by (2.6).

Now we study the asymptotic, as $\varepsilon \to 0$, of each term in (3.10). For the sake of simplicity, we divide the rest of the proof into several steps.

Step 1. We claim that

$$|\lambda_\varepsilon - \lambda_0| = O(\|V_\varepsilon\|_{L^2(Y_\varepsilon,m_\varepsilon)}), \quad \text{as } \varepsilon \to 0. \tag{3.11}$$

Letting $\mu_0 := \lambda_0^{-1}$ and $\mu_\varepsilon := \lambda_\varepsilon^{-1}$, since λ_0 is simple and $\lambda_\varepsilon \to \lambda_0$ by (A2), for ε small enough

$$|\lambda_\varepsilon - \lambda_0| = \lambda_\varepsilon \lambda_0 |\mu_\varepsilon - \mu_0| \\ \leq 2\lambda_0^2 \operatorname{dist}(\mu_0, \sigma(R_\varepsilon)) \\ \leq 2\lambda_0^2 \left(\frac{\overline{\mathcal{E}^{(\varepsilon)}}(R_\varepsilon \psi_\varepsilon - \mu_0 \psi_\varepsilon)}{\overline{\mathcal{E}^{(\varepsilon)}}(\psi_\varepsilon)} \right)^{1/2}, \tag{3.12}$$

thanks to (3.3). Since $\|\phi_0\|_{L^2(X,m)} = 1$, by (3.8) and the Cauchy-Schwarz inequality we have

$$\overline{\mathcal{E}^{(\varepsilon)}}(\psi_\varepsilon) = \lambda_0 \, (\phi_0, \psi_\varepsilon)_{L^2(Y_\varepsilon, m_\varepsilon)} \\ = \lambda_0 - \lambda_0 \left(1 - \int_{Y_\varepsilon} |\phi_0|^2 \, \mathrm{d}m_\varepsilon \right) - \lambda_0 \int_{Y_\varepsilon} \phi_0 V_\varepsilon \, \mathrm{d}m_\varepsilon \\ = \lambda_0 + o(1), \tag{3.13}$$

where in the last equality we have used (A4). By (3.2) and (3.8) tested with $R_\varepsilon \psi_\varepsilon - \mu_0 \psi_\varepsilon$,

$$\overline{\mathcal{E}^{(\varepsilon)}}(R_\varepsilon \psi_\varepsilon - \mu_0 \psi_\varepsilon) \\ = - (V_\varepsilon, R_\varepsilon \psi_\varepsilon - \mu_0 \psi_\varepsilon)_{L^2(Y_\varepsilon, m_\varepsilon)} + (\phi_0, R_\varepsilon \psi_\varepsilon - \mu_0 \psi_\varepsilon)_{L^2(Y_\varepsilon, m_\varepsilon)} \\ - \overline{\mathcal{E}^{(\varepsilon)}}(\mu_0 \psi_\varepsilon, R_\varepsilon \psi_\varepsilon - \mu_0 \psi_\varepsilon) \\ = - (V_\varepsilon, R_\varepsilon \psi_\varepsilon - \mu_0 \psi_\varepsilon)_{L^2(Y_\varepsilon, m_\varepsilon)} .$$

Hence, by the Cauchy-Schwarz inequality and Proposition 2.2,

$$\overline{\mathcal{E}^{(\varepsilon)}}(R_\varepsilon \psi_\varepsilon - \mu_0 \psi_\varepsilon) = - (V_\varepsilon, R_\varepsilon \psi_\varepsilon - \mu_0 \psi_\varepsilon)_{L^2(Y_\varepsilon, m_\varepsilon)} \\ \leq ||V_\varepsilon||_{L^2(Y_\varepsilon, m_\varepsilon)} \, ||R_\varepsilon \psi_\varepsilon - \mu_0 \psi_\varepsilon||_{L^2(Y_\varepsilon, m_\varepsilon)} \\ \leq ||V_\varepsilon||_{L^2(Y_\varepsilon, m_\varepsilon)} \, (\mu_\varepsilon \overline{\mathcal{E}^{(\varepsilon)}}(R_\varepsilon \psi_\varepsilon - \mu_0 \psi_\varepsilon))^{\frac{1}{2}}.$$

Hence

$$\left(\overline{\mathcal{E}^{(\varepsilon)}}(R_\varepsilon \psi_\varepsilon - \mu_0 \psi_\varepsilon) \right)^{1/2} = O(\| V_\varepsilon \|_{L^2(Y_\varepsilon, m_\varepsilon)}) \quad \text{as } \varepsilon \to 0. \tag{3.14}$$

Claim (3.11) follows from (3.12), (3.13), and (3.14).
Step 2. We claim that

$$\overline{\mathcal{E}^{(\varepsilon)}}(\psi_\varepsilon - \Pi_\varepsilon \psi_\varepsilon) = O\left(\| V_\varepsilon \|^2_{L^2(Y_\varepsilon, m_\varepsilon)} \right) \quad \text{as } \varepsilon \to 0. \tag{3.15}$$

Let

$$\chi_\varepsilon := \psi_\varepsilon - \Pi_\varepsilon \psi_\varepsilon \quad \text{and} \quad \xi_\varepsilon := R_\varepsilon \chi_\varepsilon - \mu_\varepsilon \chi_\varepsilon. \tag{3.16}$$

By definition we have

$$\chi_\varepsilon \in N_\varepsilon := \{ w \in Z_\varepsilon : (w, \phi_\varepsilon)_{L^2(Y_\varepsilon, m_\varepsilon)} = 0 \}$$

and, from (2.6) and (3.2), it follows that $R_\varepsilon w \in N_\varepsilon$ for all $w \in N_\varepsilon$. Hence the operator

$$\widetilde{R}_\varepsilon := R_{\varepsilon|N_\varepsilon} : N_\varepsilon \to N_\varepsilon$$

is well-defined. Furthermore, $\sigma(\widetilde{R}_\varepsilon) = \sigma(R_\varepsilon) \setminus \{\mu_\varepsilon\}$. In particular, there exists a constant $K > 0$, which does not depends on ε, such that $\big(\mathrm{dist}(\mu_\varepsilon, \sigma(\widetilde{R}_\varepsilon))\big)^2 \geq K$ (for ε small enough). Moreover, (3.3) holds for $\widetilde{R}_\varepsilon$. Then

$$
\begin{aligned}
\overline{\mathcal{E}^{(\varepsilon)}}(\chi_\varepsilon) &\leq \frac{1}{K}\big(\mathrm{dist}(\mu_\varepsilon, \sigma(\widetilde{R}_\varepsilon))\big)^2\, \overline{\mathcal{E}^{(\varepsilon)}}(\chi_\varepsilon) \\
&\leq \frac{1}{K}\, \overline{\mathcal{E}^{(\varepsilon)}}(\widetilde{R}_\varepsilon \chi_\varepsilon - \mu_\varepsilon \chi_\varepsilon) \\
&= \frac{1}{K}\, \overline{\mathcal{E}^{(\varepsilon)}}(\xi_\varepsilon),
\end{aligned}
\tag{3.17}
$$

by (3.16). In view of (3.8) and (2.6) tested with ξ_ε,

$$
\begin{aligned}
&\overline{\mathcal{E}^{(\varepsilon)}}(\chi_\varepsilon, \xi_\varepsilon) - \lambda_\varepsilon\, (\chi_\varepsilon, \xi_\varepsilon)_{L^2(Y_\varepsilon, m_\varepsilon)} \\
&\quad = \lambda_0\, (V_\varepsilon, \xi_\varepsilon)_{L^2(Y_\varepsilon, m_\varepsilon)} + (\lambda_0 - \lambda_\varepsilon)\, (\psi_\varepsilon, \xi_\varepsilon)_{L^2(Y_\varepsilon, m_\varepsilon)}\,.
\end{aligned}
\tag{3.18}
$$

Then from (3.2) and (3.18) we deduce that

$$
\begin{aligned}
\overline{\mathcal{E}^{(\varepsilon)}}(\xi_\varepsilon) &= \overline{\mathcal{E}^{(\varepsilon)}}(R_\varepsilon \chi_\varepsilon, \xi_\varepsilon) - \mu_\varepsilon \overline{\mathcal{E}^{(\varepsilon)}}(\chi_\varepsilon, \xi_\varepsilon) \\
&= -\mu_\varepsilon [\overline{\mathcal{E}^{(\varepsilon)}}(\chi_\varepsilon, \xi_\varepsilon) - \lambda_\varepsilon \overline{\mathcal{E}^{(\varepsilon)}}(R_\varepsilon \chi_\varepsilon, \xi_\varepsilon)] \\
&= -\frac{\lambda_0}{\lambda_\varepsilon}\, (V_\varepsilon, \xi_\varepsilon)_{L^2(Y_\varepsilon, m_\varepsilon)} - \frac{(\lambda_0 - \lambda_\varepsilon)}{\lambda_\varepsilon}\, (\psi_\varepsilon, \xi_\varepsilon)_{L^2(Y_\varepsilon, m_\varepsilon)}\,.
\end{aligned}
$$

From the Cauchy-Schwarz inequality and Proposition 2.2 it follows that

$$
\overline{\mathcal{E}^{(\varepsilon)}}(\xi_\varepsilon) \leq \frac{\lambda_0}{\lambda_\varepsilon^{\frac{3}{2}}} ||V_\varepsilon||_{L^2(Y_\varepsilon, m_\varepsilon)} (\overline{\mathcal{E}^{(\varepsilon)}}(\xi_\varepsilon))^{\frac{1}{2}} + \frac{|\lambda_\varepsilon - \lambda_0|}{\lambda_\varepsilon^{\frac{3}{2}}} ||\psi_\varepsilon||_{L^2(Y_\varepsilon, m_\varepsilon)} (\overline{\mathcal{E}^{(\varepsilon)}}(\xi_\varepsilon))^{\frac{1}{2}}
$$

and hence, by (A2),

$$
(\overline{\mathcal{E}^{(\varepsilon)}}(\xi_\varepsilon))^{\frac{1}{2}} \leq C\left(\|V_\varepsilon\|_{L^2(Y_\varepsilon, m_\varepsilon)} + |\lambda_\varepsilon - \lambda_0|\, \|\psi_\varepsilon\|_{L^2(Y_\varepsilon, m_\varepsilon)}\right)
\tag{3.19}
$$

for some constant $C > 0$ which does not depend on ε. Furthermore, (A6) and (A4) yield

$$
\|\psi_\varepsilon\|^2_{L^2(Y_\varepsilon, m_\varepsilon)} - 1 = o(1) \quad \text{as } \varepsilon \to 0.
$$

Then (3.15) follows from (3.11), (3.17), and (3.19). Since $\psi_\varepsilon = \phi_0 - V_\varepsilon$ we have proved (3.5).

Step 3. We claim that

$$
\|\phi_0 - \Pi_\varepsilon \psi_\varepsilon\|_{L^2(Y_\varepsilon, m_\varepsilon)} = O(\|V_\varepsilon\|_{L^2(Y_\varepsilon, m_\varepsilon)}) \quad \text{as } \varepsilon \to 0.
\tag{3.20}
$$

Indeed from the definition of ψ_ε, Proposition 2.2 and (3.15) it follows that

$$
\begin{aligned}
\|\phi_0 - \Pi_\varepsilon \psi_\varepsilon\|_{L^2(Y_\varepsilon, m_\varepsilon)} &\leq \|\phi_0 - \psi_\varepsilon\|_{L^2(Y_\varepsilon, m_\varepsilon)} + \|\psi_\varepsilon - \Pi_\varepsilon \psi_\varepsilon\|_{L^2(Y_\varepsilon, m_\varepsilon)} \\
&= O(\|V_\varepsilon\|_{L^2(Y_\varepsilon, m_\varepsilon)}) \quad \text{as } \varepsilon \to 0.
\end{aligned}
\tag{3.21}
$$

In particular we have proved (3.6). Furthermore, since $\phi_0 = V_\varepsilon + \psi_\varepsilon$,

$$
\begin{aligned}
\mathcal{E}^{(\varepsilon)}(\phi_0 - \Pi_\varepsilon \psi_\varepsilon) &= \mathcal{E}^{(\varepsilon)}(V_\varepsilon) + \mathcal{E}^{(\varepsilon)}(\psi_\varepsilon - \Pi_\varepsilon \psi_\varepsilon) + 2\mathcal{E}^{(\varepsilon)}(V_\varepsilon, \psi_\varepsilon - \Pi_\varepsilon \psi_\varepsilon) \\
&= \mathcal{E}^{(\varepsilon)}(V_\varepsilon) + O(\|V_\varepsilon\|_{L^2(Y_\varepsilon, m_\varepsilon)}\, (\mathcal{E}^{(\varepsilon)}(V_\varepsilon))^{\frac{1}{2}}) = O(\mathcal{E}^{(\varepsilon)}(V_\varepsilon)), \quad \text{as } \varepsilon \to 0,
\end{aligned}
$$

thanks to the Cauchy-Schwarz inequality, Proposition 2.2 and (3.15). Hence we have proved (3.7).

Step 4. We claim that

$$(\psi_\varepsilon, \Pi_\varepsilon \psi_\varepsilon)_{L^2(Y_\varepsilon, m_\varepsilon)} = \int_{Y_\varepsilon} |\phi_0|^2 \, \mathrm{d}m_\varepsilon + O(\|V_\varepsilon\|_{L^2(Y_\varepsilon, m_\varepsilon)}) \quad \text{as } \varepsilon \to 0. \tag{3.22}$$

We have that

$$\begin{aligned}(\psi_\varepsilon, \Pi_\varepsilon \psi_\varepsilon)_{L^2(Y_\varepsilon, m_\varepsilon)} &= (\psi_\varepsilon - \Pi_\varepsilon \psi_\varepsilon, \Pi_\varepsilon \psi_\varepsilon)_{L^2(Y_\varepsilon, m_\varepsilon)} + \|\Pi_\varepsilon \psi_\varepsilon\|^2_{L^2(Y_\varepsilon, m_\varepsilon)} \\ &= \|\Pi_\varepsilon \psi_\varepsilon\|^2_{L^2(Y_\varepsilon, m_\varepsilon)} .\end{aligned}$$

Since $\|\phi_0\|_{L^2(Y_\varepsilon, m_\varepsilon)} \xrightarrow{\varepsilon \to 0} 1$, (3.21) and the Cauchy-Schwarz inequality imply that

$$\begin{aligned}\|\Pi_\varepsilon \psi_\varepsilon\|^2_{L^2(Y_\varepsilon, m_\varepsilon)} &= \|\phi_0 - \Pi_\varepsilon \psi_\varepsilon\|^2_{L^2(Y_\varepsilon, m_\varepsilon)} + \|\phi_0\|^2_{L^2(Y_\varepsilon, m_\varepsilon)} \\ &\quad - 2\,(\phi_0 - \Pi_\varepsilon \psi_\varepsilon, \phi_0)_{L^2(Y_\varepsilon, m_\varepsilon)} \\ &= \int_{Y_\varepsilon} |\phi_0|^2 \, \mathrm{d}m_\varepsilon + O(\|V_\varepsilon\|_{L^2(Y_\varepsilon, m_\varepsilon)}) \quad \text{as } \varepsilon \to 0.\end{aligned}$$

Putting together (3.10), (3.20) and (3.22), we finally obtain

$$\lambda_\varepsilon - \lambda_0 = \frac{\lambda_0 \int_{Y_\varepsilon} \phi_0 V_\varepsilon \, \mathrm{d}m_\varepsilon + O\left(\|V_\varepsilon\|^2_{L^2(Y_\varepsilon, m_\varepsilon)}\right)}{\int_{Y_\varepsilon} |\phi_0|^2 \, \mathrm{d}m_\varepsilon + O(\|V_\varepsilon\|_{L^2(Y_\varepsilon, m_\varepsilon)})} \quad \text{as } \varepsilon \to 0,$$

thus proving (3.4). □

Remark 3.2 It is also possible to prove (3.4) and (3.6) following the approach exposed in [16, Proof of Theorem 2.3]. It is based on a abstract lemma originally proved in [10] and then revisited in [12] and [4]. See [16, Lemma 7.1] for a simplified version of this lemma suitable for simple eigenvalues and for a short proof. However, following this approach we would only obtain

$$\|\phi_0 - V_\varepsilon - \Pi_\varepsilon(v_0 - V_\varepsilon)\|_{L^2(Y_\varepsilon, m_\varepsilon)} = O\left(\|V_\varepsilon\|_{L^2(Y_\varepsilon, m_\varepsilon)}\right), \quad \text{as } \varepsilon \to 0,$$

which is a slightly weaker version of (3.5). To recover (3.5) some additional work is required, see [16, Proof of Theorem 2.3].

Remark 3.3 It is worth noticing that, as $\varepsilon \to 0$, the quotient in (3.4) can be written as

$$\begin{aligned}&\frac{\lambda_0 \int_{Y_\varepsilon} \phi_0 V_\varepsilon \, \mathrm{d}m_\varepsilon + O\left(\|V_\varepsilon\|^2_{L^2(Y_\varepsilon, m_\varepsilon)}\right)}{\int_{Y_\varepsilon} |\phi_0|^2 \, \mathrm{d}m_\varepsilon + O(\|V_\varepsilon\|_{L^2(Y_\varepsilon, m_\varepsilon)})} \\ &= \left(\lambda_0 \int_{Y_\varepsilon} \phi_0 V_\varepsilon \, \mathrm{d}m_\varepsilon + O\left(\|V_\varepsilon\|^2_{L^2(Y_\varepsilon, m_\varepsilon)}\right)\right) \\ &\left(1 + \frac{1 - \int_{Y_\varepsilon} |\phi_0|^2 \, \mathrm{d}m_\varepsilon}{\int_{Y_\varepsilon} |\phi_0|^2 \, \mathrm{d}m_\varepsilon} + O(\|V_\varepsilon\|_{L^2(Y_\varepsilon, m_\varepsilon)})\right) \\ &= \lambda_0 \int_{Y_\varepsilon} \phi_0 V_\varepsilon \, \mathrm{d}m_\varepsilon + o\left(\|V_\varepsilon\|_{L^2(Y_\varepsilon, m_\varepsilon)}\right),\end{aligned}$$

by the Cauchy-Schwarz inequality. Hence, if the rate of convergence of $||\phi_0||_{L^2(Y_\varepsilon, m_\varepsilon)} \xrightarrow{\varepsilon \to 0} 1$ can be precisely quantified, a more accurate estimate on the asymptotic behavior of $\lambda_0 - \lambda_\varepsilon$

can be recovered. In particular, if

$$1 - \int_{Y_\varepsilon} |\phi_0|^2 \, \mathrm{d}m_\varepsilon = O\left(||V_\varepsilon||_{L^2(Y_\varepsilon, m_\varepsilon)}\right) \quad \text{as } \varepsilon \to 0,$$

then by the Cauchy-Schwarz inequality

$$\lambda_\varepsilon - \lambda_0 = \lambda_0 \int_{Y_\varepsilon} \phi_0 V_\varepsilon \, \mathrm{d}m_\varepsilon + O\left(\|V_\varepsilon\|^2_{L^2(Y_\varepsilon, m_\varepsilon)}\right) \quad \text{as } \varepsilon \to 0.$$

Under some additional assumptions, we are able to identify more explicitly a sequence of eigenfunctions associated to the simple eigenvalues λ_ε, that converges in suitable sense to ϕ_0. More precisely let ϕ_ε be the eigenfunction associated to λ_ε such that

$$\int_{Y_\varepsilon} |\phi_\varepsilon|^2 \, \mathrm{d}m_\varepsilon = 1 \quad \text{and} \quad \int_{Y_\varepsilon} \phi_0 \phi_\varepsilon \, \mathrm{d}m_\varepsilon > 0. \tag{3.23}$$

Such a choice is possible, at least for small ε, since in view of (3.6), $\int_{Y_\varepsilon} \phi_0 \phi_\varepsilon \, dm \neq 0$ for any ε close enough to 0.

Proposition 3.4 *Suppose that*

$$\int_{Y_\varepsilon} |\phi_0|^2 \, \mathrm{d}m_\varepsilon = 1 + O\left(||V_\varepsilon||_{L^2(Y_\varepsilon, m_\varepsilon)}\right) \quad \text{as } \varepsilon \to 0. \tag{3.24}$$

Then

$$\mathcal{E}^{(\varepsilon)}(\phi_0 - \phi_\varepsilon) = O\left(\mathcal{E}^{(\varepsilon)}(V_\varepsilon)\right) \quad \text{as } \varepsilon \to 0^+. \tag{3.25}$$

Furthermore, if

$$\|V_\varepsilon\|^2_{L^2(Y_\varepsilon, m_\varepsilon)} = o\left(\mathcal{E}^{(\varepsilon)}(V_\varepsilon)\right) \tag{3.26}$$

then

$$\mathcal{E}^{(\varepsilon)}(\phi_0 - \phi_\varepsilon) = \mathcal{E}^{(\varepsilon)}(V_\varepsilon) + o\left(\mathcal{E}^{(\varepsilon)}(V_\varepsilon)\right) \quad \text{as } \varepsilon \to 0^+, \tag{3.27}$$

$$\|\phi_0 - \phi_\varepsilon\|^2_{L^2(Y_\varepsilon, m_\varepsilon)} = o\left(\mathcal{E}^{(\varepsilon)}(V_\varepsilon)\right) \quad \text{as } \varepsilon \to 0^+. \tag{3.28}$$

Proof Since λ_ε is simple, thanks to (3.6) and (3.23)

$$\phi_\varepsilon = \frac{\Pi_\varepsilon(\phi_0 - V_\varepsilon)}{\|\Pi_\varepsilon(\phi_0 - V_\varepsilon)\|_{L^2(Y_\varepsilon, m_\varepsilon)}}.$$

Hence

$$\mathcal{E}^{(\varepsilon)}(\phi_\varepsilon - \Pi_\varepsilon(\phi_0 - V_\varepsilon)) = \frac{|1 - \|\Pi_\varepsilon(\phi_0 - V_\varepsilon)\|_{L^2(Y_\varepsilon, m_\varepsilon)}|^2}{\|\Pi_\varepsilon(\phi_0 - V_\varepsilon)\|^2_{L^2(Y_\varepsilon, m_\varepsilon)}} \mathcal{E}^{(\varepsilon)}(\Pi_\varepsilon(\phi_0 - V_\varepsilon))$$

and, by (3.6) and (3.24),

$$\begin{aligned}
&\|\Pi_\varepsilon(\phi_0 - V_\varepsilon)\|^2_{L^2(Y_\varepsilon, m_\varepsilon)} \\
&\quad = \|\phi_0 - \Pi_\varepsilon(\phi_0 - V_\varepsilon)\|^2_{L^2(Y_\varepsilon, m_\varepsilon)} + \|\phi_0\|^2_{L^2(Y_\varepsilon, m_\varepsilon)} \\
&\quad - 2\,(\phi_0 - \Pi_\varepsilon(\phi_0 - V_\varepsilon), \phi_0)_{L^2(Y_\varepsilon, m_\varepsilon)} \\
&\quad = 1 + O(\|V_\varepsilon\|_{L^2(Y_\varepsilon, m_\varepsilon)}).
\end{aligned}$$

It follows that

$$\mathcal{E}^{(\varepsilon)}(\phi_\varepsilon - \Pi_\varepsilon(\phi_0 - V_\varepsilon)) = O(\|V_\varepsilon\|^2_{L^2(Y_\varepsilon, m_\varepsilon)}) \quad \text{as } \varepsilon \to 0^+.$$

Hence by (3.7) we have proved (3.25). Furthermore, thanks to (3.26) and (3.7),

$$\begin{aligned}
&\mathcal{E}^{(\varepsilon)}(\phi_0 - \phi_\varepsilon) \\
&\quad = \mathcal{E}^{(\varepsilon)}(V_\varepsilon) + \mathcal{E}^{(\varepsilon)}(\phi_\varepsilon - \Pi_\varepsilon(\phi_0 - V_\varepsilon)) \\
&\quad + 2\mathcal{E}^{(\varepsilon)}(\phi_\varepsilon - \Pi_\varepsilon(\phi_0 - V_\varepsilon), \Pi_\varepsilon(\phi_0 - V_\varepsilon) - \phi_0) + o\left(\mathcal{E}^{(\varepsilon)}(V_\varepsilon)\right) \\
&\quad = \mathcal{E}^{(\varepsilon)}(V_\varepsilon) + o\left(\mathcal{E}^{(\varepsilon)}(V_\varepsilon)\right) \quad \text{as } \varepsilon \to 0^+,
\end{aligned}$$

which proves (3.27). Finally, thanks to (3.6) and (3.26), we can prove (3.28) in a similar manner. □

3.2 Some additional results to quantify $\lambda_\varepsilon - \lambda_0$.

To quantify more explicitly the rate of convergences of simple eigenvalues provided by Theorem 3.1, in many situations results like [19, Proposition 4.5, Proposition 6.5] or [16, Lemma 3.1, Lemma 3.5] may be useful. Indeed they can be used to compute the order of infinitesimal of $\|V_\varepsilon\|_{L^2(Y_\varepsilon, m_\varepsilon)}$ or $\mathcal{E}^{(\varepsilon)}(V_\varepsilon)$ as $\varepsilon \to 0^+$. In this subsection we generalize to our abstract setting the aforementioned results. The first one is a characterization of the minimal value of the functional J_ε. Its interest lays in the fact that if $\|L_\varepsilon\|_{(\mathcal{F}_\varepsilon)^*} \to 0^+$ as $\varepsilon \to 0^+$ in a controlled way then the study of the asymptotic behavior as of $\mathcal{E}^{(\varepsilon)}(V_\varepsilon)$ (thus also of $\|V_\varepsilon\|_{L^2(Y_\varepsilon, m_\varepsilon)}$) can be reduced to the study of asymptotic behavior of $J_\varepsilon(V_\varepsilon)$. Indeed, testing (2.10) with V_ε, by Proposition 2.2,

$$\begin{aligned}
\lambda_{1,\varepsilon} \|V_\varepsilon\|^2_{L^2(Y_\varepsilon, m_\varepsilon)} &\le \mathcal{E}^{(\varepsilon)}(V_\varepsilon) = 2(J_\varepsilon(V_\varepsilon) + L_\varepsilon(V_\varepsilon)) \\
&\le 2(|J_\varepsilon(V_\varepsilon)| + \|L_\varepsilon\|_{(\mathcal{F}_\varepsilon)^*} (\mathcal{E}^{(\varepsilon)}(V_\varepsilon))^{\frac{1}{2}}).
\end{aligned}$$

The advantage of this approach is that competitors may be used to obtain information about the infinitesimal order of $J_\varepsilon(V_\varepsilon)$, see for example [19, Proposition 4.3].

Proposition 3.5 *For any* $\varepsilon \in [0, 1]$

$$J_\varepsilon(V_\varepsilon) = -\frac{1}{2} \sup_{w \in Z_\varepsilon \setminus \{0\}} \frac{\left(\mathcal{E}^{(\varepsilon)}(\phi_0, w) - L_\varepsilon(w)\right)^2}{\mathcal{E}^{(\varepsilon)}(w)} + \frac{1}{2}\mathcal{E}^{(\varepsilon)}(\phi_0) - L_\varepsilon(\phi_0). \tag{3.29}$$

Moreover, if $\phi_0 \in Z_\varepsilon$, *then*

$$J_\varepsilon(V_\varepsilon) = -\frac{1}{2} \sup_{w \in Z_\varepsilon \setminus \{0\}} \frac{L_\varepsilon(w)^2}{\mathcal{E}^{(\varepsilon)}(w)}.$$

Proof In view of (2.9) it follows

$$J_\varepsilon(V_\varepsilon) = \inf_{w \in Z_\varepsilon} J_\varepsilon(w + \phi_0) = \inf_{w \in Z_\varepsilon} \left(\inf_{t \in \mathbb{R}} J_\varepsilon(tw + \phi_0)\right). \tag{3.30}$$

By the definition of J_ε, we get

$$\begin{aligned} J_\varepsilon(tw+\phi_0) &= \frac{1}{2}\mathcal{E}^{(\varepsilon)}(tw+\phi_0) - L_\varepsilon(tw+\phi_0) \\ &= \frac{t^2}{2}\mathcal{E}^{(\varepsilon)}(w) + \frac{1}{2}\mathcal{E}^{(\varepsilon)}(\phi_0) + t\mathcal{E}^{(\varepsilon)}(w,\phi_0) - tL_\varepsilon(w) - L_\varepsilon(\phi_0) \end{aligned}$$

implying that for any $w \in Z_\varepsilon \setminus \{0\}$

$$\begin{aligned} \inf_{t\in\mathbb{R}} J_\varepsilon(tw+\phi_0) =& \frac{1}{2}\left(\frac{\mathcal{E}^{(\varepsilon)}(w,\phi_0) - L_\varepsilon(w)}{\mathcal{E}^{(\varepsilon)}(w)}\right)^2 \mathcal{E}^{(\varepsilon)}(w) + \frac{1}{2}\mathcal{E}^{(\varepsilon)}(\phi_0) \\ & - \frac{\mathcal{E}^{(\varepsilon)}(w,\phi_0) - L_\varepsilon(w)}{\mathcal{E}^{(\varepsilon)}(w)}\mathcal{E}^{(\varepsilon)}(w,\phi_0) \\ & + \frac{\mathcal{E}^{(\varepsilon)}(w,\phi_0) - L_\varepsilon(w)}{\mathcal{E}^{(\varepsilon)}(w)} L_\varepsilon(w) - L_\varepsilon(\phi_0). \end{aligned}$$

Hence, (3.29) follows from (3.30).

In the particular case $\phi_0 \in Z_\varepsilon$ we have that

$$\begin{aligned} J_\varepsilon(V_\varepsilon) = \inf_{w\in Z_\varepsilon} J_\varepsilon(w) &= \inf_{w\in Z_\varepsilon}\inf_{t\in\mathbb{R}} J_\varepsilon(tw) \\ &= \inf_{w\in Z_\varepsilon}\inf_{t\in\mathbb{R}}\left(\frac{t^2}{2}\mathcal{E}^{(\varepsilon)}(w) - tL_\varepsilon(w)\right) \end{aligned}$$

and so

$$J_\varepsilon(V_\varepsilon) = -\frac{1}{2}\sup_{w\in Z_\varepsilon\setminus\{0\}} \frac{L_\varepsilon(w)^2}{\mathcal{E}^{(\varepsilon)}(w)}.$$

□

The next proposition provides a sufficient condition to sharpen the asymptotic expansion in (3.4). Furthermore, in many applications it is easier to compute the infinitesimal order of $\mathcal{E}(V_\varepsilon)$ instead of $\|V_\varepsilon\|_{L^2(Y_\varepsilon,m_\varepsilon)}$, especially when blow-up arguments are involved, see for example [16, 19]. Moreover, it provides sufficient conditions for (3.26) to hold. It is a generalization of [23, Proposition 3] which dealt with the case of capacitary potentials.

Proposition 3.6 *Assume that the following holds.*

(i) For all $\varepsilon_k \to 0^+$ and $\{u_k\}_{k\in\mathbb{N}}$ with $u_k \in \mathcal{F}_{\varepsilon_k}$, $\|u_{k_n}\|_{L^2(Y_{\varepsilon_k},m_{\varepsilon_{k_n}})} = 1$ and $\mathcal{E}^{(\varepsilon_k)}(u_k) \le C$, for some constant $C > 0$ that does not depends on k, there exists $u \in \mathcal{F}_0$ and a subsequence $\{u_{k_n}\}_{n\in\mathbb{N}}$ such that:

- *(i.1) $\mathcal{E}^{(\varepsilon_{k_n})}(u_{k_n}, v) \to \mathcal{E}^{(0)}(u,v)$ for any $v \in \mathcal{F}_0$,*
- *(i.2) $\|u\|_{L^2(X,m)} = 1$.*

(ii) For any $u \in \mathcal{F}_0$ there exists $\{u_n\}_{n\in\mathbb{N}} \subset \mathcal{F}_0$ such that:

- *(ii.1) $u_n \rightharpoonup u$ weakly in $\mathcal{F}_0$ as $n \to \infty$,*
- *(ii.2) for any $\varepsilon_0 \in (0,1]$ there exists $n_0 \in \mathbb{N}$ such that $u_n \in Z_\varepsilon$ for any $n \ge n_0$ and any $\varepsilon \in (0,\varepsilon_0]$,*
- *(ii.3) $L_\varepsilon(u_n) = 0$ for any $n \ge n_0$ and any $\varepsilon \in (0,\varepsilon_0]$.*

Then

$$\|V_\varepsilon\|^2_{L^2(Y_\varepsilon,m_\varepsilon)} = o(\mathcal{E}^{(\varepsilon)}(V_\varepsilon)) \quad as\ \varepsilon \to 0.$$

Proof Suppose by contradiction that there exists a subsequence of $\{\varepsilon_k\}_{k\in\mathbb{N}}$ such that

$$\left\|V_{\varepsilon_k}\right\|^2_{L^2(Y_{\varepsilon_k},m_{\varepsilon_k})} \geq C\mathcal{E}^{(\varepsilon_k)}(V_{\varepsilon_k})$$

where $C > 0$ is a constant not depending on k. Define

$$W_k := \frac{V_{\varepsilon_k}}{\left\|V_{\varepsilon_k}\right\|_{L^2(Y_{\varepsilon_k},m_{\varepsilon_k})}}.$$

By assumption, there exists $W \in \mathcal{F}_0$ with $\|W\|_{L^2(X,m)} = 1$ and such that, up to a subsequence,

$$\mathcal{E}^{(\varepsilon_k)}(W_k, v) \to \mathcal{E}^{(0)}(W, v) \quad \text{for any } v \in \mathcal{F}_0.$$

Let $v \in \mathcal{F}_0$ and let $\{v_n\} \subset \mathcal{F}_0$ be as in (ii). Then for any $k_0 \in \mathbb{N}$ there exists a $n_0 \in \mathbb{N}$ such that

$$\mathcal{E}^{(\varepsilon_k)}(W_k, v_n) = L_\varepsilon(v_n) = 0 \quad \text{for any } n \geq n_0 \text{ and any } k \geq k_0.$$

Hence, passing to the limit as $k \to \infty$, we obtain

$$\mathcal{E}^{(0)}(W, v_n) = 0 \quad \text{for any } n \geq n_0.$$

Passing to the limit as $n \to \infty$, we conclude that

$$\mathcal{E}^{(0)}(W, v) = 0 \quad \text{for any } v \in \mathcal{F}_0.$$

Testing the above equation with W, we get $\mathcal{E}^{(0)}(W) = 0$, which is a contradiction in view of Proposition 2.2. □

3.3 A sufficient condition for spectral stability

In this subsection we provide a general criterion to prove that assumption (A2) holds, that is, that there is spectral stability. It is based on the following result, we refer to [14, Corollaries XI 9.3, XI 9.4] for a proof.

Theorem 3.7 *Let R_1, R_2 be linear bounded, non-negative, compact, self-adjoint operators on a Hilbert space $\mathcal{H}$. Let $\{\mu_{n,i}\}_{n\in\mathbb{N}\setminus\{0\}} := \sigma(R_i) \setminus \{0\}$ for $i = 1, 2$. Then*

$$|\mu_{n,1} - \mu_{n,2}| \leq \|R_1 - R_2\|_{\mathcal{L}(H)} \quad \textit{for any } n \in \mathbb{N} \setminus \{0\}.$$

The proof of the following theorem is inspired by [9, Proposition 6.1.7] and this approach to spectral stability is inspired by [35, Lemma 1.1].

Theorem 3.8 *Suppose that*

$$m_\varepsilon = m \textit{ for any } \varepsilon \in (0, 1] \textit{ and } m(X \setminus Y_\varepsilon) \to 0^+ \textit{ as } \varepsilon \to 0^+.$$

Let R_ε be as in (3.1). Assume that for any $\varepsilon \in [0, 1]$ there exists a bounded linear operator

$$P_\varepsilon : \mathcal{Z}_0 \to \mathcal{Z}_\varepsilon$$

and a bounded linear operator

$$E_\varepsilon : Z_\varepsilon \to \mathcal{Z}_0$$

such that $(E_\varepsilon u)_{|Y_\varepsilon} = u$, $P_\varepsilon E_\varepsilon u = u$ *for any* $u \in Z_\varepsilon$ *and*

$$\lim_{\varepsilon\to 0^+} (P_\varepsilon w, v)_{L^2(Y_\varepsilon,m)} = (w, v)_{L^2(X,m)} \quad \text{for any } v, w \in \mathcal{Z}_0, \tag{3.31}$$

$$\left(R_\varepsilon P_\varepsilon v, w_{|Y_\varepsilon}\right)_{L^2(Y_\varepsilon,m)} = \left(v_{|Y_\varepsilon}, R_\varepsilon P_\varepsilon w\right)_{L^2(Y_\varepsilon,m)} \quad \text{for any } v, w \in \mathcal{Z}_0. \tag{3.32}$$

Furthermore, let $(\mathcal{E}, \mathcal{F})$ *be a non-negative, densely defined, closed bilinear form and* H *the corresponding densely defined, non-negative, self-adjoint operator. Assume that* H^{-1} *is a well-defined, bounded and compact operator and that* $\mathcal{F} \subset \mathcal{Z}_0$. *Finally let*

$$\widetilde{R}_\varepsilon : \mathcal{Z}_0 \to \mathcal{Z}_0, \quad \widetilde{R}_\varepsilon := E_\varepsilon R_\varepsilon P_\varepsilon$$

and suppose $\widetilde{R}_\varepsilon : \mathcal{Z}_0 \to \mathcal{F}$ *are well-defined and equibounded, that is there exists a constant* C, *that does not depend on* ε, *such that*

$$\mathcal{E}(\widetilde{R}_\varepsilon u) \le C \|u\|^2_{L^2(X,m)} \quad \text{for any } u \in \mathcal{Z}_0 \text{ and } \varepsilon \in [0, 1]. \tag{3.33}$$

We also assume that if $\widetilde{R}_\varepsilon u \rightharpoonup w$ *weakly in* $\mathcal{F}$ *then* $w \in Z_0$ *and*

$$\lim_{\varepsilon\to 0^+} \mathcal{E}^{(\varepsilon)}(R_\varepsilon P_\varepsilon u, v) = \mathcal{E}^{(0)}(w, v), \quad \text{for any } v \in Z_0. \tag{3.34}$$

Then for any $n \in \mathbb{N}$

$$\lim_{\varepsilon\to 0^+} \lambda_{\varepsilon,n} = \lambda_{0,n}. \tag{3.35}$$

Proof For the sake of simplicity, we divide the proof in three steps.

Step 1. We have that

$$\sigma(\widetilde{R}_\varepsilon) = \sigma(R_\varepsilon) \cup \{0\} \quad \text{or} \quad \sigma(\widetilde{R}_\varepsilon) = \sigma(R_\varepsilon). \tag{3.36}$$

Indeed, if $R_\varepsilon \phi_\varepsilon = \mu_\varepsilon \phi_\varepsilon$, then

$$\widetilde{R}_\varepsilon E_\varepsilon \phi_\varepsilon = E_\varepsilon R_\varepsilon P_\varepsilon E_\varepsilon \phi_\varepsilon = E_\varepsilon R_\varepsilon \phi_\varepsilon = \mu_\varepsilon E_\varepsilon \phi_\varepsilon.$$

Since the kernel of E_ε is trivial, it follows that $\sigma(R_\varepsilon) \subseteq \sigma(\widetilde{R}_\varepsilon)$. On the other hand, if $\widetilde{R}_\varepsilon \phi_\varepsilon = \mu_\varepsilon \phi_\varepsilon$, and $\mu_\varepsilon \neq 0$, then $P_\varepsilon \phi_\varepsilon \neq 0$ and

$$E_\varepsilon R_\varepsilon (P_\varepsilon \phi_\varepsilon) = \mu_\varepsilon E_\varepsilon (P_\varepsilon \phi_\varepsilon).$$

Hence μ_ε is a eigenvalue of R_ε since E_ε is injective. We conclude that (3.36) holds.

Step 2. For any $u \in \mathcal{Z}_0$ we have that

$$\lim_{\varepsilon\to 0^+} \left\| \widetilde{R}_\varepsilon u - R_0 u \right\|_{L^2(X,m)} = 0. \tag{3.37}$$

By (3.33) and Proposition 2.1, for any $u \in \mathcal{Z}_0$ there exists a sequence $\widetilde{R}_{\varepsilon_n} u \rightharpoonup w$ weakly in $\mathcal{F}$ as $n \to \infty$, for some $w \in \mathcal{F}$. Then by (3.31), and (3.34) for any $v \in Z_0$

$$(u, v)_{L^2(X,m)} = \lim_{n\to\infty} (P_{\varepsilon_n} u, v)_{L^2(Y_\varepsilon,m)} = \lim_{\varepsilon\to 0^+} \mathcal{E}^{(\varepsilon)}(R_\varepsilon P_\varepsilon u, v) = \mathcal{E}^{(0)}(w, v).$$

Hence $\widetilde{R}_{\varepsilon_n} u \to R_0 u$, strongly in $L^2(X, m)$ as $n \to \infty$, thanks to Proposition 2.1. By the Urysohn Subsequence Principle we conclude that (3.37) holds.

Step 3. We claim that

$$\lim_{\varepsilon\to 0^+} \left\| \widetilde{R}_\varepsilon - R_0 \right\|_{\mathcal{L}(L^2(X,m))} = 0. \tag{3.38}$$

By the compactness of the operator $\widetilde{R}_\varepsilon - R_0$, for any $\varepsilon \in [0,1]$ there exists a $f_\varepsilon \in L^2(X,m)$ with $\|f_\varepsilon\|_{L^2(X,m)} = 1$ such that

$$\left\| \widetilde{R}_\varepsilon - R_0 \right\|_{\mathcal{L}(L^2(X,m))} = \left\| \widetilde{R}_\varepsilon f_\varepsilon - R_0 f_\varepsilon \right\|_{L^2(X,m)}.$$

Hence, there exists a sequence $f_{\varepsilon_n} \rightharpoonup f$ weakly for some $f \in L^2(X,m)$. By compactness of R_0 it follows that $R_0 f_{\varepsilon_n} \to R_0 f$ strongly in $L^2(X,m)$, as $n \to \infty$. By (3.33), up to a subsequence, $\widetilde{R}_{\varepsilon_n} f_{\varepsilon_n} \to g$ strongly in $L^2(X,m)$ for some $g \in \mathcal{F}$.

For any $h \in \mathcal{Z}_0$, thanks to (3.32),

$$\begin{aligned}\left(\widetilde{R}_{\varepsilon_n} f_{\varepsilon_n}, h\right)_{L^2(X,m)} &= \left(f_{\varepsilon_n}, R_{\varepsilon_n} P_{\varepsilon_n} h\right)_{L^2(Y_\varepsilon,m)} + \left(\widetilde{R}_{\varepsilon_n} f_{\varepsilon_n}, h\right)_{L^2(X\setminus Y_\varepsilon,m)} \\ &= \left(f_{\varepsilon_n}, \widetilde{R}_{\varepsilon_n} h\right)_{L^2(X,m)} - \left(f_{\varepsilon_n}, \widetilde{R}_{\varepsilon_n} h\right)_{L^2(X\setminus Y_\varepsilon,m)} + \left(\widetilde{R}_{\varepsilon_n} f_{\varepsilon_n}, h\right)_{L^2(X\setminus Y_\varepsilon,m)}.\end{aligned}$$

Thanks to the absolute continuity of the integral,

$$\left(f_{\varepsilon_n}, \widetilde{R}_{\varepsilon_n} h\right)_{L^2(X\setminus Y_\varepsilon,m)} \to 0^+, \quad \left(\widetilde{R}_{\varepsilon_n} f_{\varepsilon_n}, h\right)_{L^2(X\setminus Y_\varepsilon,m)} \to 0^+, \quad \text{as } n \to \infty.$$

By (3.37)

$$(g,h)_{L^2(X,m)} = \lim_{n\to\infty} \left(\widetilde{R}_{\varepsilon_n} f_{\varepsilon_n}, h\right)_{L^2(X,m)} = (f, R_0 h)_{L^2(X,m)} = (R_0 f, h)_{L^2(X,m)}.$$

Hence $g = R_0 f$. By Proposition 2.1, we conclude that the $\widetilde{R}_{\varepsilon_n} f_{\varepsilon_n} \to R_0 f$ strongly in $L^2(X,m)$. By Urysohn's Subsequence Principle we conclude that (3.38) holds.

In conclusion (3.35) follows from Theorem 3.7 and (3.36). □

Remark 3.9 In the applications of Theorem 3.1 presented in Section 4, we will use either Theorem 3.8 or min-max arguments to show that (A2) holds.

4 Applications

In the present section we discuss some applications of Theorem 3.1. The first two applications, a Neumann limit of a Robin problem and a conformal transformation of a Riemannian metric, could also be studied with more standard techniques, see Remark 4.3 and Remark 4.4 below. However, they are relatively simple and hence a good way to illustrate our technique, which is based instead on a minimization procedure.

In the last two applications, we study much more singular perturbations such as the removal of set of small capacity from the domain of a Dirichlet form and a non-smooth variation of the symbol of a Fourier multiplier.

4.1 A Neumann limit of a Robin problem

In what follows, let (M,g) be a complete Riemannian manifold of dimension $\dim(M) = n \geq 2$ and $\Omega \subset M$ a smooth bounded domain. We adopt the convention that Δ is the negative definite Laplace-Beltrami operator (i.e. $\Delta = \frac{d^2}{dx^2}$ in $\mathbb{R}$). Moreover, we denote by dv the Riemannian volume density of M and by da the $(n-1)$-dimensional area element. Fix $\varepsilon \in (0, 1]$ and consider the following Robin eigenvalue problem

$$\begin{cases} -\Delta u = \lambda u, & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = -\varepsilon u, & \text{on } \partial\Omega, \end{cases} \tag{4.1}$$

where ν is the outward pointing unit normal to $\partial\Omega$ and λ_ε is a simple eigenvalue. Our aim is to show that we are able to predict, at the first order, the rate of convergence of the simple eigenvalues of (4.1) to that of the limit problem

$$\begin{cases} -\Delta u = \lambda u, & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0, & \text{on } \partial\Omega \end{cases} \tag{4.2}$$

using the asymptotic expansion provided by Theorem 3.1. The complementary case, that is, the limit as $\varepsilon \to \infty$ was studied in detail in the very recent paper [33].

Since the first eigenvalue of the boundary value problem (4.2) is 0, as stressed in Section 2 it is preferable to consider the following equivalent family of problems

$$\begin{cases} -\Delta u + u = \lambda u, & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = -\varepsilon u, & \text{on } \partial\Omega, \end{cases} \tag{4.3}$$

for $\varepsilon \in [0, 1]$. Let $(\mathcal{E}^{(\varepsilon)}, \mathcal{F}_\varepsilon)$ be the Dirichlet form associated to problem (4.3)

$$\mathcal{E}^{(\varepsilon)}(u, v) := \int_\Omega [g(\nabla u, \nabla v) + uv] \, \mathrm{dv} + \varepsilon \int_{\partial\Omega} uv \, \mathrm{da}$$

with domain $\mathcal{F}_\varepsilon := H^1(\Omega)$. Following the notations of Section 2, we have for any $\varepsilon \in [0, 1]$

$$Y_\varepsilon = \Omega, \qquad Z_\varepsilon = \mathcal{F}_\varepsilon = H^1(\Omega), \qquad \mathcal{Z}_\varepsilon = L^2(\Omega).$$

Hence, letting λ_0 a simple eigenvalue of problem (4.3) with $\varepsilon = 0$ and ϕ_0 a associated normalized eigenfunction we have

$$\mathcal{E}^{(\varepsilon)}(\phi_0, u) = \lambda_0 \int_\Omega \phi_0 u \, \mathrm{dv} + \varepsilon \int_{\partial\Omega} \phi_0 u \, \mathrm{da} \quad \text{for any } u \in H^1(\Omega).$$

Denoting by $L_\varepsilon : H^1(\Omega) \to \mathbb{R}$ the linear and continuous operator

$$L_\varepsilon(u) := \varepsilon \int_{\partial\Omega} \phi_0 u \, \mathrm{da}$$

and by $J_\varepsilon : H^1(\Omega) \to \mathbb{R}$ the functional

$$\begin{aligned} J_\varepsilon(u) :&= \frac{1}{2}\mathcal{E}^{(\varepsilon)}(u) - L_\varepsilon(u) \\ &= \int_\Omega \left[|\nabla u|^2 + u^2\right] \, \mathrm{dv} + \varepsilon \int_{\partial\Omega} u^2 \, \mathrm{da} - \varepsilon \int_{\partial\Omega} \phi_0 u \, \mathrm{da}, \end{aligned}$$

by Proposition 2.9 we obtain the existence of the (unique) minimum $V_\varepsilon \in H^1(\Omega)$ of the minimization problem

$$\inf\{J_\varepsilon(u) \; : \; u \in H^1(\Omega)\}.$$

Now we show that $V_\varepsilon \xrightarrow{\varepsilon \to 0} 0$ in $H^1(\Omega)$. Indeed, by Proposition 2.9, V_ε satisfies

$$\mathcal{E}^{(\varepsilon)}(V_\varepsilon, u) = L_\varepsilon(u) \quad \text{for any } u \in H^1(\Omega),$$

i.e.

$$\int_\Omega [g(\nabla V_\varepsilon, \nabla u) + V_\varepsilon u] \text{ dv} + \varepsilon \int_{\partial\Omega} V_\varepsilon u \text{ da} = \varepsilon \int_{\partial\Omega} \phi_0 u \text{ da} \quad \text{for any } u \in H^1(\Omega). \quad (4.4)$$

Testing (4.4) with V_ε itself, it follows that

$$\int_\Omega \left[|\nabla V_\varepsilon|^2 + V_\varepsilon^2\right] \text{ dv} + \varepsilon \int_{\partial\Omega} V_\varepsilon^2 \text{ da} = \varepsilon \int_{\partial\Omega} \phi_0 V_\varepsilon \text{ da}$$

implying

$$\begin{aligned} ||V_\varepsilon||^2_{H^1(\Omega)} &\le \varepsilon \int_{\partial\Omega} \phi_0 V_\varepsilon \text{ dv} \\ &\le \varepsilon ||\phi_0||_{L^2(\partial\Omega)} ||V_\varepsilon||_{L^2(\partial\Omega)}. \end{aligned}$$

As a consequence of the following classical inequality, we get

$$||V_\varepsilon||_{H^1(\Omega)} \xrightarrow{\varepsilon\to 0} 0.$$

Proposition 4.1 *Let (M, g) be a complete Riemannian manifold and $\Omega \subset M$ a smooth domain. Then, there exists a positive constant $C_0 > 0$ so that for every $u \in H^1(\Omega)$*

$$\int_{\partial\Omega} |u|^2 \text{ da} \le C_0 \int_\Omega \left[|\nabla u|^2 + u^2\right] \text{ dv}. \quad (4.5)$$

If the dimension of M is at least 3, then the results is contained in [31, Proposition 1.1]. Otherwise, in any dimension, it can be deduced from classical trace results in smooth bounded domain in $\mathbb{R}^N$, see [30, Theorem 18.1], together with the choice of a finite cover of local charts whose closures smoothly intersect the boundary $\partial\Omega$.

Let $\lambda_{n,\varepsilon}$ be the n-th eigenvalue of problem (4.1) for any $\varepsilon \in [0, 1]$.

$$\lambda_{n,\varepsilon} = \inf_{\substack{A\subset H^1(\Omega) \\ \dim(A)=n}} \sup_{\substack{u\in A \\ ||u||_{L^2(\Omega)}=1}} \left(\int_\Omega \left[|\nabla u|^2 + u^2\right] \text{ dv} + \varepsilon \int_{\partial\Omega} u^2 \text{ da} \right),$$

where the infimum is taken over all the n-dimensional subspaces A of $H^1(\Omega)$. If $A \subset H^1(\Omega)$ is a fixed n-dimensional vector space, then by (4.5) one has

$$\begin{aligned} \int_\Omega \left[|\nabla u|^2 + u^2\right] \text{ dv} &\le \int_\Omega \left[|\nabla u|^2 + u^2\right] \text{ dv} + \varepsilon \int_{\partial\Omega} u^2 \text{ da} \\ &\le (1 + \varepsilon C) \int_\Omega \left[|\nabla u|^2 + u^2\right] \text{ dv} \qquad \text{for any } u \in A, \end{aligned}$$

obtaining

$$\begin{aligned} \sup_{\substack{u\in A \\ ||u||_{L^2(\Omega)}=1}} \int_\Omega \left[|\nabla u|^2 + u^2\right] \text{ dv} &\le \sup_{\substack{u\in A \\ ||u||_{L^2(\Omega)}=1}} \left(\int_\Omega \left[|\nabla u|^2 + u^2\right] \text{ dv} + \varepsilon \int_{\partial\Omega} u^2 \text{ da} \right) \\ &\le (1 + \varepsilon C) \sup_{\substack{u\in A \\ ||u||_{L^2(\Omega)}=1}} \int_\Omega \left[|\nabla u|^2 + u^2\right] \text{ dv}. \end{aligned}$$

Taking the infimum over all the n-dimensional subspaces $A \subset H^1(\Omega)$, it follows

$$\lambda_{0,n} \le \lambda_{\varepsilon,n} \le (1 + \varepsilon C)\lambda_{0,n}$$

that implies

$$\lim_{\varepsilon\to 0}\lambda_{\varepsilon,n}=\lambda_{0,n}.$$

As a consequence, by Theorem 3.1, we get

$$\lambda_{\varepsilon,n}-\lambda_{0,n}=\lambda_{0,n}\int_\Omega V_\varepsilon\phi_0\,\mathrm{dv}+O\left(||V_\varepsilon||^2_{L^2(\Omega)}\right)\quad\text{as }\varepsilon\to 0.$$

We study the asymptotic behavior of V_ε in order to obtain a more explicit description of the eigenvalue variation. To this aim, let us define

$$\widetilde{V}_\varepsilon(x):=\frac{V_\varepsilon(x)}{\varepsilon}.$$

Firstly, we observe that by (4.4)

$$\int_\Omega\left[g(\nabla\widetilde{V}_\varepsilon,\nabla u)+\widetilde{V}_\varepsilon u\right]\,\mathrm{dv}+\varepsilon\int_{\partial\Omega}\widetilde{V}_\varepsilon u\,\mathrm{da}=\int_{\partial\Omega}\phi_0 u\,\mathrm{da}\tag{4.6}$$

for every $u\in H^1(\Omega)$, implying, by testing the equation with $u=\widetilde{V}_\varepsilon$,

$$\int_\Omega\left[\left|\nabla\widetilde{V}_\varepsilon\right|^2+\widetilde{V}_\varepsilon^2\right]\,\mathrm{dv}\le\int_{\partial\Omega}\phi_0\widetilde{V}_\varepsilon\,\mathrm{da}.$$

By Hölder inequality and (4.5), we get

$$\begin{aligned}||\widetilde{V}_\varepsilon||^2_{H^1(\Omega)}&\le||\phi_0||_{L^2(\partial\Omega)}||\widetilde{V}_\varepsilon||_{L^2(\partial\Omega)}\\&\le C||\phi_0||_{L^2(\partial\Omega)}||\widetilde{V}_\varepsilon||_{H^1(\Omega)}\end{aligned}$$

i.e.

$$||\widetilde{V}_\varepsilon||_{H^1(\Omega)}\le C||\phi_0||_{L^2(\partial\Omega)},$$

thus the family $\{\widetilde{V}_\varepsilon\}_\varepsilon$ is bounded in $H^1(\Omega)$. Whence, there exists a sequence $\varepsilon_n\to 0^+$ as $n\to\infty$ and a function $\widetilde{V}\in H^1(\Omega)$ such that $\widetilde{V}_{\varepsilon_n}\xrightarrow{H^1(\Omega)}\widetilde{V}$ weakly in $H^1(\Omega)$ as $n\to\infty$.

Passing to the limit in (4.6), it follows that

$$\int_\Omega\left[g(\nabla\widetilde{V},\nabla u)+\widetilde{V}u\right]\,\mathrm{dv}=\int_{\partial\Omega}\phi_0 u\,\mathrm{da}\tag{4.7}$$

for every $u\in H^1(\Omega)$.

Remark 4.2 We stress that (4.7) has at most one solution. Indeed, if $\widetilde{V}$ and $\widetilde{W}$ are two functions satisfying the equation above, then the function $\widetilde{U}:=\widetilde{V}-\widetilde{W}$ satisfies

$$\int_\Omega\left[g(\nabla\widetilde{U},\nabla u)+\widetilde{U}u\right]\,\mathrm{dv}=0$$

for every $u\in H^1(\Omega)$, implying $\widetilde{U}=0$ and hence $\widetilde{V}=\widetilde{W}$.

As a consequence of Remark 4.2, it follows that every subsequence $\{\varepsilon_n\}_n$ so that $\{\widetilde{V}_{\varepsilon_n}\}_n$ converges weakly must have $\widetilde{V}$ as limit function. In particular, by the Urysohn Subsequence

Principle it follows that $\widetilde{V}_\varepsilon \xrightarrow{H^1(\Omega)} \widetilde{V}$ weakly in $H^1(\Omega)$ and not only along the sequence $\{\varepsilon_n\}_n$. By (4.6) we get

$$\begin{aligned}\lim_{\varepsilon\to 0^+} ||\widetilde{V}_\varepsilon||^2_{H^1(\Omega)} &= \lim_{\varepsilon\to 0^+}\left[\int_{\partial\Omega}\phi_0\widetilde{V}_\varepsilon \,\mathrm{da} - \varepsilon\int_{\partial\Omega}|\widetilde{V}_\varepsilon|^2\,\mathrm{da}\right]\\ &= \int_{\partial\Omega}\phi_0\widetilde{V}\,\mathrm{da}\\ &= ||\widetilde{V}||^2_{H^1(\Omega)},\end{aligned}$$

where the last equality follows by (4.7). We also observe that, by the very definition of $\widetilde{V}_\varepsilon$, this exactly means that

$$||V_\varepsilon||^2_{H^1(\Omega)} = O(\varepsilon^2) \quad \text{as } \varepsilon \to 0. \tag{4.8}$$

Moreover, since in any Hilbert space the weak convergence together with the convergence of the norm implies the strong convergence, we get that

$$\widetilde{V}_\varepsilon \xrightarrow{\varepsilon\to 0} \widetilde{V} \quad \text{strongly in } H^1(\Omega).$$

Lastly, we stress that by (4.7)

$$\begin{aligned}\lim_{\varepsilon\to 0}\lambda_0\int_\Omega \varepsilon^{-1}V_\varepsilon\phi_0\,\mathrm{dv} = \lambda_0\int_\Omega\widetilde{V}\phi_0\,\mathrm{dv}\\ &= \int_\Omega\left[g(\nabla\widetilde{V},\nabla\phi_0)+\widetilde{V}\phi_0\right]\mathrm{dv}\\ &= \int_\Omega|\phi_0|^2\,\mathrm{da}\end{aligned}$$

which, together with (4.8), provides the following expression

$$\lambda_{\varepsilon,n} - \lambda_{0,n} = \varepsilon\int_{\partial\Omega}|\phi_0|^2\,\mathrm{da} + O(\varepsilon^2) \quad \text{as } \varepsilon \to 0.$$

Remark 4.3 We stress that the first order approximation obtained in the present subsection is not new in literature, at least for the first eigenvalue in the Euclidean setting (see [27, Chapter 4]). As mentioned at the beginning of this section, the same result can be recovered using more standard techniques, involving for instance the Feynman-Hellmann formula for families of analytic forms in the sense of Kato (see [28]). This example is hence aimed to provide a simple but clear model of the way our approach can be applied to a wide variety of different problems.

4.2 Conformal transformations

Let (M, g) be a complete Riemannian manifold and $\Omega \subseteq M$ a compact domain. Fixed a smooth function

$$\begin{aligned}\Phi : [0,1]\times M &\to \mathbb{R}\\ (\varepsilon, p) &\mapsto \Phi_\varepsilon(p)\end{aligned}$$

so that $\|\Phi_\varepsilon\|_{L^\infty(\Omega)} \to 0$ as $\varepsilon \to 0^+$, consider the following family of metrics on M

$$g_\varepsilon := e^{2\Phi_\varepsilon} g$$

which are conformal to $g = g_0$.

To every $\varepsilon \in [0, 1]$ we can associate the following eigenvalue problems

$$\begin{cases} -\Delta_\varepsilon u = \lambda u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{4.9}$$

where by Δ_ε we mean the Laplace-Beltrami operator associated to the metric g_ε.

For any fixed $\varepsilon \in [0, 1]$, the bilinear form associated to the problem (4.9) is $(\mathcal{E}^{(\varepsilon)}, \mathcal{F}_\varepsilon)$, where $\mathcal{F}_\varepsilon = H^1_0(\Omega, \mathrm{dv}_\varepsilon)$ and

$$\mathcal{E}^{(\varepsilon)}(u, v) := \int_\Omega g_\varepsilon(\nabla_\varepsilon u, \nabla_\varepsilon v)\, \mathrm{dv}_\varepsilon \quad \text{for any } u, v \in \mathcal{F}_\varepsilon.$$

Since Ω is compact, it is a standard fact (see [26]) that $H^1_0(\Omega, \mathrm{dv}_\varepsilon)$ does not depend on the metric g_ε. Hence, in the following we consider the bilinear forms $\mathcal{E}^{(\varepsilon)}$ as acting over the same domain $\mathcal{F} := \mathcal{F}_0$. Similarly to Subsection 4.1, we have that for any $\varepsilon \in [0, 1]$

$$Y_\varepsilon = \Omega, \qquad Z_\varepsilon = \mathcal{F}_\varepsilon, \qquad \mathcal{Z}_\varepsilon = L^2(\Omega).$$

Clearly, assumptions (A1), (A3) and (A5) are satisfied.

We recall the following useful identities

$$g_\varepsilon^{-1} = e^{-2\Phi_\varepsilon} g^{-1}, \qquad \nabla_\varepsilon = e^{-2\Phi_\varepsilon}\nabla \qquad \text{and} \qquad \mathrm{dv}_\varepsilon = e^{n\Phi_\varepsilon}\, \mathrm{dv}.$$

As a consequence, using the fact that Φ_ε converges uniformly to 0 on Ω, we have the following control on the H^1_0 norms

$$e^{-(n-2)}\, \|u\|_{H^1_0(\Omega,\, \mathrm{dv}_\varepsilon)} \le \|u\|_{H^1_0(\Omega,\, \mathrm{dv})} \le e^{n-2}\, \|u\|_{H^1_0(\Omega,\, \mathrm{dv}_\varepsilon)} \tag{4.10}$$

for ε small enough.

Fix λ_0 a simple eigenvalue (with associated normalized eigenfunction ϕ_0) to the limit problem

$$\begin{cases} -\Delta_0 u = \lambda_0 u, & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

In particular, for every $\varepsilon \in [0, 1]$

$$\begin{aligned} \mathcal{E}^{(\varepsilon)}(\phi_0, u) = \lambda_0 \int_\Omega \phi_0 u\, \mathrm{dv} &+ \int_\Omega g_\varepsilon(\nabla_\varepsilon \phi_0, \nabla_\varepsilon u)\, \mathrm{dv}_\varepsilon \\ &- \int_\Omega g(\nabla\phi_0, \nabla u)\, \mathrm{dv} \quad \text{for any } u \in H^1_0(\Omega), \end{aligned}$$

showing that condition (2.7) is satisfied, where $L_\varepsilon : H^1_0(\Omega) \to \mathbb{R}$ is given by the linear and continuous operator

$$L_\varepsilon(u) := \int_\Omega g_\varepsilon(\nabla_\varepsilon \phi_0, \nabla_\varepsilon u)\, \mathrm{dv}_\varepsilon - \int_\Omega g(\nabla\phi_0, \nabla u)\, \mathrm{dv}.$$

Denoting by $J_\varepsilon : H^1_0(\Omega) \to \mathbb{R}$ the functional

$$\begin{aligned} J_\varepsilon(u) :&= \frac{1}{2}\mathcal{E}^{(\varepsilon)}(u) - L_\varepsilon(u) \\ &= \frac{1}{2}\int_\Omega g_\varepsilon(\nabla_\varepsilon u, \nabla_\varepsilon u)\, \mathrm{dv}_\varepsilon - \int_\Omega g_\varepsilon(\nabla_\varepsilon \phi_0, \nabla_\varepsilon u)\, \mathrm{dv}_\varepsilon + \int_\Omega g(\nabla\phi_0, \nabla u)\, \mathrm{dv}, \end{aligned}$$

let $V_\varepsilon \in H_0^1(\Omega)$ be the function provided by Proposition 2.5. We start by showing that

$$\lim_{\varepsilon \to 0} ||V_\varepsilon||_{L^2(M,\mathrm{dv}_\varepsilon)} = 0.$$

By Proposition 2.5, we have

$$\mathcal{E}^{(\varepsilon)}(V_\varepsilon, u) = L_\varepsilon(u) \quad \text{for any } u \in H_0^1(M)$$

and hence

$$\begin{aligned} \int_\Omega g_\varepsilon(\nabla_\varepsilon V_\varepsilon, \nabla_\varepsilon u)\, \mathrm{dv}_\varepsilon &= \int_\Omega g_\varepsilon(\nabla_\varepsilon \phi_0, \nabla_\varepsilon u)\, \mathrm{dv}_\varepsilon - \int_\Omega g(\nabla \phi_0, \nabla u)\, \mathrm{dv} \\ &= \int_\Omega \left(e^{(n-2)\Phi_\varepsilon} - 1\right) g(\nabla \phi_0, \nabla u)\, \mathrm{dv} \end{aligned} \tag{4.11}$$

for any $u \in H_0^1(\Omega)$. Testing the previous equality with $u = V_\varepsilon$, we get

$$\begin{aligned} &\int_\Omega g_\varepsilon(\nabla_\varepsilon V_\varepsilon, \nabla_\varepsilon V_\varepsilon)\, \mathrm{dv}_\varepsilon \\ &\leq \int_\Omega \left|e^{(n-2)\Phi_\varepsilon} - 1\right| |g(\nabla \phi_0, \nabla V_\varepsilon)|\, \mathrm{dv} \\ &\leq \max_{\overline{\Omega}} \left|e^{(n-2)\Phi_\varepsilon} - 1\right| \lambda_0^{\frac{1}{2}} \left(\int_\Omega g(\nabla V_\varepsilon, \nabla V_\varepsilon)\, \mathrm{dv}\right)^{\frac{1}{2}} \\ &= \max_{\overline{\Omega}} \left|e^{(n-2)\Phi_\varepsilon} - 1\right| \lambda_0^{\frac{1}{2}} \left(\int_\Omega \frac{e^{(n-2)\Phi_\varepsilon}}{e^{(n-2)\Phi_\varepsilon}} g(\nabla V_\varepsilon, \nabla V_\varepsilon)\, \mathrm{dv}\right)^{\frac{1}{2}} \\ &\leq \max_{\overline{\Omega}} \left|e^{(n-2)\Phi_\varepsilon} - 1\right| \lambda_0^{\frac{1}{2}} \frac{1}{e^{\left[\frac{n-2}{2} \min_{\overline{\Omega}} \Phi_\varepsilon\right]}} \left(\int_\Omega g_\varepsilon(\nabla_\varepsilon V_\varepsilon, \nabla_\varepsilon V_\varepsilon)\, \mathrm{dv}_\varepsilon\right)^{\frac{1}{2}} \end{aligned}$$

that implies

$$\left(\int_\Omega g_\varepsilon(\nabla_\varepsilon V_\varepsilon, \nabla_\varepsilon V_\varepsilon)\, \mathrm{dv}_\varepsilon\right)^{\frac{1}{2}} \leq \max_{\overline{\Omega}} \left|e^{(n-2)\Phi_\varepsilon} - 1\right| \frac{1}{e^{\left[\frac{n-2}{2} \min_{\overline{\Omega}} \Phi_\varepsilon\right]}} \lambda_0^{\frac{1}{2}} \tag{4.12}$$

and, by Proposition 2.2,

$$\begin{aligned} ||V_\varepsilon||_{L^2(\Omega,\mathrm{dv}_\varepsilon)} &\leq \max_{\overline{\Omega}} \left|e^{(n-2)\Phi_\varepsilon} - 1\right| \frac{1}{e^{\left[\frac{n-2}{2} \min_{\overline{\Omega}} \Phi_\varepsilon\right]}} \left(\lambda_\varepsilon^{-1}\lambda_0\right)^{\frac{1}{2}} \\ &\leq \left(e^{(n-2)\|\Phi_\varepsilon\|_{L^\infty(\Omega)}} - 1\right) \frac{1}{e^{\left[\frac{n-2}{2} \min_{\overline{\Omega}} \Phi_\varepsilon\right]}} \left(\lambda_\varepsilon^{-1}\lambda_0\right)^{\frac{1}{2}} \\ &= O\left(e^{(n-2)\|\Phi_\varepsilon\|_{L^\infty(\Omega)}} - 1\right) \qquad \text{as } \varepsilon \to 0 \\ &= O\left(\|\Phi_\varepsilon\|_{L^\infty(\Omega)}\right) \qquad \text{as } \varepsilon \to 0, \end{aligned}$$

since $\|\phi_\varepsilon\|_{L^\infty(\Omega)} \xrightarrow{\varepsilon \to 0} 0$, implying that condition (A6) is satisfied.

To prove the stability of the spectrum, i.e. condition (A2), we start by observing that

$$\frac{e^{(n-2)\min_{\overline{\Omega}}\Phi_\varepsilon}}{e^{n\max_{\overline{\Omega}}\Phi_\varepsilon}}\frac{\int_\Omega g(\nabla u,\nabla u)\,\mathrm{dv}}{\int_\Omega u^2\,\mathrm{dv}} \le \frac{\int_\Omega g_\varepsilon(\nabla_\varepsilon u,\nabla_\varepsilon u)\,\mathrm{dv}_\varepsilon}{\int_\Omega u^2\,\mathrm{dv}_\varepsilon}$$
$$\le \frac{e^{(n-2)\max_{\overline{\Omega}}\Phi_\varepsilon}}{e^{n\min_{\overline{\Omega}}\Phi_\varepsilon}}\frac{\int_\Omega g(\nabla u,\nabla u)\,\mathrm{dv}}{\int_\Omega u^2\,\mathrm{dv}}$$

Hence with min-max argument,

$$\frac{e^{(n-2)\min_{\overline{\Omega}}\Phi_\varepsilon}}{e^{n\max_{\overline{\Omega}}\Phi_\varepsilon}}\lambda_{n,0} \le \lambda_{n,\varepsilon} \le \frac{e^{(n-2)\max_{\overline{\Omega}}\Phi_\varepsilon}}{e^{n\min_{\overline{\Omega}}\Phi_\varepsilon}}\lambda_{n,0}$$

and so $\lambda_{n,\varepsilon} \to \lambda_{n,0}$ as $\varepsilon \to 0$ for any $n \in \mathbb{N}$.

If λ_0 is simple, by Theorem 3.1 we have that

$$\lambda_\varepsilon - \lambda_0 = \frac{\lambda_0\int_\Omega \phi_0 V_\varepsilon\,\mathrm{dv}_\varepsilon + O\left(||V_\varepsilon||^2_{L^2(\Omega,\mathrm{dv}_\varepsilon)}\right)}{\int_\Omega \phi_0^2\,\mathrm{dv}_\varepsilon + O\left(||V_\varepsilon||_{L^2(\Omega,\mathrm{dv}_\varepsilon)}\right)} \qquad \text{as } \varepsilon \to 0.$$

Noticing that

$$\left|1-\int_\Omega \phi_0^2\,\mathrm{dv}_\varepsilon\right| = \left|\int_\Omega (1-e^{n\Phi_\varepsilon})\phi_0^2\,\mathrm{dv}\right|$$
$$\le \max_{\overline{\Omega}}\left|1-e^{n\Phi_\varepsilon}\right|\int_\Omega \phi_0^2\,\mathrm{dv}$$
$$= \max_{\overline{\Omega}}\left|1-e^{n\Phi_\varepsilon}\right|$$
$$= O\left(\|\Phi_\varepsilon\|_{L^\infty(\Omega)}\right) \qquad \text{as } \varepsilon \to 0,$$

as in Remark 3.3 it follows that

$$\lambda_\varepsilon - \lambda_0 = \left[\lambda_0\int_\Omega \phi_0 V_\varepsilon\,\mathrm{dv}_\varepsilon + O\left(\|\Phi_\varepsilon\|^2_{L^\infty(\Omega)}\right)\right]\left(1+O\left(\|\Phi_\varepsilon\|_{L^\infty(\Omega)}\right)\right)$$
$$= \lambda_0\int_\Omega \phi_0 V_\varepsilon\,\mathrm{dv}_\varepsilon + O\left(\|\Phi_\varepsilon\|^2_{L^\infty(\Omega)}\right) \qquad \text{as } \varepsilon \to 0.$$

Under additional assumption on Φ_ε, we can obtain a more precise precise result. Let us suppose that

$$\|\Phi_\varepsilon\|_{L^\infty(\Omega)} = o\left(\sqrt{\varepsilon}\right) \quad \text{as } \varepsilon \to 0, \tag{4.13}$$

and

$$\left|\frac{e^{(n-2)\Phi_\varepsilon(x)}-1}{\varepsilon}\right| \le h(x) \quad \text{for a.e. } x \in \Omega, \text{ and any } \varepsilon \text{ small}, \tag{4.14}$$

where $h \in L^2(\Omega)$. Furthermore, we require that for any $x \in \Omega$ there exists the limit

$$\lim_{\varepsilon\to 0^+}\frac{\Phi_\varepsilon(x)}{\varepsilon} =: \Psi(x) \tag{4.15}$$

and that $\Psi \in L^\infty(\Omega)$. For instance, the above assumptions hold if $\frac{\Phi_\varepsilon}{\varepsilon} \to \Psi$ in $L^\infty(\Omega)$ as $\varepsilon \to 0^+$.

In view of (4.10), (4.12) and (4.15) we have that the family $\left\{\frac{V_\varepsilon}{\varepsilon}\right\}_{\varepsilon\in[0,1]}$ is bounded in $H_0^1(\Omega)$. In particular, there exists a subsequence $\frac{V_{\varepsilon_n}}{\varepsilon_n}$ and $\widetilde{V}\in H_0^1(\Omega)$ such that $\frac{V_{\varepsilon_n}}{\varepsilon_n}\rightharpoonup\widetilde{V}$ weakly in $H_0^1(\Omega)$ as $n\to\infty$.

Then by the Dominated Convergence Theorem, (4.14) and (4.15) we can pass to the limit in (4.11). It follows that $\tilde{V}$ solves the equation

$$\int_\Omega g(\nabla\tilde{V},\nabla u)\,\mathrm{dv}=(n-2)\int_\Omega \Psi\, g(\nabla\phi_0,\nabla u)\,\mathrm{dv}.$$

Since the solution to the equation above is unique, by the Urysonh subsequence principle, we can see that $\frac{V_\varepsilon}{\varepsilon}\rightharpoonup\widetilde{V}$ weakly in $H_0^1(\Omega)$ as $\varepsilon\to0^+$.

Finally by the Dominated Convergence Theorem

$$\begin{aligned}\lim_{\varepsilon\to0^+}\lambda_0\int_\Omega\phi_0\frac{V_\varepsilon}{\varepsilon}\,\mathrm{dv}_\varepsilon&=\lambda_0\int_\Omega\phi_0\widetilde{V}\,\mathrm{dv}\\&=\int_\Omega g(\nabla\widetilde{V},\nabla\phi_0)\,\mathrm{dv}\\&=(n-2)\int_\Omega\Psi\,g(\nabla\phi_0,\nabla\phi_0)\,\mathrm{dv}.\end{aligned}$$

Hence

$$\lambda_\varepsilon-\lambda_0=\varepsilon(n-2)\int_\Omega\Psi\,g(\nabla\phi_0,\nabla\phi_0)\,\mathrm{dv}+o(\varepsilon)\quad\text{as }\varepsilon\to0^+,\tag{4.16}$$

in view of (4.13).

Remark 4.4 At least formally, it would be possible to derive the (4.16) with more classical methods, as the already cited Feynman-Hellmann formula, see [28]. For example, whenever $\{g_\varepsilon\}_{\varepsilon\in[0,1]}$ is analytic with respect to ε then the family of associated eigenvalues $\{\lambda_{\varepsilon,n}\}_{\varepsilon\in[0,1]}$ and eigenfunctions $\{\phi_{\varepsilon,n}\}_{\varepsilon\in[0,1]}$ are smooth with respect to ε and a second order estimate was obtained in [6]. See also [15].

4.3 Dirichlet forms

Let (X,d,m) be a locally compact and separable measure metric space, where m is a positive Radon measure defined on the Borel σ-algebra $\mathcal{B}$ of X. Consider $(\mathcal{E},\mathcal{F})$ a Dirichlet form associated to the linear, non-negative, self-adjoint and densely defined operator H. Moreover, suppose that H has a discrete spectrum $\{\lambda_n\}_{n\in\mathbb{N}}$ with associated orthonormal basis of eigenfunctions $\{\phi_n\}_{n\in\mathbb{N}}$ of $L^2(X,m)$. In view of Remark 3.3, it is not restrictive to suppose that $\lambda_1>0$, and so $R:=H^{-1}$ is a well defined, non-negative, self-adjoint, bounded and compact operator. Let

$$C_c(X):=\{u:X\to\mathbb{R}\;:\;u\text{ is continuous with compact support}\}.$$

We assume that

$$C_c(X)\cap\mathcal{F}\text{ is dense in }\mathcal{F}\text{ with respect to the norm }\mathcal{E}_1.\tag{4.17}$$

Letting $(\mathcal{E},\mathcal{F})$ be a Dirichlet form, there is a classical notion of capacity of sets relative to $(\mathcal{E},\mathcal{F})$, see for example [24]. Given $f\in\mathcal{F}$, a natural generalization of capacity is the definition of f-capacity of a set $K\subset X$ relative to $(\mathcal{E},\mathcal{F})$, that is,

$$\mathrm{Cap}_f(K):=\inf\left\{\mathcal{E}(u)\;:\;u\in\mathcal{F},\;\widetilde{u}\geq\widetilde{f}\text{ q.e. in }K\right\},\tag{4.18}$$

where by $\widetilde{u}$ we mean the quasi-continuous representative of the function $u \in \mathcal{F}$. Here and in the following, by *q.e. in* K we mean that a certain condition holds out of a subset E of K of 0 capacity. Similarly, by *quasi-continuous* we mean that the preimage of any open set by $\widetilde{u}$ is quasi-open, that is, it is open up to a set of 0 capacity. For the existence of a quasi-continuous representative of any $u \in \mathcal{F}$ we refer again to [24].

We recall that every element of $\mathcal{F}$ has a quasi-continuous representative with respect to the classical notion of capacity associated to $(\mathcal{E}, \mathcal{F})$, see for example [24].

In the spirit of [1], we consider family $\{K_\varepsilon\}_{\varepsilon\in[0,1]}$ of compact subsets of X such that:

(i) $\mathrm{Cap}(K_0) := \mathrm{Cap}_1(K_0) = 0$
(ii) the family $\{K_\varepsilon\}_{\varepsilon\in(0,1]}$ concentrates at K_0, that is, for every open neighbourhood U of K_0 there exists $\varepsilon_0 \in (0, 1]$ so that $K_\varepsilon \subset U$ for every $\varepsilon < \varepsilon_0$.

Heuristically, the simpler example of concentrating family of compact sets are shrinking holes but the assumption that $\{K_\varepsilon\}_{\varepsilon\in(0,1]}$ concentrates at K_0 holds in many other cases. For example if $K_\varepsilon \to K_0$ in the sense of Hausdorff as $\varepsilon \to 0^+$, then it is easy to see that K_ε concentrate at K_0. On the other hand concentration of sets is a more general notion, since for example if $\{K_\varepsilon\}_{\varepsilon\in(0,1]}$ concentrates at K_0 then it also concentrate to any set $\tilde{K}_0$ such that $K_0 \subseteq \tilde{K}_0$ while the limit in sense of Hausdorff is unique on compact sets.

Let

$$Z_\varepsilon := \{u \in \mathcal{F} \ : \tilde{u} = 0 \text{ q.e. on } K_\varepsilon\}.$$

It is easy to see that $\mathcal{Z}_\varepsilon$, the closure of Z_ε with respect to the norm of $L^2(X, m)$, is

$$\{u \in L^2(X, m) \ : \ u = 0 \text{ a.e. on } K_\varepsilon\}.$$

$(\mathcal{E}, Z_\varepsilon)$ is still a Dirichlet form, see [24], and we denote by H_ε its non-negative, densely defined (in $\mathcal{Z}_\varepsilon$), self-adjoint operator. We may suppose that $R_\varepsilon := H_\varepsilon^{-1}$ is a well defined and bounded operator thanks to Remark 3.3. In particular, since R_ε takes values in $\mathcal{F}$ by Proposition 2.1 it follows that

$$R_\varepsilon : \mathcal{Z}_\varepsilon \to \mathcal{Z}_\varepsilon$$

is compact. Hence its spectrum is discrete. Let $\{\lambda_{\varepsilon,n}\}_{n\in\mathbb{N}}$ be the discrete spectrum of H_ε.

With the same notations of Section 2, for any $\varepsilon \in [0, 1]$

$$Y_\varepsilon = X, \qquad \mathcal{F}_\varepsilon = \mathcal{F} \quad \text{and} \quad L_\varepsilon = 0.$$

Fix a simple eigenvalue $\lambda := \lambda_n$ of $(\mathcal{E}, \mathcal{F})$ and let ϕ be a corresponding eigenfunction with $\|\phi\|_{L^2(X,m)} = 1$. Clearly for any $w \in Z_\varepsilon$

$$\mathcal{E}(\phi, w) = \lambda(\phi, w)_{L^2(X,m)} \tag{4.19}$$

and so $L_\varepsilon = 0$. Let J_ε be as in (2.8) and consider the function $V_\varepsilon \in \mathcal{F}$ given by Proposition 2.5. V_ε coincides with the classical capacitary potential V_ε^ϕ associated to the ϕ-capacity $\mathrm{Cap}_\phi(K_\varepsilon)$. Indeed V_ε^ϕ solves the minimization problem (4.18) and so, since $V_\varepsilon^\phi = \phi$ on K_ε, it solves (2.9), which admits a unique solution. We remark that V_ε satisfies

$$\mathcal{E}(V_\varepsilon, w) = 0 \quad \text{for any } w \in Z_\varepsilon. \tag{4.20}$$

In order to prove the stability of the spectrum of H and the validity of (A6), we start with the following technical lemma.

Lemma 4.5 *Let* K *be a compact subset of* X *and suppose that* $\mathrm{Cap}(K) = 0$. *Then the set* $Z_K := \{u \in \mathcal{F} : K \cap \operatorname{supp} u = \emptyset\}$ *is dense in* $\mathcal{F}$ *endowed with its weak topology.*

Proof Fix $u \in C_c(X) \cap \mathcal{F}$ and consider a sequence $\{u_n\}_{n\in\mathbb{N}} \subset \mathcal{F}$ so that

$$\mathcal{E}(u_n) \xrightarrow{n\to+\infty} 0 \quad \text{and} \quad u_n = 1 \text{ a.e. in } U_n,$$

where U_n is a neighbourhood of K. In particular we may suppose that $u_n \le 1$ a.e. in X in view of the Markovianity of $\mathcal{E}$, see [24, Subsection 1.1]. Then $(1-u_n)u \in Z_K$ and

$$\mathcal{E}(u-(1-u_n)u) = \mathcal{E}(u_n u) \le 2\left(\|u_n\|^2_{L^\infty(X,m)}\,\mathcal{E}(u) + \|u\|^2_{L^\infty(X,m)}\,\mathcal{E}(u_n)\right) \text{ for any } n \in \mathbb{N}.$$

Hence $\{uu_n\}_{n\in\mathbb{N}}$ is bounded in $\mathcal{F}$ in view of Proposition 2.2. In particular, up to a subsequence, there exists $w \in \mathcal{F}$ such that $uu_n \rightharpoonup w$ weakly in $\mathcal{F}$ as $n \to \infty$. By Proposition 2.1 it follows that $uu_n \to w$ strongly in $L^2(X,m)$. Furthermore by Proposition 2.2

$$\int_X |uu_n|^2\,dm \le \|u\|^2_{L^\infty(X,m)}\,\|u_n\|^2_{L^2(X,m)} \le \lambda_0^{-1}\,\|u\|^2_{L^\infty(X,m)}\,\mathcal{E}(u_n) \to 0^+, \text{ as } n \to \infty.$$

We conclude that $w = 0$ and so $(1-u_n)u \rightharpoonup u$ weakly in $\mathcal{F}$. Then the claim follows from (4.17). □

In the spirit of [17, Proposition 3.8], we have the following lemma which, together with Proposition 2.2, proves that (A6) holds. We recall that V_ε^f is the classical capacitary potential associated to the capacity $\mathrm{Cap}_f(K_\varepsilon)$.

Lemma 4.6 *Under the assumptions above on K_ε and K_0, for any $f \in \mathcal{F}$*

$$\mathrm{Cap}_f(K_\varepsilon) \to 0^+, \quad \textit{as } \varepsilon \to 0^+, \tag{4.21}$$

$$V_\varepsilon^f \to 0 \textit{ strongly in } \mathcal{F}, \quad \textit{as } \varepsilon \to 0^+, \tag{4.22}$$

$$m(K_\varepsilon) \to 0^+, \quad \textit{as } \varepsilon \to 0^+. \tag{4.23}$$

Proof For any $\varepsilon \in (0,1]$ the potential V_ε^f solves the equation

$$\mathcal{E}(V_\varepsilon^f, w) = 0 \quad \text{for any } w \in Z_\varepsilon. \tag{4.24}$$

Testing the equation above with $V_\varepsilon^f - f$ we conclude that, by the Cauchy-Schwarz inequality,

$$\mathcal{E}(V_\varepsilon^f) \le \mathcal{E}(f) \quad \text{for any } \varepsilon \in (0,1].$$

Hence there exist a sequence $\{V_{\varepsilon_n}^f\}_{n\in\mathbb{N}}$ and $V \in \mathcal{F}$ such that $V_{\varepsilon_n}^f \rightharpoonup V$ weakly in $\mathcal{F}$. Let $\psi \in Z_{K_0}$ and U a neighbourhood of K_0 such that $\operatorname{supp}\psi \cap U = \emptyset$. Let ε_0 be such that $K_\varepsilon \subset U$ for any $\varepsilon \in [0,\varepsilon_0]$. Testing (4.24) with ψ and passing to the limit as $n \to \infty$ we conclude that $\mathcal{E}(V,\psi) = 0$ and so

$$\mathcal{E}(V, w) = 0 \quad \text{for any } w \in Z_{K_0}.$$

Hence by Lemma 4.5 it follows that $V = V_{K_0}^f = 0$. Moreover,

$$0 = \mathrm{Cap}_f(K_0) = \mathcal{E}(V) = \mathcal{E}(V,f) = \lim_{n\to\infty} \mathcal{E}(V_{\varepsilon_n}, f) = \lim_{n\to\infty} \mathrm{Cap}_f(K_{\varepsilon_n}).$$

By the Urysohn Subsequence Principle we conclude that (4.21) and (4.22) hold.

We recall that for any set $K \subset X$

$$\mathrm{Cap}(K) = \inf\{\mathrm{Cap}(U) \;:\; U \subseteq X \text{ open}, \; K \subset U\}.$$

Fix $\delta > 0$ small enough and $U \subseteq X$ open so that $K_0 \subset U$ and $\mathrm{Cap}(U) \leq \delta$. For $\varepsilon_0 \in (0, 1]$ small enough, $K_\varepsilon \subset U$ for every $\varepsilon \in (0, \varepsilon_0]$ and so, thanks to [24, Subsection 2.1] and Proposition 2.2,

$$m(K_\varepsilon) \leq m(U) \leq (1 + \lambda_1^{-1})\mathrm{Cap}(U) \leq (1 + \lambda_1^{-1})\delta.$$

Passing to the limit as $\varepsilon \to 0^+$ we conclude that

$$\limsup_{\varepsilon \to 0^+} m(K_\varepsilon) \leq (1 + \lambda_1^{-1})\delta,$$

for any $\delta > 0$ and so (4.23) holds. □

Proposition 4.7 *For any* $n \in \mathbb{N}$

$$\lim_{\varepsilon \to 0^+} \lambda_{\varepsilon,n} = \lambda_n. \tag{4.25}$$

Proof We are going to prove that the assumptions of Theorem 3.8 are satisfied. Let us define the linear operator $P_\varepsilon : L^2(X, m) \to \mathcal{Z}_\varepsilon$ as

$$P_\varepsilon(u)(x) := \begin{cases} u(x), & \text{if } x \in X \setminus K_\varepsilon, \\ 0, & \text{if } x \in K_\varepsilon. \end{cases}$$

Then for any $u \in L^2(X, m)$, thanks to Proposition 2.2,

$$\begin{aligned} \mathcal{E}(R_\varepsilon P_\varepsilon u) &= (P_\varepsilon u, R_\varepsilon P_\varepsilon u)_{L^2(X,m)} \\ &\leq \|u\|_{L^2(X,m)} \|R_\varepsilon P_\varepsilon u\|_{L^2(X,m)} \\ &\leq \lambda_1^{-\frac{1}{2}} \|u\|_{L^2(X,m)} \mathcal{E}(R_\varepsilon P_\varepsilon u)^{\frac{1}{2}}. \end{aligned}$$

Furthermore if $R_{\varepsilon_n} P_{\varepsilon_n} u \rightharpoonup w$ weakly in $\mathcal{F}$ for some $w \in \mathcal{F}$ as $n \to \infty$, then clearly

$$\lim_{n \to \infty} \mathcal{E}(R_{\varepsilon_n} P_{\varepsilon_n} u, v) = \mathcal{E}(w, v) \quad \forall v \in Z_0.$$

We conclude that (4.25) holds in view of Theorem 3.8 □

We have proved that all the hypotheses of Theorem 3.1 are satisfied. Furthermore by Lemma 4.5 and Proposition 3.6

$$\|V_\varepsilon\|^2_{L^2(X,m)} = o(\mathcal{E}(V_\varepsilon)) = o(\mathrm{Cap}_\phi(K_\varepsilon)), \quad \text{as } \varepsilon \to 0^+,$$

and by (4.20) tested with $V_\varepsilon - \phi$ and (4.19) tested with V_ε

$$\mathrm{Cap}_\phi(K_\varepsilon) = \mathcal{E}(V_\varepsilon, V_\varepsilon) = \lambda(V_\varepsilon, \phi).$$

In conclusion, by Theorem 3.1, we have shown that

$$\lambda_\varepsilon - \lambda = \mathrm{Cap}_\phi(K_\varepsilon) + o(\mathrm{Cap}_\phi(K_\varepsilon)), \quad \text{as } \varepsilon \to 0^+,$$

where we are denoting with λ_ε the simple eigenvalue $\lambda_{\varepsilon,n}$ for any $\varepsilon \in (0, 1]$.

4.4 Fourier-multipliers

Let $\Omega \subset \mathbb{R}^N$ be a bounded Lipschitz domain. Consider a family $\{H_\varepsilon\}_\varepsilon$ of Fourier-multipliers with domains

$$L_0^2(\Omega) := \{u \in L^2(\mathbb{R}^N) : u = 0 \text{ in } \mathbb{R}^N \setminus \Omega\}$$
$$\mathrm{Dom}(H_\varepsilon) := \{u \in L_0^2(\Omega) \; : \; f_\varepsilon \widehat{u} \in L^2(\mathbb{R}^N)\},$$

where $\widehat{u}$ is the Fourier transform of u and $f_\varepsilon \geq 0$ denotes the symbol of H_ε. We will also use $\mathfrak{F}$ to denote the Fourier transform. On the symbol f_ε we assume that

$$f_\varepsilon \in L^2_{loc}(\mathbb{R}^N) \quad \text{and} \quad f_\varepsilon(\xi) = O(|\xi|^m), \text{ for some } m \in \mathbb{N} \text{ as } |\xi| \to \infty.$$

Let $\mathcal{E}^{(\varepsilon)}$ be the bilinear form associated to H_ε

$$\mathcal{E}^{(\varepsilon)}(u, v) := \int_{\mathbb{R}^N} f_\varepsilon(\xi)\widehat{u}(\xi)\overline{\widehat{v}(\xi)}\, \mathrm{d}\xi$$

with domain

$$\mathcal{F}_\varepsilon := \{u \in L_0^2(\Omega) \; : \; f_\varepsilon^{1/2}\widehat{u} \in L^2(\mathbb{R}^N)\}.$$

In the same notation of Section 2, for $\varepsilon \in [0, 1]$ we have

$$Y_\varepsilon = \Omega, \qquad Z_\varepsilon = \mathcal{F}_\varepsilon \qquad \text{and} \qquad \mathcal{Z}_\varepsilon = L_0^2(\Omega).$$

In what follows we assume that for every $\varepsilon \in [0, 1]$ the symbol f_ε of the operator H_ε satisfies the following properties:

(1) $f_\varepsilon > 1$ almost everywhere in $\mathbb{R}^N$;
(2) there exists a positive constants $C_1, C_0 > 0$ such that

$$C_1 f_1(\xi) \leq f_\varepsilon(\xi) \leq C_0 f_0(\xi) \quad \text{almost everywhere in } \mathbb{R}^N. \tag{4.26}$$

Moreover, we require that

$$f_\varepsilon \xrightarrow{\varepsilon \to 0} f_0 \quad \text{almost everywhere in } \mathbb{R}^N. \tag{4.27}$$

The first assumption is not restrictive, as observed after assumption (A3), and will ensure that the bottom of the spectrum is bigger or equal than 1. In particular the resolvent operator $R_\varepsilon = H_\varepsilon^{-1}$ is well defined. We also assume that R_ε is compact. A simple criterion to ensure the validity of this assumption is given in the next proposition.

Proposition 4.8 *Suppose that*

$$\int_{\mathbb{R}^N} e^{-t f_\varepsilon^{\frac{1}{2}}}\, \mathrm{d}\xi < +\infty \quad \textit{for any } t > 0. \tag{4.28}$$

Then the embedding $\mathcal{F}_\varepsilon \hookrightarrow L_0^2(\Omega)$ is compact. In particular, R_ε is a compact operator.

Proof Let $u \in \mathcal{F}_\varepsilon$ and $t > 0$. We define

$$e^{-t\sqrt{H_\varepsilon}} u := \mathfrak{F}^{-1}\left(e^{-t f_\varepsilon^{\frac{1}{2}}(\xi)}\widehat{u}(\xi)\right).$$

We divide the proof in two steps.

Step 1. We claim that for any $t > 0$ and $u \in \mathcal{F}_\varepsilon$

$$\left\| u - e^{-t\sqrt{H_\varepsilon}} u \right\|_{L^2(\mathbb{R}^N)} \leq t(\mathcal{E}^{(\varepsilon)}(u))^{\frac{1}{2}}.$$

Indeed by the Plancherel identity

$$\int_{\mathbb{R}^N} |u - e^{-t\sqrt{H_\varepsilon}} u|^2 \,\mathrm{d}x = \int_{\mathbb{R}^N} |1 - e^{-t(f_\varepsilon(\xi))^{\frac{1}{2}}}|^2 |\widehat{u}(\xi)|^2 \,\mathrm{d}\xi$$
$$\leq t^2 \int_{\mathbb{R}^N} |f_\varepsilon(\xi)||\widehat{u}(\xi)|^2 \,\mathrm{d}\xi.$$

Step 2. Let $\{u_n\}_{n\in\mathbb{N}}$ and u be such that $u_n \rightharpoonup u$ weakly in $\mathcal{F}_\varepsilon$ as $n \to \infty$. We claim that $u_n \to u$ strongly $L^2_0(\Omega)$ as $n \to \infty$. To this end we notice that

$$\|u - u_n\|_{L^2(\Omega)} \leq \left\| u - e^{-t\sqrt{H_\varepsilon}} u \right\|_{L^2(\Omega)} + \left\| u_n - e^{-t\sqrt{H_\varepsilon}} u_n \right\|_{L^2(\Omega)}$$
$$+ \left\| e^{-t\sqrt{H_\varepsilon}} (u - u_n) \right\|_{L^2(\Omega)}$$

and so by Step 1 and the fact that $\{u_n\}$ is bounded in $\mathcal{F}_\varepsilon$, it is enough to prove that

$$\lim_{n\to\infty} \left\| e^{-t\sqrt{H_\varepsilon}} (u - u_n) \right\|_{L^2(\Omega)} = 0. \tag{4.29}$$

We notice that for any $v \in \mathcal{F}_\varepsilon$

$$e^{-t\sqrt{H_\varepsilon}} v = \mathfrak{F}^{-1} \left(e^{-t f_\varepsilon^{\frac{1}{2}}(\xi)} \right) * v \tag{4.30}$$

and so by the Young inequality and (4.28)

$$\left\| e^{-t\sqrt{H_\varepsilon}} v \right\|_{L^\infty(\mathbb{R}^N)} \leq \|v\|_{L^2(\mathbb{R}^N)} \left\| e^{-t f_\varepsilon^{\frac{1}{2}}} \right\|_{L^2(\mathbb{R}^N)}.$$

Furthermore, since $u_n \rightharpoonup u$ weakly in $L^2_0(\Omega)$ as $n \to \infty$, by (4.30), it follows that

$$e^{-t\sqrt{H_\varepsilon}} u_n \to e^{-t\sqrt{H_\varepsilon}} u \quad \text{a.e. in } \mathbb{R}^N.$$

Then by Dominated Convergence Theorem we conclude that (4.29) holds. Since $R_\varepsilon : L^2_0(\Omega) \to \mathcal{F}_\varepsilon$ is continuous then is clear that $R_\varepsilon : L^2_0(\Omega) \to L^2_0(\Omega)$ is compact. □

Remark 4.9 The assumption (4.28) is actually related to Sobolev type inequalities for the operator $\sqrt{H_\varepsilon}$. Indeed, by the Young inequality

$$\left\| e^{-t\sqrt{H_\varepsilon}} v \right\|_{L^\infty(\mathbb{R}^N)} \leq \|v\|_{L^2(\mathbb{R}^N)} \left\| e^{-t f_\varepsilon^{\frac{1}{2}}} \right\|_{L^2(\mathbb{R}^N)},$$

for any $v \in L^2(\mathbb{R}^N)$. Hence, by [13, Theorem 2.4.2], for any $\mu > 2$, the Sobolev-type inequality

$$\|v\|^2_{L^{\frac{2\mu}{\mu-2}}(\mathbb{R}^N)} \leq C_1 \int_{\mathbb{R}^N} f_\varepsilon^{\frac{1}{2}}(\xi) |\widehat{v}(\xi)|^2 \,\mathrm{d}\xi$$

is equivalent to

$$\left\| e^{-tf_\varepsilon^{\frac{1}{2}}} \right\|_{L^2(\mathbb{R}^N)} \leq C_2 t^{-\frac{\mu}{4}}$$

with $C_1, C_2 > 0$.

As usual we denote with $\{\lambda_{\varepsilon,n}\}_{n\in\mathbb{N}}$ the spectrum of H_ε for $\varepsilon \in [0, 1]$. Similarly we denote with $\{\phi_{0,n}\}_{n\in\mathbb{N}}$ an orthonormal basis of eigenfunctions of H_0 of $L^2_0(\Omega)$.

Proposition 4.10 *For any $n \in \mathbb{N}$*

$$\lim_{\varepsilon\to 0^+} \lambda_{\varepsilon,n} = \lambda_{0,n}. \tag{4.31}$$

Proof Let us check that the assumptions of Theorem 3.8 hold. Following the notation of Theorem 3.8, it is enough to trivially define $P_\varepsilon = E_\varepsilon = \mathrm{Id}$, where Id is the identity functional. Furthermore, if we consider the bilinear form $(\mathcal{E}^{(1)}, \mathcal{F}_1)$, thanks to (4.26), for any $u \in L^2_0(\Omega)$

$$\mathcal{E}^{(1)}(R_\varepsilon u) \leq \frac{1}{C_1}\mathcal{E}^{(\varepsilon)}(R_\varepsilon u) = \frac{1}{C_1}(u, R_\varepsilon u)_{L^2(\Omega)} \leq \frac{1}{C_1}\|u\|^2_{L^2(\Omega)}$$

and, in view of Proposition 4.8, the associated self-adjoint operator H_ε has a compact resolvent $R_\varepsilon := H_\varepsilon^{-1}$. Furthermore if $R_{\varepsilon_n} u \rightharpoonup w$ weakly in $\mathcal{F}_1$ for some $w \in \mathcal{F}_1$ as $n \to \infty$, then, up to a subsequence,

$$\int_{\mathbb{R}^N} f_0 |\widehat{w}|^2 \,\mathrm{d}\xi \leq \liminf_{n\to\infty} \int_{\mathbb{R}^N} f_{\varepsilon_n} |\widehat{R_{\varepsilon_n} u}|^2 \,\mathrm{d}\xi \leq \frac{1}{C_1}\|u\|^2_{L^2(\Omega)},$$

by Fatou's Lemma and Proposition 4.8. Hence $w \in \mathcal{F}_0 = Z_0$. Since, up to a subsequence, there exists a function $g \in L^2(\mathbb{R}^N)$ such that $|\widehat{R_{\varepsilon_n} u}| \leq g$ a.e. in $\mathbb{R}^N$, by the Dominated Convergence Theorem

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} f_{\varepsilon_n} \widehat{R_{\varepsilon_n} u}\overline{\widehat{v}} \,\mathrm{d}\xi = \int_{\mathbb{R}^N} f_0 \widehat{w}\overline{\widehat{v}} \,\mathrm{d}\xi,$$

that is, we have proved that $\mathcal{E}^\varepsilon(R_{\varepsilon_n} u, v) \to \mathcal{E}^{(0)}(w, u)$ for any $u \in \mathcal{F}_0$. By Theorem 3.8, we conclude that (4.31) holds. □

Fix a simple eigenvalue $\lambda_0 := \lambda_{n_0,0}$ and a corresponding eigenfunction $\phi_0 := \phi_{n_0,0}$.

Proposition 4.11 *For any $u \in \mathcal{F}_\varepsilon$ we have that*

$$\mathcal{E}^{(\varepsilon)}(\phi_0, u) = \lambda_0 \int_\Omega \widehat{\phi_0}\overline{\widehat{u}} \,\mathrm{d}\xi + L_\varepsilon(u), \tag{4.32}$$

where

$$L_\varepsilon(u) = \int_{\mathbb{R}^N} (f_\varepsilon - f_0)\widehat{\phi_0}\overline{\widehat{u}} \,\mathrm{d}\xi.$$

Proof Let $u \in \mathcal{F}_\varepsilon$. Then clearly

$$\mathcal{E}^{(\varepsilon)}(\phi_0, u) = \int_{\mathbb{R}^N} f_0 \widehat{\phi_0}\overline{\widehat{u}} \,\mathrm{d}\xi + \int_{\mathbb{R}^N} (f_\varepsilon - f_0)\widehat{\phi_0}\overline{\widehat{u}} \,\mathrm{d}\xi$$

and the previous equation makes sense in view of Proposition 2.3. Let $\{u_n\}_{n\in\mathbb{N}} \subset C_c^\infty(\Omega)$ be a sequence of functions such that $u_n \to u$ strongly in $L^2(\Omega)$. Hence, by Proposition 2.3,

$$\int_{\mathbb{R}^N} f_0\widehat{\phi}_0\overline{\widehat{u}}\, \mathrm{d}\xi = \lim_{n\to\infty}\int_{\mathbb{R}^N} f_0\widehat{\phi}_0\overline{\widehat{u_n}}\, \mathrm{d}\xi = \lim_{n\to\infty}\lambda_0\int_{\mathbb{R}^N}\phi_0 u_n\, \mathrm{d}x = \lambda_0\int_{\mathbb{R}^N}\phi_0 u\, \mathrm{d}\xi$$

which proves (4.32). □

Let us define

$$J_\varepsilon(u) = \frac{1}{2}\mathcal{E}^{(\varepsilon)}(u) - L_\varepsilon(u).$$

Then, in view of Proposition 2.5, there exists a function $V_\varepsilon \in \mathcal{F}_\varepsilon$ that minimizes J_ε over $\mathcal{F}_\varepsilon$.

Proposition 4.12 *We have that*

$$\lim_{\varepsilon\to 0^+}\mathcal{E}^{(\varepsilon)}(V_\varepsilon) = 0. \tag{4.33}$$

Proof Testing (2.10) with V_ε we obtain

$$\mathcal{E}^{(\varepsilon)}(V_\varepsilon) \le \left(\int_{\mathbb{R}^N}|f_\varepsilon - f_0|^2|\widehat{\phi}_0|^2\, \mathrm{d}\xi\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}^N} f_\varepsilon|V_\varepsilon|^2\, \mathrm{d}\xi\right)^{\frac{1}{2}}, \tag{4.34}$$

in view of the Hölder inequality and the fact that $f_\varepsilon > 1$. By Proposition 2.3, the Dominated Convergence Theorem, and (4.27) we conclude that (4.33) holds. □

In conclusion, we are in position to apply Theorem 3.1 thus obtaining

$$\lambda_\varepsilon - \lambda_0 = \lambda_0\int_\Omega \phi_0 V_\varepsilon\, \mathrm{d}x + O(\|V_\varepsilon\|^2_{L^2(\Omega)}) \quad \text{as } \varepsilon \to 0^+,$$

in view of Remark 3.3.

If the rate of convergence of $f_\varepsilon \to f_0$ can be quantified, we can compute the vanishing order of $\lambda_0 \int_\Omega \phi_0 V_\varepsilon\, dx$. More precisely, we assume that there exists $\varepsilon_0 \in (0, 1]$ such that

$$\lim_{\varepsilon\to 0^+}\frac{f_\varepsilon(\xi) - f_0(\xi)}{\varepsilon} = h(\xi), \tag{4.35}$$

$$\left|\frac{f_\varepsilon(\xi) - f_0(\xi)}{\varepsilon}\right| \le C(1 + |h(\xi)|) \quad \text{for a.e. } \xi \in \mathbb{R}^N, \text{ for any } \varepsilon \in (0, \varepsilon_0), \tag{4.36}$$

and

$$|h(\xi)|^2 \le Cf_0(\xi)^2 f_\varepsilon(\xi) \quad \text{for a.e. } \xi \in \mathbb{R}^N, \text{ for any } \varepsilon \in [0, \varepsilon_0) \tag{4.37}$$

for some measurable functions $h : \mathbb{R}^N \to \mathbb{R}$ and constant $C > 0$. By the Lagrange Theorem, the assumption $\left|\frac{f_\varepsilon(\xi)-f_0(\xi)}{\varepsilon}\right| \le C(1 + |h(\xi)|)$ is verified, when, for example, $\varepsilon \to f_\varepsilon(\xi)$ is derivable for any $\varepsilon \in (0, \varepsilon_0)$, and $\left|\frac{\partial f_\varepsilon}{\partial\varepsilon}(\xi)\right| \le C(1 + |h(\xi)|)$ for some constant $C > 0$ that does not depend on ε.

Proposition 4.13 *There exists* $V \in \mathcal{F}_0$ *such that* $\frac{V_\varepsilon}{\varepsilon} \rightharpoonup V$ *weakly in* $\mathcal{F}_1$ *and* V *solves the equation*

$$\int_{\mathbb{R}^N} f_0\widehat{V}\overline{\widehat{u}}\, \mathrm{d}\xi = \int_{\mathbb{R}^N} h\widehat{\phi}_0\overline{\widehat{u}}\, \mathrm{d}\xi, \quad \text{for any } u \in \mathrm{Dom}[H_0].$$

Proof Thanks to (4.26), (4.35), Proposition 2.3, and (4.34) the family $\left\{\frac{V_\varepsilon}{\varepsilon}\right\}_{\varepsilon\in[0,1]}$ is bounded in $\mathcal{F}_1$. In particular there exists a sequence $\varepsilon_n \to 0^+$ and $V \in \mathcal{F}_1$ such that $\frac{V_\varepsilon}{\varepsilon} \rightharpoonup V$ weakly in $\mathcal{F}_1$ as $n \to \infty$ and so, by Proposition 2.1, $\frac{V_\varepsilon}{\varepsilon} \to V$ strongly in $L^2(\mathbb{R}^N)$.

Equation (2.10) in this case is

$$\int_{\mathbb{R}^N} f_\varepsilon \widehat{V_\varepsilon} \overline{\widehat{u}}\, \mathrm{d}\xi = \int_{\mathbb{R}^N} (f_\varepsilon - f_0)\widehat{\phi_0}\overline{\widehat{u}}\, \mathrm{d}\xi \quad \text{for any } u \in \mathcal{F}_\varepsilon. \tag{4.38}$$

Furthermore, up to pass to a further subsequence, by Fatou's Lemma,

$$\int_{\mathbb{R}^N} f_0 |\widehat{V}|^2\, d\xi \le \liminf_{n\to\infty} \int_{\mathbb{R}^N} f_\varepsilon \frac{|\widehat{V_{\varepsilon_n}}|^2}{{\varepsilon_n}^2}\, \mathrm{d}\xi \le \liminf_{n\to\infty} \int_{\mathbb{R}^N} \frac{|f_{\varepsilon_n} - f_0|}{\varepsilon_n} \frac{|\widehat{V_{\varepsilon_n}}|}{\varepsilon_n} |\widehat{\phi_0}|\, \mathrm{d}\xi$$

and hence, by the Cauchy-Schwarz inequality,

$$\liminf_{\varepsilon_n\to 0^+} \int_{\mathbb{R}^N} f_{\varepsilon_n} \frac{|\widehat{V_{\varepsilon_n}}|^2}{{\varepsilon_n}^2}\, \mathrm{d}\xi \le \lim_{n\to\infty} \left(\int_{\mathbb{R}^N} \frac{(f_0 - f_{\varepsilon_n})^2}{\varepsilon_n^2} f_{\varepsilon_n}^{-1} |\widehat{\phi_0}|^2\, \mathrm{d}\xi \right)^{\frac{1}{2}}.$$

We conclude that $V \in \mathcal{F}_0$ in view of (4.35), (4.36), (4.37) and Proposition 2.3. Multiplying by ε_n^{-1} and passing to the limit as $n \to \infty$ in (4.38) we obtain

$$\int_{\mathbb{R}^N} f_0 \widehat{V} \overline{\widehat{u}}\, \mathrm{d}\xi = \int_{\mathbb{R}^N} h \widehat{\phi_0} \overline{\widehat{u}}\, \mathrm{d}\xi \quad \text{for any } u \in \mathcal{F}_0,$$

thanks to (4.35), (4.36), (4.37) and the Dominated Convergence Theorem. Since the solution of the above equation is unique, we conclude that $\frac{V_\varepsilon}{\varepsilon} \rightharpoonup V$ weakly in $\mathcal{F}_1$ by the Urysohn Subsequence Principle. □

In conclusion, by the Plancherel identity and Proposition 4.13,

$$\frac{\lambda_0}{\varepsilon} \int_\Omega \phi_0 V_\varepsilon\, \mathrm{d}x = \lambda_0 \int_{\mathbb{R}^N} \widehat{\phi_0} \frac{\overline{\widehat{V_\varepsilon}}}{\varepsilon}\, \mathrm{d}\xi \to \lambda_0 \int_{\mathbb{R}^N} \widehat{\phi_0} \overline{\widehat{V}}\, \mathrm{d}\xi = \int_{\mathbb{R}^N} f_0 \widehat{\phi_0} \overline{\widehat{V}}\, \mathrm{d}\xi = \int_{\mathbb{R}^N} h |\widehat{\phi_0}|^2\, \mathrm{d}\xi.$$

Hence

$$\lambda_\varepsilon - \lambda_0 = \varepsilon \int_{\mathbb{R}^N} h |\widehat{\phi_0}|^2\, \mathrm{d}\xi + O(\varepsilon^2) \quad \text{as } \varepsilon \to 0^+.$$

Example 4.14 Let $\phi_{n,0}$ be the eigenfunction associated to the simple eigenvalue $\lambda_{n,0}$ of the Dirichlet Laplacian on a bounded domain Ω. Choosing the symbol $f_\varepsilon(\xi) := 1 + |\xi|^{2-2\varepsilon}$ for any $\varepsilon \in [0, 1]$, we have that $h(\xi) = -2\log(|\xi|)|\xi|^2$, where h is as in (4.35). Furthermore

$$\left| \frac{\partial f_\varepsilon(\xi)}{\partial \varepsilon} \right| = -2\log(|\xi|)|\xi|^{2-2\varepsilon} \le C(1 + 2|\log(|\xi|)|\,|\xi|^2), \quad \text{for any } \xi \in \mathbb{R}^N$$

for some positive constant $C > 0$. We conclude that (4.35) holds.

Hence we have the following asymptotic for any simple eigenvalue $\lambda_{n,\varepsilon}$ of the fractional Laplacian $(-\Delta)^{1-\varepsilon}$ on a bounded domain with Dirichlet boundary conditions:

$$\lambda_{n,\varepsilon} - \lambda_{n,0} = -2\varepsilon \int_{\mathbb{R}^N} \log(|\xi|)|\xi|^2 |\widehat{\phi_{n,0}}(\xi)|^2\, \mathrm{d}\xi + O(\varepsilon^2) \quad \text{as } \varepsilon \to 0^+.$$

In conclusion, we have quantified the spectral stability result obtained in [8] in the special case of the fractional Laplacian.

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