The Scalar Curvature Problem on S^n: an approach via Morse Theory

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Abstract

We prove the existence of positive solutions for the equation on S^n $-4\frac{(n-1)}{(n-2)}\Delta_{g_0}u + n(n-1)u = (1+\varepsilon K_0(x))u^{2^*-1}$, where Δ_{g_0} is the Laplace-Beltrami operator on S^n, 2^* is the critical Sobolev exponent, and ε is a small parameter. The problem can be reduced to a finite dimensional study which is performed via Morse theory.

Key words: Elliptic equations, Critical Sobolev exponent, Scalar Curvature, Perturbation method, Morse Theory.

1 Introduction

If (M, g) is a Riemannian manifold of dimension $n \geq 3$, with scalar curvature R, and one considers the conformal metric $g' = u^{\frac{4}{n-2}}g$, where u is a smooth positive function, then the scalar curvature R' of (M, g') is given by the following relation, see [3]

$$-4\frac{(n-1)}{(n-2)}\Delta_g u + Ru = R'u^{\frac{n+2}{n-2}}. \tag{1.1}$$

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Here Δ_g denotes the Laplace-Beltrami operator on (M, g). We also recall that for $n = 2$, if one sets $g' = e^{2u}g$, then

$$-\Delta_g u + R = R' e^u. \tag{1.2}$$

The problem of *prescribing scalar curvature* is the following: assigned a function S on M, one looks for a metric g' conformal to g, for which $R' \equiv S$. Equivalently, one has to find a positive solution u to equation (1.1) or (1.2). This problem is quite delicate: for example, in [14] or [19] some non existence results on S^n and on $\mathbb{R}^n$ are shown. The Scalar Curvature Problem on S^n has been mainly faced under two types of assumptions

1) Assumptions of global type

2) Assumptions at prescribed levels

In the case *1)*, the hypotheses involve the critical points of S at all levels. Roughly, a typical result says that a solution exists provided S is a Morse function with $\Delta S \neq 0$ at every critical point, and

$$\sum_{x \in Crit(S), \Delta S(x) < 0} (-1)^{m(S,x)} \neq (-1)^n. \tag{1.3}$$

Here $m(S, x)$ is the Morse index of S at x. For $n = 2$ this result has been given in [9], and in [4] for $n = 3$, see also [7]. For $n \geq 4$ the situation is more delicate, and, in general one has to require a flatness condition. More precisely, see [15], [16], for every $x_i \in Crit(S)$, it is assumed to exist $\beta_i \in (n-2, n)$ such that in some orthonormal coordinates (y_j) centered at x_i it is

$$S(y) = S(0) + \sum_{j=1}^{n} a_j |y_j|^{\beta_i} + o(|y|^{\beta_i}), \tag{1.4}$$

with $a_j \neq 0$ and $\sum_{j=1}^n a_j \neq 0$. Suitable flatness conditions on the derivatives of S are also required. For every $x \in Crit(S)$, set $\tilde{\Delta} S(x) = \sum_{j=1}^n a_j$, and $i(x) = \sharp a_j : a_j < 0$. Then solutions of (1.1) are obtained provided

$$\sum_{x \in Crit(S), \tilde{\Delta} S(x) < 0} (-1)^{i(x)} \neq (-1)^n. \tag{1.5}$$

The case *2)* deals with assumptions at some prescribed levels of S. Typically, S must possess two maxima x_0 and x_1 which are connected by some path $x(t)$, and

$$x \text{ saddle point for } S, \ \inf_t S(x(t)) \leq S(x) < S(x_0) \ \Rightarrow \ \Delta S(x) > 0. \tag{1.6}$$

Results of this kind have been obtained in [8], [13] for $n = 2$, and in [5] for $n \geq 3$.

Morse Theory has been used in [12] for $n = 2$, and in [20] for $n = 3$. In particular, in [20] it is shown that a solution of (1.1) exists provided S is a Morse function and

$$D_0 - D_1 + D_2 \neq 1, \qquad \text{or} \qquad D_0 - D_1 > 1. \tag{1.7}$$

Here $D_q = \sharp\{x \in Crit(S) : m(S,x) = 3 - q, \Delta S(x) < 0\}$. Note that the first condition in (1.7) is equivalent to (1.3).

In our paper we consider the case $(M, g) = (S^n, g_0)$, $n \geq 3$, and S close to a constant, i.e. S of the form $S = 1 + \varepsilon K_0(x)$, for $|\varepsilon|$ small. So we are reduced to study the problem on S^n

$$-4\frac{(n-1)}{(n-2)}\Delta_{g_0} u + n(n-1)u = (1 + \varepsilon K_0)u^{\frac{n+2}{n-2}}, \quad u > 0. \tag{1.8}$$

Our main results are given in Section 4. The first one, Theorem 4.1, deals with case *1)*. Under suitable non-degeneracy assumptions on K_0, existence of solutions is found if

$$\sum_{j=0}^{q}(-1)^{q-j}F_j - (-1)^q \leq -1, \qquad \text{for some } q = 1, \dots, n-1. \tag{1.9}$$

Here $F_j = \sharp\{x \in Crit(K_0) : m(K_0, x) = j, \Delta K_0(x) > 0\}$. When $\varepsilon < 0$, Theorem 4.1 extends, in the perturbative setting, the results in [20] to all dimensions, see Remark 4.2. Our second main result, Theorem 4.4, and its generalization Theorem 4.6, deals with case *2)*. The main difference with respect to [5] is that we require condition (1.6) to hold just for the saddle point of Morse index $n - 1$. Remark 4.5 gives precise comparisons with the results in [5], [8], and [13].

Our approach follows that of [1] and [2], where functionals of the form $f_\varepsilon = f_0 - \varepsilon G$, $|\varepsilon|$ small, are studied. In particular, also [2] deals with problem (1.8), and recovers existence under condition (1.5) for an order of flatness $\beta \in (1, n)$. See also [10] for other perturbation results.

In the present case f_0 possesses a manifold of critical points $Z \sim \mathbb{R}_+ \times \mathbb{R}^n = \{(\mu, \xi) : \mu > 0, \xi \in \mathbb{R}^n\}$. One can show that Z perturbs to another manifold $Z_\varepsilon \simeq Z$ which is a natural constraint for f_ε. Moreover, it turns out that $f_\varepsilon|_{Z_\varepsilon} = b_0 - \varepsilon G(z) + o(\varepsilon)$, where b_0 is a constant. In this way, one is led to study the finite-dimensional functional $\Gamma = G|_Z$. In Proposition 3.5 it is shown that, from the properties on Γ at $\mu = 0$ and at infinity, we can apply Morse Theory under general boundary condition, see [18]. Using this technique, we can treat the cases *1)* and *2)* with the same approach.

In Section 5 we state some generalizations of the above discussed results, which also include conditions of the type (1.5).

The above results are stated in the preliminary note [17].

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Notations

We will work mainly in the space

$$E = D^1(\mathbb{R}^n) = \left\{ u \in L^{2^*}(\mathbb{R}^n) : \int |\nabla u|^2 < +\infty \right\},$$

which coincides with the completion of $C_c^\infty(\mathbb{R}^n)$ with respect to the Dirichlet norm. Given a function $f : X \to \mathbb{R}$, where X is an Hilbert space or a Riemannian manifold, we denote with f' or with ∇f its gradient, and we set $Crit(f) = \{x : f'(x) = 0\}$; if f is of class C^2, and if $x \in Crit(f)$, $m(f, x)$ is the Morse index of f at x. Given $a, b \in \mathbb{R}$, we set also $f^a = \{x \in X : f(x) \leq a\}$, and $f_a^b = \{x \in X : a \leq f(x) \leq b\}$. $B_r^m(y)$ stands for the m-dimensional closed ball of radius r centred at $y \in \mathbb{R}^m$, while B_R is $B_R = \{u \in E \ : \ \|u\| \leq R\}$. Embedding S^n in $\mathbb{R}^{n+1}$ as $S^n = \{x \in \mathbb{R}^{n+1} : \|x\| = 1\}$, we denote by $\sigma : S^n \to \mathbb{R}^n$ the stereographic projection through the north pole P_N of S^n, $P_N = (0, \dots, 0, 1)$, and we define $\mathcal{R} : S^n \to S^n$ to be the reflection through the hyperplane $x_{n+1} = 0$. Given $y \in \mathbb{R}_+^{n+1}$, we denote by $y_1, \dots, y_{n+1}$ its components. The function $\Pi : \mathbb{R}_+^{n+1} \to \mathbb{R}^n$ denotes the projection onto the last n coordinates, and $\bar{\mathbb{R}}_+^{n+1}$ is the closure of $\mathbb{R}_+^{n+1}$.

2 Preliminaries

The abstract perturbation method

In this section we recall the abstract perturbation method developed in [1]. Let E be an Hilbert space, and let $f_0, G \in C^2(E, \mathbb{R})$. Our aim is to find critical points of the perturbed functional

$$f_\varepsilon(u) = f_0(u) - \varepsilon\, G(u), \qquad u \in E. \tag{2.1}$$

The fundamental tool is the following Theorem (see [1], Lemmas 2 and 4).

Theorem 2.1 *Suppose f_0 satisfies the following conditions*

f1) f_0 possesses a finite dimensional manifold of critical points Z; let $b_0 = f_0(z)$, for all $z \in Z$;

f2) $f_0''(z)$ is a Fredholm operator of index zero for all $z \in Z$;

f3) for all $z \in Z$, it is $T_z Z = Ker f_0''(z)$.

Then, given $R > 0$, there exist $\varepsilon_0 > 0$, and a C^1 function $w = w(z, \varepsilon) : N = Z \cap B_R \times (-\varepsilon_0, \varepsilon_0) \to E$ which satisfies the following properties

i) $w(z, 0) = 0$ for all $z \in Z \cap B_R$;

ii) $w(z, \varepsilon)$ is orthogonal to $T_z Z$, $\qquad \forall (z, \varepsilon) \in N$;

iii) the manifold

$$Z_\varepsilon = \{z + w(z, \varepsilon) : z \in Z \cap B_R\}$$

is a natural constraint for f'_ε, namely: if $u \in Z_\varepsilon$ and $f'_\varepsilon|_{Z\varepsilon}(u) = 0$, then it is also $f'_\varepsilon(u) = 0$.

The inclusion $T_z Z \subseteq Ker f''_0(z)$ is always true: *f3)* is a non-degeneracy condition which allows to apply the Implicit Function Theorem. Since in our case the elements of Z are positive functions, one can deduce that the critical points of f_ε on Z_ε are non-negative functions. Using standard regularity arguments and the maximum principle, see for example [21] Appendix B, it is possible to prove that the critical points of $f_\varepsilon|_{Z_\varepsilon}$ are actually regular and positive functions.

Morse theory for manifolds with boundary

For a complete treatment about this topic we refer to [11], where a refinement of the theory in [18] is presented. Let M be a Riemannian manifold, and let $f \in C^1(M)$. If p is an isolated critical point of f, with $f(p) = c$, and if $q \geq 0$, the q^{th} *critical group* of f at p is

$$C_q(f, p) = H_q(f^c \cap U_p, (f^c \setminus \{p\}) \cap U_p),$$

where U_p is a neighborhood of p such that $Crit(f) \cap U_p = \{p\}$, and H_q are the singular homology groups. By the excision property, the critical groups are well defined, i.e. they do not depend on U_p. If f is of class C^2 and p is a non-degenerate critical point, then clearly $C_q(f, p) \simeq \mathbb{Z}$ for $q = m(f, p)$, and $C_q(f, p) = 0$ otherwise. If M possesses a smooth boundary $\partial M = \Sigma$, which is an oriented submanifold with codimension 1, then the outward unit normal $\nu(x)$ at $x \in \Sigma$ is well defined.

Definition 2.2 *$f \in C^1(M)$ is said to satisfy the* general boundary condition *on f_a^b if the following two properties hold*

(i) $Crit(f) \cap (\Sigma \cap f_a^b) = \emptyset$;

(ii) the restriction $f|_\Sigma \cap f_a^b$ has only isolated critical points.

Let $(\Sigma_-)_a^b = \left\{x \in \Sigma \cap f_a^b : (f'(x), \nu(x)) \leq 0\right\}$, and suppose that f has only isolated critical points in f_a^b. Let $\{x_1, \ldots, x_j, \ldots\}$ be the critical points of f_a^b, and $\{y_1, \ldots, y_j, \ldots\}$ those of $f|_{(\Sigma_-)_a^b}$; the *Morse type numbers* of f on f_a^b and on $(\Sigma_-)_a^b$ are respectively defined as follows:

$$m_q = \sum_{j=1}^{\infty} rank\ C_q(f, x_j), \qquad q = 0, 1, 2, \ldots,$$

$$\mu_q = \sum_{j=1}^{\infty} rank\ C_q(f|_{\Sigma_-}, y_j), \qquad q = 0, 1, 2, \ldots.$$

The *augmented Morse type numbers* are

$$M_q = m_q + \mu_q, \qquad q = 0, 1, 2, \dots.$$

The following Theorem is a version of the Morse inequalities for manifolds with boundary.

Theorem 2.3 *Let $f \in C^1(M)$, and let a, b be regular values for f and for $f|_\Sigma$. Suppose that f has only isolated critical points and that satisfies the general boundary condition on f_a^b. Then the following version of the Morse inequalities holds*

$$\sum_{j\geq 0} M_j t^j = \sum_{j\geq 0} \beta_j t^j + (1+t)Q(t), \tag{2.2}$$

where $\beta_q = rank\ H_q(f^b, f^a)$, $j = 0, 1, 2, \dots,$ and $Q(t)$ is a formal power series with non-negative coefficients.

We recall that the meaning of (2.2) is the following

$$\sum_{j=0}^{q}(-1)^{q-j} M_j \geq \sum_{j=0}^{q}(-1)^{q-j}\beta_j, \qquad q = 0, 1, 2, \dots; \tag{2.3}$$

$$\sum_{q=0}^{\infty}(-1)^q M_q = \sum_{q=0}^{\infty}(-1)^q \beta_q. \qquad q = 0, 1, 2, \dots\ . \tag{2.4}$$

3 Application to the scalar curvature problem

Solutions of equation (1.1), with $(M, g) = (S^n, g_0)$, can be obtained by variational methods, as critical points of the functional

$$J(u) = 2\frac{(n-1)}{(n-2)}\int_{S^n} |\nabla_{g_0} u|^2 + \frac{1}{2}n(n-1)\int_{S^n} u^2 - \frac{1}{2^*}\int_{S^n} S|u|^{2^*}, \qquad u \in H^1(S^n), \tag{3.1}$$

where $2^* = \frac{2n}{(n-2)}$ is the critical Sobolev exponent. For $n \geq 3$, let $E = D^1(\mathbb{R}^n)$, and denote with

$$z_0 = c_n \frac{1}{(1+|x|^2)^{\frac{n-2}{2}}}, \qquad c_n = [4n(n-1)]^{\frac{n-2}{4}}, \tag{3.2}$$

the unique (up to dilation and translation) solution to the problem

$$\begin{cases} -4\frac{(n-1)}{(n-2)}\Delta u = u^{\frac{n+2}{n-2}} & \text{in } \mathbb{R}^n \\ u > 0,\ u \in E. \end{cases} \tag{3.3}$$

The stereographic projection $\sigma : S^n \to \mathbb{R}^n$ induces an isomorphism $\iota : H^1(S^n) \to E$ given by $(\iota u)(x) = z_0(x)u(\sigma^{-1}(x))$, and $J(u) = f(\iota u)$, where

$$f(u) = 2\frac{(n-1)}{(n-2)}\int_{\mathbb{R}^n} |\nabla u|^2 - \frac{1}{2^*}\int_{\mathbb{R}^n} \bar{S}|u|^{2^*}, \qquad u \in E.$$

Here $\bar{S} : \mathbb{R}^n \to \mathbb{R}$ is the function given by $\bar{S}(x) = S(\sigma^{-1}(x))$.

Since we consider the case $S = 1 + \varepsilon K_0$, it is $\bar{S}(x) = 1 + \varepsilon K(x)$, with $K(x) = K_0(\sigma^{-1}(x))$. Thus we are reduced to find solutions of

$$\begin{cases} -4\frac{(n-1)}{(n-2)}\Delta u = (1+\varepsilon K(x))u^{\frac{n+2}{n-2}} & \text{in } \mathbb{R}^n \\ u > 0, \ u \in E. \end{cases} \tag{3.4}$$

Throughout this paper, it will be always understood that $K(x)$ originates from a smooth function K_0 defined on S^n, and we suppose that $\nabla_{g_0} K_0(P_N) \neq 0$.

This problem has been recently tackled in [2] by using an abstract perturbation result developed in [1]. $f(u)$ can be written as $f(u) = f_\varepsilon(u) = f_0(u) - \varepsilon\, G(u)$, where

$$f_0(u) = 2\frac{(n-1)}{(n-2)}\int_{\mathbb{R}^n} |\nabla u|^2 - \frac{1}{2^*}\int |u|^{2^*}; \qquad G(u) = \frac{1}{2^*}\int_{\mathbb{R}^n} K|u|^{2^*}, \quad u \in E.$$

The functional f_0 possesses a manifold Z of critical points given by

$$Z = \left\{ z_{\mu,\xi} = \mu^{-\frac{n-2}{2}} z_0\left(\frac{x-\xi}{\mu}\right), \mu > 0, \xi \in \mathbb{R}^n \right\} \simeq \mathbb{R}_+ \times \mathbb{R}^n.$$

Z is an $(n+1)$-dimensional manifold which is homeomorphic to the half space $\mathbb{R}^{n+1}_+ = \{(\mu,\xi) : \mu > 0, \xi \in \mathbb{R}^n\}$, so hypothesis *f1)* in Theorem 2.1 is satisfied with $b_0 = f_0(z_0)$. Condition *f2)* holds too, since $f_0''(z) = I - C, C$ compact for every $z \in Z$, while *f3)* is consequence of the following Lemma (see [2] or [6]).

Lemma 3.1 *For every $z_{\mu,\xi} \in Z$, it is $T_{z_{\mu,\xi}} Z = Ker f_0''(z_{\mu,\xi})$. Namely if $u \in E$ solves*

$$-4\frac{(n-1)}{(n-2)}\Delta u = (2^*-1)\, z_{\mu,\xi}^{2^*-2} u,$$

then there holds

$$u = \alpha D_\mu z_{\mu,\xi} + \langle \nabla_x z_{\mu,\xi}, \beta\rangle, \quad \textit{for some } \alpha \in \mathbb{R},\ \beta \in \mathbb{R}^n.$$

By Theorem 2.1 *iii)*, critical points of $f_\varepsilon|_{Z_\varepsilon}$ are also critical points of f_ε. The following Proposition, proved in [2], is very useful for the study of the reduced functional $f_\varepsilon|_{Z_\varepsilon}$.

Proposition 3.2 *Setting $\varphi_\varepsilon(\mu,\xi) := f_\varepsilon(z_{\mu,\xi} + w(\varepsilon, z_{\mu,\xi}))$, there holds*

$$\varphi_\varepsilon(\mu,\xi) = b_0 - \varepsilon\Gamma(\mu,\xi) + o(\varepsilon), \quad \varphi'_\varepsilon(\mu,\xi) = -\varepsilon\Gamma'(\mu,\xi) + o(\varepsilon), \quad \textit{as } \varepsilon \to 0, \tag{3.5}$$

where $\varepsilon^{-1}o(\varepsilon) \to 0$ as $\varepsilon \to 0$ uniformly on the compact sets of $\mathbb{R}^{n+1}_+$, and

$$\Gamma(\mu,\xi) = \frac{1}{2^*}\int K(x) z_{\mu,\xi}^{2^*}(x)dx. \tag{3.6}$$

Since $w(\varepsilon, z_{\mu,\xi})$ is constructed using local inversion theorem, the functions φ_ε is defined only on a compact set of $\mathbb{R}^{n+1}_+$, depending on ε. However, as $\varepsilon \to 0$, the domain of φ_ε invade all of $\mathbb{R}^{n+1}_+$. The behavior of the function Γ has been studied in [2]: we collect the main features of this study in the following Proposition .

Proposition 3.3 *The function Γ is of class C^2, and can be extended to the hyperplane $\{\mu = 0\}$ by setting*

$$\Gamma(0,\xi) = \frac{1}{2^*}\int K(\xi) z_0^{2^*}(x)dx \equiv c_0 K(\xi), \tag{3.7}$$

where $2^*\, c_0 = \int z_0^{2^*}$. Moreover the first and the second derivatives of Γ at $\{\mu = 0\}$ are given by

$$\Gamma_\mu(0,\xi) = 0, \quad \Gamma_{\mu\xi_i}(0,\xi) = 0, \quad \Gamma_{\mu\mu}(0,\xi) = c_1 \Delta K(\xi); \qquad \forall \xi \in \mathbb{R}^n, \tag{3.8}$$

where $n\, 2^*\, c_1 = \int |x|^2 z_0^{2^*}(x)dx$.

It is also useful to study the function Γ at *infinity*, i.e. for $\mu + |\xi|$ large; this can be done by using the Kelvin transform $y \to \frac{y}{|y|^2}$ in $\mathbb{R}^n$. Define the function $\hat{K} : \mathbb{R}^n \to \mathbb{R}$ as $\hat{K}(x) = K\left(\frac{x}{|x|^2}\right)$, and consider the functional $\hat{\Gamma}(\mu,\xi) = \frac{1}{2^*}\int \hat{K}(x) z_{\mu,\xi}^{2^*}(x)dx$, which is the counterpart of $\Gamma(\mu,\xi)$ for the function $\hat{K}$.

Lemma 3.4 *There holds*

$$\Gamma(\mu,\xi) = \hat{\Gamma}(\bar{\mu},\bar{\xi}), \qquad \bar{\mu} = \frac{\mu}{\mu^2+\xi^2},\ \bar{\xi} = \frac{\xi}{\mu^2+\xi^2},$$

i.e. the function Γ modifies by means of a Kelvin transform in $\mathbb{R}^{n+1}_+$.

Proof. It is immediate to check that, from the relation between K and K_0, there holds $K_0 \circ \mathcal{R} = \hat{K} \circ \sigma$. If one substitutes K with $\hat{K}$, and considers the corresponding functionals $\hat{f}, \hat{G}$, it can be easily deduced that $f(u) = \hat{f}(u^*), G(u) = \hat{G}(u^*)$, where $u \in E$ and $u^*(x) = |x|^{2-n} u\left(\frac{x}{|x|^2}\right)$. An easy computation shows that

$$(z_{\mu,\xi})^* = z_{\bar{\mu},\bar{\xi}}, \qquad \bar{\mu} = \frac{\mu}{\mu^2+\xi^2},\ \bar{\xi} = \frac{\xi}{\mu^2+\xi^2}.$$

Since $\Gamma(\mu,\xi) = G(z_{\mu,\xi})$, it follows that

$$\Gamma(\mu,\xi) = G(z_{\mu,\xi}) = \hat{G}(z_{\bar{\mu},\bar{\xi}}) = \hat{\Gamma}(\bar{\mu},\bar{\xi}),$$

and this concludes the proof of the Lemma. □

Since $(\bar{\mu}, \bar{\xi}) \to (0,0)$ when $\mu + |\xi| \to +\infty$, the problem of studying Γ at infinity becomes equivalent to study $\hat{\Gamma}$ near $(0,0)$.

Proposition 3.5 *Suppose $K \in C^2(\mathbb{R}^n)$ is a Morse function such that*

$$(L) \qquad x \in Crit(K) \quad \Rightarrow \quad \Delta K(x) \neq 0.$$

For $s > 0$, let $\tilde{B}_s = B^{n+1}_{\frac{s^2-1}{2s}}\left(\frac{s^2+1}{2s}, 0\right)$. Then, for s sufficiently large, Γ satisfies the general boundary condition on $\tilde{B}_s$.

Proof. Note that $\partial\tilde{B}_s$ is the n-dimensional sphere centred on the axis $\xi = 0$, which intersects this axis at the points s and $\frac{1}{s}$. In particular, it follows that $\partial\tilde{B}_s$ is invariant under the Kelvin transform $x \to \frac{x}{|x|^2}$ in $\mathbb{R}^{n+1}$. It is clear that $\Gamma \in C^1(\tilde{B}_s)$. We will prove (i) of Definition 2.2 by estimating the component of $\nabla\Gamma$ normal to $\partial\tilde{B}_s$ near the points $(0,\xi)$, $\xi \in Crit(K)$, and the tangent component on the remainder of $\partial\tilde{B}_s$. From expression (3.6), using the change of variables $x = \mu y + \xi$, one infers that $\Gamma(\mu,\xi) = \frac{1}{2^*}\int K(\mu y + \xi) z_0^{2^*}(y)dy$; the Dominated Convergence Theorem implies

$$\Gamma_{\xi_i}(\mu,\xi) \to K_{\xi_i}(\xi), \quad \Gamma_{\xi_i\xi_j}(\mu,\xi) \to K_{\xi_i\xi_j}(\xi) \qquad \text{as } \mu \to 0, \tag{3.9}$$

uniformly for ξ in a fixed compact subset of $\mathbb{R}^n$. Fixed $r > 0$, $\partial\tilde{B}_s \cap B^{n+1}_{2r}(0)$ is the graph of a smooth function $h_s : \Pi(\partial\tilde{B}_s \cap B^{n+1}_{2r}(0)) \to \mathbb{R}$ for s large. Moreover, $h_s \to 0$ in the C^2 norm as $s \to +\infty$, so from (3.9) it follows that

$$\nabla\Gamma(x, h_s(x)) \to (0, \nabla K(x)), \qquad \text{as } s \to +\infty,$$

for $x \in B^n_r(0)$, since $B^n_r(0) \subseteq \Pi(\partial\tilde{B}_s \cap B^{n+1}_{2r}(0))$. In particular we deduce that

$$x \in B^n_r(0), \nabla K(x) \neq 0 \;\Rightarrow\; \nabla_\xi\Gamma(x, h_s(x)) \neq 0 \quad \text{for } s \text{ large.} \tag{3.10}$$

From equation (3.8) we have also

$$\Gamma_\mu(x, h_s(x)) = c_1 \Delta K(x)\, h_s(x)\,(1 + o(1)), \qquad \forall x \in B^n_r(0),$$

where $o(1) \to 0$ uniformly as $s \to +\infty$. Hence

$$x \in B^n_r(0), \Delta K(x) \neq 0 \;\Rightarrow\; \nabla_\mu\Gamma(x, h_s(x)) \neq 0 \quad \text{for } s \text{ large.} \tag{3.11}$$

From condition (L) and using (3.10) and (3.11), it follows that $\nabla\Gamma(h_s(x), x) \neq 0$ for every $x \in B^n_r$. To prove (i) it is sufficient to show that $\nabla\Gamma$ is different from 0 on $\partial\tilde{B}_s$ also for $\mu + |\xi|$ large.

Since, as noticed before, $\tilde{B}_s$ is invariant under the Kelvin transform, the problem of studying $\nabla\Gamma$ on $\partial\tilde{B}_s$ at infinity becomes equivalent to study $\nabla\hat{\Gamma}$ on $\partial\tilde{B}_s$ near the origin. In particular, since

$$\nabla\Gamma(\mu,\xi) = \frac{1}{|(\bar{\mu},\bar{\xi})|^2}\left[\nabla\hat{\Gamma}(\bar{\mu},\bar{\xi}) - \frac{2}{|(\bar{\mu},\bar{\xi})|^2}\left(\nabla\hat{\Gamma}(\bar{\mu},\bar{\xi})\,(\bar{\mu},\bar{\xi})\right)(\bar{\mu},\bar{\xi})\right],$$

then $\nabla\Gamma(\mu,\xi) = 0$ if and only if $\nabla\hat{\Gamma}(\bar{\mu},\bar{\xi}) = 0$. But $\nabla K_0(P_N) \neq 0$, hence $\nabla\hat{K}(0) \neq 0$. This implies that also $\nabla\hat{\Gamma} \neq 0$ on $\partial\tilde{B}_s$ near the point $(0,0)$, thus (i) is proved. Condition (ii) follows from (3.9) and from the fact that the critical points of K are non degenerate. □

Remark 3.6 *Suppose a and b are regular values for K, and that K satisfies*

$$(L_a^b) \qquad x \in K_a^b, x \in Crit(K) \ \Rightarrow \ x \text{ is non-degenerate, and } \Delta K(x) \neq 0.$$

Then, for s sufficiently large, a, b are regular values for Γ *on* ∂B_s*, and* Γ *satisfies the general boundary condition on* $\partial \tilde{B}_s \cap \Gamma_a^b$*. The proof follows that of Proposition 3.5.*

In the following, for brevity, we denote with B a large ball $\tilde{B}_s$ for which Proposition 3.5 holds. We also set $\Gamma_B = \Gamma|_B$.

4 Main results

We will prove existence of solutions to (3.4) by means of Theorem 2.1, finding critical points (μ, ξ) of φ_ε with $\mu > 0$. Arguing by contradiction, we will assume throughout this section that φ_ε possesses no such a critical point.

Theorem 4.1 *Suppose* $K \in C^2(\mathbb{R}^n)$ *is a Morse function which satisfies (L). For* $j = 0, \dots, n-1$*, let* F_j *be the number of critical points of K with Morse index j and with* $\Delta K > 0$*. Suppose also that condition (1.9) holds. Then for* $|\varepsilon|$ *sufficiently small, problem (3.4) has solution.*

Proof. We first consider the case $\varepsilon > 0$: the case $\varepsilon < 0$ requires just some little modification. Proposition 3.5 cannot be directly applied to φ_ε on B, since condition (ii) of Definition 2.2 does not hold in general. In fact we cannot deduce a non-degeneracy condition for the critical points of φ_ε on ∂B, because we can provide C^1 estimates only. Thus we slightly modify φ_ε near the critical points $\{z_1, \dots, z_h\}$ of $\Gamma|_{\partial B}$, in order to deal with a function which satisfies the general boundary condition. Fix $\delta > 0$ so small that the balls $B_\delta^{n+1}(z_1), \dots B_\delta^{n+1}(z_h)$ are all disjoint, compactly contained in $\mathbb{R}_+^{n+1}$, and such that $\nabla \Gamma \neq 0$ in $B_\delta^{n+1}(z_i)$. Such a choice is possible because Γ satisfies the general boundary condition in B.

Choose a smooth cut-off function ψ_δ such that $\psi_\delta \equiv 1$ in $B_{\frac{\delta}{2}}^{n+1}(z_i)$, and such that $\psi_\delta \equiv 0$ outside each $B_\delta^{n+1}(z_i)$. Consider the functions $\varphi_\varepsilon^\delta = \psi_\delta \, \varepsilon \Gamma + (1 - \psi_\delta)\,(b_0 - \varphi_\varepsilon)$. For ε sufficiently small, $\varphi_\varepsilon^\delta$ is defined on the whole B; moreover, by (3.5) it is $\nabla \varphi_\varepsilon^\delta \neq 0$ on ∂B, since also $\nabla \Gamma \neq 0$ there. For ε small, the function $\varphi_\varepsilon^\delta|_{\partial B}$ possesses only isolated critical points, which, by construction, are those of $\Gamma|_{\partial B}$. Thus $\varphi_\varepsilon^\delta$ satisfies the general boundary condition on B. Hence, we can apply Theorem 2.3 to $\varphi_\varepsilon^\delta$ on B, with $a = \varepsilon c_a$, and $b = \varepsilon c_b$, where $c_a < c_0 \inf K$, and $c_b > c_0 \sup K$. From (3.6) we deduce that $c_0 \inf K \leq \Gamma \leq c_0 \sup K$, so, if ε and δ are sufficiently small, a and b are regular values for $\varphi_\varepsilon^\delta$ on B and on ∂B, and also $(\Gamma_B)_a^b = B$. Since we are assuming that φ_ε possesses no critical point, then neither $\varphi_\varepsilon^\delta$ has interior critical point in B, so it is $m_j = 0$ for all j. Moreover, since the only critical points of $\varphi_\varepsilon^\delta$ on ∂B are those of Γ, from (3.9) it follows that they are non-degenerate, and

$\mu_j = F_j$ for all j. Since B is contractible, $\beta_0(B) = 1$, and $\beta_j(B) = 0$ for all $j \geq 1$; applying (2.3) to the present case we obtain

$$\sum_{j=0}^{q}(-1)^{q-j}F_j = \sum_{j=0}^{q}(-1)^{q-j}\mu_j = \sum_{j=0}^{q}(-1)^{q-j}M_j \geq \sum_{j=0}^{q}(-1)^{q-j}\beta_j = (-1)^q,$$

but this is in contradiction with (1.9). □

Remark 4.2 a) *Set $S = 1 + \varepsilon K_0$, and let $D_j = \sharp\{x \in Crit(S) : M(S,x) = n-j, \Delta S(x) < 0\}$. Then for $q = 1$ and $\varepsilon < 0$, condition (1.9) becomes $D_0 - D_1 > 1$. Thus Theorem 4.1 extends, in the perturbative case, the result in [20].*

b) *With the same arguments as before, we can prove existence also under condition (1.3). In fact, since $\sum_{x_i \in Crit(S)}(-1)^{m(S,x_i)} = 1 + (-1)^n$, hypothesis (1.3) is equivalent to*

$$\sum_{x_i \in Crit(S), \Delta S(x_i) > 0} (-1)^{m(S,x_i)} \neq 1. \tag{4.1}$$

In order to get existence, we have only to repeat the proof of Theorem 4.1 and to use equation (2.4) instead of (2.3) at the end, to deduce

$$1 = \chi(B) = \chi((\Gamma_B)_a^b) = \sum_{q \geq 0}(-1)^q M_q = \sum_{x_i \in Crit(K), \Delta K(x_i) > 0} (-1)^{m(K,x_i)},$$

which is in contradiction with (4.1).

c) *For $n = 2$, in [8] is proved existence of solutions of (1.2) if $D_0 - D_1 \neq 1$. Theorem 4.1 with $\varepsilon < 0$ and $q = 1$, partially extends this result, but in a different way from (1.3).*

We can use relative homology to study the topological changes in the sublevels of Γ. This allows us to provide some existence results under some "localized" hypotheses on K, of the type *2)*.

Lemma 4.3 *Let $a \in \mathbb{R}$ be a regular value for Γ and for $\Gamma|_{\partial B}$. Let $f_n : \mathbb{R}_+^{n+1} \to \mathbb{R}$, $f_n \to \Gamma$ on B in the C^1 norm. Then $(f_n)^a \cap B$ is homeomorphic to $(\Gamma_B)^a$ for n sufficiently large.*

Proof. We just give a sketch, details are left to the reader. First of all we consider the case in which Γ is a function with compact support in $\mathbb{R}^{n+1}$, and $a \neq 0$. For $\rho > 0$, let $a_\rho = \{x \in \mathbb{R}^{n+1} : |x - \Gamma^{-1}(a)| < \rho\}$: we take $\rho > 0$ so small that $\nabla\Gamma \neq 0$ in a_ρ. Since a is a regular value for Γ, and since f_n converge to Γ uniformly, the sets $(f_n)^a$ and Γ^a coincide outside a_ρ. In a_ρ we can consider the flow $\dot{x} = \nabla\Gamma(x)$, which is well defined because $\Gamma \in C^2$. For n large, since $f_n \to \Gamma$ in C^1, the levels $\{f_n = a\}$ are transversal to $\nabla\Gamma$; so we can continuously deform them into $\{\Gamma = a\}$ using the gradient flow of Γ. When Γ is just defined on B, it is sufficient to substitute $\nabla\Gamma$ with a suitable pseudo-gradient field γ for Γ near $\{\Gamma = a\}$ which leaves ∂B invariant. □

The following Theorem improves, in the perturbative and non-degenerate case, a result in [5]: in fact, we make assumptions only on the saddle points with Morse index 1.

Theorem 4.4 *Suppose K has a local minimum x_0, and that there exists $x_1 \neq x_0$ with $K(x_1) \leq K(x_0)$. Let $x(t) : [0,1] \to \mathbb{R}^n$ be a curve with $x(0) = x_0, x(1) = x_1$, and set $a = K(x_0), b = \max_t K(x(t))$. Suppose also that K satisfies (L_a^b), and that the following condition holds*

$$(H) \qquad x \in Crit(K), a < K(x) \leq b, m(x,K) = 1 \quad \Rightarrow \quad \Delta K(x) < 0.$$

Then for $|\varepsilon|$ small, problem (3.4) admits a solution.

Proof. We can suppose that $x(t)$ is a smooth curve of "mountain pass type", i.e. the supremum of K on $x(t)$ is not greater than the supremum of K on any curve joining x_0 to x_1. In particular, the supremum on $x(t)$ is attained only at a finite number of points $x_2, \dots x_h$ whose Morse index is 1. From Proposition 3.3 we have that $\Gamma(0, x_0) = c_0\, a$, and since $\Delta K(x_0) > 0$, from (3.8) it follows that $(0, x_0)$ is a strict local minimum for Γ. Moreover (3.7) implies that $\Gamma(0, x_1) = c_0\, K(x_1) \leq c_0\, a$, so Γ possesses a mountain pass geometry at x_0. Let $\mathcal{C}$ be the class of curves $\mathcal{C} = \{c : [0,1] \to \bar{\mathbb{R}}^{n+1}_+ \,|\, c(0) = x_0, c(1) = x_1\}$, and set

$$\bar{\Gamma} = \inf_{c \in \mathcal{C}} \sup_t \Gamma(c(t)).$$

We claim that $c_0\, a < \bar{\Gamma} < c_0\, b$. The first inequality is trivial; in order to prove the second, consider the family of curves $y(t) : [0,1] \to \mathbb{R}^{n+1}_+$, depending on a parameter $\eta > 0$, defined in the following way

$$y(t) = \begin{cases} (0, x(t)) & \text{if } |x(t) - x_j| \geq \eta, \text{ for all } j = 2, \dots, h; \\ (\sqrt{1 - |x(t) - x_j|^2}, x(t)) & \text{if } |x(t) - x_j| < \eta, \text{ for some } j = 2, \dots, h. \end{cases}$$

Equation (3.8) implies $\Gamma(y(t)) = K(x(t)) + c_1 \Delta K(x(t))\, (y_1(t))^2\, (1 + o(1))$; moreover, from hypothesis (H), it follows that $\Delta K(x_j) < 0$, for $j = 2, \dots, h$. Hence for η sufficiently small, there holds $\sup_t \Gamma(y(t)) < \sup_t K(x(t))$, so our claim is proved. We can choose $\tau > 0$ such that the components of $A_\tau = \{\Gamma < c_0\, a + \tau\}$ containing x_0 and x_1 are different. Let $c(\cdot) = (y_1(\cdot) + \omega, y_2(\cdot), \dots, y_{n+1}(\cdot))$. By continuity, for $\omega > 0$ sufficiently small, we obtain $c(t) \subseteq \mathbb{R}^{n+1}_+$, $c(0), c(1) \in A_\tau$, and $\sup_t \Gamma(c(t)) < c_0\, b$. Let B be the ball given by Remark 3.6; we can take B so large that the range of c is contained in B, and B intersects both the components of A_τ containing x_0 and x_1. Fix $\tilde{a}$ regular for Γ and for $\Gamma|_{\partial B}$, such that $\max\{\Gamma(c(0)), \Gamma(c(1))\} < \tilde{a} < c_0\, K(x_0) + \tau$, and $\tilde{b}$ regular for Γ and for $\Gamma|_{\partial B}$, with $\bar{\Gamma} < \tilde{b} < c_0\, b$. We need again some small modification to get the general boundary condition; so we define the functions $\varphi_\varepsilon^\delta$ as in the proof of Theorem 4.1. If δ and ε are sufficiently small, then $\varepsilon\tilde{a}$ and $\varepsilon\tilde{b}$ are regular values for $\varphi_\varepsilon^\delta$, and from Lemma 4.3, it is $H_1((\varphi_\varepsilon^\delta)^{\varepsilon\tilde{b}}, (\varphi_\varepsilon^\delta)^{\varepsilon\tilde{a}}) \simeq H_1(\Gamma^{\tilde{b}}, \Gamma^{\tilde{a}})$, since $\varepsilon^{-1}(b_0 - \varphi_\varepsilon) \to \Gamma$ in C^1 on B. Moreover, by the definition of $\bar{\Gamma}, \tilde{a}$ and $\tilde{b}$, it is

$\Gamma(c(1)) < \tilde{a}$, hence $[c(1)-c(0)]$ is a 0-cycle in $\Gamma^{\tilde{a}}$, and is the boundary of the 1-chain $[c]$ in $\Gamma_B^{\tilde{b}}$. On the other hand, by our choice of τ, $[c(1)-c(0)]$ is not a boundary in $\Gamma_B^{\tilde{a}}$; it follows that $[c(1)-c(0)] \neq 0$ as a 0-cycle in $\Gamma_B^{\tilde{a}}$. Hence, if i is the inclusion $i : \Gamma_B^{\tilde{a}} \to \Gamma_B^{\tilde{b}}$, it follows that $i_*([c(1)-c(0)]) = 0$. By the exactness of the homology sequence of the pair $(\Gamma_B^{\tilde{b}}, \Gamma_B^{\tilde{a}})$

$$\cdots \longrightarrow H_1(\Gamma_B^{\tilde{b}}, \Gamma_B^{\tilde{a}}) \xrightarrow{\partial_*} H_0(\Gamma_B^{\tilde{a}}) \xrightarrow{i_*} H_0(\Gamma_B^{\tilde{b}}) \longrightarrow \cdots,$$

and from the fact that i_* has a nontrivial kernel, we deduce that $H_1(\Gamma_B^{\tilde{b}}, \Gamma_B^{\tilde{a}}) \neq 0$. It follows that also $H_1((\varphi_\varepsilon^\delta)^{\varepsilon\tilde{b}}, (\varphi_\varepsilon^\delta)^{\varepsilon\tilde{a}}) \neq 0$. The Morse type number μ_1 of $\varphi_\varepsilon^\delta$ on $(\partial B)_{\varepsilon\tilde{a}}^{\varepsilon\tilde{b}}$ coincides with that of Γ, which is zero by assumption (H). The function $\varphi_\varepsilon^\delta|_B$ satisfies the general boundary condition on $B_{\varepsilon\tilde{a}}^{\varepsilon\tilde{b}}$, and there holds $\mu_1 = 0$, $\beta_1 > 0$, so, using (2.3) with $q = 1$ and then with $q = 0$, we obtain

$$0 = m_1 = M_1 \geq \beta_1 + M_0 - \beta_0 \geq 1, \tag{4.2}$$

which is a contradiction. □

Remark 4.5 a) *A related result is given in [8], where S is assumed to be a Morse function with non-zero Laplacian at each of its critical points. A solution is found if S possesses at least two local maximum points, and at all saddle points it is $\Delta S > 0$. For $\varepsilon < 0$ we are in the same situation, but our assumptions involve only the saddle points whose level is between a and b.*

b) *Suppose x_0 and x_1 are global minima for K, possibly degenerate, and let $x(t)$ be a "mountain pass type" curve joining x_0 to x_1. Set $a = \inf K$, $b = \sup_t K(x(t))$, and assume that*

$$Crit(K) \cap \{a < K < b\} = \emptyset; \quad y \in x([0,1]), K(x) = b \Rightarrow \Delta K(x) < 0,$$

Also in this case Γ has the mountain pass geometry at $(0, x_0)$, and reasoning as above, one can find $\tilde{a}$ and $\tilde{b}$, $c_0\, a < \tilde{a} < \tilde{b} < c_0\, b$ such that $H_1(\Gamma_B^{\tilde{b}}, \Gamma_B^{\tilde{a}}) \neq 0$. Since there is no critical value of K in $K_{\tilde{a}}^{\tilde{b}}$, the Morse type numbers of Γ on $(\partial B)_{\tilde{a}}^{\tilde{b}}$ are zero, and we are led again to (4.2). Hence, for $\varepsilon < 0$, we recover the existence result under the assumptions in [13].

c) *For $n \geq 3$, in [5] it is assumed that S possesses only a finite number of critical points, and that, again, two maxima x_0 and x_1 are connected by a curve $x(t)$. Moreover, at every saddle point of S between $\inf_{x([0,1])} S$ and $S(x_0)$ it must be $\Delta S > 0$. Here the main difference, is that we make assumptions only at saddle points with prescribed Morse index.*

d) *For $\varepsilon > 0$, Theorem 4.4 has no known counterpart.*

Theorem 4.4 can be easily generalized in the following way.

Theorem 4.6 *Suppose K possesses a local minimum x_0 and l connected components $A_1, \dots, A_l$ of $(K^a \setminus x_0)$, where $a = K(x_0)$. For $j = 1, \dots, l$, let $c_j : [0,1] \to S^n$ be a curve with $c_j(0) = x_0, c_j(1) \in A_j$, and set $b = \max_j \sup_t K(c_j(t))$. Suppose K satisfies condition (L_a^b), and that it possesses at most $l-1$ saddle points of Morse index 1 in $\{a < K \leq b\}$. Then for $|\varepsilon|$ small, problem (3.4) admits a solution.*

Proof. It is sufficient to reason as in the proof of Theorem 4.4. Define $\mathcal{C}_j = \{c : [0,1] \to \mathbb{R}_+^{n+1} \mid c(0) = x_0, c(1) \in A_j\}$, and set

$$\tilde{\Gamma} = \max_j \{ \inf_{c \in \mathcal{C}_j} \sup_t \Gamma(c_j(t))\}.$$

Then, again, one proves that $\tilde{\Gamma} < c_0 \, b$. In this case, choosing $\tilde{a}$ and $\tilde{b}$ appropriately, it turns out that $\beta_1 > l$; so the result again follows from the Morse inequalities. □

5 Further results

Isolated critical points with non-null Laplacian

We can recover the general boundary condition also when K possesses isolated critical points, which can be possibly degenerate, if the Laplacian at these points has a definite sign. In order to do this, for $s > 0$ we set

$$G_s(\mu,\xi) = \Gamma(2\mu,\xi) - \Gamma(2\mu - h_s(\xi),\xi) + \Gamma(\mu,\xi) - \Gamma(2h_s(\xi),\xi) + \Gamma(0,\xi). \tag{5.1}$$

Here ξ belongs to a fixed compact set of $\mathbb{R}^n$, and $2\mu > h_s(\xi)$, where h_s is the function defined in the proof of Proposition 3.5. Let ψ_δ be a cut-off function as in the proof of Theorem 4.1 centred at the points $(0, x_j)$, where $x_1, \dots, x_h$ are the critical points of K. We can suppose that $|\nabla \psi_\delta| < \frac{4}{\delta}$.

Theorem 5.1 *Suppose that K possesses isolated critical points $x_1, \dots, x_h$, and that $\Delta K(x_j) \neq 0$, for $j = 1, \dots, h$. Assume that for any $j = 1, \dots, h$, rank $C_q(K, x_j) = 0$ for q sufficiently large. Set $\mathcal{F}_q = \sum_{j, \Delta K(x_j) > 0}$ rank $C_q(K, x_j)$, and suppose that*

$$\sum_{j=0}^{n} (-1)^j \mathcal{F}_j \neq 1, \quad or \quad \sum_{j=0}^{q} (-1)^{q-j} \mathcal{F}_j - (-1)^q \leq -1, \tag{5.2}$$

for some $q = 0, \dots, n$. Then for $|\varepsilon|$ small, problem (3.4) admits a solution.

Proof. Again, we assume by contradiction that φ_ε does not possess any critical point (μ, ξ) with $\mu > 0$. We show that the function $\varphi_\varepsilon^{\delta,s} = \varepsilon \psi_\delta G_s + (1 - \psi_\delta)(b_0 - \varphi_\varepsilon)$ satisfies the general boundary condition on B for suitable values of δ, s and for ε arbitrarily small. First we prove that $\varphi_\varepsilon^{\delta,s}$ does not possess any critical point (μ, ξ) with $\mu > 0$, so in particular condition (i) in Definition 2.2 holds.

Given $\delta > 0$, we set

$$U_\delta = \bigcup_{j=1}^{h} \left(B_\delta^{n+1}((0,x_j)) \setminus B_{\frac{\delta}{2}}^{n+1}((0,x_j)) \right); \qquad C_\delta = \inf_{U_\delta} |\nabla \Gamma|.$$

Since $\Delta K(x_j) \neq 0$, and since x_j is an isolated critical point of K, we can deduce from formulas (3.8) and (3.9) that $C_\delta > 0$. From the definition of G_s it follows that

$$\begin{array}{rcl} G_s(h_s(\xi),\xi) & = & \Gamma(0,\xi), \\ (G_s(h_s(\xi),\xi))_\mu = 2\Gamma_\mu(2h_s(\xi),\xi) & - & 2\Gamma_\mu(h_s(\xi),\xi) + \Gamma_\mu(h_s(\xi),\xi), \end{array} \tag{5.3}$$

and that $G_s(\mu,\xi) \to \Gamma(\mu,\xi)$, as $s \to +\infty$, C^1-uniformly on bounded sets, since Γ and $\nabla\Gamma$ are Lipschitz functions. We also know from (3.8) that the following estimate holds

$$\Gamma_\mu(\mu,\xi) = c_1 \Delta K(\xi) \mu \,(1 + o(1)), \tag{5.4}$$

where $(\mu,\xi) \in B_\delta^{n+1}(0,x_j)$, and where $o(1) \to 0$ uniformly as $\delta \to 0$. We can choose δ to be so small that

$$16\,|\Delta K(\xi) - \Delta K(x_j)| < |\Delta K(x_j)|, \ |\xi - x_j| < \delta; \qquad 16\,|o(1)| < 1. \tag{5.5}$$

Next, if s is sufficiently large, there holds

$$20 \sup_{U_\delta} |\nabla G_s - \nabla \Gamma| < C_\delta; \qquad 40 \sup_{U_\delta} |G_s - \Gamma| < \delta\, C_\delta. \tag{5.6}$$

Hence, from elementary computations we deduce

$$\begin{array}{rcl} |\nabla \varphi_\varepsilon^{\delta,s} - \varepsilon \nabla \Gamma| & \leq & |\nabla \psi_\delta| \left(|\varepsilon G_s - \varepsilon \Gamma| + |\varepsilon \Gamma - (b_0 - \varphi_\varepsilon)| \right) \\ & + & |\psi_\delta \varepsilon \nabla G_s + (1-\psi_\delta)\nabla \varphi_\varepsilon - \varepsilon \nabla \Gamma|. \end{array}$$

Since $|\nabla \psi_\delta| < \frac{4}{\delta}$, using (5.6) it follows that

$$|\nabla \varphi_\varepsilon^{\delta,s} - \varepsilon \nabla \Gamma| \leq \frac{1}{10} \varepsilon C_\delta + \frac{1}{20} \varepsilon C_\delta + |\nabla \psi_\delta|\, |\varepsilon \Gamma - (b_0 - \varphi_\varepsilon)| + |\nabla \varphi_\varepsilon + \varepsilon \nabla \Gamma|.$$

Now take ε so small that $10 \sup_{U_\delta} |\nabla \varphi_\varepsilon + \varepsilon \nabla \Gamma| < \varepsilon C_\delta$, and such that $40 \sup_{U_\delta} |\varepsilon \Gamma - (b_0 - \varphi_\varepsilon)| < \varepsilon \delta C_\delta$. Taking into account (5.6), we deduce that $2\,|\nabla \varphi_\varepsilon^{\delta,s} - \varepsilon \nabla \Gamma| \leq \varepsilon C_\delta$, so by the definition of C_δ, it follows that $\nabla \varphi_\varepsilon^{\delta,s} \neq 0$ in U_δ. Using equations (5.4) and (5.5), one proves that $2\Delta K(x_j)\,(G_s(\mu,\xi))_\mu \geq c_1\,(\Delta K(x_j))^2\,(\mu + h_s(\xi))$, for $(\mu,\xi) \in B_\delta^{n+1}(0,x_j)$, so $\nabla \varphi_\varepsilon^{\delta,s} \neq 0$ also in $B_{\frac{\delta}{2}}^{n+1}(0,x_j)$. Since $\varphi_\varepsilon^{\delta,s} = b_0 - \varphi_\varepsilon$ outside each $B_\delta^{n+1}(0,x_j)$, we conclude that $\nabla \varphi_\varepsilon^{\delta,s}$ never vanishes, and (i) is proved. As far as (ii), if s is sufficiently large, and if ε is sufficiently small, then $\nabla \varphi_\varepsilon \neq 0$ on $O_\delta = \partial B \setminus \cup_j B_\delta^{n+1}(0,x_j)$. Since $\varphi_\varepsilon^{\delta,s}$ equals $b_0 - \varphi_\varepsilon$ on O_δ, all the critical points of $\varphi_\varepsilon^{\delta,s}|_{\partial B}$ must be contained in the balls $B_{\frac{\delta}{2}}^{n+1}(0,x_j)$. But here, see the first formula in (5.3), $\varphi_\varepsilon^{\delta,s} = \varepsilon G_s$ coincide with $\varepsilon\,\Gamma(0,\xi) = \varepsilon c_0\, K(\cdot)$, so its critical points are isolated. This proves (ii), and thus $\varphi_\varepsilon^{\delta,s}$ satisfies the general boundary condition on B. From the above computations it follows that the flow of $\nabla\varphi_\varepsilon^{\delta,s}$ near its critical points is inward B, reps. outward, if also $\nabla\Gamma$ is inward, resp. outward. Hence, the j-th Morse type number μ_j of $\nabla\varphi_\varepsilon^{\delta,s}$ on ∂B coincides with $\mathcal{F}_j$, and moreover $m_j = 0$ for all j, since $\nabla\varphi_\varepsilon^{\delta,s}$ does not possess interior critical points. Now we conclude as in Theorem 4.1 and in Remark 4.2. □

β-degeneracy

The following Lemma, see [2] describes the behavior of Γ when a critical point of K admits some degeneracy.

Lemma 5.2 *Given $\xi \in \mathbb{R}^n$, suppose that there exist $\beta_\xi \in (1,n)$, and $Q_\xi : \mathbb{R}^n \to \mathbb{R}$ such that*

1) $Q_\xi(\lambda y) = \lambda^{\beta_\xi} Q_\xi(y), \quad y \in \mathbb{R}^n, \lambda > 0$;

2) $K(x) = K(\xi) + Q_\xi(x-\xi) + o(|x-\xi|^{\beta_\xi})$, *as* $x \to \xi$,

and let $T_\xi = \frac{1}{2^*}\int Q_\xi z_0^{2^*}$. *Then*

$$\lim_{\mu\to 0^+} \frac{\Gamma(\mu,\xi) - \Gamma(\mu,0)}{\mu^\beta} = T_\xi. \tag{5.7}$$

Moreover $q = (0,\xi)$ *is an isolated critical point of* Γ.

This Lemma can be applied when K admits an expansion as in (1.4) near its critical points. In fact, the topological structure of the sublevels of K is analogous to the non-degenerate case. There exists a unique nonnegative number q for which $C_q(K, x_i) \neq 0$; this number coincides with $i(x)$, and is the corresponding of the Morse index. Moreover it turns out that $T_{x_i} = C_{\beta_i}\,\tilde\Delta K$, where $C_{\beta_i} > 0$, and, we recall, $\tilde\Delta K = \sum_{j=1}^n a_j$. Hence, the quantity $\tilde\Delta K$ plays the role of the Laplacian in the non-degenerate case. So, Theorems 4.1, 4.4 and 4.6 can be stated with obvious changes in the case when K is degenerate of order $\beta_i \in (1,n)$ at its critical points x_i.

Higher dimensions

The following result generalizes Theorem 4.4, substituting m-dimensional balls to one dimensional curves.

Theorem 5.3 *Suppose there exists a positive integer $r < n$, and a smooth embedding $h_0 : S^r \to S^n$ such that the maximum of K on $h_0(S^r)$ is attained at some critical point $x_0 \in Crit(K)$, with $\Delta K(x_0) > 0$, and $m(K, x_0) = r$. Let $h : B_1^{r+1} \to S^n$*

with $h|_{\partial B_1^{r+1}=S^r}$*, and let* $a = K(x_0)$*,* $b = \max_{y \in B_1^{r+1}} K(h(y))$*. Suppose* K *satisfies condition* (L_a^b)*, and that*

$$(B_r) \qquad z \in Crit(K) \cap K_a^b, m(z,K) = r+1 \quad \Rightarrow \quad \Delta K(z) < 0.$$

Then for $|\varepsilon|$ *small, problem (3.4) has a solution.*

Proof. The proof follows that of Theorem 4.4, but here the mountain pass construction is substituted by a linking scheme. By the non-degeneracy of x_0, there exists an $(n-r)$-dimensional subspace H of $\mathbb{R}^n$ where $K''(x_0)$ is positive definite. Given $\zeta > 0$, we define the half-sphere V_ζ to be

$$V_\zeta = \{ z \in \mathbb{R} \times H : |z - (0, x_0)| = \zeta \} .$$

Taking into account (3.8), we deduce that for ζ sufficiently small it is $\inf_{z \in V_\zeta} K(z) > K(x_0)$. If we choose B appropriately, we find an homeomorphism $\tilde{h} : S^r \to B$ such that $\sup_{\tilde{h}(S^r)} \Gamma < \inf_{V_\zeta} \Gamma$, and such that $\tilde{h}(S^r)$ and V_ζ homotopically link. It turns out that $\tilde{h}(S^r)$ and V_ζ also homologically link, see [11], Chapter 2.1, so we can find $\tilde{a}$ and $\tilde{b}$ such that $H_r(\Gamma_B^{\tilde{b}}, \Gamma_B^{\tilde{a}}) \neq 0$. To conclude, it is sufficient to use (2.3) with $q = r$ and $r+1$. □

Remark 5.4 *If* $r = n-1$*, then condition* (B_r) *is automatically satisfied. For* $n = 2$ *and* $r = 1$ *this result has been obtained in [12].*

Remark 5.5 *If* Γ *turns out to be a Morse function, then also some multiplicity results can be obtained. In fact, the local degree of* Γ *at each of its critical points is different from* 0*. From expression (3.5) one deduces that* φ_ε *possesses as many stationary points as* Γ*. A lower bound for this number, see [2], can be found via Degree Theory, and is given by* $\left| \sum_{q=0}^{n-1} (-1)^q F_q - (-1)^n \right|$*. See also [20] for related multiplicity results.*

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