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ORIGINAL RESEARCH

Hintikka's conception of syntheticity as the introduction of new individuals

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Abstract

In a series of papers published in the sixties and seventies, Jaakko Hintikka, drawing upon Kant's conception, defines an argument to be analytic whenever it does not introduce new individuals into the discussion and argues that there exists a class of arguments in polyadic first-order logic that are to be synthetic according to this sense. His work has been utterly overlooked in the literature. In this paper, I claim that the value of Hintikka's contribution has been obscured by his formalisation of the original definition. Therefore, I provide (i) a brief reconstruction of the historical framework of the problem and the revolutionary import of Hintikka's contribution, (ii) a clarification of the most complicated steps of Hintikka's elaboration of his insight, (iii) a criticism of several features that play a fundamental role in Hintikka's formalisation and (iv) a selection from Hintikka's own material of some valuable suggestions towards a clear and workable formalisation. As for the *pars construens*, I isolate in the approach of depth-bounded first-order logics (D'Agostino et al. 2021) an alternative formalisation of the notion of syntheticity as the introduction of new individuals in the reasoning, and I show that it is not affected by the same difficulties as Hintikka's proposal. In so doing, I hope to have contributed to the realisation of the project of rehabilitating Kant's analytic–synthetic distinction in the context of modern first-order logic with the purpose of showing, against the logical empiricist movement, that logic is not analytic.

Keywords Hintikka · Kant · Analytic-synthetic distinction · Existential instantiation · Depth-bounded first-order logics

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1 Introduction

In his paper *An Analysis of Analyticity*,[1] Hintikka discusses four central notions of analyticity:

i. Analytic truths as conceptual truths
ii. Analytic arguments as arguments satisfying some form of the subformula property
iii. Analytic arguments as arguments that do not introduce new individuals into the discussion
iv. Analytic truths as tautological truths

Hintikka's work has some indisputable merits. It debates traditional senses of the term analyticity, such as in notions (i) and (iv), and points out definitions that deserve more attention from the philosophical side, such as in notion (ii). Nevertheless, the kernel of Hintikka's contribution is contained in his third sense of analyticity. Hintikka elaborates on this notion by providing further specifications of it, which he understands as variations (marked by an increasing level of logical formality) of the same basic idea. The elaboration of this sense is as follows:

iii. A (valid) argument step is analytic if it does not introduce new individuals into the discussion.
iii.a. An analytic argument cannot lead from the existence of an individual to the existence of a different individual.
iii.b. A (valid) argument step is analytic if it does not increase the number of individuals one is considering in relation to each other.
iii.c. A (valid) argument step is analytic if the degree of the conclusion is no greater than the degree of at least one of the premises.
iii.d. An argument is analytic if all its steps are analytic in sense (iii.c).
iii.e. A (valid) proof of the sentence F_1 from F_0 is analytic if none of the sentences occurring as intermediate stages of this proof has a higher degree than F_0 and F_1. (Hintikka, 1973, pp. 148–149)

Explicit form of sense (iii): A sentence F_1 follows from F_0 analytically if and only if the distributive normal form of F_0 will become a part of the normal form of F_1 as soon as trivially inconsistent constituents are eliminated from it. (Hintikka, 1973, p. 185).

Sense (iii.a) does not seem to distance itself significantly from (iii). It simply rests on the plausible assumption that introducing an individual into an argument is tantamount to affirming its existence. On the contrary, sense (iii.b) introduces the notion of *related individuals* into the reasoning and specifies, which is against sense (iii), that argument steps in which either unrelated individuals are introduced or the number of new related individuals that are introduced is the same as the number of old related individuals that are removed still count as analytic.

However, it is only in sense (iii.c) that Hintikka defines the maximal number of individuals that are in relation to each other in a certain sentence F, which he calls the *degree of* F, as the sum of two numbers; namely, the number of the free singular terms of F and the maximum length of nested sequences of quantifiers in F, called

[1] Hintikka (1966, 1973, pp. 123–149). References will always be made to the latter version of the text.

the *depth of* F (Hintikka, 1973, pp. 141–142). The depth of a sentence F, which is the number of different layers of quantifiers in F at its deepest level, is just the number of bound individual symbols that are needed to understand F, provided that quantifiers with overlapping scopes always have different variables bound to them.

Sense (iii.d) extends sense (iii.c) from argument steps to longer arguments. Unlike sense (iii.d), sense (iii.e) does not regard the direction of the argument but takes into account the individuals that occur in the conclusion. As a result, any inference in which the degree of the conclusion is higher than the degree of the premise is synthetic according to sense (iii.d) but may be analytic according to sense (iii.e). Last, the explicit form of sense (iii) is based on Hintikka's theory of distributive normal forms for first-order logic. This is an extension of the corresponding theory for propositional and monadic logic and provides a description of possible worlds, where the complexity of the configurations of individuals that can be considered is limited.

The importance and ambitions of Hintikka's contribution to the analytic–synthetic distinction can hardly be overrated. As I shall point out below, not only does his definition qualify as an extension of Kant's notion beyond the boundaries of categorical judgments, but it also provides good reasons to reject the logical empiricists' tenet of the analyticity of logic and vindicate Kant's position on the syntheticity of mathematics. However, despite the relevance of these results, the significance of Hintikka's theory has been utterly overlooked in the literature. Moreover, scholars who took into consideration Hintikka's work usually focused on a specific aspect of the theory while usually leaving the overall picture aside.

For example, Rantala (1987) and Rantala and Tselishchev (1987) concentrate on the theory of distributive normal forms and the related notion of information, and Sequoiah-Grayson (2008) criticises Hintikka's measure of the information yielded by deductive inference. Van Benthem (1974) focuses on some formal aspects of sense (iii), while de Jong (1997) and Russell (1990) target Hintikka's reading of the Kantian materials. An attempt to provide a comprehensive assessment of Hintikka's work, taking into account both the philosophical and formal parts of his theory, is given in Larese (2019, 2020a), where a series of objections are also pointed out but not fully developed.

In this paper, I argue that the reason for this critical misfortune is that the value of sense (iii) has been obscured by Hintikka's further elaboration and formalisation of the original idea. As I will point out in detail, these are marred by numerous problems: several unclear conceptual passages, inaccurate definitions and complex formalism. However, I claim that Hintikka's work does not need to stand or fall as a whole. On the contrary, his original idea, namely sense (iii), may be so relevant to the debate on analyticity and logic that it deserves to be rid of the difficulties connected to Hintikka's elaboration, expressed according to alternative formal tools, and used again.

The main contributions of this paper are the following. First, I provide a brief reconstruction of the historical framework of the problem and highlight the revolutionary import of Hintikka's contribution. Second, I offer clarification of the most obscure steps of Hintikka's elaboration of sense (iii), which I believe are responsible for its lack of critical fortune. Third, I criticise several features of Hintikka's construction and show that the difficulties that arise are substantial to the fact that the formal results are in need of a complete revision. Fourth, I isolate some suggestions in Hintikka's material that I believe are useful for an alternative formalisation of sense (iii). Last, I show why the work presented by D'Agostino et al. (2021) appears to be a good candidate to provide an alternative formalisation of sense (iii).

This paper is organised as follows. Section 2 offers a brief reconstruction of the historical background of the relation between analyticity and logic. Section 3 points out the importance of Hintikka's contribution. Sections 4 to 8 then each focus on a particular aspect of Hintikka's elaboration of sense (iii) and find the difficulties that affect each aspect. In particular, Sect. 4 criticises Hintikka's treatment of propositional and monadic arguments. Section 5 focuses on the representation Hintikka proposes for the number of individuals considered together in the premise, and Sect. 6 discusses his three definitions of the degree of a formula. Section 7 is devoted to analysing the problems that affect sense (iii.e), and Sect. 8 uncovers the ambiguities in his work about the role of existential instantiation.

Section 9 proposes an overall evaluation of Hintikka's development of sense (iii) that organises the criticisms of the previous sections into a coherent whole. Sections 10 to 12 offer a sketch of the alternative approach provided by D'Agostino et al. (2021) that aims to show that their approach does not seem to be marred by the problems that affect Hintikka's theory. Finally, Sect. 13 concludes the paper.

2 Analyticity and categorical judgments

The concept of analyticity is central to the historical development of the analytic tradition in philosophy. Its origin is indissolubly connected to the work of Kant. This is not because he was the first to introduce it, for there are several precursors in this sense (Kant, 1997, p. 22), but because of the use he made of this concept and the key role played by the analytic–synthetic distinction in his theoretical construction, the main purpose of which was to show the possibility of synthetic a priori knowledge. Although the *Critique of Pure Reason* apparently provides no less than four definitions of analyticity,[2] for Kant, the fundamental criterion to distinguish between analytic and synthetic judgments is based on the notion of containment, which is:

> In all judgments in which the relation of a subject to the predicate is thought (if I consider only affirmative judgments, since the application to negative ones is easy), this relation is possible in two different ways. Either the predicate B belongs to the subject A as something that is (covertly) contained in this concept A or B lies entirely outside the concept A, though to be sure it stands in connection with it. In the first case, I call the judgment **analytic**, in the second **synthetic**. (Kant, 1998, pp. A6–7/B10)

This definition soon became a touchstone for all philosophers who wished to use the analytic–synthetic distinction in their work, which gave rise to an intense debate on the details of Kant's conception. There are three criticisms against the notion of analyticity construed in terms of the containment criterion that are most recurrent. First, Kant's definition has been accused of psychologism, i.e. that the analyticity or syntheticity of

[2] Containment, clarification, identity and contradiction. See Kant (1998, A6–7/B10–11 and A151–2/B190–1).

a certain judgment would be dependent on the subject that considers that judgment. In particular, it would rely on the features that the individual involved associates both with the subject and with the predicate.

The second criticism, which has been put forward by Kant's contemporaries and frequently evoked even in recent times, argues that the containment criterion is nothing more than an obscure and metaphorical way of speaking. For example, Bolzano (1973, Sect. 148, p. 196), in his *Wissenschaftslehre*, states that 'the explication of this distinction... still fall[s] somewhat short of logical precision' and adds that 'these are in part merely figurative forms of expression that do not analyse the concept to be defined, in part expressions that admit of too wide an interpretation'. The same charge became stereotypical after Quine's (1951, p. 21) influential attack in his *Two Dogmas of Empiricism*: 'This [Kant's] formulation... appeals to a notion of containment which is left at a metaphorical level'.

Third, the analytic–synthetic distinction formulated via the containment criterion can only be applied to categorical judgments, i.e. judgments of the subject–predicate form. While both the first and second criticisms can be easily dismissed,[3] the third is probably the most serious issue of Kant's theory of analyticity; that is, its formulation soon appeared too narrow. For example, in his *Foundations of Arithmetic*, Frege (1960, Sect. 88, pp. 99–100) argues that this characteristic feature of the containment criterion stems from Kant's misunderstanding of the nature of arithmetical judgments: 'Kant obviously—as a result, no doubt, of defining them too narrowly—underestimated the value of analytic judgments'. Many scholars found this restriction to categorical judgments so unpalatable that they tried to deny it *in toto*, either by claiming that the containment criterion is nothing but a part of Kant's theory of analyticity, which is subsumed by more comprehensive definitions (e.g. Hanna, 2001, p. 145), or by providing more or less convincing readings of other passages of Kant's *Critique* (e.g. Anderson, 2015, p. 20).

However, recent scholars, such as De Jong (1995) and Proops (2005), have shown that these attempts are bound to fail; therefore, we must accept that Kant's distinction via containment is not exhaustive, and, as a consequence, there are some judgements that are neither analytic nor synthetic. This controversial feature of the notion of analyticity that emerges in the *Critique* can be explained by examining the contexts for which Kant devised his distinction. First, as Proops (2005, p. 610) underlines, Kant's 'chief concern is to argue for the *syntheticity* of certain judgments', such as the claims of mathematics, natural sciences and metaphysics, 'that in his days would have been assumed to have subject–predicate form'. Second, as Anderson (2015) points out, Kant's distinction hides a substantial thesis against German rationalist metaphysics, especially against Leibniz and the Wolffian tradition: truth is not exhausted by the containment relation; on the contrary, important cognition cannot be explained in terms of analytic (in the Kantian sense) judgments and thus falls on the synthetic side.

A strong contribution to the idea that Kant's definition, although foundational, should be improved through an extension beyond categorical judgments is given by Frege. There are two turning points in his work that interest us most here. First, while

[3] For the former, see Hanna (2001, p. 155 and ff.); for the latter, see Anderson (2015, Part I) and De Jong (1995).

for Kant, pure general logic was a restricted version of Aristotelian syllogistic with the addition of a simple theory of disjunctive and hypothetical propositions, Frege's *Begriffsschrift* proposed essentially what is known today as classical second-order logic, with the identity embracing both the logic of propositions and the logic of quantification.

An indispensable premise of this revolutionary achievement is Frege's replacement of the traditional analysis of judgments into subject and predicate concepts with the innovative notion of analysis in terms of function and argument. According to this new conception of analysis, propositions are decomposed into a variable part and a constant part, and this analysis can be carried out in different ways. As an example, Frege considers the proposition 'Cato killed Cato' and reasons:

> If we here think of 'Cato' as replaceable at its first occurrence, 'to kill Cato' is the function; if we think of 'Cato' as replaceable at its second occurrence, 'to be killed by Cato' is the function; if, finally, we think of 'Cato' as replaceable at both occurrences 'to kill oneself' is the function. (Frege, 1972, Sect. 9, p. 21)

A result of Frege's innovative approach is that categorical judgments lose their traditional centrality in favour of more complex expressions involving relations and nested quantifiers, which could not be properly treated by the logic available to Kant. The logic of the *Begriffsschrift* is not only capable of dealing with propositions with a greater expressive power but is also endowed with the crucial role in Frege's logicist programme of reducing arithmetic to logic. A natural consequence of the new content and new function of modern logic is that Frege cannot be satisfied with Kant's analytic–synthetic distinction based on the containment criterion that applies only to categorical judgment. In the *Foundations of Arithmetic*, we find the following conception:

> The problem becomes, in fact, that of finding the proof of the proposition and of following it up right back to the primitive truths. If, in carrying out this process, we come only on general logical laws and on definitions, then the truth is an analytic one, bearing in mind that we must take account also of all the propositions upon which the admissibility of any of the definitions depends. If, however, it is impossible to give the proof without making use of truths which are not of a general logical nature, but belong to the sphere of the special science, then the proposition is a synthetic one. (Frege, 1960, Sect. 3, p. 4)

Although Frege (1960, Sect. 3, p. 3, footnote) did not fully realise the distance between his and Kant's definitions, the two disagree in essential respects, such as the treatment of non-categorical propositions, but fit the theoretical scenarios that differ for several characteristics, such as the role assigned to logic. As far as the latter is concerned, it is worth underlining that logic is, in Frege's theory, analytic. To be more precise, the laws of logic that are chosen as axioms of the system are analytic because of their self-evidence—logical theorems are instead analytic in that they can be proved through logical laws only. Frege is never explicit on this point because, as Proust (1989, p. 112) observes, the 'analyticity of *logical* propositions is not itself in question, but it is rather presupposed by the problem Frege has to solve'. It is with Frege's work that modern logic starts becoming the paradigmatic example of analytic discipline and the yardstick to measure the status of other disciplines with respect to the analytic–synthetic distinction. At the same time, Kant's conception of analyticity based on the containment criterion loses its appeal.

The outcome of this process is evident in the conception of analyticity held by the logical empiricist movement. A fairly accurate account of this conception might be found in a work that famously criticises it, namely Quine's *Two Dogmas*, which asserts:

> Statements which are analytic by general philosophical acclaim [...] fall into two classes. Those of the first class, which may be called logically true, are typified by: (1) No unmarried man is married. [B]ut there is also a second class of analytic statements, typified by: (2) No bachelor is married. The characteristic of such a statement is that it can be turned into a logical truth by putting synonyms for synonyms; thus, (2) can be turned into (1) by putting 'unmarried man' for its synonym 'bachelor'. (Quine, 1951, p. 23)

In other words, a statement is said to be analytic if it is either a logical truth or 'can be turned into a logical truth by putting synonyms for synonyms'. Although this definition is clearly different from Frege's, it shares both the applicability to non-categorical statements and the privileged role of logic, which is taken to be the case of analyticity *par excellence*. For Quine, the problem arises with the so-called material analyticities, but his attack utterly spares the first class of analyticities, namely, logical truths. By considering logical truths as simply non-questionable cases of analyticities, Quine agrees with his critical target, the logical empiricist movement and Carnap's perspective in particular:

> The scientific world-conception knows only empirical statements about things of all kinds and analytical statements of logic and mathematics. (Carnap et al., 1973, p. 308)
>
> By means of the concept 'analytic', an exact understanding of what is usually designated as 'logically valid' or 'true on logical grounds' is achieved. (Carnap, 1959, p. 41)

3 Hintikka's vindication of Kant's distinction

As we saw in the introduction, Hintikka (1973) defines analytic arguments as arguments that do not introduce new individuals into the discussion. In his work, he further argues that sense (iii) 'approximates rather closely Kant's notion of analyticity'[4] and suggests that it represents a fair reconstruction in modern terms of Kant's conception. Although the philological accuracy of Hintikka's proposal is not my concern here,[5] it is worth mentioning that this claim may be objected to on several grounds. For example, it rests upon certain interpretational premises that have proved to be controversial, such as that intuitions, for Kant, are nothing more than representations of singular

[4] Hintikka (1973, p. 137).

[5] See Larese (2020a).

objects[6] or that Kant is 'an heir to the constructional sense of analysis'.[7] Nevertheless, for the purposes of this paper, I am happy with the much less ambitious claim that Hintikka's sense (iii) is strongly Kantian in spirit.

To see that this is the case, recall that, according to the definition of the *Critique of Pure Reason*, the concept of the predicate of synthetic judgments is not contained in the concept of the subject. The connection between the two concepts involved, which is necessary for grounding the truth of the judgment, can only be indirect in the sense that it must link the two concepts to one another by connecting them to a third element. The third element that is always necessary for the truth of synthetic judgments is, for Kant, an object, and the relation between concepts and objects must always be mediated by intuitions, which are representations of individual objects (Kant, 1998, A155/B194 and ff.). Now, the familiarity between Kant's and Hintikka's definitions should be clear: the individuals introduced in synthetic arguments according to Hintikka's sense (iii) mirror the intuitions that characterise Kant's synthetic judgments.

What is crucial for our reasoning here is to underline that Hintikka's sense (iii) is, at the same time, both inspired by Kant in a strong (yet to be measured precisely) sense and applicable, unlike Kant's original definition, to not only categorical judgements but also non-categorical judgements. To be more precise, as we will see later, Hintikka's sense (iii) applies to every judgement that can be expressed through the means of modern first-order logic. I argue that this is probably the most fundamental reason for the importance of Hintikka's contribution to the debate on the analytic–synthetic distinction. In taking Kant's original definition, (more or less faithfully) translating it into modern terms and extending it to non-categorical judgments, Hintikka drops one of the hardest charges against the Kantian idea and restores it to its pride of place not only in the history of the concept but also in the contemporary debate. Through Hintikka's insight, a Kantian-inspired definition is also a viable option after the invention of modern first-order logic.

Bringing Kant's definition to its former glory does not merely have an archaeological interest. It offers new solutions to long-standing questions, such as the status of logic and mathematics in relation to the analytic–synthetic distinction. To begin with, consider the status of logic. It is a commonplace view in the history of the discipline that logic is exceptional in the sense that it is somehow special when compared to other sciences.[8] Among the properties that make it unique, we find analyticity. However, the idea that logic is analytic, despite it seeming to be an ahistorical tenet, is relatively young. As we have seen in the previous section, it is Frege's work that laid the basis for this principle, which acquired its importance only later on with the logical empiricists' movement. The movement used the property of analyticity as a justification for its necessity once the logical empiricists rejected metaphysics as the source of unconditionally valid knowledge.

However, it is worth mentioning that before Frege, the analyticity of logic was not taken for granted. For example, Kant was not interested in applying his analytic–synthetic distinction to logic at all because he conceived of the discipline as a canon and

[6] See Parsons (1969), Parsons (1980), Parsons (1983) and Friedman (2000).

[7] Hintikka (1973, p. 205). See De Jong (1997).

[8] For a recent discussion on the anti-exceptionalism about logic, see Hjortland and Martin (2022).

not as a body of truths. As a result, in the *Critique of Pure Reason*, logical judgments are neither analytic nor synthetic (De Jong, 2010; Larese, 2020b). The case of Bernard Bolzano is different. Assuming that logic is a deductive science and that deductive sciences are mainly synthetic a priori, the *Wissenschaftslehre* argues that logic is synthetic a priori, at least as far as the *ordo essendi*, as opposed to the *ordo cognoscendi*, is concerned (Bolzano, 1973, Sect. 12). Another perspective is offered by John Stuart Mill's radical empiricism in epistemology, which led him to believe that logic, like the other sciences, is grounded inductively on experience and is thus synthetic (Mill, 2002).

Now, Hintikka's restoration of Kant's definition leads him to reject the logical empiricists' principle of the analyticity of logic together with one of its supposed consequences; namely, the idea that logic is tautological and informationally trivial. More details on this position can be found in the next section, but I anticipate the main point here. Hintikka applies his Kantian analytic–synthetic distinction to modern first-order logic and concludes that there exists a class of polyadic truths of first-order logic that are synthetic a priori according to his sense (iii). Moreover, Hintikka offers a good reason, which is independent of his formalism, that supports his claim: the undecidability of first-order logic.

Consider now the status of mathematics. As it is well-known, the *Critique of Pure Reason* defended the revolutionary thesis that mathematics, including arithmetic, is synthetic a priori and searches the conditions of its possibility. Frege famously disagrees with Kant on this point and argues for the analyticity of arithmetic while believing that geometry is synthetic a priori. This position was an inescapable result of Frege's project of reducing arithmetic to logic together with his thesis on the analyticity of logic. After Frege, the synthetic a priori, already impoverished by the logicist programme, was rejected *in toto* by the logical empiricists' movement.

Hintikka's work vindicates not only the fruitfulness of Kant's analytic–synthetic distinction but also Kant's thesis that mathematical arguments are synthetic a priori. Hintikka explains:

> We can now vindicate Kant. What he meant when he held that mathematical arguments are normally synthetic was quite right. By mathematical arguments, he primarily meant modes of reasoning which are now treated in quantification theory. But it was just seen that many quantificational modes of reasoning are inevitably synthetic in a natural sense of the word. This sense is, moreover, closely related to Kant's intentions, for it was pointed out in Chapter VI above that the group of senses III of analyticity may be taken to be a very good reconstruction of Kant's notion of analyticity as he used it in his philosophy of mathematics. (Hintikka, 1973, p. 182)

The point is that the contemporary boundaries between mathematics and logic are not the Kantian ones. Kant distinguished between pure general logic, which consisted of the Aristotelian syllogistic and some propositional pattern of reasoning, and mathematics, the inferences of which are paradigmatic examples of the synthetic reasoning. Similarly, according to Hintikka, the inferences of modern first-order logic can be divided into two classes: the analytic inferences of monadic logic and the synthetic inferences that translate a typical mathematical way of reasoning.

4 The status of logic: on propositional and monadic arguments

In Sect. 2, we have seen that the idea that logic is analytic finds its most fertile ground in the philosophical movement of logical empiricism. In this cultural milieu, logic is not only said to be analytic but also tautological, i.e. devoid of any informational content. This seems to be a somewhat inescapable consequence of the so-called paradox of analysis (Langford, 1992, p. 323), which states that analysis cannot be sound and informative at the same time, for if it is sound, the analysed and the analysandum are equivalent and analysis cannot be augmentative, but if it is informative, then the analysed and the analysandum are not equivalent, and the analysis is incorrect. Logical empiricists give up the informativeness of logical analysis in favour of its soundness and admit that '[l]ogical investigation... leads to the result that all thought and inference consists of nothing but a transition from statements to other statements that contain nothing that was already in the former (tautological transformation)' (Carnap et al., 1973, p. 308).

The principle of analyticity and tautologicity of logic caught on and became part of logical folklore. Nevertheless, this principle is highly counterintuitive; that is, the conclusion obtained through a long deductive chain might appear as an actual novelty with respect to its premises, and the recognition that a particularly complex sentence is a valid truth might appear as an unexpected discovery. Hintikka describes this situation as a true 'scandal of deduction', that is:

> C.D. Broad has called the unsolved problems concerning induction a scandal of philosophy. It seems to me that in addition to this scandal of induction, there is an equally disquieting scandal of deduction. Its urgency can be brought home to each of us by any clever freshman who asks, upon being told that deductive reasoning is 'tautological' or 'analytical' and that logical truths have no 'empirical content' and cannot be used to make 'factual assertions': in what other sense, then, does deductive reasoning give us new information? Is it not perfectly obvious there is some such sense, for what point would there otherwise be to logic and mathematics? (Hintikka, 1973, p. 222)

Hintikka argues against the scandal of deduction that there exists a class of logical arguments (or truths) in first-order logic that are not only synthetic but also informative. As he explains, the ultimate reason that supports his thesis is the undecidability of first-order logic (Church, 1936; Turing, 1937). Hintikka states:

> In propositional logic and in monadic first-order logic, distributive normal forms yield a decision method: if a formula has a non-empty normal form, it is satisfiable, and vice versa; it is logically true if and only if its normal form contains all the constituents with the same parameters as it. In view of Church's undecidability result, they cannot do this in the full first-order logic (with or without identity). It is easily seen that this failure is possible only if some of our constituents are in this case inconsistent. In fact, the decision problem of first-order logic is seen to be equivalent to the problem of deciding which constituents are inconsistent. More explicitly, the decision problem for formulae with certain

> fixed parameters is equivalent to the problem of deciding which constituents with these parameters are inconsistent. (Hintikka, 1973, p. 255)

Hintikka thus distinguishes between inconsistent constituents that are trivially inconsistent and inconsistent constituents that are not trivially inconsistent. While the former is blatantly self-contradictory, the inconsistency of the latter can be detected only by increasing their depth. This means that for every inconsistent constituent of depth d, there is some natural number e such that all the subordinate constituents of depth $d + e$ are trivially inconsistent. The point is that we do not know which depth we should reach in order to acknowledge that a certain constituent is inconsistent because first-order logic is undecidable.

As the quotation above clarifies and which has already been noticed in the literature (Sequoiah-Grayson, 2008, p. 88 and ff.; D'Agostino & Floridi, 2009, p. 278), the class of analytic arguments (or truths) is broader than it might first appear. It includes, beyond many polyadic deductions, the entire set of not only propositional but also monadic arguments. Because propositional and monadic calculi contain only consistent constituents, the inferences included in this set fail to be synthetic and thus increase deductive information. However, is the principle of analyticity and tautologicity of propositional and monadic logic not an 'equally disquieting scandal of deduction'? Is Hintikka's thesis not liable to the same accusations that the Finnish logician directed against the Vienna Circle? Is his proposal not only a *partial* solution?

Holding that propositional and monadic calculi are analytic and tautological is no less counterintuitive than arguing that full first-order logic is not informative. Moreover, these doubts seem to find confirmation in the theory of computational complexity, a branch of the theory of computation in theoretical computer science that at the time of Hintikka's proposal was at the beginning of its flourishing (Garey & Johnson, 1979). In this context, decision problems can be classified according to their resource-based complexity. Class P includes all the decision problems that can be solved in polynomial time by a non-deterministic Turing machine. The most important unsolved problem in theoretical computer science concerns the relationship between these two classes and asks whether P is identical to NP or not. It is widely assumed that the two classes are not identical (P $\neq$ NP) and that no deterministic Turing machine can be found to solve problems in NP.

As far as Boolean logic is concerned, it is possible to identify three decision problems that are strictly connected. First, the Boolean satisfiability problem, which is the problem of determining whether there exists an interpretation that satisfies a given propositional formula, was proven to be NP-complete (Cook, 1971), that is to say, one of the most difficult problems in NP. Second, the problem of determining whether a given Boolean formula is a tautology is NP-hard; that is to say, it is not known whether it belongs to NP, but it is known that every problem in NP can be reduced to it in polynomial time. Third, the problem of determining whether a Boolean inference is correct or not can be reduced to the tautology problem from both a deterministic and non-deterministic point of view. This means that, if the widely accepted conjecture P $\neq$ NP is true, then the satisfiability problem, the tautology problem and the inference problem are intractable, viz. not decidable in practice. As D'Agostino (2010) underlines, this amounts to saying that any real agent, even if equipped with an up-to-date computer running a decision procedure for Boolean logic, may never be able to feasibly recognise that certain Boolean sentences logically follow from sentences that one regards as true.

Hintikka considered the undecidability of first-order logic as a strong reason to hold that polyadic logical truths are not analytic. Similarly, the P $\neq$ NP conjecture is a reasonable justification to reject the logical positivists and Hintikka's thesis on propositional logic; i.e. if the decision problem for Boolean logic is (most probably) intractable, how is it possible to maintain that it is uninformative and analytic?

5 On the number of individuals considered together in the premise

Unfortunately, concerns about Hintikka's work are not confined to the monadic fragment of first-order logic but also regard the treatment of the logic of quantification as a whole. This is because, as I shall argue in what follows, Hintikka's detailing of his sense (iii) is problematic in several respects.

To begin with, Hintikka (1973, p. 123 and ff.) explains that an argument (or an argument step) is usually said to be analytic if, and only if, the conclusion is obtained by merely analysing what the premise gives us. However, what does the premise of an argument give us? While the traditional answer is a number of *concepts* put together in a definite way, Hintikka's key insight is that the premise allows us to analyse a number of *individuals*. I claim that, in elaborating on this intuition further, Hintikka seems to oscillate between the following two notions:

N_{prem} = the number of individuals that must be considered together in a given premise.
N_{rel} = the maximal number of individuals considered in their relation to each other in the given premise.

The first notion amounts to the number of individuals that are necessary in order to grasp the premise or, equivalently, to the number of individuals that have to be thought of in considering a given premise. Hintikka seems to privilege N_{prem} when speaking in informal terms and when philosophical explanations are required.

However, the kernel of Hintikka's work is the second notion, which crucially rests on the conception of the *relation* between individuals. N_{rel} is at the core of Hintikka's expression in formal terms of his theory of the analytic–synthetic distinction and is called upon whenever he wishes to give technical explanations or needs to make explicit the details of his sense (iii) of analyticity. To this end, Hintikka identifies N_{rel} with the degree of the given premise F, defined as the sum of two numbers[9]:

1. The number of the free singular terms of F and
2. The maximum length of nested sequences of quantifiers in F, called the *depth of* F, $d(F)$, recursively defined as follows:

   - $d(F) = 0$ for F atomic
   - $d(\neg F) = d(F)$
   - $d(F_1 \wedge F_2) = d(F_1 \vee F_2) = d(F_1 \rightarrow F_2) = \max(d(F_1), d(F_2))$
   - $d(\forall x F) = d(\exists x F) = d(F) + 1$

[9] Hintikka (1973, pp. 141–2), with minor modifications.

This definition will be discussed in detail in the next section.

In his writings, Hintikka shifts the focus away from N_{prem} and N_{rel} with a certain ease and naturalness. A telling example of this *modus operandi* is provided by a key passage of his paper *An Analysis of Analyticity*, where N_{rel} is introduced for the first time (Hintikka, 1973, p. 138 and ff.). Here, he starts suggesting that N_{prem} is the fundamental basis for his sense (iii) of analyticity:

> What is it that a premiss gives us to be analysed? Perhaps the most concrete answer to this question is to say that it gives us a number of individuals to be considered. If this answer is accepted, then a step of argument is analytic if and only if it does not introduce any new individuals. (Hintikka, 1973, p. 136)

In order to define N_{prem}, Hintikka holds that not only do free singular terms but also quantifiers (and the variables they bind) invite us to consider individuals (Hintikka, 1973, pp. 138–139). After that, he dwells on the relation between free singular terms and variables bound by quantifiers (Hintikka, 1973, p. 140), which I will discuss in the next section. Finally, Hintikka feels ready to leap from N_{prem} to N_{rel}, saying that the restriction of bound variables by the scope of the quantifier has the effect that

Parallel quantifiers (i.e., quantifiers whose scopes do not overlap) cannot be said to add to the number of individuals we have to consider in their relation to one another. They may introduce new cases to be considered, but they do not complicate the complexes of individuals we have to take into account. (Hintikka, 1973, p. 140)

This completes Hintikka's reasoning, and the text goes on with the proposal of the degree of a formula as a formal definition of N_{rel}. As this reconstruction makes clear, N_{prem} and N_{rel} are treated as interchangeable notions with no apparent justifying reason to support this.

The excerpt quoted above allows me to make a preliminary distinction that will be useful when clarifying N_{prem} and N_{rel}. Hintikka is aware that, in general, N_{rel} (and also N_{prem}, as a consequence of the observation above) is different from a third notion that we might call N_{ment} and make explicit as follows:

$$N_{ment} = \text{the number of individuals that are mentioned in a given premise.}$$

Recall that, according to Hintikka, the individuals represented by variables bound by parallel quantifiers do not add to the number of individuals considered in relation to each other. As a result, $N_{rel} < N_{ment}$ whenever parallel quantifiers are involved. This is because N_{rel} does not take into account the quantifiers whose scopes do not overlap but only the maximum length of the nested sequences of quantifiers.

To see this point, consider the formula $\varphi_1 = \forall x Fx \wedge \forall y Gy$. The names of the distinct individuals mentioned in φ_1 are x and y, thus, $N_{ment}(\varphi_1) = 2$. However, the degree of φ_1 is equal to 1, given that the two universal quantifiers are not nested. Similarly, the number of distinct individuals that must be considered together to grasp φ_1 is also equal to 1. This is because, to grasp that every individual has property F and every individual has property G, we only need to consider that one single generic individual has both properties (in formal terms, $\varphi_1 \equiv \forall x(Fx \wedge Gx)$). Thus, we have that $N_{prem}(\varphi_1) = N_{rel}(\varphi_1) = 1$ while $N_{ment}(\varphi_1) = 2$.

Despite Hintikka's treatment of the two notions, I claim that N_{prem} and N_{rel} are different in a substantial way and that this is because the identification between 'variables bound by parallel quantifiers' and 'individuals that are superfluous to grasp the premise' is not true in general. First, it may be the case that $N_{prem} > N_{rel}$, which will happen in the given premise whenever some parallel quantifiers are not superfluous to grasping the premise. Consider, as an example, the formula $\varphi_2 = \exists x Fx \wedge \exists y Gy$. While $N_{rel}(\varphi_2) = 1$, I argue that $N_{prem}(\varphi_2) = 2$. The reason why we are forced to take into account two distinct individuals is that φ_2 does not specify whether the individual with property F is the same individual with property G (in formal terms, it is not the case that $\exists x(Fx \wedge Gy)$ follows from φ_2). Therefore, formulae like φ_2 (as well as, for example, $\forall x Fx \vee \forall y Gy$) make clear that, in general, parallel quantifiers are not superfluous; in some cases, there are some individuals that necessarily have to be considered together in order to grasp the premise even if they are not related in Hintikka's sense.

Second, it may also be the case that $N_{prem} < N_{rel}$, which will happen whenever some nested quantifiers are superfluous to grasping the premise. Consider, as an example, the formula $\varphi_3 = \forall x \forall y \forall z(Rxy \wedge Syz)$. At first sight, it might seem that $N_{prem}(\varphi_3) = N_{rel}(\varphi_3) = 3$. Nevertheless, a closer inspection reveals that either the former or the latter universal quantifier that occurs in φ_3 is superfluous for grasping the meaning of the formula; in order to understand that every couple of individuals is in the relation R and the relation S, two, not three, individuals are needed (in formal terms, $\forall x \forall y \forall z(Rxy \wedge Syz) \equiv \forall x \forall y(Rxy \wedge Syx)$). Formulae like φ_3 point out that, in some cases, there are some related individuals that are superfluous to grasping the premise, and, in general, the set of individuals related in a certain premise is not a subset of individuals that have to be considered in that premise.

Up to this point, I have argued that, in elaborating his conception of the analytic–synthetic distinction, Hintikka mainly thinks of N_{rel}, the maximal number of individuals considered in their relation to each other in the given premise. However, he easily goes back to N_{prem}, the number of individuals that must be considered together in the given premise, when speaking in informal terms. Moreover, I have shown that the two notions differ from each other and that both of them differ from a third notion, N_{ment}, which is the number of individuals that are mentioned in a given premise.

N_{rel} is an interesting notion. Nevertheless, Hintikka explains neither the necessity of its introduction in his picture nor his preference for it over N_{prem}. It could be guessed that he has been driven to it due to technical reasons, such as his distributive normal forms and his aim of providing a description of possible worlds alternative to Carnap's state-descriptions (Hintikka, 1973, pp. 154–63), or due to philosophical motivations, such as his reading of the ancient method of analysis as described by Pappus, which is based on the idea of the configuration of individuals (Hintikka & Remes, 1976, p. 266). In any case, I argue that N_{prem} is a much more natural notion than N_{rel} (and this probably explains why Hintikka employs the latter in informal contexts) and that an immediate and formal translation of N_{prem}, capable of making all of its features explicit, is possible (see Sects. 10 to 12).

6 On the degree of a formula

In the previous section, we have seen that Hintikka oscillates between two different notions, namely N_{prem} and N_{rel}. In what follows, I shall argue against Hintikka (1973, p. 141) that neither the former nor the latter is identical to the notion of the degree of a formula. I start by considering N_{rel}, i.e. the maximal number of individuals considered in their relation to each other in a certain premise.

As it has already been noticed in the literature (see Van Benthem, 1974, p. 422; Rantala, 1987, pp. 72–73), *the notion of the degree of a formula adds to the sum of related individuals also unconnected individuals represented through bound variables*. To see the point, consider the formula $\varphi_4 = \forall x \forall y(Fx \wedge Gy)$. Following the definition (see Sect. 5), the degree of φ_4 is 2, because the quantifiers binding x and y are nested. Nevertheless, x and y do not seem to be related in φ_4 in any reasonable sense of the word. I think that the real problem does not lie in the struggle between the formal notion of degree and our intuitive grasp of the term *relation* (for philosophical analysis is also devised to correct our intuitions, if necessary) but rather in the fact that Hintikka's notion of degree dismisses parallel quantifiers as superfluous, while admitting nested and unconnected quantifiers: as a result, the degree of $\varphi_1 = \forall x Fx \wedge \forall y Gy$ is one, and the degree of $\varphi_4 = \forall x \forall y(Fx \wedge Gy)$ is two, while the two formulae are equivalent.

Hintikka himself, in a footnote of his *An Analysis of Analyticity*, discusses the issue of unconnected quantifiers and proposes a sharper definition of degree that takes into account.

> Only such bound variables x_1, x_k as inviting us to consider individuals that are related to each other in the sentence in the sense that there is a sequence of bound variables $x_1, x_2, \ldots, x_{k-1}, x_k$ with the following properties: for each $i = 1, 2, \ldots, k-1$, x_i and x_{i+1} occur in the same atomic subsentence or identity of F; each variable x_i is bound to a quantifier occurring within the scope of the wider of the two quantifiers to which x_1 and x_k are bound. Let us call such variables and the quantifiers to which they are bound *connected*. [...] The maximal number of nested and connected quantifiers in F is now called the depth of F. (Hintikka, 1973, p. 142, n. 33).

Following Hintikka's suggestion, let the *degree*$^+$ of a formula F be the sum of the number of the free singular terms of F and the maximal number of nested *and connected* quantifiers in F. As it is clear, the degree$^+$ of φ_4 is 1, because the two quantifiers are not connected in that formula. Thus, at first glance, the new definition seems to work well.

Unfortunately, this is not the case. I argue that the notion of degree$^+$ is affected by a similar problem, this time concerning free variables instead of bound variables. As I anticipated above, according to Hintikka (1973, pp. 138–139), two different expressions invite us to consider individuals: (1) free singular terms and (2) quantifiers with the variables bound to them. Moreover, he explicitly recognises that the difference between these two kinds of expressions is minimal:

> The intimate relation of quantifiers and of the variables bound to them to free singular terms is shown by the fact that for many purposes quantifiers may be

> omitted and the variables bound to them replaced by suitable free singular terms. Of course, this is just what one tries to do in applying natural deduction methods. Roughly speaking, there usually is little to choose between a free singular term and a variable which is bound to a quantifier whose scope comprises the whole formula (with the possible exception of a string of initial quantifiers), in the sense that the logical operations one is allowed to perform are to most intents and purposes exactly the same in the two cases. Hence we must conclude that if individuals are introduced into one's logical arguments by free singular terms, they are likewise introduced by quantifiers, too. What really distinguishes a bound variable from a free singular term, one might almost say, is not so much its being bound to a quantifier as its being restricted by the scope of the quantifier. (Hintikka, 1973, p. 140)

After all, an open formula is satisfiable if and only if its existential closure is true, and valid if and only if its universal closure is true. Thus, according to Hintikka, there is 'little to choose' between an open formula such as $\varphi_5 = Fx \wedge Gy$ and its closure $\varphi_5' = Q_1 x Q_2 y (Fx \wedge Gy)$, where Q_1 and Q_2 are two quantifiers.

However, we find that, while the degree$^+$ of φ_5' is one, because x and y are not connected in the closed formula, the degree$^+$ of φ_5 is two. This example points at a more general problem regarding the notion of degree$^+$, namely, that *while individuals that are expressed through* bound *variables are required to be connected in order to be considered, individuals that are expressed through* free *variables count no matter what*, even if they are not related in any reasonable sense of the word. This clashes with Hintikka's insight that individuals represented through bound variables are almost identical to individuals represented through free variables.

Notice that the fact that individuals expressed through free variables add to the number of related individuals independently of the actual relations between them affects not only the notion of degree$^+$, but also that of degree. Moreover, the same difficulty persists even in a third definition of degree, which is sharper than degree$^+$ and originates from the suggestion that follows:

> Let us consider two quantifiers occurring in F such that the latter occurs within the scope of the former and that the bound variables occurring in them are x and y, respectively [...] we have related to each other the individuals introduced by these two quantifiers if x and y occur (bound to the quantifiers in question) in one and the same atomic part in F. In this case we shall say that the quantifiers (and the variables bound to them) are *immediately related* to each other in F (Hintikka, 1973, pp. 18–19).

According to this hint, let the *degree** of a formula F be the sum of the number of the free singular terms of F and the maximal number of nested *and immediately related* quantifiers in F. This notion is never discussed in detail by Hintikka (1973). Probably it is too strict for representing an intuitive notion of relation (for example, the degree* of $\varphi_6 = \exists x \exists y \exists z (Rxy \wedge Syz)$ is two and not three).

Thus, I can conclude that neither the notion of degree of a formula F nor the notions of degree$^+$ and degree* of a formula F are adequate translations of N_{rel}, namely, the maximal number of individuals considered in their relation to each other in F. At this point, we might ask ourselves: isn't it the case that one of Hintikka's notions of degree is an adequate translation of N_{prem} (instead of N_{rel})? This question is relevant because, as I argued in Sect. 5, Hintikka seems to consider the difference between N_{prem} and N_{rel} to be minimal. But unfortunately, I think that the answer is negative also in this case.

First of all, the notion of degree is not an adequate translation of N_{prem} because *there might be some parallel individuals in F that have to be considered together in F and some nested individuals in F that do not have to be considered together in F*. For the former case, the example is $\varphi_2 = \exists x Fx \wedge \exists y Gy$. Here, the degree of φ_2 is 1, but it is necessary to consider two individuals to grasp the meaning of φ_2. For the latter case, the example is $\varphi_3 = \forall x \forall y \forall z(Rxy \wedge Syz)$. Here, the degree of φ_3 is 3, but it is sufficient to consider only two individuals to grasp the meaning of φ_3.

Second, the notion of degree$^+$ is not an adequate translation of N_{prem}. This is because *there might be some unconnected individuals in F that have to be considered together in F, and some connected individuals in F that do not have to be considered together in F*. The example for the latter case is φ_3, as above. For the former, while it is true that unconnected individuals are never related, it is not always the case that unconnected individuals do not add to the number of individuals that must be considered together in a given formula. Consider, as an example, the formula $\varphi_7 = \exists x \exists y(Fx \wedge Gy)$. Here, the degree$^+$ of φ_7 is 1, while both individuals, although unconnected, are needed to grasp the meaning of φ_7 (for, again, we cannot assume that the same individual has both the property *F* and the property *G*).

Last, the notion of degree* is not an adequate translation of N_{prem}. Although it manages to exclude from the computation all those related individuals that are superfluous to grasp the premise, such in the example of φ_3 examined above, it might also exclude related individuals that are not superfluous, as in $\varphi_6 = \exists x \exists y \exists z(Rxy \wedge Syz)$.

Thus, the previous conclusion can be strengthened by the following claim: Hintikka's notions of the degree of a formula are adequate translations neither of N_{prem} nor of N_{rel}.

7 On sense (iii.e)

As I explained in the introduction, Hintikka (1973, p. 145) regards senses (iii.a) to (iii.e) as 'variations of one and the same basic idea' expressed by sense (iii). Among them, sense (iii.e) has a prominent place. It is said to be 'the more interesting and important' (Hintikka, 1973, p. 193) of the two senses (iii.d) and (iii.e) because it is defined for arguments and not for argument steps, while its competitor, sense (iii.d), 'appears rather unsatisfactory' (Hintikka, 1973, p. 194). Moreover, sense (iii.e) is at the core of several Hintikkian papers.[10] Nevertheless, I think that three key features of sense (iii.e) are problematic, either because they depart in a substantial way from sense (iii) (beyond the fact that sense [iii.e] applies to arguments as a whole and sense [iii.d] to single argument steps) or because they are not general enough.

[10] For example, *Kant Vindicated* in Hintikka (1973, pp. 174–98).

First, sense (iii) says that a valid argument step is synthetic whenever new individuals are introduced into the discussion, and sense (iii.e) provides a method to determine whether new individuals are introduced into an argument. This recipe says: compute the degree of the premise, the conclusion and each of the intermediate stages of the argument. If the degree of one of the intermediate stages is higher than the degree of the premise or the degree of the conclusion, one can deduce that the resulting argument is synthetic.

My worry is that the way of counting new individuals indicated by sense (iii.e) is not correct. It may be the case that in a certain derivation, it is necessary to introduce a number of distinct individuals n such that n is higher than the degree of the premise and the conclusion, yet no intermediate stage has degree n. How is this possible? It may happen[11] that the n individuals are never mentioned together in the same step or, as Hintikka (1973, p. 193) himself recognises, that some intermediate stage 'is a compressed form of a longer chain of arguments, some intermediate stages of which are of a very high degree indeed'. Thus, the recipe given by sense (iii.e) may not be adequate to count the number of new individuals introduced into the argument. The same conclusion can quickly be drawn for sense (iii.d) in a similar manner.

Second, sense (iii.e) differs from sense (iii.d) in that only the former prescribes to take into account not only the premise but also the conclusion of an argument in determining whether it is analytic or synthetic. Hintikka then explains:

> That this sense is more natural than sense III (d) is suggested by the fact that by contrapositing all the steps of a proof of F_2 from F_1, we obtain a proof of $\sim F_1$ from $\sim F_2$. In sense III (e), the old and the new proofs are synthetic or analytic simultaneously, as one seems to be entitled to expect on the basis of the fact that the difference between the two proofs seems inessential for our purposes. In sense III (d), however, one of them may be synthetic and the other analytic, which seems rather odd. (Hintikka, 1973, p. 144).

It is not clear why this consequence should be especially welcome. First of all, because, as van Benthem (1974, pp. 424–425) has rightly noticed, every immediate inference (i.e. one-step argument) becomes analytic. Moreover, there are even some conceptual difficulties for longer arguments. Consider the example in which the premise P is given by $\forall x \exists y Rxy$ and the conclusion C is given by $\forall x \exists y \exists z (Rxy \wedge Ryz)$. Here, the degrees of P and C are two and three, respectively. Now, in cases like this one, why should a derivation and its contrapositive be either both analytic or both synthetic? Concluding C from P requires introducing a new individual; in contrast, the proof from $\neg C$ to $\neg P$ needs only to analyse the three individuals occurring in $\neg C$. This is the difference between these two proofs, and it cannot be regarded as inessential from the perspective of sense (iii).

Third, sense (iii.e) is determined only for one-premise arguments. Although Hintikka does not explicitly indicate how to extend this definition, it seems reasonable to follow what he says about the method of model set construction: 'the degree of a finite set of sentences may be defined simply as the degree of the conjunction of all

[11] An example is given in D'Agostino, Larese and Modgil (2021, p. 447).

its members' (Hintikka, 1973, p. 184). However, this suggestion is misleading, since it inherits the difficulties regarding parallel quantifiers highlighted in Sects. 5 and 6.

For example, let $\Gamma = \{\exists x Fx, \exists x Gx\}$ be a set premise. Following Hintikka's prescription, the degree of Γ is the degree of the conjunction $\exists x Fx \wedge \exists x Gx$, which is equal to one. However, for the reasons analysed earlier, the number of individuals we have to think of to understand that conjunction is two, not one because we cannot assume that it is the same individual that has both the property F and the property G.

I claim that these three problems that affect sense (iii.e) all stem from the same source. Although Hintikka is not very explicit on this point, he seems to regard his sense (iii) and all of its variations as general, in the sense that they are independent of the proof system in which derivations are carried out[12] and that they are not entirely defined until a certain proof system has been fully specified. Hintikka states:

> In many of the senses we have defined, the analyticity of a logical truth or the possibility of deriving a sentence from another analytically depends on the underlying selection of axioms and of rules of proof. In order to specify these senses of analyticity more closely, we shall therefore have to say more about the selection of axioms and of rules of inference. Only when the principles of this selection are described more carefully can we say that we have fully defined the relevant senses of analyticity. (Hintikka, 1973, p. 144)

Yet, I argue that the three difficulties described above show that the opposite is true: sense (iii.e) is not general at all but is shaped rather on the proof procedure based on Hintikka's distributive normal forms. This hypothesis would explain, first of all, the role of the degree of the intermediate stages of an argument in establishing whether a certain proof is analytic or synthetic. From the way it has been defined, there is always an intermediate stage in which all the individuals needed in the proof occur together. Second, it would clarify Hintikka's choice of taking into account not only the premise but also the conclusion of a given argument—the proof procedure based on distributive normal forms essentially consists of the combination of the parameters of the premise and the conclusion and the comparison of the non-trivially inconsistent constituents of their expanded versions. Third, this hypothesis also illustrates the restriction to one-premise arguments.

This should be enough to conclude that Hintikka's sense (iii.e) is essentially the same as the explicit form of sense (iii), except that the former is described with fewer technical details. Now, the existing literature has correctly underlined several unsatisfactory features of Hintikka's analytic–synthetic distinction based on the theory of distributive normal forms. The major problem is the complexity of the proof procedure. Rantala and Tselishchev (1987, p. 89) admit that 'as an actual method, the use of normal forms is not very practical', while Lampert, using the results obtained by Nelte (1997, Sect. 4.1), provides a fairly clear picture of the extent of this unpracticality. He says:

> Even if one considers only formulas of pure FOL without names and functions, only one binary predicate, and formulas of depth 2, this leads to FOLDNFs with

[12] This is confirmed, for example, by Hintikka's argument of the inevitability of synthetic elements in first-order logic independently of the proof system. See Hintikka (1973, pp. 178–185).

> 2512 disjuncts, where each disjunct contains 512 conjuncts. Thus, the length of Hintikka's distributive normal form for even the simple formula $\exists x \exists y Fxy$ is 2^{512}. Merely increasing the depth by one already results in $2^{21+2^{35}}$ possible disjuncts. (Lampert, 2017, p. 326).

Sure enough, Hintikka's aim in formulating his proof procedure based on distributive normal forms is different from that of Gentzen in his seminal paper on natural deduction: Gentzen is neither interested in setting up a formal system that comes as close as possible to actual reasoning (Gentzen, 1935, p. 176) nor concerned with the complexity of this proof system. On the contrary, the goals of his theory are of a more theoretical nature, for example, to provide a description of possible worlds as exhaustive as our limited resources allow us to give or to define a measure of information that might increase through deductive practice. Yet, the extreme difficulty of distributive normal forms for first-order logic is an insurmountable obstacle to using sense (iii.e) as a workable definition of the analytic–synthetic distinction.

8 On the rule of existential instantiation

According to Proclus, the proof of a theorem and the solution to a problem in Euclid's *Elements* comprise six main parts.[13] The second main part is called *ecthesis* and immediately follows the general enunciation of the proposition in question. It consists of the exhibition of a particular figure that sets out the geometrical entities with which the general enunciation deals. This step (together with further determinations and constructions) allows the geometer to carry out the proof proper or *apodeixis* on that particular figure and to conclude by extending the result to the general case, given that the particular determinations of the specific figure are utterly indifferent to the proof of the proposition.

The notion of *ecthesis* is the starting point of Hintikka's reconstruction of Kant's analytic–synthetic distinction. Here, we are not interested in the details of this complex reconstruction but rather in two of its major turning points. First, Hintikka holds that, according to Kant, it is mainly the use of ecthesis, which amounts to the exhibition of a certain figure, that makes the geometrical method synthetic. The figure introduced through ecthesis goes 'beyond the given concept in order to consider something entirely different from what is thought in it as in a relation to it' (Kant, 1998, A154/B193), and yet it is necessary to carry out the demonstration. Second, Hintikka generalises the reasoning over the geometrical field and maintains that, for Kant, the use of constructions, in general, makes any kind of argument synthetic. For Hintikka's interpretation of Kant, constructions are the a priori exhibition of intuitions, which are nothing more than singular representations. This is the meaning of Hintikka's (1973, p. 205) claim that Kant is 'an heir to the constructional sense of analysis'.

Against the backdrop of his historical reconstruction of Kant's thought, Hintikka pushes this reasoning further. First, he claims that for all practical purposes, ecthesis is identical to existential instantiation (from now on, *ExistInst*) in first-order logic (Hintikka, 1967, pp. 368–369; 1973, p. 111). This pattern of reasoning allows us to

[13] For an extended discussion and critical analysis of this issue, see Acerbi (2019).

infer from an existentially quantified sentence $\exists x Fx$, a sentence instantiating it, e.g. $F(x/a)$, where a is a free individual symbol that does not occur earlier in the argument, and $F(x/a)$ is the result of replacing x with a in F. Second, Hintikka argues that the rule of ExistInst is the paradigmatic example of synthetic argument steps, in which new individuals are introduced into quantified arguments. In fact, he even holds that 'in suitable formulations', synthetic arguments 'can be boiled down to existential instantiation' (Hintikka, 1973, p. 211).

However, which formulations are 'suitable'? Hintikka takes into account the issue of ExistInst in the context of three different proof systems, two of which have been formulated by the Finnish logician himself: the method of model set constructions (Hintikka, 1973, pp. 7–18; Smullyan, 1995, pp. 27, 57) and the method based on distributive normal forms (Hintikka, 1973, pp. 242–286). For the former, Hintikka claims that the rule of ExistInst, which enables the addition of $F(x/a)$ to a set μ provided that $\exists x Fx \in \mu$ and that a is new, always introduces a new individual into the argument. Moreover, ExistInst is the only synthetic rule, since the rule of universal generalisation, which enables the addition of $F(x/a)$ to a set μ provided that $\forall x Fx \in \mu$, requires that a already occurs in the sentences of μ.

For the latter, namely, the method based on distributive normal forms, Hintikka argues that 'there appears to be a close connection between the applications of this rule and the increase of depth in the expansion process' (Hintikka, 1973, p. 184). Without getting into technicalities, the expansion process is what is necessary to recognise that a certain formula is synthetically derivable from a given premise or, which amounts to the same thing, that all the non-trivially inconsistent constituents of the expanded premises are among the non-trivially inconsistent constituents of the expanded conclusion. Nevertheless, Hintikka (1973, p. 184) admits that 'the details of this connection are in need of closer study', and, might I add, the very existence of this connection is in need of any sort of proof whatsoever.

When it comes to the discussion of ExistInst in the third proof system, namely, natural deduction, Hintikka is, I think, even more ambiguous. An immediate objection against the identification of ExistInst with the synthetic argument step par excellence is that the premise and the conclusion of the application of this rule always have the same degree. This is because, once the rule has been applied, the individual referred to by the existential quantifier is represented by the fresh individual symbol that replaces the free variable in the open formula. Therefore, according to this reasoning, ExistInst cannot be a synthetic argument step and cannot introduce any new individual. Hintikka seems to have this objection in mind when he argues:

> When in a system of natural deduction one goes from $\exists x Fx$ to an expression of the form $F(x/a)$ (i.e., F with 'x' replaced by 'a'), the legitimacy of this step depends on the choice of the free singular term 'a'. The usual requirement is that 'a' must not coincide with any of the free singular terms occurring earlier in the proof, that it must be a *new* term. Hence this step depends, not just on the sentence (formula) $\exists x Fx$, but also on all the free singular terms which do not occur in it but which occur at earlier stages of the proof. If their number is added to the degree of $\exists x Fx$, we have a parameter which is a more realistic measure of the number of individuals we are considering in the step of proof in question. (Hintikka, 1973, p. 183).

Hintikka's point is that ExistInst does not satisfy the requirement of being invariant with respect to the permutation of free singular terms. As a result, a more accurate measure of the number of individuals that have to be considered in a step of ExistInst is given by the sum of $d(\exists x Fx)$, which is the degree of the premise, and $s(\exists x Fx)$, which is the number of free singular terms that do not occur in $\exists x Fx$ but occur at earlier stages of the proof.

It seems fair to say that Hintikka's hint, which unfortunately is never developed any further, has the effect of establishing a distinction between analytic and synthetic applications of ExistInst. In the analytic ones, $s(\exists x Fx) = 0$, and the number of individuals that have to be considered in the premise of ExistInst is the same as the number of individuals of the conclusion—it simply amounts to the degree of the formulae involved. In the synthetic ones, the number of individuals that have to be considered in the premise of ExistInst is greater than the number of individuals in the conclusion because there are some individuals that occur in the proof above but do not occur in the premise of that step, i.e. $s(\exists x Fx) \neq 0$. My contention is that although there is a distinction between analytic and synthetic applications of ExistInst, Hintikka's hint goes in the wrong direction. The point is that $s(\exists x Fx)$ is too broad; there may be free singular terms that occur at earlier stages of the proof but not in $\exists x Fx$. These free singular terms are not relevant to the issue of establishing whether an application of ExistInst is analytic or not.

To sum up, the rule of ExistInst is crucial in Hintikka's system and is said to be almost identical to *ecthesis* and to the use of constructions, and represents the paradigmatic example of a synthetic argument step. Yet, Hintikka is not clear enough on the role played by this rule either in the context of his explicit form of sense (iii) based on distributive normal forms or in the systems of natural deduction; i.e. his explanation of when an application of ExistInst introduces a new individual into the argument (and is thus synthetic) and when it does not is not satisfying.

9 An overall evaluation

We have seen that one of Hintikka's most fruitful intuitions concerning the analytic-synthetic distinction is to investigate the theoretical consequences of using individuals, instead of concepts, as the object of analysis provided by a set of premises. As I showed above, this shift leads to formulating his sense (iii), according to which an argument is analytic if it does not introduce new individuals into the discussion and synthetic otherwise. Section 3 pointed out the reasons why Hintikka's contribution deserves to be taken into serious account in the debate on the analytic-synthetic distinction, namely, the rehabilitation it offers of Kant's definition in the context of modern first-order logic and the positions it defends concerning the status of logic and mathematics. Sections 4 to 8 identified the conceptual problems that arise when Hintikka tries to formalise his sense (iii).

To be more precise, I now point out the questions and the road map that Hintikka subtly individuates to achieve his aim. Moreover, I highlight the precious suggestions that he puts forward towards his purpose. Then, I recapitulate the difficulties that affect his answers to these questions, which I think are substantial to the effect that Hintikka's results are in need of a complete revision.

The first question towards a formalisation of Hintikka's sense (iii) is: *how should the number of individuals considered together in the premises be represented?* Despite important observations, such as that not all the individuals mentioned together in a given premise are necessary to grasp it, I have argued that Hintikka's answer is marred by the following problems, which concern the adequacy of both N_{rel} and the notions of the degree of a formula:

- Problem P1 (see Sect. 5): Hintikka oscillates between N_{prem} (used in informal and philosophical explanations) and N_{rel} (the kernel of his theory). These two notions are not equivalent.

  P1.1 In some cases, there are some individuals that necessarily have to be considered together in order to grasp the premise even if they are not related (example: $\varphi_2 = \exists x Fx \wedge \exists y Gy$).
  P1.2 In some cases, there are some related individuals that are superfluous in order to grasp the premise (example: $\varphi_3 = \forall x \forall y \forall z(Rxy \wedge Syz)$).

- Problem P2 (see Sect. 6): Neither the notion of degree of a formula F nor the notions of degree$^+$ and degree* of a formula F are adequate translations in formal terms of N_{rel}.

  P2.1 The notion of the degree of a formula adds to the sum of related individuals also *unconnected* individuals represented through bound variables (example: $\varphi_4 = \forall x \forall y(Fx \wedge Gy)$).
  P2.2 The notions of degree, degree$^+$ and degree* of a formula add to the sum of related individuals also unconnected individuals represented through free variables (example: $\varphi_5 = Fx \wedge Gy$).

- Problem P3 (see Sect. 6): Neither the notion of degree of a formula F nor the notions of degree$^+$ and degree* of a formula F are adequate translations in formal terms of N_{prem}.

  P3.1 The notion of degree is not an adequate translation of N_{prem}, because there might be some parallel individuals in F that have to be considered together in F and some nested individuals in F that do not have to be considered together in F (examples: $\varphi_2 = \exists x Fx \wedge \exists y Gy$ and $\varphi_3 = \forall x \forall y \forall z(Rxy \wedge Syz)$).
  P3.2 The notion of degree$^+$ is not an adequate translation of N_{prem}, because there might be some unconnected individuals in F that have to be considered together in F, and there might be some connected individuals in F that do not have to be considered together in$F$ (examples: $\varphi_7 = \exists x \exists y(Fx \wedge Gy)$ and $\varphi_3 = \forall x \forall y \forall z(Rxy \wedge Syz)$).
  P3.3 The notion of degree* is not an adequate translation of N_{prem}, because there might be some not immediately related individuals in F that have to be considered together in F (example: $\varphi_6 = \exists x \exists y \exists z(Rxy \wedge Syz)$).

The second question towards formalising sense (iii) is: *when are new individuals introduced into the arguments?* Although Hintikka realises that to establish whether an argument is analytic or not, a comparison with the number of individuals occurring in the premise is needed, I have claimed that Hintikka's proposal is affected by several problems, which mainly have to do with his sense (iii.e) and its unwelcome consequences:

- Problem P4 (see Sect. 7): Sense (iii.e), Hintikka's most favourite variation of sense (iii), is not proof-system independent, but is shaped instead on the proof procedure based on the theory of distributive normal forms.

  P4.1 Sense (iii.e) cannot recognise as synthetic all the arguments in which new individuals are introduced.
  P4.2 In determining whether an argument is analytic or synthetic, sense (iii.e) takes into account the degree not only of the premise but also of the conclusion of that argument.
  P4.3 Sense (iii.e) is determined only for one-premise arguments, and its immediate extension inherits problems P1.1 and P1.2.

- Problem P5 (see Sect. 4): Hintikka classifies propositional and monadic arguments as analytic. This is highly counterintuitive and does not consider the probable intractability of Boolean logic.

The third and last question towards a formalisation of sense (iii) is: *how can we distinguish between analytic and synthetic rules?* Again, despite Hintikka's insight that there are both analytic and synthetic applications of *ExistsInst*, I have shown some problems in his reasoning, which concern the role of existential instantiation and the complexity of distributive normal forms:

- Problem P6 (see Sect. 8): The rule of existential instantiation is crucial in Hintikka's system. Yet, his explanation of when an application of the rule of existential instantiation introduces a new individual into the argument (and is thus synthetic) and when it does not is not satisfying.
- Problem P7 (see Sect. 7): The explicit form of sense (iii), which at the end almost coincides with sense (iii.e), is too complex, and fails to distinguish between analytic and synthetic arguments.

As for the *pars construens* of this paper, I now follow the road map hinted at by Hintikka's work, together with his profitable suggestions. In Sects. 10 to 12, I select from the approach put forward by D'Agostino et al. (2021) alternative answers to the three questions raised above. I claim that the solution proposed according to this plan avoids the difficulties that make sense (iii.e) flawed and offers a *workable* and Kantian-inspired notion of syntheticity in the context of first-order logic. In so doing, I hope that Hintikka's key insight, once released from the problems affecting his further elaboration, could seize the place it deserves in the debate on the analytic-synthetic distinction (see Sect. 3).

D'Agostino et al. (2021) provide a natural deduction proof-system for a hierarchy of decidable depth-bounded approximations of classical first-order logic that expands the hierarchy of tractable approximations of Boolean logic known as Depth-Bounded Boolean Logics investigated in D'Agostino and Floridi (2009), D'Agostino et al. (2013), D'Agostino (2015). The basic element of the hierarchy, logic $\vdash_0$, admit only *analytic* proofs, while the other elements in the hierarchy, logics $\vdash_k$ for $k > 0$, also admits *synthetic* proofs to an increasing depth.

10 Towards an alternative representation of the number of individuals considered together in the premises

I have shown in Sect. 5 that Hintikka oscillates between N_{prem} and N_{rel}, and that the latter notion cannot be taken as a formal translation of the former, because the two differ in a substantial way. In this section, I propose an alternative representation in formal terms of N_{prem}, which I think is a much more natural and intuitive notion than N_{rel}. Recall that N_{prem} is the minimal number of individuals that have to be considered together in a given premise or, equivalently, the number of necessary and sufficient distinct individuals that have to be taken into account to grasp or represent that premise.

D'Agostino et al. (2021) provide a good candidate for a formal definition of N_{prem}. First of all, recall that a formula is in *prenex normal form* (PNF) if it has the form.

$$Q_1 x_1 \ldots Q_n x_n G[x_1, \ldots, x_n],$$

where each Q_i is either an occurrence of $\forall$ or an occurrence of $\exists$, G is quantifier-free and all variables in G are bound by some quantifier in the prefix. Then, a formula is said to be in *minimal prenex normal form* (min-PNF) if there is no logically equivalent formula with the same matrix and a lower number of occurrences of quantifiers in the prefix. Given these notions, N_{prem} might be defined in the following way:

Definition 1: The minimal number of individuals that have to be considered together in a given closed formula F is the number of occurrences of quantifiers in the prefix in one of the min-PNFs of F (see D'Agostino et al., 2021, p. 433).

Consider the following four examples:

1. Let φ_1 be, as in Sect. 5, the closed formula $\forall x Fx \wedge \forall y Gy$, which is not in PNF. One of the min-PNFs of φ_1 is $\forall x(Fx \wedge Gx)$, and the number of occurrences of quantifiers in its prefix is one.
2. Let φ_2 be, as in Sect. 5, the closed formula $\exists x Fx \wedge \exists y Gy$, which is not in PNF. One of the min-PNFs of φ_2 is $\exists x \exists y(Fx \wedge Gy)$, and the number of occurrences of quantifiers in its prefix is two.
3. Let φ_3 be, as in Sect. 5, the closed formula $\forall x \forall y \forall z(Rxy \wedge Syz)$. φ_3 is in PNF but not in min-PNF because one of the quantifiers is redundant. One of the min-PNF of φ_3 is $\forall x \forall y(Rxy \wedge Syx)$, and the number of occurrences of quantifiers in its prefix is two.

I argue that Def. 1 avoids all the difficulties that, I have claimed, affect Hintikka's representation of N_{prem}. On the one hand, as shown in Sect. 5, Hintikka excludes from the computation individuals represented through parallel quantifiers. However, I pointed out that not all parallel quantifiers are bad, i.e. superfluous for grasping the meaning of the formula involved (see P1.1), and not all nested quantifiers are good, i.e. essential for grasping the meaning of the formula involved (see P1.2).

On the other hand, according to Definition 1, every quantifier in one of the min-PNFs of F is nested, i.e. there are no parallel quantifiers to exclude from the computation. Yet *all and only* superfluous quantifiers, whether parallel or not in F, are removed in the min-PNFs of F. Thus, Def. 1 solves problem P1. Example 1 shows a case in which a superfluous parallel quantifier is removed; example 2 illustrates a case in which a parallel but essential quantifier is included in the computation; example 3 exhibits a case in which nested but superfluous quantifiers are removed.

Moreover, as we have seen in Sect. 6, Hintikka refines his definition through the notion of degree$^+$ and excludes from the computation individuals represented by unconnected quantifiers. However, I argued that *some but not all* unconnected quantifiers are bad (see P3.1 and P3.2). Def. 1 overcomes this obstacle because, again, it removes from the min-PNF of F *all and only* unconnected quantifiers in F that are superfluous. For example, one of the min-PNF of $\varphi_4 = \forall x \forall y(Fx \wedge Gy)$ is $\forall x(Fx \wedge Gx)$, while one of the min-PNFs of $\varphi_7 = \exists x \exists y(Fx \wedge Gy)$ is φ_7 itself. Something similar happens for individuals represented through connected but not immediately related quantifiers (against P3.3).

Last, Def. 1 applies only to closed formulae: as a result, it is more restricted than Hintikka's notions of degree, but, at the same time, it succeeds in avoiding problems, such as P2.2, that might arise considering free individual variables.

This definition also outperforms Hintikka's proposal because, against P4.3, it might be extended naturally from one-premise arguments to multiple-premise arguments. D'Agostino et al. (2021) define a set of formulae Γ to be in *perfect prenex normal form* (PPNF) if and only if:

- every formula in Γ is in min-PNF; and
- all occurrences of existential quantifiers in Γ bind variables that are different from each other and from all the universally quantified variables; and
- the number of distinct universally quantified variables occurring in Γ is minimal.

Every set Γ of formulae in min-PNF can be easily transformed into a set Γ' in PPNF, by renaming of variables. Then, the definition of N_1 might be extended in the following way:

Definition 2: The minimal number of individuals that have to be considered together in a given set Γ of closed formulae is the number of distinct variables that occur in Γ, called the Q-complexity of Γ (D'Agostino et al., 2021, pp. 433–435).

This extension to the multiple-premise case is natural because the Q-complexity of a set Γ of formulae in PPNF is nothing but the number of occurrences of quantifiers in the prefix in one of the min-PNFs of a conjunction of the formulae in Γ (D'Agostino et al., 2021, p. 434).

To make an example, consider the set $\Gamma = \{\exists x Fx, \exists x Gx\}$ discussed in Sect. 7. Γ is not in PPNF, because there are two occurrences of existential quantifiers binding the same variable. Γ might be transformed into a set $\Gamma' = \{\exists x Fx, \exists y Gy\}$, the Q-complexity of which is two. On the other hand, a set like $\Delta = \{\forall x Gx, \forall x Fx\}$ is in PPNF, and its Q-complexity is equal to one.

11 Towards an alternative way of specifying when new individuals are introduced into the arguments

In the previous section, I argued that the notion of Q-complexity of a set of closed formulae should replace Hintikka's notions of the degree of a formula for representing the number of individuals considered together in a given set of premises. Now, I turn to the next step of Hintikka's road map: is it possible to formulate a distinction between analytic and synthetic arguments in first-order logic, which avoids the difficulties affecting Hintikka's approach? In other words, is it possible to specify in a clear way when it is really necessary to introduce new individuals into the arguments? D'Agostino et al. (2021) offer the following definition:

Definition 3: A proof is said to be analytic if and only if: 1. it does not introduce new individuals into the argument, and 2. it does not use any piece of virtual information. A proof is said to be synthetic otherwise.

I start focusing on the first necessary condition of Def. 3, which is more closely related to Hintikka's work. In order to facilitate the individuation of cases in which new individuals are introduced into proofs, two assumptions are made by D'Agostino et al. (2021). First, the set of premises of an argument is required to be in PPNF.[14] Second, the language is assumed to contain no constants and to be equipped with a set of parameters $a, b, c, \ldots$ that may occur in the proof, but neither in the premises nor in the conclusion. Then, D'Agostino et al. (2021) assume that:

Definition 4: New individuals are introduced into a proof if and only if the number of distinct parameters in it exceeds the initial premises' Q-complexity.

I now prove that this definition overcomes the problems affecting Hintikka's sense (iii.e) highlighted in Sect. 7. First, in order to find whether new individuals are introduced into an argument, Hintikka's senses (iii.d) and (iii.e) on the one hand and Def. 4 on the other follow two different strategies: the former compares the degrees of certain formulae occurring in the proof, while the latter compares the number of distinct parameters with the Q-complexity of the set of premises. In this way, problem P4.1, which affects Hintikka's proposal, is entirely overcome by Def. 4. This is because, to be sure that every synthetic argument is isolated as such, the former strategy requires that all the new individuals necessary for the argument occur in the same argument step, while the latter does not.

Second, Def. 4, unlike Hintikka's sense (iii.e), does not take the individuals mentioned in the conclusion as given (see P4.2). As a result, the analytic-synthetic distinction proposed is sensible to the difference between the argument from P to C and the argument from $\neg C$ to $\neg P$ whenever the two formulae have a different Q-complexity. Third, against problem P4.3, the definition above is determined not only for one-premise arguments but also for multiple-premise arguments, thanks to the natural step that leads from min-PNFs to sets in PPNF (see Sect. 10).

[14] This involves no loss of generality, and allows an easier formulation of the rules for the quantifiers (see Sect. 12).

The second necessary condition of Def. 3 for a proof to be analytic is taken from the approach of Depth-Bounded Boolean Logics. Virtual information, which is forbidden in analytic proofs, is information that is by no means contained in the premises but must nevertheless be considered in order to obtain the conclusion. Consider the following examples (D'Agostino et al., 2021, p. 429):

1	$P \vee Q \ \ (Premise)$	1 $P \vee Q \ \ (Premise)$	
2	$Q \rightarrow R \ \ (Premise)$	2 $P \rightarrow Q \ \ (Premise)$	
3	$\neg P \ \ (Premise)$	3.1 Suppose that P	3.2 Suppose that $\neg P$
4	Q (from 1 and 3)	4.1 Q (from 2 and 3.1)	4.2 $\neg Q$ (from 1 and 3.2)
5	R (from 2 and 4)	5 Q	

The former proof is analytic because each step uses information that is actually possessed. In contrast, the latter proof is synthetic because it makes essential use of information that is not actually possessed, but yet it is introduced and subsequently discharged for the sake of the argument. In this case, we simulate information states that are richer than the actual one and consider the two possible outcomes of acquiring such information. As D'Agostino (2013, p. 55) explains:

> This use of virtual information, which is not contained in the data and so may not be actually held by any agent who holds the information carried by the data, appears to qualify this kind of argument as "synthetic" in a sense close to Kant's sense, in that it forces the agent to consider potential information that is not included in the information "given" to him.

D'Agostino (2014, p. 410) shows that the greater the number of pieces of virtual information needed in an inference, the greater the 'cognitive effort' required by the agent to recognize its validity and the computational resources that need to be consumed for this task, to the effect that an unbound use of virtual information, such as in Boolean logic, leads to the intractability of the corresponding decision problem.

An extremely welcome consequence of this second condition imposed on the notion of analyticity is the solution of problem P5 that affected Hintikka's theory, i.e. *according to this approach, it is not the case that propositional and monadic arguments are, in general, analytic*. As the second example shows, arguments that make essential use of virtual information are synthetic according to this definition independently of whether relations occur in them or not. Thus, the approach put forward in D'Agostino et al. (2021) provides a complete criticism of the neo-empiricist tenet that logic is analytic and tautological, which includes both propositional and monadic logic.

12 Towards an alternative way of distinguishing between analytic and synthetic logical rules

The previous sections have specified a definition of the analytic-synthetic distinction that endorses Hintikka's most fundamental idea—namely that synthetic arguments, unlike analytic ones, introduce new individuals into the discussion—and at the same time avoids the main difficulties that affect Hintikka's sense (iii.e.). The next problem of Hintikka's road map is then: can we formulate a set of analytic logical rules in the sense specified in Sect. 11?

D'Agostino et al. (2021) provide a natural-deduction system for logic $\vdash_0$ defined by a set of introduction and elimination rules for the connectives and the quantifiers, which treat a formula and its negation symmetrically. The proof-system for logic $\vdash_0$ allows to prove all and only arguments that are analytic in the sense of Def. 3: neither the introduction of new individuals nor the use of virtual information is allowed. This means that, for every proof in logic $\vdash_0$, the number of distinct parameters that occur in it never exceeds the Q-complexity of its initial premises.

We are not interested here in giving the formal definitions of this system and discussing its rules, but rather in these crucial questions: 1. how can distinct parameters be introduced into a proof? 2. And how can this be accomplished in such a way that their number does not exceed the Q-complexity of the premise? The answer to the first question should not be surprising[15]: *parameters might be introduced into the proof through the elimination rules for the quantifiers only*. While Hintikka considers the rule of existential instantiation to be the sole paradigmatic example of *ecthesis* and is interested in proof-systems in which synthetic arguments can be boiled down to applications of that rule (see Sect. 8), the present approach takes into account also the rule of universal instantiation as a good candidate to introduce new individuals into the reasoning.

The answer to our second question is astonishingly simple. Recall that this approach assumes that the premises are in PPNF. Then, *the variable bound by a universal or an existential quantifier can be instantiated at most once by a* new *parameter; moreover, every* universal *quantifier can also be instantiated by all the parameters that already occur in the proof.* Without entering the formal details, consider the following two examples (D'Agostino et al., 2021, pp. 443, 440):

1	$\forall x \exists y Rxy$	Premise	1	$\forall x \exists y Rxy$	Premise
2	$\forall x \forall z (Rxz \to Rzx)$	Premise	2	$\exists y Ray$	$\forall E, 1$
3	$\exists y Ray$	$\forall E, 1$	3	Rab	$\exists E, 2$
4	$\forall z (Raz \to Rza)$	$\forall E, 2$	4	$\exists y Rby$	$\forall E, 1$
5	Rab	$\exists E, 3$	5	$\boldsymbol{Rbc}$	$\exists E, 4$

15 Recall that this approach assumes that the language contains no constants but is equipped with a set of parameters that may occur in the proof, but neither in the premises nor in the conclusion.

6	$Rab \rightarrow Rba$	$\forall E, 4$	6	$Rab \wedge Rbc$	$\wedge I, 3, 5$
7	Rba	$\rightarrow E, 5, 6$	7	$\exists z(Rab \wedge Rbz)$	$\exists I, 6$
8	$Rab \wedge Rba$	$\wedge I, 5, 7$	8	$\exists y \exists z(Ray \wedge Ryz)$	$\exists I, 7$
9	$\exists y(Ray \wedge Rya)$	$\exists I, 8$	9	$\forall x \exists y \exists z(Rxy \wedge Ryz)$	$\forall I, 8$
10	$\forall x \exists y(Rxy \wedge Ryx)$	$\forall I, 9$			

The former example shows an analytic argument. At step 3, the variable x occurring in the first premise is instantiated by a new parameter, a, and at step 4 the variable x, this time occurring in the second premise, is instantiated again by a (which is not new anymore at this point of the proof). At step 5, y is instantiated by a new parameter, b, which is used to instantiate z at step 6. The latter example shows a *wrong* application of the rule of existential instantiation: at step 3, y has been instantiated by the new parameter b and, for this reason, it cannot be instantiated again at step 5. Thus, the proof from $\forall x \exists y Rxy$ to $\forall x \exists y \exists z(Rxy \wedge Ryz)$ is synthetic because a new individual must be introduced to reach the conclusion from the premise.

The problem of gradually retrieving the full deductive power of classical first-order logic is addressed by means of a bound recursive use of a structural rule called RB after 'rule of bivalence'. This rule governs the use of virtual information and allows to introduce new individuals into the arguments. RB is the only discharge rule of the system and takes the following form: *if* $\Gamma \cup \{A\} \vdash_k B$ *and* $\Delta \cup \{\neg A\} \vdash_k B$, *then* $\Gamma \cup \Delta \vdash_{k+1} B$. The second of the examples above might be proved in logic $\vdash_1$. This is because logic $\vdash_0$ proves that $\forall x \exists y \exists z(Rxy \wedge Ryz)$ is the case from both premise $\forall x \exists y Rxy$ together with the piece of virtual information that Rbc and premise $\forall x \exists y Rxy$ together with the piece of virtual information that $\neg Rbc$ (in the latter case, the conclusion follows from a contradiction).

I claim this approach outperforms Hintikka's proposal in at least two respects. First, against problem P6, it specifies in a clear way the distinction between applications of the rule of existential instantiation that do introduce new individuals into the arguments from the ones that do not. The first kind of application is simulated through the use of the structural rule RB, while the second through the rule of existential instantiation, where the restriction shown above is respected. The idea behind this solution is remarkably intuitive once the notion of sets in PPNF has been devised: an application of the rule of existential instantiation is analytic if and only if the variable bound by that existential quantifier has never been instantiated by a new parameter before.

Second, the approach put forward by D'Agostino et al. (2021) is simple: against P7 and Hintikka's explicit form of sense (iii), it can be easily used to distinguish between analytic and synthetic arguments.

13 Conclusions

The relevance of Hintikka's work on the analytic-synthetic distinction is, I think, the effort to build a bridge between Kant's conception of the synthetic a priori and the post-Fregean debate on the status of logic. But this is also the source of the difficulties that affect his picture. In trying to keep these two banks together, Hintikka stretches the Kantian materials to make Kant speak the same modern language of the logical empiricist movement. But, at the same time, Hintikka offers an excessively complex formalism that demands to incorporate somehow Kant's original insights.[16] The result is a theory that, high hopes notwithstanding, did not manage to impact on the subsequent debate.

In this paper, I have shown that Hintikka's work does not need to stand or fall as a whole. In my analysis, I have distinguished the merits of his approach, in particular of his fundamental idea expressed by sense (iii), from the difficulties of his further elaboration and formalism. Moreover, I have individuated an alternative formalisation of syntheticity as the introduction of new individuals that, I have claimed, is not marred by the same problems affecting Hintikka's proposal. In so doing, I hope to have contributed to the realisation of the project of rehabilitating Kant's analytic-synthetic distinction in the context of modern first-order logic with the purpose of showing, against the logical empiricist movement, that logic is not analytic.

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References

Acerbi, F. (2019). Mathematical generality, letter-labels, and all that. *Phronesis, 65*(1), 1–49.
Anderson, R. L. (2015). *The poverty of conceptual truth*. Oxford University Press.
Bolzano, B. (1973). *Theory of science*. Reidel.

[16] This is mirrored, for example, in the oscillation between N_{prem} and N_{rel} (Sect. 5).

Carnap, R. (1959). *The logical syntax of language*. Littlefield Adams & Co.
Carnap, R., Hahn, H., & Neurath, O. (1973). The scientific conception of the World: The Vienna circle. In O. Neurath, M. Neurath, & R. S. Cohen (Eds.), *Empiricism and sociology* (pp. 299–328). Reidel.
Church, A. (1936). An unsolvable problem of elementary number theory. *American Journal of Mathematics, 58*(2), 345–363.
Cook, S. (1971). The complexity of theorem proving procedures. In: *Proceedings of the third annual ACM symposium on theory of computing*, (pp. 151–158).
D'Agostino, M. (2010). Tractable depth-bounded logics and the problem of logical omniscience. In F. Montagna & H. Hosni (Eds.), *Probability, uncertainty and rationality.* Edizioni della Normale.
D'Agostino, M. (2013). Semantic information and the trivialization of logic: Floridi on the scandal of deduction. *Information, 4*(1), 33–59.
D'Agostino, M. (2014). Analytic inference and the informational meaning of the logical operators. *Logique Et Analyse, 57*(227), 407–437.
D'Agostino, M. (2015). An informational view of classical logic. *Theoretical Computer Science, 606*, 79–97.
D'Agostino, M., Finger, M., & Gabbay, D. (2013). Semantics and proof-theory of depth bounded boolean logics. *Theoretical Computer Science, 480*, 43–68.
D'Agostino, M., & Floridi, L. (2009). The enduring scandal of deduction. Is propositional logic really uninformative? *Synthese, 167*(2), 271–315.
D'Agostino, M., Larese, C., & Modgil, S. (2021). Towards depth-bounded natural deduction for classical first-order logic. *Journal of Applied Logics, 8*(2), 423–451.
De Jong, W. R. (1995). Kant's analytic judgments and the traditional theory of concepts. *Journal of the History of Philosophy, 33*, 631–641.
De Jong, W. R. (1997). Kant's theory of geometrical reasoning and the analytic-synthetic distinction on Hintikka's interpretation of Kant's philosophy of mathematics. *Studies in History and Philosophy of Science, 28*(1), 141–166.
De Jong, W. R. (2010). The analytic-synthetic distinction and the classical model of science: Kant Bolzano and Frege. *Synthese, 174*(2), 237–261.
Doets, K. (1994). *From logic to programming*. The MIT Press.
Frege, G. (1960). *The Foundations of Arithmetic*, Austin, J.L., trans., Northwestern University Press.
Frege, G. (1972). *Conceptual notation and related articles*. Oxford University Press.
Friedman, M. (2000). Geometry, construction, and intuition in kant and his successors. In G. Sher & R. L. Tieszen (Eds.), *Between logic and intuition: Essays in honor of Charles Parsons* (pp. 186–218). Cambridge, Cambridge University.
Garey, M. R., & Johnson, D. S. (1979). *Computers and intractability: A guide to the theory of NP-completeness*. W.H. Freeman and Company.
Gentzen, G. (1935). Untersuchungen über das logische Schließen. *Mathematische Zeitschrift, 39*, 176–210.
Hanna, R. (2001). *Kant and the foundations of analytic philosophy*. Clarendon Press.
Hintikka, K. J. J. (1965). Are logical truths analytic? *Philosophical Review, 74*, 178–203.
Hintikka, K. J. J. (1966). An analysis of analyticity. In P. Weingartner (Ed.), *Deskription, Analitizität und Existenz* (pp. 193–214). Pustet.
Hintikka, K. J. J. (1967). Kant on the mathematical method. *The Monist, 51*(3), 352–375.
Hintikka, K. J. J. (1968). Are mathematical truths synthetic a priori? *The Journal of Philosophy, 65*(20), 640–651.
Hintikka, K. J. J. (1970). Surface information and depth information. In J. K. K. Hintikka & P. Suppes (Eds.), *Information and inference* (pp. 263–297). North Holland Publishing.
Hintikka, K. J. J. (1973). *Logic Kantian themes in the philosophy of logic*. Oxford University Press.
Hintikka, K. J. J., & Remes, U. (1976). Ancient geometrical analysis and modern logic. In R. S. Cohen, P. K. Feyerabend, & M. W. Wartofsky (Eds.), *Essays in memory of Imre Lakatos* (pp. 253–276). D. Reidel Publishing Co.
Hjortland, O. T., & Martin, B. (2022). Anti-exceptionalism about logic as tradition rejection. *Synthese, 200*, 148.
Kant, I. (1997). *Prolegomena to Any Future Metaphysics that May Be Able to Come forward as a Science*, Hatfield, G., ed. and trans., Cambridge University Press.
Kant, I. (1998). *Critique of pure reason*. Guyer, P., and Wood, A., ed. and trans., Cambridge University Press.

Lampert, T. (2017). Minimizing disjunctive normal forms of pure first-order logic. *Logic Journal of the IGPL, 25*(3), 325–347.

Langford, C. H. (1992). The notion of analysis in Moore's philosophy. In P. A. Schilpp (Ed.), *The philosophy of GE Moore* (pp. 321–342). Open Court.

Larese (2019). The Principle of Analyticity of Logic. *PhD thesis*, Scuola Normale Superiore, Pisa.

Larese, C. (2020a). Notes on Hintikka's Analytic-Synthetic Distinction. *Isonomia – Epistemologica*, http://isonomia.uniurb.it/epistemologica

Larese, C. (2020b). Kant on the analyticity of logic. *Argumenta*. https://doi.org/10.14275/2465-2334/20200.lar

Mill, J. S. (2002). *A system of logic*. University Press of the Pacific.

Nelte, K. (1997). *Formulas of first-order logic in distributive normal form*. University of Cape Town. https://open.uct.ac.za/handle/11427/9648

Parsons, C. (1969). Kant's philosophy of arithmetic. In S. Morgenbesser, P. Suppes, & M. White (Eds.), *Philosophy and method: Essays in honor of Ernst Nagel* (pp. 568–594). St. Martin's Press.

Parsons, C. (1980). Mathematical intuition. *Proceedings of the Aristotelian Society, 80*, 145–168.

Parsons, C., et al. (1983). Kant's philosophy of arithmetic. In S. Morgenbesser (Ed.), *Philosophy, science and method: Essays in honor of Ernest Nagel* (pp. 568–594). New York.

Proops, I. (2005). Kant's conception of analytic judgment. *Philosophy and Phenomenological Research, 70*(3), 588–612.

Proust, J. (1989). *Questions of form. Logic and the analytic proposition from Kant to Carnap*, Brenner, A.A., trans., Minneapolis, University of Minnesota Press.

Quine, W. V. O. (1951). Two dogmas of empiricism. *The Philosophical Review, 60*, 20–43.

Rantala, V. (1987). Constituents. In R. J. Bogdan (Ed.), *Jaakko Hintikka: A profile* (pp. 43–76). D. Reidel Publishing Co.

Rantala, V., & Tselishchev, V. (1987). Surface information and analyticity. In R. J. Bogdan (Ed.), *Jaakko Hintikka: A profile* (pp. 77–90). D. Reidel Publishing Co.

Russell, C. (1990). Hintikka on Kant and logic. *Erkenntnis, 33*, 23–38.

Sequoiah-Grayson, S. (2008). The scandal of deduction. Hintikka on the information yield of deductive inferences. *The Journal of Philosophical Logic, 37*(1), 67–94.

Smullyan, R. (1995). *First-order logic*. Dover.

Turing, A. M. (1937). On computable numbers, with an application to the *Entscheidungsproblem*. *Proceedings of the London Mathematical Society II, 42*(1), 230–265.

Van Benthem, J. F. A. K. (1974). Hintikka on analyticity. *Journal of Philosophical Logic, 3*(4), 419–431.