Scuola Normale Superiore di Pisa e Université Paris Sud

The Khinchin theorem for interval exchange transformations and its consequences for the Teichmüller flow.

Tesi di Perfezionamento in Matematica

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Discussione alla Scuola Normale Superiore di Pisa
venerdi 23 ottobre 2009
alla presenza della commissione d'esame:

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I would like to tank Jean-Christophe Yoccoz and Stefano Marmi for everything they did for me in these years.
I would like to thank also the institutions that made my work possible. In particular the Scuola Normale Superiore di Pisa, for providing the main financial support for this thesis. The Vinci *grant, offered by the Università italo-francese, has been an important contribution. The last of the four years was entirely covered by the* Eiffel *scholarship, provided by the Ministère Francais des Affaires étrangères.*

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CHAPTER 1

Introduction

The classical Khinchin theorem concerns analytic number theory. For any real number x we denote with $\{x\}$ its fractionary part, that is $\{x\} := \min\{x - k; k \in \mathbb{Z}, k \leq x\}$. Let us consider $\alpha \in \mathbb{R}$. The most general diophantine condition on α can be defined considering a positive sequence $\varphi : \mathbb{N} \to \mathbb{R}_+$ and looking at the equation

$$\{n\alpha\} \leq \varphi(n). \tag{1.1}$$

Khinchin proved the following classical result, known as Khinchin teorem (see [**K**]).

THEOREM 1.0.1. *Let $\varphi : \mathbb{N} \to \mathbb{R}_+$ be a positive sequence such that $n\varphi(n)$ is monotone decreasing.*

- *If $\sum_{n \in \mathbb{N}} \varphi(n) < \infty$ then equation (1.1) has just finitely many solutions $n \in \mathbb{N}$ for almost any $\alpha \in \mathbb{R}_+$*
- *If $\sum_{n \in \mathbb{N}} \varphi(n) = \infty$ then for almost any α equation (1.1) has infinitely many solutions $n \in \mathbb{N}$.*

A rotation can be thought as a map R_α of the interval $[0, 1)$ to itself, where $\alpha \in [0, 1)$ denotes the *rotation number* of R_α, in this way it takes the form:

$$R_\alpha x = x + \alpha \mod \mathbb{Z}.$$

The dynamics of a rotation is strictly linked to the diophantine properties of its rotation number. A first well known fact is that the rotation is periodic if and only if the rotation number is rational and conversely all irrational rotations are minimal. A solution of equation (1.1) can be thought as a rational rotation $R_{p/q}$ with rotation number p/q (where $(p, q) = 1$) at C^0 distance less that $\varphi(q)/q$ from the rotation R_α. Khinchin theorem therefore has a natural dynamical interpretation in terms of estimation of the recurrence of irrational rotations. A very classical tool to make good rational approximations of irrational numbers is the *Gauss map*. It is the map $G : (0, 1) \to [0, 1)$ defined by

$$G(\alpha) := \{\alpha^{-1}\},$$

which generates the *continued fraction algorithm*. Khinchin result gives a quantitative estimation about the maximal speed of approximation of generic rotations with rational ones.

It is also quite well known that the continued fraction algorithm has a strict relation with the geodesic flow g_t on the *modular surface* $\mathbb{H}/\mathrm{PSL}(2, \mathbb{Z})$ (see for example [**Ser**], even if there are many other references). In particular one of the consequences of Khinchin theorem is the *logarithmic law* for the geodesic flow on the modular surface.

THEOREM 1.0.2. *Let z_0 and z be any pair of points in $\mathbb{H}/\mathrm{PSL}(2,\mathbb{Z})$. For almost any unit vector v at z, if we denote with g_t be the geodesic passing from z in the direction of v, in terms of the* Poincaré distance d *we have*

$$\limsup_{t\to\infty}\frac{d(g_t z, z_0)}{\log t} = 1/2.$$

1.1. Interval exchange transformations.

An alphabet is a finite set $\mathcal{A}$ with $d \geq 2$ elements. An *interval exchange transformation* (also called i.e.t.) is a map T from an interval I to itself such that I admits two finite partitions $\mathcal{P}_t := \{I^t_\alpha\}_{\alpha\in\mathcal{A}}$ and $\mathcal{P}_b := \{I^b_\beta\}_{\beta\in\mathcal{A}}$ into d open intervals and for any $\alpha \in \mathcal{A}$ the restriction of T to the interval I^t_α is a translation with image the interval I^b_α. The map $T : I \to I$ is therefore defined by rearranging (via translations) the intervals of the partition $\mathcal{P}_t$ in a new order inside I given by $\mathcal{P}_b$ and is entirely defined by the following data:

(1) The lengths of the intervals,
(2) The order before and after rearranging.

The first are called *length data*, and are given by a vector $\lambda \in \mathbb{R}^{\mathcal{A}}_+$, where λ_α denotes the length of I^t_α (which is equal to the length of I^b_α) for any $\alpha \in \mathcal{A}$. The second are called *combinatorial data* and are given by a pair of bijections $\pi = (\pi^t, \pi^b)$ from $\mathcal{A}$ to $\{1, .., d\}$. The meaning of π is that for any $\alpha \in \mathcal{A}$, if we count starting from the left, the interval I^t_α is in $\pi^t(\alpha)$-th position in $\mathcal{P}_t$ and I^b_α is in $\pi^b(\alpha)$-th position in $\mathcal{P}_b$. For any combinatorial datum π let us call $\Delta_\pi := \{\pi\} \times \mathbb{R}^{\mathcal{A}}_+$ the set of all the possible i.e.t. with combinatorial datum π. Let us consider any $T \in \Delta_\pi$. For any $\alpha \in \mathcal{A}$ such that $\pi^t(\alpha) > 1$ we call u^t_α the left endpoint of I^t_α. In general T is not continuous at u^t_α. Similarly for any $\beta \in \mathcal{A}$ such that $\pi^b(\beta) > 1$ we call u^b_β the left endpoint of I^b_β. In general the inverse T^{-1} of T is not continuous at u^b_β. For any $\alpha, \beta \in \mathcal{A}$ with $\pi^t(\alpha)$ and $\pi^b(\beta) > 1$ the position of the associated singularities depend from the data (π, λ) with the relation:

$$u^t_\alpha := \sum_{\pi^t(\alpha')<\pi^t(\alpha)} \lambda_{\alpha'} \ \ \texttt{and} \ \ u^b_\beta := \sum_{\pi^b(\beta')<\pi^b(\beta)} \lambda_{\beta'}. \tag{1.2}$$

We say that the combinatorial datum π is *admissible* if there is no proper subset $\mathcal{A}' \subset \mathcal{A}$ with $k < d$ elements such that $\pi^t(\mathcal{A}') = \pi^b(\mathcal{A}') = \{1, .., k\}$. We always suppose that the combinatorial datum is admissible since otherwise the dynamics of an i.e.t. decomposes into two simpler parts that can be studied separately.

DEFINITION 1.1.1. A *connection* for $T : I \to I$ is a triple (β, α, n), where $n \in \mathbb{N}$ and $\alpha, \beta \in \mathcal{A}$ satisfy $\pi^t(\alpha), \pi^b(\beta) > 1$ and $T^n u^t_\alpha = u^b_\beta$

I.e.t.s with connections are the generalization of rational rotations. A first reason to believe it is the following theorem, due to Keane:

THEOREM 1.1.2. *If T has no connections, then it is minimal.*

We remark anyway that there is not an true dichotomy as for rotations, since there exist minimal i.e.t.s with connections. Lemma 2.1.4 in paragraph 2.1.3 gives a deeper reason for this interpretation. Now as Khinchin did for irrational numbers, we define a general diophantine condition for i.e.t.s. Let us consider a positive sequence $\varphi : \mathbb{N} \to \mathbb{R}_+$ such that $n\varphi(n)$ is decreasing monotone.

DEFINITION 1.1.3. Let π be an admissible combinatorial datum and let T be an i.e.t. in Δ_π be a i.e.t. without connections. A mod φ connection for T is a triple (β, α, n), where $n \in \mathbb{N}$ and $\beta, \alpha \in \mathcal{A}$ satisfy $\pi^t(\alpha)$ and $\pi^b(\beta) > 1$ that is a solution of

$$\left|T^n(u^b_\beta) - u^t_\alpha\right| < \varphi(n). \tag{1.3}$$

Let us fix any i.e.t. $T : I \to I$ with combinatorial-length data $(\pi, \lambda) \in \Delta_\pi$ and π admissible. For any $\beta, \alpha \in \mathcal{A}$ with $\pi^t(\alpha)$ and $\pi^b(\beta) > 1$ and for any $n \in \mathbb{N}$ let us call $I(\beta, \alpha, n)$ the open subinterval of I whose endpoints are $T^n(u^b_\beta)$ and u^t_α (doesn't matter their reciprocal order).

DEFINITION 1.1.4. Let π be admissible. Given the i.e.t. T with combinatorial-length data $(\pi, \lambda) \in \Delta_\pi$ let us consider a triple (β, α, n) with $n \in \mathbb{N}$ and $\pi^t(\alpha), \pi^b(\beta) > 1$. We say that (β, α, n) is a *reduced triple* for T if for any $k \in \{0, ..n\}$ the pre-image $T^{-k}(I(\beta, \alpha, n))$ of $I(\beta, \alpha, n)$ does not contain in its interior any singularity $u^t_{\alpha'}$ for T or any singularity $u^b_{\beta'}$ for T^{-1} with $\alpha', \beta' \in \mathcal{A}$.

DEFINITION 1.1.5. Given a function $\varphi : \mathbb{N} \to \mathbb{R}_+$ as before and an admissible π, an interval exchange transformation $T \in \Delta_\pi$ is said

- modφ-Diophantine if equation (1.3) has just finitely many solutions.
- modφ-Liouville if for any pair of letters β, α with $\pi^t(\alpha)$ and $\pi^b(\beta) > 1$ there exists infinitely many triples (β, α, n) reduced for T that are solution of equation (1.3).

We proved the following generalization of Khinchin theorem.

THEOREM 1.1.6. *Let us consider a positive sequence $\varphi : \mathbb{N} \to \mathbb{R}_+$ such that $n\varphi(n) : \mathbb{N} \to \mathbb{R}_+$ is decreasing monotone. For any admissible combinatorial datum π we have the following dichotomy:*

a: *If $\sum_{n=1}^{+\infty} \varphi(n) < +\infty$ then almost any i.e.t. $T \in \Delta_\pi$ is* modφ*-Diophantine.*
b: *If $\sum_{n=1}^{+\infty} \varphi(n) = +\infty$ then almost any i.e.t. $T \in \Delta_\pi$ is* modφ*-Liouville.*

1.2. Translation surfaces.

Let us consider M a smooth, compact, orientable, boundaryless surface of genus g. Let $\Sigma = \{p_1, .., p_r\}$ be a finite subset of M with r elements and $k_1, .., k_r$ non-negative integers. Let $\mathcal{H}(k_1, .., k_r)$ be the (*stratum* of the) moduli space of pairs (S, w), where S is a Riemann surface structure on M and w is a holomorphic abelian differential with respect to the structure S having a zero of order k_i at any point $p_i \in \Sigma$. The genus of M and the order of zeros of w are related by the formula $k_1 + .. + k_r = 2g - 2$. The datum $X = (S, w)$ is equivalent to the datum of a flat metric on $M \setminus \Sigma$ with cone singularities with angles $2\pi(k_i + 1)$ at points $p_i \in \Sigma$ (the zeros of w) and is also called a *translation surface*. Any $\mathcal{H}(k_1, .., k_r)$ is a complex orbifold with $\dim_{\mathbb{C}} = 2g + r - 1$ (see proposition 2.3.5 in the background), in general zorichit is neither compact nor connected. The area of a flat structure is given by

$$\mathrm{Area}(X) := \int_{M \setminus \Sigma} w \wedge \bar{w},$$

and the normalization $\mathrm{Area}(X) = 1$ defines an hypersurface $\mathcal{H}^{(1)}(k_1, .., k_r)$ which has finite Lebesgue measure (theorem of Masur and Veech, see [**Ma1**] and [**Ve**]).

Let us fix $X = (S, w)$ in some stratum $\mathcal{H}(k_1, .., k_r)$. Since the angles at the singularities are multiples of 2π then X determines a parallel constant vector field ∂_y on $M \setminus \Sigma$, called *vertical vector field*. A *saddle connection* γ for the translation surface X is a geodesic for the flat metric starting and ending at zeros of w and with no other zeros in its interior. If γ is a saddle connection for X, at any point of γ the angle with the vertical direction ∂_y is constant and we denote it $\text{angle}(\gamma, \partial_y)$. It is possible to see that the condition $\text{angle}(\gamma, \partial_y) = 0$, that is that the saddle connection γ is parallel to the vertical direction ∂_y on X, forces X to stay in some codimension one immersed sub-manifold of $\mathcal{H}(k_1, .., k_r)$. Since the saddle connections are countably many it follows that for a generic X they are never vertical. Nevertheless $\text{angle}(\gamma, \partial_y)$ accumulate at 0 when γ varies in the set of all saddle connections of a fixed X. We ask how good the approximation is with respect to the length $|\gamma|$ of the saddle connection γ with respect to the flat metric on X. This motivates the following definition:

Definition 1.2.1. Let us consider a positive function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ bounded from above such that $t\varphi(t)$ is decreasing monotone. A $\mod \varphi$ vertical connection for a translation structure X is a saddle connection γ such that

$$|\tan \text{angle}(\gamma, \partial_y)| \leq \frac{\varphi(|\gamma|)}{|\gamma|}. \tag{1.4}$$

Let us fix a translation surface X. A point $p_i \in \Sigma$ is a zero of w of order k_i, therefore is the starting point of $k_i + 1$ trajectories of the vertical field ∂_y associated to X. These trajectories are called *outgoing vertical separatrices* and can be labeled with an integer l in $\{0, .., k_i\}$ in the following way: a separatrix $V^{start}_{p_i,0}$ is arbitrarily chosen, then we call $V^{start}_{p_i,l}$ the l-th separatrix that we meet moving in counterclockwise sense around p_i. Similarly there are k_i+1 trajectories of ∂_y that end in p_i, they are called *ingoing vertical separatrices*. We chose arbitrarily a separatrix $V^{end}_{p_i,0}$ and for any l in $\{0, .., k_i\}$ we call $V^{end}_{p_i,l}$ the l-th separatrix that we meet moving in counterclockwise sense around p_i.

For any saddle connection γ on X we can specify the points p_j and p_i in Σ where γ respectively starts and ends. We can also determine uniquely a pair of vertical trajectories $V^{start}_{p_j,m}$ and $V^{end}_{p_i,l}$ respectively starting in p_j and ending in p_i such that

$$-\pi \leq \text{angle}(\gamma, V^{start}_{p_i,l}) < \pi \quad \texttt{and} \quad -\pi \leq \text{angle}(\gamma, V^{end}_{p_j,m}) < \pi. \tag{1.5}$$

On the other hand let us fix some X, a pair of points $p_i, p_j \in \Sigma$ and two integers $l \in \{0, .., k_i\}$ and $m \in \{0, .., k_j\}$. We define the *configuration* $\mathcal{C}^{(p_i,p_j,l,m)}(X)$ as the set of saddle connections γ for X that satisfy the relation in equation (1.5) for the fixed data (p_i, p_j, l, m).

The vertical separatrices at singular points can be labeled in a way such that their names vary *coherently* when X moves in some small open set in $\mathcal{H}(k_1, .., k_r)$. This is to say that if $V = V^{start}_{p_j,l}$ is any outgoing vertical separatricx for X and X' is sufficiently close to X, then the outgoing vertical separatrix V' for X' which corresponds to V has name $V'^{start}_{p_j,l}$. The same obviously holds for any ingoing vertical separatrix. It follows that if γ is a saddle connection for X in configuration $\mathcal{C}^{(p_i,p_j,l,m)}(X)$ that still is a saddle connection for X', then γ is in configuration $\mathcal{C}^{(p_i,p_j,l,m)}(X')$ as saddle connection for X'. Unfortunately this is no more true globally: we can find loops in $\mathcal{H}(k_1, .., k_r)$ based at some X such that when X

comes back to itself its saddle connections have changed their configurations. To make configurations vary globally in a coherent way we have to introduce the space $\widehat{\mathcal{H}}(k_1,..,k_r)$ whose elements are the data $\hat{X}=(X,S_1,..,S_r)$, where X is a translation surface and for any singular point p_i we have chosen an horizontal separatrix S_i for X starting at p_i. We call such a datum a *totally marked translation surface*. The moduli space of totally marked translation surfaces is denoted $\widehat{\mathcal{H}}(k_1,..,k_r)$, with the trivial projection $\hat{X}\mapsto X$ it forms a finite cover of $\mathcal{H}(k_1,..,k_r)$. Details are given in paragraph 3.1.2, conceptually the construction is the same as the one of the orientable double covering of a non-orientable manifold. Exactly as any stratum $\mathcal{H}(k_1,..,k_r)$, its covering $\widehat{\mathcal{H}}(k_1,..,k_r)$ is a complex orbifold with $\dim_{\mathbb{C}}=2g+r-1$, it is neither compact nor connected, nevertheless the hypersurface $\widehat{\mathcal{H}}^{(1)}(k_1,..,k_r)$ of totally marked area one translation surfaces has finite Lebesgue measure (see paragraph 3.1.2). Since for us it is important to preserve the configurations when X moves in the moduli space, we will state our results for totally marked translation surfaces.

DEFINITION 1.2.2. Let $\varphi:\mathbb{R}_+\to\mathbb{R}_+$ be a positive function as in definition 1.2.1 such that $t\varphi(t)$ is decreasing monotone. An element $\hat{X}=(X,S_1,..,S_r)$ in $\widehat{\mathcal{H}}(k_1,..,k_r)$ is said

- modφ-Diophantine if there are just finitely many saddle connections γ for X that are solutions of equation (1.4).
- modφ-Liouville if any configuration $\mathcal{C}^{(p_i,p_j,l,m)}(\hat{X})$ contains infinitely many saddle connections γ that are solutions of equation (1.4).

For (totally marked) translation surfaces we proved the following version of theorem 1.1.6.

THEOREM 1.2.3. *Let us consider any stratum $\widehat{\mathcal{H}}(k_1,..,k_r)$ of the moduli space of totally marked translation surfaces and let $\varphi:\mathbb{R}_+\to\mathbb{R}_+$ be a function as in definition 1.2.1.*

a: *If $\int_0^\infty\varphi(t)dt<\infty$ then almost any $\hat{X}\in\widehat{\mathcal{H}}(k_1,..,k_r)$ is* modφ*-Diophantine.*
b: *If $\int_0^\infty\varphi(t)dt=\infty$ then almost any $\hat{X}\in\widehat{\mathcal{H}}(k_1,..,k_r)$ is* modφ*-Liouville.*

Theorems 1.1.6 and 1.2.3 are *Borel-Cantelli* type result. Part **a** of both theorems corresponds to the easy half of the argument. A natural construction for translation surfaces (totally marked or not) is to fix any $\hat{X}\in\widehat{\mathcal{H}}(k_1,..,k_r)$ and rotate its vertical direction ∂_y. If θ is the angle of the rotation we call $\hat{X}_\theta$ the rotated translation surface. We denote $\mathrm{SO}(2,\mathbb{R})\hat{X}$ the image in the moduli space of the application $\theta\mapsto\hat{X}_\theta$: it is often a circle, except some cases when it can have a finite number of self-intersections. The following proposition (proved in paragraph 3.4) is another version of the easy half of the Borel-Cantelli argument. In section 3.5 we show that it implies the converging part of theorems 1.1.6 and 1.2.3.

PROPOSITION 1.2.4. *Let us consider a function $\varphi:\mathbb{R}_+\to\mathbb{R}_+$ as before and suppose that $\int_0^\infty\varphi(t)dt<\infty$. For any $\hat{X}$ in $\widehat{\mathcal{H}}(k_1,..,k_r)$, almost any $\hat{X}_\theta\in\mathrm{SO}(2,\mathbb{R})\hat{X}$ is* modφ*-Diophantine.*

Part **b** of theorems 1.1.6 and 1.2.3 correspond to the difficult part of the Borel-Cantelli argument and follows from the result that we develop in section 4.

1.3. Generalized logarithmic law for abelian differentials

Let $\mathcal{F}_t$ denote the *Teichmüller flow* (see paragraph 2.3.4) on the stratum $\widehat{\mathcal{H}}(k_1,..,k_r)$ with zeros of orders $k_1,..,k_r$. Since it preserves the area of flat surfaces we can consider its restriction to the hypersurface $\widehat{\mathcal{H}}^{(1)}(k_1,..,k_r)$ of area one totally marked translation surfaces. For any $\hat{X}=(X,S_1,..,S_r)$ in $\widehat{\mathcal{H}}(k_1,..,k_r)$, any pair of points $p_i,p_j\in\Sigma$, any $l\in\{0,..,k_i\}$ and any $m\in\{0,..,k_j\}$ we put

$$\mathtt{Sys}^{(p_i,p_j,l,m)}(\hat{X}):=\min\{\mathrm{length}(\gamma);\gamma\in\mathcal{C}^{(p_i,p_j,l,m)}(\hat{X})\},$$

that is the length of the shortest geodesic in the configuration $\mathcal{C}^{(p_i,p_j,l,m)}(\hat{X})$ (see paragraph 3.1.2). The classic systole function $\mathtt{Sys}(X)$ is defined for elements X in $\mathcal{H}(k_1,..,k_r)$ as the length(γ_{min}), where γ_{min} is the shortest saddle connection for X. Its property is that a sequence X_n in $\mathcal{H}(k_1,..,k_r)$ (or in $\mathcal{H}^{(1)}(k_1,..,k_r)$) diverges, that is it escapes from any compact set, if and only if $\mathtt{Sys}(X_n)\to 0$. For sequences $\hat{X}_n\in\widehat{\mathcal{H}}^{(1)}(k_1,..,k_r)$ and for any datum (p_i,p_j,l,m) as before, the condition $\mathtt{Sys}^{(p_i,p_j,l,m)}(\hat{X}_n)\to 0$ establish therefore a finer criterion to describe the divergence, since in some sense to be specified it says that the sequence $\hat{X}_n$ is escaping towards some fixed direction in the boundary of $\widehat{\mathcal{H}}^{(1)}(k_1,..,k_s)$. By recurrence of the Teichmüller flow, for a generic $\hat{X}$ neither the Teichmüller geodesic passing from $\hat{X}$ is diverging, nor it stays in any compact set. Equivalently, for any datum (p_i,p_j,l,m) as before, the quantity $\mathtt{Sys}^{(p_i,p_j,l,m)}(\mathcal{F}_t\hat{X})$ stays bounded away from 0 for the most of the time, but there are arbitrary big instants t_n such that $\mathtt{Sys}^{(p_i,p_j,l,m)}(\mathcal{F}_{t_n}\hat{X})$ becomes small. We proved the following theorem, that gives the (optimal) quantitative description of this phenomenon.

THEOREM 1.3.1. *Let $\psi:\mathbb{R}_+\to\mathbb{R}_+$ be a monotone decreasing positive function bounded from above.*

- *If $\int_0^\infty\psi(t)dt<+\infty$ then for almost any $\hat{X}\in\widehat{\mathcal{H}}^{(1)}(k_1,..,k_r)$ we have:*

$$\lim_{t\to\infty}\frac{\mathit{Sys}(\mathcal{F}_t\hat{X})}{\sqrt{\psi(t)}}=\infty. \tag{1.6}$$

- *If $\int_0^\infty\psi(t)dt=+\infty$ then for any pair of points $p_i,p_j\in\Sigma$, any $l\in\{0,..,k_i\}$, any $m\in\{0,..,k_j\}$ and for almost any $\hat{X}\in\widehat{\mathcal{H}}^{(1)}(k_1,..,k_r)$ we have:*

$$\liminf_{t\to\infty}\frac{\mathit{Sys}^{(p_i,p_j,l,m)}(\mathcal{F}_t\hat{X})}{\sqrt{\psi(t)}}=0. \tag{1.7}$$

The estimate in theorem 1.3.1 is a consequence of theorem 1.2.3. For the one parameter family $\psi(t):=t^{-r}$ with $r\geq 1$ theorem 1.3.1 implies the following corollary

COROLLARY 1.3.2. *For any pair of points $p_i,p_j\in\Sigma$, for any $l\in\{0,..,k_i\}$ and any $m\in\{0,..,k_j\}$ and almost any $\hat{X}\in\widehat{\mathcal{H}}^{(1)}(k_1,..,k_r)$ we have:*

$$\limsup_{t\to\infty}\frac{-\log\mathit{Sys}^{(p_i,p_j,l,m)}(\mathcal{F}_t\hat{X})}{\log t}=\frac{1}{2}. \tag{1.8}$$

Corollary 1.3.2 is the classical Masur's logarithm law for abelian differentials (see [**Ma3**]). We get it in a quite direct and natural way, that is via a Khinchin theorem. Moreover we are able to prove it not only for the classical systole function

$\mathtt{Sys}(X)$, but for all the functions $\mathtt{Sys}^{(p_i,p_j,l,m)}(\hat{X})$ for any pair of points $p_i, p_j \in \Sigma$ and for any $l \in \{0,..,k_i\}$ and any $m \in \{0,..,k_j\}$. On the other hand Masur's result holds in the most general setting of quadratic differentials.

QUESTION 1.3.3. I.e.t.'s have been generalized with linear involutions by Danthony and Nogueira (See [**DaNo**]) and the latter have been related to quadratic differentials by Boissy and Lanneau (see [**BoLa**]). We ask if it is possible to extend theorem 1.1.6 to linear involutions. The consequence of a Khinchin theorem for linear involutions would be the generalization of theorem 1.3.1 to the setting of quadratic differentials

1.4. Back to rotations.

The orbit foliation for the action of $\mathrm{SO}(2,\mathbb{R})$ (also called the $\mathrm{SO}(2,\mathbb{R})$-foliation) is regular, thus when a property holds almost everywhere on $\widehat{\mathcal{H}}(k_1,..,k_r)$ it automatically holds almost everywhere on almost any leaf of the foliation. In this case a natural question is to ask if the property holds almost everywhere on any leaf. Proposition 1.2.4 gives an example of positive answer. Once we proved the difficult half of the Borel-Cantelli argument on $\widehat{\mathcal{H}}(k_1,..,k_r)$, that is that almost any $\hat{X} \in \widehat{\mathcal{H}}(k_1,..,k_r)$ is modφ-Liouville if $\int_0^\infty \varphi(t)dt = \infty$, a natural task is to show the same result on any leaf of the $\mathrm{SO}(2,\mathbb{R})$ foliation. We have some partial result in this direction when the genus of the surface is one, that is for translation structures on a topological torus M.

Before stating our result we observe that an holomorphic one form cannot vanish on a complex torus, therefore if the set of singularities Σ has r elements then the only admissible stratum is $\mathcal{H}(0,..,0)$, where the coefficient 0 appears r times. The angles at points in Σ are all 2π, therefore these points are not real cone singularities for the flat metric and each one is the starting point of just one horizontal sepatratrix. It follows that the covering $\widehat{\mathcal{H}}(0,..,0)$ coincides with $\mathcal{H}(0,..,0)$ (and is a fiber bundle over the modular surface). In particular for any $X \in \mathcal{H}(0,..,0)$ a configuration of saddle connection is specified simply fixing the pair of points p_i, p_j of Σ where the saddle connections start and end respectively, and is simply denoted $\mathcal{C}^{(p_j,p_i)}(X)$ or $\mathcal{C}^{(j,i)}(X)$. Finally we remark that even if in this case points in Σ have a flat neighborhood, we still want saddle connections not have them as interior points. Let $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ be a positive function bounded from above such that $t\varphi(t)$ is decreasing monotone. We proved the following theorem.

THEOREM 1.4.1. *If $\int_0^\infty \varphi(t)dt = \infty$ then for any $X \in \mathcal{H}(0,..,0)$ and for almost any $\theta \in S^1$ the rotated marked torus X_θ has at least $2r-1$ different configurations each one containing infinitely many saddle connection γ which are solutions of equation (1.4).*

Note: As it is explained by remark 5.2.1 in paragraph 5.2, for an arbitrary marked torus X we cannot expect that almost any X_θ in its $\mathrm{SO}(2,\mathbb{R})$ orbit is modφ-Liouville. Moreover we still do not know it the result in theorem 1.4.1 is sharp or not, that is if for any X and for generic θ the rotated marked torus X_θ has more than $2r-1$ configurations containing each one infinitely many solutions of equation (1.4).

The first return map of the vertical flow to a properly chosen horizontal segment defines a rotation, thus the analysis of complex tori can be reduced to analysis of irrational rotations. Then we can pass to analytic number theory via the rotation number. On a flat torus X with just one marked point the set of saddle connections corresponds to the set of closed geodesics. If α is the rotation number of the rotation associated to X (once a transversal is fixed), at the n-th return of a vertical trajectory to the horizontal transversal the error from having a closed geodesic is given by the the quantity $\{n\alpha\}$. If Σ contains more than one element we have to consider the relative position between more than one marked point. The diophantine condition that arises is the following generalization of equation (1.1): let us fix any vector $v = (x, y)$ in $\mathbb{R}^2$ and for any positive real number α let us consider the following equation:

$$\{(n + x)\alpha - y\} < \varphi(n). \tag{1.9}$$

We proved the following statement.

THEOREM 1.4.2. *Let $\varphi : \mathbb{N} \to \mathbb{R}_+$ be a positive sequence such that $n\varphi(n)$ is decreasing monotone and let $v = (x, y)$ be any vector in $\mathbb{R}^2$.*

- **a:** *If $\sum_{n=1}^{\infty} \varphi(n) < +\infty$, then for almost every $\alpha \in \mathbb{R}_+$ there exist finitely many solutions $n \in \mathbb{N}$ of equation (1.9).*
- **b:** *If $\sum_{n=1}^{\infty} \varphi(n) = \infty$ then for almost any α equation (1.9) has infinitely many solutions $n \in \mathbb{N}$.*

We observe that for $v \in \mathbb{Z}^2$ theorem 1.4.2 gives the Khinchin theorem.

1.5. Structure of this text.

The results of this thesis are theorems 1.1.6, 1.2.3, 1.3.1, 1.4.1 and 1.4.2, together with proposition 1.2.4. They are all either Borel-Cantelli arguments or direct consequences. We can group our results into two main categories: arbitrary genus and genus one. Theorems 1.1.6, 1.2.3, 1.3.1 and proposition 1.2.4 belong to the first category, theorems 1.4.1 and 1.4.2 belong to the second one.

In arbitrary genus we have two direct results: proposition 1.2.4 and part b) of theorem 1.1.6. We call them direct since their proof is a Borel-Cantelli type argument. Proposition 1.2.4 corresponds to the easy half of the argument and its proof takes less than one page. Part b) of theorem 1.1.6 is the difficult half of the argument and the main task is to prove a form of weak independence for a family of sets in the parameter space of all interval exchange transformations. The proof takes the entire section 4. All other results in arbitrary genus are consequence of these two.

In genus one the direct result is theorem 1.4.2, which can be seen as an other generalization of Khinchin theorem.

We put in evidence two more results that does not appear in the introduction: they are proposition 4.2.1 and theorem 4.6.1. They both appear in section 4.

1.5.1. Chapter 2. This section is a brief survey of the background theory.

In paragraph 2.1 we introduce the algorithm of Rauzy-Veech-Zorich and its basic properties.

Paragraph 2.2 is the description of a result in [**A,G,Y**] about the control of the distortion of the Lebesgue measure under the iteration of the algorithm. A combinatorial operation called *reduction* is also described.

In paragraph 2.3 we define translation surfaces, their moduli space, the Teichmüller flow and its invariant measure.

In paragraph 2.4 we describe the Veech construction. It's role is to cover a full measure subset of the moduli space with local charts where the machinery of i.e.t. can be applied.

1.5.2. Chapter 3. This paragraph contains the proofs of all results in arbitrary genus, except the prove of part b) of theorem 1.1.6, which is the subject of section 4.

In paragraph 3.1 we introduce the moduli space $\widehat{\mathcal{H}}(k_1, .., k_r)$ of totally marked translation surfaces. The choice of an horizontal saddle connection starting at any singular point in Σ plays the role of reference frame and allows us to define the configurations of saddle connections. We show that for any $\hat{X} \in \widehat{\mathcal{H}}(k_1, .., k_r)$ the configurations of its saddle connections vary coherently when $\hat{X}$ moves along the orbit of the Teichmüller flow (lemma 3.1.3).

In paragraph 3.2 we develop the relation between the diophantine condition that we study for i.e.t. (equation (1.3)) and the one for translation surfaces (equation (1.4)). The tool is the Veech construction, the main result is proposition 3.2.3, that says how to pass from a solution of equation (1.3) to a solution for equation (1.4).

In paragraph 3.3 we give an equivalent formulation of theorem 1.1.6 restricting to the set of i.e.t. with total length normalized to one (proposition 3.3.1). We need such a normalization sice the Rauzy-Veech-Zorich alogorithm has interesting properties just at projective level. In the same paragraph we give also an equivalent formulation of theorem 1.2.3 considering the restriction to the moduli space of area one totally marked translation surfaces (proposition 3.3.2).

Paragraph 3.4 contains the proof of proposition 1.2.4.

In paragraph 3.5 we show the relation between theorem 1.1.6 and theorem 1.2.3. In particular we give the prove of part a) of theorem a) applying proposition1.2.4. Then via a Fubini argument we prove part a) of theorem 1.1.6. On the other hand, with the same Fubini argument, we show that part b) of theorem 1.1.6 implies part b) of theorem 1.2.3.

Finally in paragraph 3.6 we deduce theorem 1.3.1 from theorem 1.2.3.

1.5.3. Chapter 4. This is the main section in this thesis, it consist in the proof of part b) of theorem 1.1.6.

The main idea in paragraph 4.1 is to express the differences $|T^n u^b_\beta - u^t_\alpha|$ as lengths of some interval for the k-th step $T^{(k)} : I^{(k)} \to I^{(k)}$ of the Rauzy-Veech algorithm (for some good instant $k \in \mathbb{N}$). The idea is developed in lemmas 4.1.2 and 4.1.3. They work in parallel, since they holds under some combinatorial conditions and one applies when the other fails and vice-versa. A non-evident combinatorial property of Rauzy classes is necessary. The main result is proposition 4.1.12, which converts part b) of theorem 1.1.6 into a shrinking target property for the (normalized) Rauzy-Veech algorithm.

Paragraph 4.2 entirely concerns the combinatorics of Rauzy classes. We prove proposition 4.2.1, which provides the combinatorial property that we need in order to apply either lemma 4.1.2 or lemma 4.1.3.

In the sufficient condition formulated by proposition 4.1.12 we have a non-uniform speed of shrinking. In paragraph 4.3 we apply some classical ergodic theory for the Rauzy-Veech-Zorich algorithm in order to have an uniform control of such a speed. The main result is proposition 4.3.2.

The main problem in the diverging part of the Borel-Cantelli argument (the difficult half) is to prove some kind of weak independence for a family of sets with divergent sum of measures. In paragraph 4.4 we explain some local constructions in the parameter space of normalized i.e.t. that we use to provide the independence.

Paragraph 4.5 is just a reformulation of a sufficient condition (proposition 4.5.1) putting together the results of the preceding paragraphs.

Paragraph 4.6 contains the proof of our main distortion estimate, that is theorem 4.6.1.

Finally in paragraph 4.7 we complete the Borel-Cantelli argument in the divergent case.

1.5.4. Chapter 5. In this section we prove the results in genus one, that is theorem 1.4.1 and theorem 1.4.2.

In paragraph 5.1 we collect two results holding in the general setting of i.e.t.s and translation surfaces that anyway we use only in genus one. The first is lemma 5.1.1, which describes how the projective length data vary when we rotate a translation surface $X(\pi, \lambda, \tau)$. The second one is lemma 5.1.2 (together with corollary 5.1.3) and explains how to get reduced triples starting from non-reduced ones.

In paragraph 5.2 we assume the arithmetic statement, that is theorem 1.4.2, and we prove theorem 1.4.1.

Paragraph 5.3 is purely arithmetical and consists in the proof of theorem 1.4.2. Since the converging part of the theorem is straightforward (we give anyway a short argument) we are only concerned with diverging part (part b)). We start recalling the geometric interpretation of the classical continued fraction algorithm in terms of action of $\mathrm{SL}(2, \mathbb{Z})$. Then we develop geometrically another approximation procedure that we call the *twisted continued fraction algorithm.* In terms of the twisted approximation we give a sufficient condition to have infinitely many solutions of equation (1.9). Finally prove that our sufficient condition has total measure. The main step is lemma 5.3.8.

CHAPTER 2

Background material

This chapter is an essential survey of the theory of interval exchange transformations and translations surfaces. We follow the presentation given in [**A,G,Y**], [**M,M,Y**] and [**Y1**].

2.1. Interval exchange transformations

2.1.1. Rauzy-Veech algorithm and Rauzy diagrams. Let $\pi = (\pi^t, \pi^b)$ and λ define an interval exchange transformation $T : I \to I$. Let $\epsilon \in \{t, b\}$, where the letter t stands for *top* and the letter b for *bottom*. If $\epsilon = t$ we put $1 - \epsilon := b$ and if $\epsilon = b$ we put $1 - \epsilon := t$. Let us call α_t and α_b the two letters in $\mathcal{A}$ such that respectively $\pi^t(\alpha_t) = d$ and $\pi^b(\alpha_b) = d$. The rightmost singularity of T is therefore $u^t_{\alpha_t}$ and the rightmost singularity of T^{-1} is $u^b_{\alpha_b}$. We suppose that

$$u^t_{\alpha_t} \neq u^b_{\alpha_b} \tag{2.1}$$

and we consider the value of $\epsilon \in \{t, b\}$ such that

$$u^\epsilon_{\alpha_\epsilon} < u^{1-\epsilon}_{\alpha_{1-\epsilon}}.$$

With this definition of ϵ we say that the interval exchange transformation $T : I \to I$ is of *type* ϵ. We also say that the letter α_ϵ is the *winner* of $T : I \to I$ and $\alpha_{1-\epsilon}$ is the *loser*. We define the subinterval $\tilde{I}$ of I by

$$\tilde{I} := I \cap (0, u^{1-\epsilon}_{\alpha_{1-\epsilon}})$$

and we consider the first return map $\tilde{T}$ of T to the subinterval $\tilde{I}$ of I. It is just a matter of combinatorial computations to check that $\tilde{T} : \tilde{I} \to \tilde{I}$ is an interval exchange transformation. The combinatorial datum $\tilde{\pi} = (\tilde{\pi}^t, \tilde{\pi}^b)$ of $\tilde{T}$ is given by:

$$\begin{aligned}
&\tilde{\pi}^\epsilon(\alpha) = \pi^\epsilon(\alpha)\,\forall \alpha \in \mathcal{A} \\[1em]
&\tilde{\pi}^{1-\epsilon}(\alpha) = \pi^{1-\epsilon}(\alpha) \text{ if } \pi^{1-\epsilon}(\alpha) \leq \pi^{1-\epsilon}(\alpha_\epsilon) \\
&\tilde{\pi}^{1-\epsilon}(\alpha_{1-\epsilon}) = \pi^{1-\epsilon}(\alpha_\epsilon) + 1 \\
&\tilde{\pi}^{1-\epsilon}(\alpha) = \pi^{1-\epsilon}(\alpha) + 1 \text{ if } \pi^{1-\epsilon}(\alpha_\epsilon) < \pi^{1-\epsilon}(\alpha) < d.
\end{aligned} \tag{2.2}$$

The length datum $\tilde{\lambda}$ of $\tilde{T}$ is given by:

$$\begin{aligned}
&\tilde{\lambda}_\alpha = \lambda_\alpha \text{ if } \alpha \neq \alpha_\epsilon \\
&\tilde{\lambda}_{\alpha_\epsilon} = \lambda_{\alpha_\epsilon} - \lambda_{\alpha_{1-\epsilon}}.
\end{aligned} \tag{2.3}$$

When $T = (\pi, \lambda)$ satisfy the condition in equation (2.1), equations (2.2) and (2.3) define a map

$$(\pi, \lambda) \mapsto Q(\pi, \lambda) := (\tilde{\pi}, \tilde{\lambda})$$

that we call the *Rauzy-Veech induction map*. We also introduce the combinatorial operations R^t and R^b such that, if ϵ is the type of the pair (π, λ), then for the combinatorial datum $\tilde{\pi}$ defined in equation (2.2) we can write

$$\tilde{\pi} = R^{\epsilon}(\pi). \tag{2.4}$$

LEMMA 2.1.1. *If π is an admissible combinatorial datum then both $R^t(\pi)$ and $R^b(\pi)$ are admissible.*

Proof: The proof is just an exercise, it is enough to apply the definition of $R^t(\pi)$ and $R^b(\pi)$ in equation (2.2). □

DEFINITION 2.1.2. Let's call $\mathfrak{S}$ the set of all the admissible combinatorial data π over some alphabet $\mathcal{A}$. The two maps R^t and R^b from $\mathfrak{S}$ to itself are called the *Rauzy elementary operations*. A *Rauzy class* is a minimal non-empty subset $\mathcal{R}$ of $\mathfrak{S}$ which is invariant under R^t and R^b. If we look to elements in $\mathcal{R}$ as vertexes of a graph, any elementary operation $\pi \mapsto R^{t/b}(\pi)$ defines an oriented arc, or *arrow*, and we define a *Rauzy diagram* $\mathcal{D}$ as the maximal connected graph obtained taking all the elements of some Rauzy class and all the arrows between them. A concatenation of compatible arrows in a Rauzy diagram is called a *Rauzy path*. A Rauzy path that has as first vertex π and as last vertex π' is denoted by $\gamma : \pi \mapsto \pi'$. We denote by $\Pi(\mathcal{R})$ the set of all Rauzy paths connecting elements of $\mathcal{R}$.

2.1.2. Linear action. For any Rauzy class $\mathcal{R}$ and any path $\gamma \in \Pi(\mathcal{R})$ we define a linear map $B_\gamma \in \mathrm{SL}(\mathrm{d}, \mathbb{Z})$ as follows. If γ is trivial then $B_\gamma = id$. If γ is an arrow with winner α and loser β then $B_\gamma e_x = e_x$ for all $x \in \mathcal{A} \setminus \{\alpha\}$ and $B_\gamma e_\alpha = e_\alpha + e_\beta$, where $\{e_x\}_{x \in \mathcal{A}}$ is the canonical basis of $\mathbb{R}^{\mathcal{A}}$. We extend the definition to paths so that $B_{\gamma_1 \gamma_2} = B_{\gamma_2} B_{\gamma_1}$.

We recall that for any combinatorial datum π we defined $\Delta_\pi := \{\pi\} \times \mathbb{R}^{\mathcal{A}}_+$. For any $\gamma \in \Pi(\mathcal{R})$ starting at some $\pi \in \mathcal{R}$ we define the simplicial sub-cone $\Delta_\gamma \subset \Delta_\pi$ by

$$\Delta_\gamma =^t B_\gamma(\mathbb{R}^{\mathcal{A}}_+) \times \{\pi\},$$

where ${}^t B_\gamma$ denotes the trasposed of the matrix B_γ defined above. For the same γ we also define the vector $q^\gamma \in \mathbb{N}^{\mathcal{A}}$ by

$$q^\gamma := B_\gamma \vec{1},$$

where $\vec{1}$ denotes the vector of $\mathbb{N}^{\mathcal{A}}$ that has all entries equals to 1.

2.1.3. Iteration of the algorithm. The set of all the possible intervals exchange transformations arising from a Rauzy class $\mathcal{R}$ is

$$\Delta(\mathcal{R}) := \bigcup_{\pi \in \mathcal{R}} \Delta_\pi.$$

When $T \in \Delta(\mathcal{R})$ is such that the n-th iterated of Q is defined we have an explicit formula for $Q^n(T)$.

LEMMA 2.1.3. *Let $\gamma \in \Pi(\mathcal{R})$ be a path in the Rauzy diagram starting at π and with length n, that is a path which is concatenation of n arrows. Let B_γ and Δ_γ be respectively the matrix and the simplicial cone defined in paragraph 2.1.2. Then for any $T \in \Delta_\gamma$ the n-th iterated of Q is defined and the length datum $\lambda^{(n)}$ of $Q^n(T)$ is given by the formula*

$$\lambda^{(n)} =^t B_\gamma^{-1} \lambda. \tag{2.5}$$

Proof: If γ is an arrow in the Rauzy diagram and $T \in \Delta_\gamma$, then the relation between the length data λ and $\lambda^{(1)}$ respectively of T and $Q(T)$ is given by equation (2.3) in paragraph 2.1.1, that proves the lemma for paths of length one. If γ is a concatenation of n arrows $\gamma_1 ... \gamma_n$, since the definition of the matrix B_γ is such that $B_\gamma = B_{\gamma_n} ... B_{\gamma_1}$, then equation (2.5) follows by iterative application of the formula in equation (2.3). The lemma is proved. □

The domain of definition of Q is the set of all $(\pi, \lambda) \in \Delta(\mathcal{R})$ such that $\lambda_{\alpha_\epsilon} \neq \lambda_{\alpha_{1-\epsilon}}$ and is denoted by $\Delta_1(\mathcal{R})$. The connected components Δ_π of $\Delta(\mathcal{R})$ are naturally labeled by elements of $\mathcal{R}$ and the connected components Δ_γ of $\Delta_1(\mathcal{R})$ are naturally labeled by arrows, that is paths γ of length 1. One easily checks that each connected component of $\Delta_1(\mathcal{R})$ is mapped homeomorphically to some connected component of $\Delta(\mathcal{R})$. Let Q^n be the n-th iterated of the Rauzy induction map and let $\Delta_n(\mathcal{R})$ be its domain. The connected components of $\Delta_n(\mathcal{R})$ are naturally labeled by paths in $\Pi(\mathcal{R})$ of length n. If γ is obtained by following a sequence of arrows $\gamma_1 ... \gamma_n$ then $\Delta_\gamma = \{\lambda \in \Delta(\mathcal{R})\,;\, Q^{k-1} \in \Delta_{\gamma_k}\, \forall 1 \leq k \leq n\}$. Furthermore if γ ends at π' then $Q^n : \Delta_\gamma \to \Delta_{\pi'}$ is a homeomorphism. The set $\Delta_\infty(\mathcal{R}) := \bigcap_{n \in \mathbb{N}} \Delta_n(\mathcal{R})$ is the set of those i.e.t. T to which the Rauzy algorithm can be applied infinitely many times. Being the intersection of countably many sets of full lebesgue measure, $\Delta_\infty(\mathcal{R})$ has full lebesgue measure. Its elements are those $T = (\pi, \lambda)$ such that for any $n \in \mathbb{N}$, the n-th step $T^{(n)} = (\pi^{(n)}, \lambda^{(n)})$ of the algorithm satisfy the condition in equation (2.1). For the points where the algorithm stops the following characterization holds (see [**Y1**] for a proof).

LEMMA 2.1.4. *When applied to some $(\pi, \lambda) \in \Delta(\mathcal{R})$ the Rauzy induction algorithm Q eventually stops if and only if $T = (\pi, \lambda)$ has a connection.*

2.1.4. Normalized Rauzy Veech algorithm and Zorich's acceleration. The Rauzy-Veech algorithm has interesting recurrence properties just at projective level. In order to see these properties we introduce a normalization on the sum of the lengths of the intervals of interval exchange transformations. For any vector $\lambda \in \mathbb{R}_+^{\mathcal{A}}$ we introduce the notation

$$\|\lambda\| := \sum_{\alpha \in \mathcal{A}} \lambda_\alpha \texttt{ and } \hat{\lambda} := \frac{\lambda}{\|\lambda\|}$$

and for any combinatorial datum π over some alphabet $\mathcal{A}$ we write

$$\Delta_\pi^{(1)} := \{(\pi, \lambda) \in \Delta_\pi; \|\lambda\| = 1\}.$$

Interval exchange transformations $T \in \Delta_\pi^{(1)}$ will be often denoted $(\pi, \hat{\lambda})$. For any Rauzy class $\mathcal{R}$ the set of all the possible normalized intervals exchange transformations with combinatorial datum in $\mathcal{R}$ is denoted $\Delta^{(1)}(\mathcal{R}) := \bigcup_{\pi \in \mathcal{R}} \Delta_\pi^{(1)}$.

DEFINITION 2.1.5. Let $\mathcal{R}$ be a Rauzy class over an alphabet $\mathcal{A}$. The *normalized Rauzy-Veech algorithm* is the map $\mathbb{P}Q : \Delta^{(1)}(\mathcal{R}) \to \Delta^{(1)}(\mathcal{R})$ defined by

$$\mathbb{P}Q(\pi, \lambda) := \frac{Q(\pi, \lambda)}{\|Q(\pi, \lambda)\|}, \tag{2.6}$$

where Q is the Rauzy-Veech algorithm introduced in paragraph 2.1.1.

It is possible to see that the map $\mathbb{P}Q$ has an unique invariant measure which is absolutely continuous with respect to the lebesgue measure, nevertheless this measure is not finite (the original proof appears in [**Ve**], see also [**Y1**]). Following Zorich (see [**Z**]) we define an *acceleration* of the Rauzy-Veech algorithm as follows. Let T be an i.e.t. with combinatorial-length data (π, λ) in the domain of Q and consider the first step $T \mapsto Q(T)$ of the Rauzy-Veech algorithm applied to T. We recall the definition of the type of the i.e.t. T given at the beginning of paragraph 2.1.1: we say that T is of *type top* (or "t") if the combinatorial datum of $Q(T)$ is $R^t(\pi)$, on the other hand T is of *type bottom* (or "b") if the combinatorial datum of $Q(T)$ is $R^b(\pi)$. Then for any $T \in \Delta(\mathcal{R})$ (even not normalized) we define

$$N(T) := \min\{n > 0; \texttt{type}(T) \neq \texttt{type}(Q^n T)\}.$$

DEFINITION 2.1.6. The *Zorich's acceleration* is the map $\mathcal{Z} : \Delta^{(1)}(\mathcal{R}) \to \Delta^{(1)}(\mathcal{R})$ defined by

$$\mathcal{Z}(\pi, \lambda) := \mathbb{P}Q^{N(\pi,\lambda)}(\pi, \lambda), \tag{2.7}$$

where Q^N is the N-th iterated of the Rauzy-Veech algorithm introduced in paragraph 2.1.1 and $\mathbb{P}Q^N$ is its normalization.

The basic result in the ergodic theory of interval exchange transformation is the following theorem (see [**Z**] or [**Y1**]).

THEOREM 2.1.7. *For any Rauzy class $\mathcal{R}$ the Zorich's acceleration map $\mathcal{Z}$ defined by equation (2.7) has an unique invariant measure μ which is absolutely continuous with respect to the lebesgue measure on $\Delta^{(1)}(\mathcal{R})$. Moreover μ is finite and ergodic.*

Since the normalized Rauzy-Veech algorithm $\mathbb{P}Q : \Delta^{(1)}(\mathcal{R}) \to \Delta^{(1)}(\mathcal{R})$ has an ergodic acceleration (the Zorich's one), then it is recurrent. Quite often it is useful to consider the *first return map* of $\mathbb{P}Q : \Delta^{(1)}(\mathcal{R}) \to \Delta^{(1)}(\mathcal{R})$ to some subset of $\Delta^{(1)}(\mathcal{R})$. We end this paragraph developing some basic facts that we are going to use in the following.

Let us fix any Rauzy path $\eta : \pi_{start} \to \pi_{end}$ starting at π_{start} and ending at π_{end} and consider the sub-simplex $\Delta^{(1)}_\eta$ of $\Delta^{(1)}_{\pi_{start}}$ defined by

$$\Delta^{(1)}_\eta := \Delta_\eta \cap \Delta^{(1)}_{\pi_{start}},$$

that is the set of those $T = (\pi_{start}, \lambda)$ whose Rauzy path begins with η (the simplicial sub-cone Δ_η has be defined in paragraph 2.1.2). For any $T \in \Delta^{(1)}_\eta$ we write

$$r(T) := \min\{n > 0; Q^{r(T)}(T) \in \Delta_\eta\}.$$

The first return map of the Rauzy-Veech algorithm to the sub-simplex $\Delta^{(1)}_\eta$ is the map $R_\eta : \Delta^{(1)}_\eta \to \Delta^{(1)}_\eta$ defined by

$$R_\eta(T) := \mathbb{P}Q^{r(T)}(T).$$

Since $\mathbb{P}Q$ is recurrent the map R_η is defined almost everywhere on $\Delta^{(1)}_\eta$. On each connected component of its domain it acts with a linear map composed with the projection $\mathbb{P} : \Delta_\eta \to \Delta^{(1)}_\eta$. In order to get e better description of the connected components of the domain of R_η let us introduce the following partial order on the set $\Pi(\mathcal{R})$ of paths in the Rauzy diagram of $\mathcal{R}$:

$$\nu \prec \gamma \texttt{ iff } \gamma \texttt{ begins with } \nu. \tag{2.8}$$

Let Γ_η be the set of Rauzy paths $\gamma : \pi_{start} \to \pi_{end}$ both beginning and ending with η and minimal with this property with respect to the ordering $\prec$ introduced in equation (2.8). In other words for any such γ there exists two sub-paths γ' and γ'' such that we can write

$$\gamma : \eta\gamma' \text{ and } \gamma = \gamma''\eta \tag{2.9}$$

and for proper no sub-path ν with $\nu \prec \gamma$ the two decomposition in equation are possible. The connected components of the domain of R_η are exactly the sub-simplices $\Delta_\gamma^{(1)}$ of $\Delta_\eta^{(1)}$ with $\gamma \in \Gamma_\eta$. For any such γ, written according to the decomposition $\gamma = \gamma'\eta$ as above, and for any $T = (\pi_{start}, \lambda) \in \Delta_\gamma^{(1)}$ we have

$$R_\eta(T) = (\pi_{start}, \frac{{}^tB_{\gamma'}^{-1}\lambda}{\|{}^tB_{\gamma'}^{-1}\lambda\|}). \tag{2.10}$$

Since R_η is defined almost everywhere on $\Delta_\eta^{(1)}$ we have $\Delta_\eta^{(1)} = \bigsqcup_{\gamma\in\Gamma_\eta} \Delta_\gamma^{(1)} \mod 0$.

2.2. Reduction of Rauzy classes and distortion estimate.

This paragraph still concerns interval exchange transformations. It contains a fine estimate on the distortion of the lebesgue measure under iteration of the normalized Rauzy-Vecch algorithm. The proof of the estimate is based on a combinatorial operation on Rauzy classes called *reduction*. Both the estimate and the reduction operation are obtained in [**A,G,Y**], the latter being a generalization of a similar combinatorial operation introduced in [**A,V**]. The distortion estimate is quite technical and we will only use it in the proof of theorem 4.6.1 in chapter 4, for this reason we just state it (theorem 2.2.3) and recommend [**A,G,Y**] for details. The reduction operation appears more frequently in our work, for this reason we give here a complete description of it. We closely follow §5 of [**A,G,Y**], the reader may skip this paragraph at the first lecture and come back to it after having attached chapter 4.

2.2.1. Decorated Rauzy classes. Let $\mathcal{R}$ be a Rauzy class with alphabet $\mathcal{A}$ and $\mathcal{A}' \subset \mathcal{A}$ be a proper subset. An arrow is called $\mathcal{A}'$-*colored* if its winner belongs to $\mathcal{A}'$. A path $\gamma \in \Pi(\mathcal{R})$ is $\mathcal{A}'$-colored if it is a concatenation of $\mathcal{A}'-$ colored arrows.

For an element $\pi \in \mathcal{R}$ we say that π is $\mathcal{A}'$-*trivial* if the last letters on both the top and the bottom rows of π do not belong to $\mathcal{A}'$, π is $\mathcal{A}'$-*intermediate* if exactly one of those letters belongs to $\mathcal{A}'$ and finally π is $\mathcal{A}'$-*essential* if both letters belong to $\mathcal{A}'$. An $\mathcal{A}'$-*decorated Rauzy class* $\mathcal{R}_* \subset \mathcal{R}$ is a maximal subset whose elements can be joined by an $\mathcal{A}'-$ colored path. We let $\Pi_*(\mathcal{R}_*)$ be the set of $\mathcal{A}'-$ colored paths starting (and ending) at permutations in $\mathcal{R}_*$.

A decorated Rauzy class is called *trivial* if it contains a trivial element π, in this case $\mathcal{R}_* = \{\pi\}$ and $\Pi_*(\mathcal{R}_*) = \{\pi\}$, recalling that vertices are identified with zero-length paths. A decorated Rauzy class is called *essential* if it contains an essential element. Any essential decorated Rauzy class contains intermediate elements.

Let $\mathcal{R}_*$ be an essential decorated Rauzy class and let $\mathcal{R}_*^{ess} \subset \mathcal{R}_*$ be the subset of essential elements. Let $\Pi_*^{ess}(\mathcal{R}_*)$ be the set of paths that start and end at an element of $\mathcal{R}_*^{ess}$. An *arc* is a minimal non-trivial path in $\mathcal{R}_*^{ess}$, all arrows in the same arc are of the same type and have the same winner, so winner and type of an arc are well defined. Any element of $\mathcal{R}_*^{ess}$ is thus the start and end of one top arc and one bottom arc. The losers in an arc are all distinct, moreover the first loser is in $\mathcal{A}'$ and the others are not.

If $\gamma \in \Pi_*(\mathcal{R}_*)$ is an arrow then there exist unique paths $\gamma_s, \gamma_e \in \Pi_*(\mathcal{R}_*)$ such that $\gamma_s\gamma\gamma_e$ is an arc, called the *completion* of γ. If π is intermediate there exists a single arc passing through π, the completion of the arrow starting (or ending) at π.

If $\pi \in \mathcal{R}_*$ we define π^{ess} as follows. If π is essential then $\pi^{ess} = \pi$, if π is intermediate let π^{ess} be the end of the arc passing through π.

To $\gamma \in \Pi_*(\mathcal{R}_*)$ we associate an element $\gamma^{ess} \in \Pi_*^{ess}(\mathcal{R}_*)$ as follows. For a trivial path $\pi \in \mathcal{R}_*$ we use the previous definition of π^{ess}. Assuming that γ is an arrow we distinguish two cases:

(1) If γ starts at an essential element, we let γ^{ess} be the completion of γ.
(2) Otherwise, we let γ^{ess} be the endpoint of the completion of γ.

We extend the definition to paths $\gamma \in \Pi_*(\mathcal{R}_*)$ by concatenation. Notice that if $\gamma \in \Pi_*^{ess}(\mathcal{R}_*)$ then $\gamma^{ess} = \gamma$.

2.2.2. Reduction of Rauzy classes. Given a permutation π on the alphabet $\mathcal{A}$, even not admissible, whose top and bottom rows end with differen letters, we obtain the *admissible end* of π by deleting as many letters from the top and bottom rows of π as necessary to obtain an admissible permutation. The resulting permutation belongs to some Rauzy class $\mathcal{R}''$ on some alphabet $\mathcal{A}'' \subset \mathcal{A}$.

Let $\mathcal{R}_*$ be an essential decorated Rauzy class, and let $\pi \in \mathcal{R}_*^{ess}$. Delete all the letters not belonging to $\mathcal{A}'$ from the top and bottom rows of π. The resulting permutation π' is not necessary admissible, but since π is essential the letters in the end of the top and bottom rows of π' are distinct. Let π^{red} be the admissible end of π'. We call π^{red} the *reduction* of π. We extend the operation of reduction from $\mathcal{R}_*^{ess}$ to the whole $\mathcal{R}_*$ by taking the reduction of an element $\pi \in \mathcal{R}_*$ as the reduction of π^{ess}.

If $\gamma \in \Pi_*^{ess}(\mathcal{R}_*)$ is an arc starting at π_s and ending at π_e, then the reductions of π_s and π_e belong to the same Rauzy class and are joined by an arrow γ^{red}. The arrow γ^{red} has the same type and the same winning letter of the arc γ and its losing letter is the same as the one of the first letter of γ. It follows that the set of reductions of all $\pi \in \mathcal{R}_*$ is a Rauzy class $\mathcal{R}^{red}$ on some alphabet $\mathcal{A}'' \subset \mathcal{A}' \subset \mathcal{A}$. We define the reduction of a path $\gamma \in \Pi_*(\mathcal{R}_*)$ as follows. If γ is a trivial (zero-length) path or an arc, it is defined as above. We extend the definition to the case $\gamma \in \Pi_*^{ess}(\mathcal{R}_*)$ by concatenation. In general we let the reduction of γ to be equal to the reduction of γ^{ess}.

Restricted to essential elements the operation of reduction give a bijection $red : \mathcal{R}_*^{ess} \to \mathcal{R}^{red}$. If we think to elements $\pi \in \mathcal{R}$ as trivial paths we can extend the previous operation to a bijection compatible with concatenation on the set of arcs

$$red : \Pi_*^{ess}(\mathcal{R}_*) \to \Pi(\mathcal{R}^{red}).$$

The following lemma holds.

LEMMA 2.2.1. *Let $\mathcal{R}_*$ be an essential $\mathcal{A}'$−decorated Rauzy class and let $\mathcal{R}^{red}$ be its reduction, with alphabet $\mathcal{A}'' \subset \mathcal{A}'$. For any $\gamma \in \Pi_*(\mathcal{R}_*)$ we have two commutative diagrams*

$$\begin{array}{ccc} \mathbb{R}_+^{\mathcal{A}} & \to B_\gamma \to & \mathbb{R}_+^{\mathcal{A}} \\ \downarrow & & \downarrow \\ \mathbb{R}_+^{\mathcal{A}''} & \to B_{\gamma^{red}} \to & \mathbb{R}_+^{\mathcal{A}''} \end{array} \qquad \begin{array}{ccc} \mathbb{R}_+^{\mathcal{A}} & \to B_\gamma \to & \mathbb{R}_+^{\mathcal{A}} \\ \downarrow & & \downarrow \\ \mathbb{R}_+^{\mathcal{A}'\setminus\mathcal{A}''} & \to Id \to & \mathbb{R}_+^{\mathcal{A}'\setminus\mathcal{A}''}. \end{array}$$

Proof: See [**A,G,Y**], page 174. □

2.2.3. Drift in essential decorated Rauzy classes. Let $\mathcal{R}_* \subset \mathcal{R}$ be an essential $\mathcal{A}'$-decorated Rauzy class. For $\pi \in \mathcal{R}_*$ let $\alpha_t(\pi)$ (respectively $\alpha_b(\pi)$) be the rightmost letter in the top (respectively in the bottom) row of π that belongs to $\mathcal{A} \setminus \mathcal{A}'$. Let $d_t(\pi)$ (respectively $d_b(\pi)$) be the position of $\alpha_t(\pi)$ (respectively of $\alpha_b(\pi)$). Let $d(\pi) := d_t(\pi) + d_b(\pi)$. An essential element of $\mathcal{R}_*$ is thus some π such that $d_t(\pi), d_b(\pi) < d$. If π_s is an essential element of $\mathcal{R}_*$ and γ is an arrow starting at π_s and ending at π_e then

(1) $d_t(\pi_e) = d_t(\pi_s)$ or $d_t(\pi_e) = d_t(\pi_s) + 1$, the second possibility happening if and only if γ is a bottom whose winner precedes $\alpha_t(\pi_s)$ in the top of π_s.
(2) $d_b(\pi_e) = d_b(\pi_s)$ or $d_b(\pi_e) = d_b(\pi_s) + 1$, the second possibility happening if and only if γ is a top whose winner precedes $\alpha_b(\pi_s)$ in the bottom of π_s.

In particular $d(\pi_e) = d(\pi_s)$ or $d(\pi_e) = d(\pi_s) + 1$. In the second case we say that γ is *drifting*. Let $\mathcal{R}^{red}$ be the reduction of $\mathcal{R}_*$ and let $\mathcal{A}'' \subset \mathcal{A}' \subset \mathcal{A}$ be the alphabet of $\mathcal{R}^{red}$. If $\pi \in \mathcal{R}_*$ is essential, then there exists some $\alpha \in \mathcal{A}''$ that either precedes $\alpha_t(\pi)$ in the top row of π or precedes $\alpha_b(\pi)$ in the bottom row of π, we call such an α *good*. Indeed, if $\gamma \in \Pi_*(\mathcal{R}_*)$ is a path starting at π, ending with a drifting arrow and minimal with this property then the winner of the last arrow of γ belongs to $\mathcal{A}''$ and either precedes $\alpha_t(\pi)$ in the top of π (if the drifting arrow is a bottom) or precedes $\alpha_b(\pi)$ in the bottom of π (if the drifting arrow is a top).

Note that if $\gamma \in \Pi_*^{ess}(\mathcal{R}_*)$ is an arrow starting and ending at essential elements π_s, π_e then a good letter for π_s is also a good letter for π_e. Moreover, if γ is not drifting then the winner of γ is not a good letter for π_s.

2.2.4. Standard decomposition of separated paths. An arrow is called $(\mathcal{A} \setminus \mathcal{A}')$-*separated* if both its winner and its loser belong to $\mathcal{A}'$. A path $\gamma \in \Pi_*(\mathcal{R}_*)$ is $(\mathcal{A} \setminus \mathcal{A}')$-separated if it is a concatenation of $(\mathcal{A} \setminus \mathcal{A}')-$separated arrows. We also say that a Rauzy path γ is *complete* (or $\mathcal{A}$-*complete*) if for any letter $\alpha \in \mathcal{A}$ there exists an arrow composing γ having α as winner.

If $\gamma \in \Pi(\mathcal{R})$ is a non-trivial maximal $(\mathcal{A} \setminus \mathcal{A}')$-separated path then there exists an essential $\mathcal{A}'$-decorated Rauzy class $\mathcal{R}_* \subset \mathcal{R}$ such that $\gamma \in \Pi_*(\mathcal{R}_*)$. Moreover, if $\gamma = \gamma_1 ... \gamma_n$ then any arrow γ_i starts at an essential element $\pi_i \in \mathcal{R}_*^{ess}$ (and γ_n ends at an intermediate element of $\mathcal{R}_*$ by maximality).

Remark 2.2.2. Let $r := d(\pi_n) - d(\pi_1)$. Let $\gamma = \gamma^{(1)}\gamma^1 ... \gamma^{(r)}\gamma^r$, where the γ^i are drifting arrows and $\gamma^{(i)}$ are (possibly trivial) concatenation of non drifting arrows. If α is a good letter for π_1, then it follows that α is not the winner of any arrow in any $\gamma^{(i)}$. The reduction of any $\gamma^{(i)}$ are therefore non-complete paths in $\Pi(\mathcal{R}^{red})$.

2.2.5. The distortion estimate. For $q \in \mathbb{R}_+^{\mathcal{A}}$ let $\Lambda_q := \{\lambda \in \mathbb{R}_+^{\mathcal{A}}\,;\, \langle \lambda, q \rangle < 1\}$. Let $\mathcal{R}$ a Rauzy class and let $\gamma \in \Pi(\mathcal{R})$. We let $\Lambda_{q,\gamma} :=^t B_\gamma \Lambda_{B_\gamma q}$. The definition is so that $\{\pi\} \times \Lambda_{q,\gamma} = (\{\pi\} \times \Lambda_q) \cap \Delta_\gamma$, where π is the starting point of γ.

For $\mathcal{A}' \subset \mathcal{A}$ and $q \in \mathbb{R}_+^{\mathcal{A}}$ we put $N_{\mathcal{A}'}(q) := \prod_{\alpha \in \mathcal{A}'} q_\alpha$ and $M_{\mathcal{A}'}(q) := \max_{\alpha \in \mathcal{A}'} q_\alpha$. In the trivial case $\mathcal{A}' = \mathcal{A}$ we simply denote $N(q) := N_{\mathcal{A}}(q)$ and $M(q) := M_{\mathcal{A}}(q)$. We have $\mathrm{Leb}(\Lambda_q) = \frac{1}{d! N(q)}$.

If $\Gamma \subset \Pi(\mathcal{R})$ is a set of paths starting at the same $\pi \in \mathcal{R}$ let $\Lambda_{q,\Gamma} := \bigcup_{\gamma\in\Gamma} \Lambda_{q,\gamma}$. Given $\gamma \in \Pi(\mathcal{R})$ let $\Gamma_\gamma \subset \Gamma$ be the set of paths starting with γ. Let's define

$$P_q(\Gamma|\gamma) := \frac{\mathrm{Leb}(\Lambda_{q,\Gamma_\gamma})}{\mathrm{Leb}(\Lambda_{q,\gamma})}. \tag{2.11}$$

Let $\mathcal{R}_*$ be an essential $\mathcal{A}'-$decorated Rauzy class and let $\mathcal{R}^{red}$ be its reduction, with alphabet $\mathcal{A}'' \subset \mathcal{A}'$. Let q^{red} be the canonical projection of q o $\mathbb{R}_+^{\mathcal{A}''}$ obtained by forgetting the coordinates in $\mathcal{A} - \mathcal{A}''$. For any $\gamma \in \Pi_*(\mathcal{R}_*)$ lemma 2.2.1 in paragraph 2.2.2 implies that

$$\frac{P_q(\gamma|\pi)}{P_{q^{red}}(\gamma^{red}|\pi^{red})} = \frac{N_{\mathcal{A}-\mathcal{A}'}(q)}{N_{\mathcal{A}-\mathcal{A}'}(B_\gamma q)}. \tag{2.12}$$

When the vector q is a vector q^γ for a Rauzy path γ (defined in paragraph 2.1.2) we will write P_γ instead of P_{q^γ}. The following distortion estimate holds (See theorem 5.4 and proposition 5.7 in[**A,G,Y**] for a proof and for more details).

Theorem 2.2.3. *There exist a pair of constant $C > 0$ and $\theta > 0$ depending only on $d = \sharp(\mathcal{A})$ with the following property. Let $\mathcal{A}' \subset \mathcal{A}$ be a non-empty proper subset, $0 \le m \le M$ be integers, $q \in \mathbb{R}_+^{\mathcal{A}}$. Then for every $\pi \in \mathcal{R}$ we have*

(2.13)

$$P_q\{\gamma \in \Pi(\mathcal{R})\,;\, M(B_\gamma q) > 2^M M(q) \;\; \texttt{and} \;\; M_{\mathcal{A}'}(B_\gamma q) < 2^{M-m} M(q)|\pi\} \le C(m+1)^\theta 2^{-m}$$

and

(2.14)

$$P_q\{\gamma \in \Pi(\mathcal{R}) \;\; \texttt{is not complete} \;\; ; M_{\mathcal{A}'}(B_\gamma q) > 2^M M(q)|\pi\} \le C(M+1)^\theta 2^{-M}.$$

Note. If fact the complete result in [**A,G,Y**] contains two more distortion estimates similar to these ones, but we don't need them in our work.

2.3. Translation surfaces

We consider a triple (M, Σ, k), where M is a compact, boundaryless, orientable topological surface of genus g, Σ is a finite subset $\{p_1, ..., p_r\}$ of M with r elements and $k = (k_1, .., k_r) \in \mathbb{N}^r$ is an integer vector whose entries satisfy the relation:

$$\sum_{i=1}^{r} k_i = 2g - 2. \tag{2.15}$$

A *translation structure* ξ on (M, Σ, k) is given by

(1) A maximal atlas on $M \setminus \Sigma$ such that the changes of charts are translations.
(2) For each marked point $p_i \in \Sigma$ a neighborhood V_i of p_i, a neighborhood W of 0 in $\mathbb{C}$ and a ramified covering $\rho : (V_i, p_i) \to (W, 0)$ of degree $k_i + 1$ such that the local sections of ρ are charts of the atlas.

We call $\texttt{Transl}(M, \Sigma, k)$ the set of translation structures on the triple (M, Σ, k). A translation atlas ξ on (M, Σ, k) provides us also with the following structures:

(1) A Riemann surface structure on M. (Not only on $M \setminus \Sigma$)
(2) A holomorphic 1-form w_ξ on M, that in a translation chart is given by dz. The 1-form w_ξ is newer 0 on $M \setminus \Sigma$ and in a chart around a marked point p_i it takes the form $w_\xi = z^{k_i} dz$, that is it has a zero of order k_i. The combinatorial relation (2.15) therefore corresponds to the well known topological relation for the zeroes of a closed one form on (M, Σ).

(3) A flat metric $g_\xi := |dz|^2$ defined in $M \setminus \Sigma$ with cone singularities of total angle $2(k_i+1)\pi$ at any point $p_i \in \Sigma$.
(4) Two vertical and horizontal vector fields ∂_y and ∂_x on $M \setminus \Sigma$. These fields are not complete since their trajectories stop when they arrive at a marked point. We call *outgoing vertical separatrices* the trajectories of ∂_y starting from a marked point and *ingoing vertical separatrices* the trajectories of ∂_y ending in a marked point. For the horizontal vector field ∂_x we define similarly its outgoing or ingoing *horizontal separatrices*. Observe that a marked point $p_i \in \Sigma$ of degree k_i is the end-point of exactly k_i+1 outgoing vertical separatrices and k_i+1 outgoing horizontal separatrices.
(5) An area form $dz \wedge d\bar{z}$ on $M \setminus \Sigma$.

The total number of vertical separatrices equals the sum of the angles at singularities (modulo a factor 2π) and is equal to $\sum_{p_i \in \Sigma}(k_i+1) = 2g+r-2$. We recall from the introduction the notion of *saddle connection* for a translation surface ξ: it is a path $\gamma : [0,T] \to M$ such that $\gamma^{-1}(\Sigma) = \{0,T\}$ (it starts end ends at marked points in Σ and do not contains other such points in its interior) and that is geodetic for the flat metric on $M \setminus \Sigma$.

2.3.1. Triangulations. Let $\xi \in \texttt{Transl}(M,\Sigma,k)$ be a translations structure. A *triangulation* for ξ is a cellular decomposition

$$M = X_0 \cup X_1 \cup X_2$$

of M where the 0-*skeleton* X_0 is the set Σ, the 1-*skeleton* X_1 is a finite set $\{\gamma_1,..,\gamma_L\}$ of saddle connection (without endpoints) for ξ and the 2-*skeleton* X_2 is a finite set $\{T_1,..,T_N\}$ of open triangles having as sides the saddle connections in X_1. Any translation surface admits such a triangulation (see [**KeMaS**] or [**E,M**]). For any $T \in X_2$ there is an homeomorphism $\varphi : T \to D$ onto an open triangle $D \subset \mathbb{C}$ which is a chart of the translation atlas. We have therefore a family $\{D_1,..,D_N\}$ of open triangles in $\mathbb{C}$. Let $T \in X_2$ be any triangle of the triangulation and let D be its correspondent euclidean triangle in $\mathbb{C}$. We assign to any saddle connection γ in the boundary of T an open segment L in $\mathbb{C}$ in the boundary of D. Any pair of neighboring triangles T,T' produces a pair L,L' of parallel segments of the same length. Let γ be any element in X_1 and let L be the correspondent segment in $\mathbb{C}$. We associate to any endpoint p of γ a vertex v in $\mathbb{C}$ which is an endpoint of L.

We cut M along a proper subset of saddle connections in the 1-skeleton X_1 in order to obtain a surface with boundary which can be embedded in $\mathbb{C}$ (that is a polygon). The embedding is obtained pasting together the triangles D_i with identifications between sides of the same length. Several vertices of the triangles are in this way identified to the same point. We get a closed polygon $\mathcal{P} = \mathcal{P}(\xi) \subset \mathbb{C}$, which has a cellular decomposition $\mathcal{P} = \mathcal{P}_0 \cup \mathcal{P}_1 \cup \mathcal{P}_2$, where

$$\mathcal{P}_0 = \{v_1,..,v_V\}$$
$$\mathcal{P}_1 = \{L_1,..,L_S\}$$
$$\mathcal{P}_2 = \{D_1,..,D_F\}.$$

As a consequence of the identifications some vertex v_i may fall in the interior of $\mathcal{P}$, this can happen if v_i corresponds to a marked point $p_i \in \Sigma$ which is not a zero for the form. Since the triangles pasted together are at least two, there will always be some side $L_j \in \mathcal{P}_1$ in the interior of $\mathcal{P}$. All the other sides L_k are in $\partial\mathcal{P}$ and correspond exactly to those saddle connections $\gamma_j \in X_1$ that have been cut to construct the embedding. These sides therefore arrears in pairs (L, L'), where L and L' are parallel and have the same length.

DEFINITION 2.3.1. A *developing graph* for ξ is a planar cellular complex $\mathcal{P} = \mathcal{P}(\xi)$ constructed as above.

LEMMA 2.3.2. *Any translation structure ξ admits a developing graph $\mathcal{P}(\xi)$ without horizontal sides in its boundary.*

Proof: Let us suppose that we have a developing graph $\mathcal{P}(\xi)$ for the translation structure ξ that has an horizontal side L in its boundary. Let us call D the unique triangle in $\mathcal{P}(\xi)$ that has L as side. Since L is a boundary side, $\mathcal{P}(\xi)$ has in its boundary another side L' parallel to L and with the same length. Let D' the unique triangle in $\mathcal{P}(\xi)$ that has L' as side. We can construct another developing graph $\mathcal{P}'(\xi)$ for ξ cutting the triangle D away from $\mathcal{P}(\xi)$ and pasting it to the triangle D' along the side L'. Since in any triangle the sum of the angle is π then this operation does not make appear new horizontal sides, on the other hand the sides L and L' are no more in $\partial\mathcal{P}'(\xi)$ but they are identified to some side L'' in the interior of $\mathcal{P}'(\xi)$. Therefore the new developing graph $\mathcal{P}'(\xi)$ has two horizontal sides less than $\mathcal{P}(\xi)$. Iterating the procedure we can eliminate all the horizontal sides in the boundary. The lemma is proved. □

Let ξ be a translation structure and let $\mathcal{P} = \mathcal{P}(\xi)$ be a developing graph for ξ with no horizontal sides in its boundary. Let us introduce the subset $\mathcal{V}(\xi) \subset \mathcal{P}_0$ of those vertices v in $\mathcal{P}_0$ such that there exists some $\epsilon > 0$ and $\theta > 0$ such that the angular sector $\{v + re^{is}; 0 \leq r < \epsilon, |s| < \theta\}$ is contained in the intersection $B(v, \epsilon) \cap \mathcal{P}$ of $\mathcal{P}$ with a small disk around v of radius ϵ.

LEMMA 2.3.3. *Let ξ be a translation structure in $\texttt{Transl}(M, \Sigma, k)$ and let $\mathcal{P} = \mathcal{P}(\xi)$ be a developing graph without horizontal sides in its boundary. Let $\mathcal{V} = \mathcal{V}(\xi)$ the set of vertices defined above. There is a bijection between the points of $\mathcal{V}$ and the pairs (p_i, S), where p_i is a point in Σ and S is an horizontal separatrix for ξ starting at p_i.*

Proof: Elements v in $\mathcal{V}(\xi)$ are exactly the vertices in $\mathcal{P}(\xi)$ such that the initial segment of the horizontal half line starting from v is contained in the interior of $\mathcal{P}(\xi)$. Since open sets inside $\mathcal{P}(\xi)$ project to open sets in the translation surface ξ we have a well defined application

$$v \mapsto (p_i, S_i)$$

assigning to any $v \in \mathcal{V}(\xi)$ the datum of a point $p_i \in \Sigma$ and an horizontal separatrix S_i starting at p_i.

For any pair (p_i, S_i) either there exists a triangle $T \in X_2$ that has p_i as vertex and contains the beginning of S_i or there is a pair of triangles $T, T' \in X_2$ that share a side $\gamma \in X_1$ and such that $T \cup \gamma \cup T'$ satisfy the same property. It follows that the application above is surjective.

On the other hand a triangle $T \in X_2$ can contain the beginning of just one horizontal separatrix starting at one of its vertices, this because the sum of its angles is π. The same properties follows for the triangles D that compose $\mathcal{P}(\xi)$. The application above therefore is injective too and the lemma is proved. □

2.3.2. Teichmüller space of abelian differentials and its strata. Let $\texttt{Diff}^+(M,\Sigma)$ denote the set of orientation preserving diffeomorphisms of M that are the identity on Σ. Elements ξ in $\texttt{Transl}(M,\Sigma,k)$ are holomorphic atlas on M, each atlas being a maximal family $\xi = \{\varphi_i\}$ of compatible homeomorphisms $\varphi_i : U_i \to V_i$ from opens sets U_i in M to open sets V_i in $\mathbb{C}$. For any $f \in \texttt{Diff}^+(M,\Sigma)$ any local chart φ of ξ can be composed with f, giving an other local chart $\varphi \circ f$. It is easy to check that when φ varies among all the charts of ξ, the set of charts $\varphi \circ f$ is a new translation atlas $f_*\xi$, moreover at any singular point $p_j \in \Sigma$ the total angles respectively for the structures ξ and $f_*\xi$ are the same. In other words $\texttt{Diff}^+(M,\Sigma)$ acts on the right on $\texttt{Transl}(M,\Sigma,k)$:

$$\begin{array}{ccc} \texttt{Transl}(M,\Sigma,k) \times \texttt{Diff}^+(M,\Sigma) & \to & \texttt{Transl}(M,\Sigma,k) \\ (\xi, f) & \mapsto & f_*\xi. \end{array} \tag{2.16}$$

Let $\texttt{Diff}_0^+(M,\Sigma)$ denote the set of those f in $\texttt{Diff}^+(M,\Sigma)$ that are isotopic to the identity. It acts on $\texttt{Transl}(M,\Sigma,k)$ as a subgroup of $\texttt{Diff}^+(M,\Sigma)$. The quotient space

$$\mathcal{T}(M,\Sigma,k) := \texttt{Transl}(M,\Sigma,k)/\texttt{Diff}_0^+(M,\Sigma). \tag{2.17}$$

is the set of isotopy classes $[\xi]$ of translation structures on the triple (M,Σ,k) and is a *stratum* of the so called *Teichmüller space* of abelian differentials on M. It turns out that the set $\mathcal{T}(M,\Sigma,k)$ can be provided by a complex manifold structure. We briefly describe the construction on local charts.

Let us fix a singular point $q \in \Sigma$ and consider a (differentiable) universal covering $\rho : (M^*, q^*) \to (M, q)$ of the topological surface M. Let ξ be any translation structure on (M,Σ,k). The associated 1-form w_ξ can be lifted to a 1-form $w_\xi^* := \rho^* w_\xi$ on M^*. Since w_ξ is closed, w_ξ^* is exact on M^* and we get a C^∞ map $\varphi_\xi : M^* \to \mathbb{C}$ defined by $x \mapsto \int_q^x w_\xi^*$ for any $x \in M^*$. Note that the value of the integral is unchanged when we change ξ with $f_*\xi$, where $f \in \texttt{Diff}_0^+(M,\Sigma)$ is an isotopy. Therefore φ_ξ depends only on the isotopy class $[\xi]$ of ξ and we get a map

$$\begin{array}{ccc} \mathcal{T}(M,\Sigma,k) & \to & C^\infty(M^*,\mathbb{C}) \\ \xi & \mapsto & \varphi_\xi \end{array} \tag{2.18}$$

The topology we put on $\mathcal{T}(M,\Sigma,k)$ is the pull-back of the *compact-open* topology on $C^\infty(M^*,\mathbb{C})$.

Local charts on $\mathcal{T}(M,\Sigma,k)$ are introduced through the so called *period map*. For any translation structure ξ on (M,Σ,k), the integral of the associated 1-form w_ξ on relative homology classes

$$[\gamma] \mapsto \int_\gamma w_\xi$$

defines an element in $\hom(H_1(M,\Sigma,\mathbb{Z}),\mathbb{C})$ that is in $H^1(M,\Sigma,\mathbb{C}) = \mathbb{C}^{2g+r-1}$. Moreover for any $[\gamma] \in H_1(M,\Sigma,\mathbb{Z})$ the integral above keeps unchanged when changing w_ξ with $f_* w_\xi$ for $f \in \texttt{Diff}_0^+(M,\Sigma)$. Denoting with $\Theta(\xi)$ the linear operator associated to ξ as above we get a map

$$\begin{array}{cccc} \Theta : & \mathcal{T}(M,\Sigma,k) & \to & H^1(M,\Sigma,\mathbb{C}) \\ & \xi & \mapsto & \Theta(\xi) \end{array} \tag{2.19}$$

which is called *period map*.

LEMMA 2.3.4. *The map Θ defined in equation (2.19) is a local homeomorphism.*

Proof: From the definition of the topology on $\mathcal{T}(M,\Sigma,k)$ given above it is evident that the map Θ is continuous, therefore in order to prove the statement it is enough to prove that for any $[\xi]\in\mathcal{T}(M,\Sigma,k)$ there exists an open subset $U\subset\mathcal{T}(M,\Sigma,k)$ with $[\xi]\in U$ such that:

- the image $\Theta(U)$ is open in $H^1(M,\Sigma,\mathbb{C})=\mathbb{C}^{2g+r-1}$,
- the restriction $\Theta|_U$ of the period map to U is injective.

Let $[\xi]$ be any element in $\mathcal{T}(M,\Sigma,k)$ and let $\{\gamma_1,..,\gamma_n\}$ be saddle connections for the translation structure ξ that define a triangulation. They are smooth curves $\gamma_i:[0,T_i]\to M$ with $\gamma_i^{-1}(\Sigma)=\{0,T_i\}$ which are geodesic for the flat metric induced by ξ on $M\setminus\Sigma$. In particular they define n homotopy classes $[\gamma_1],..,[\gamma_n]$ with fixed endpoints for the pair (M,Σ). Let $\mathcal{P}=\mathcal{P}(\xi)$ be a developing graph for ξ induced by the triangulation above. Pasting together the sides in $\partial\mathcal{P}$ parallel and with the same length we obtain a map $\mathcal{P}\to M$ that induces the translation structure ξ on M.

We consider an open subset U in $\mathcal{T}(M,\Sigma,k)$ such that for any $[\xi']\in U$ any homothopy class $[\gamma_i]$ defined above contains a representant γ_i' which is a saddle connection for ξ'. For any $[\xi']\in U$ the family of curves $\{\gamma_1',..,\gamma_n'\}$ obtained in this way defines a triangulation for ξ'. With the same cuts used to construct the developing graph $\mathcal{P}$ for ξ we obtain a developing graph $\mathcal{P}'=\mathcal{P}'(\xi')$ for ξ'.

We first show that $\Theta(U)$ is open in $\mathbb{C}^{2g+r-1}$. It is sufficient to prove that the image is open at $\Theta([\xi])$. The datum $(z_1,..,z_d)=\Theta([\xi])\in\mathbb{C}^{d=2g+r-1}$ corresponds to a choice of a basis $\{\gamma_{i(1)},..,\gamma_{i(d)}\}$ of $H_1(M,\Sigma,\mathbb{Z})$ whose elements are sides of the fixed triangulation for ξ. The values $z_1=\int_{\gamma_{i(1)}}w_\xi,..,z_d=\int_{\gamma_{i(d)}}w_\xi$ determine the geometry of the developing graph $\mathcal{P}$ for ξ. Once the combinatorics is fixed, the condition of forming a graph is open in $z_1,..,z_d$, therefore choosing $z_1',..,z_d'$ sufficiently close to $z_1,..,z_d$ we can form a graph $\mathcal{P}'$. Then pasting together the sides in $\partial\mathcal{P}'$ that are parallel and have the same length we obtain a translation surface ξ' in U with $\Theta([\xi'])=(z_1',..,z_d')$. The openness of the image is proved.

Now we prove that the restriction $\Theta|_U$ is injective. Let us consider any pair $[\xi],[\xi']\in U$. By definition of U the translation surfaces ξ and ξ' both admit a triangulation with sides in the fixed homotopy classes $[\gamma_1],..,[\gamma_d]$ defined above. Therefore, cutting along the same sides, ξ and ξ' admit two developing graphs $\mathcal{P}$ and $\mathcal{P}'$ with the same combinatorics. Once the combinatorics is fixed the value of the period map determines the developing graph, therefore the condition $\Theta([\xi])=\Theta([\xi'])$ implies $\mathcal{P}=\mathcal{P}'$. We have a commutative diagram

$$\begin{array}{ccc}\mathcal{P} & \rightarrow & \mathcal{P}' \\ \downarrow & & \downarrow \\ \xi & \dashrightarrow & \xi'\end{array}$$

where the upper horizontal arrow is the identity. The lower horizontal arrow therefore gives an element $f\in\mathtt{Diff}_0^+(M,\Sigma)$ such that $[f_*\xi]=[\xi]$, the restriction $\Theta|_U$ is therefore injective. The lemma is proved.

□

Proposition 2.3.5. *For any closed compact surface M, for any non-empty finite subset Σ of it with r elements and for any integer vector $k=(k_1,..,k_r)$ with non-negative entries such that $k_1+..+k_r=2g-2$, the stratum $\mathcal{T}(M,\Sigma,k)$ of* *the Teichmüller space is a non-compact complex manifold of complex dimension* $d = 2g + r - 1$.

Proof: Local charts are given by the restriction of the period map in equation (2.18) to the small open sets where it acts as an homeomorphism. □

Let's fix any integer vector $k = (k_1, .., k_r)$ with non-negative entries such that $k_1 + .. + k_r = 2g - 2$. For any two pairs (M, Σ) and (M', Σ') of closed compact surfaces of genus g and finite subsets of the same cardinality r, we can find a diffeomorphism of pairs $h : (M, \Sigma) \to (M', \Sigma')$. For any translation structure ξ for (M, Σ, k) its push-forward $h_*\xi$ is a translation structure for (M', Σ', k). Obviously isotopic structures are sent to isotopic structures, thus we have a well defined application between strata:

$$h_* : \mathcal{T}(M, \Sigma, k) \to \mathcal{T}(M', \Sigma', k).$$

Moreover in the carts given by the period maps the action of h_* correspond to the action of h on relative homology groups

$$H_1(M, \Sigma, \mathbb{Z}) \to H_1(M', \Sigma', \mathbb{Z}),$$

which is a $\mathbb{Z}$-linear isomorphism, therefore complex linear. The strata $\mathcal{T}(M, \Sigma, k)$ and $\mathcal{T}(M', \Sigma', k)$ therefore differ for a complex diffeomorphism h_* and they can be identified. The information about the pair (M, Σ) can be therefore omitted and for any positive integer g and any r non-negative integers $k_1, .., k_r$ such that $\sum_{i=1}^{r} k_i = 2g - 2$ we denote

$$\mathcal{T}(k_1, .., k_r)$$

the corresponding stratum of the Teichmüller space of abelian differentials.

2.3.3. Action of mapping class group and $\mathrm{GL}(2, \mathbb{R})$**, area, volume form.** Given a closed compact oriented surface M of genus g and a finite subset Σ of it with cardinality r, the *mapping class group* of the pair (M, Σ) is the quotient

$$\mathrm{Mod}(M, \Sigma) := \mathtt{Diff}^+(M, \Sigma)/\mathtt{Diff}^+_0(M, \Sigma). \tag{2.20}$$

Consider any two pairs (M, Σ) and (M', Σ') as above. A diffeomorphism of pairs $h : (M, \Sigma) \to (M', \Sigma')$ induces (by conjugation) an isomorphism of groups respectively between $\mathtt{Diff}^+(M, \Sigma)$ and $\mathtt{Diff}^+(M', \Sigma')$ and between $\mathtt{Diff}^+_0(M, \Sigma)$ and $\mathtt{Diff}^+_0(M', \Sigma')$. The corresponding mapping class groups are therefore isomorphic and we can omit the dependence on the pair (M, Σ) and simply write $\mathrm{Mod}(g, r)$ for $\mathrm{Mod}(M, \Sigma)$. The action of $\mathtt{Diff}^+(M, \Sigma)$ on $\mathtt{Transl}(M, \Sigma, k)$ induces an action of $\mathrm{Mod}(g, r)$ on $\mathcal{T}(k_1, .., k_r)$

$$\begin{array}{ccc} \mathcal{T}(k_1, .., k_r) \times \mathrm{Mod}(g, r) & \to & \mathcal{T}(k_1, .., k_r) \\ ([\xi], [f]) & \mapsto & [f_*\xi]. \end{array} \tag{2.21}$$

The action defined in equation (2.21) is proper, but not free, since there are translation structure that admit conformal automorphisms.

The group $\mathrm{GL}(2, \mathbb{R})$ acts on the Teichmüller space $\mathcal{T}(k_1, .., k_r)$ as follows. Given a pair (M, Σ) and a translation structure $\xi \in \mathtt{Transl}(M, \Sigma, k)$ let us look at ξ as to a family of local charts $\{\varphi_i\}$, where any $\varphi_i : U_i \to V_i$ is an homeomorphism from an open subset U_i of M to an open subset V_i of $\mathbb{C}$. For any $G \in \mathrm{GL}(2, \mathbb{R})$ we consider the family $\{G \circ \varphi_i\}$. Since the translation subgroup is normal in the group of complex affine automorphisms of $\mathbb{C}$, the family $\{G \circ \varphi_i\}$ is still a translation atlas and we denote it $G\xi$. We get an action on the left

$$\begin{array}{ccc} \mathrm{GL}(2,\mathbb{R}) \times \mathtt{Transl}(M,\Sigma,k) & \to & \mathtt{Transl}(M,\Sigma,k) \\ (G,\xi) & \mapsto & G\xi. \end{array}$$

Since $\mathtt{Diff}^+(M,\Sigma)$ acts on the right on $\mathtt{Transl}(M,\Sigma,k)$ and $\mathrm{GL}(2,\mathbb{R})$ on the left then the two actions commute and we get an action on the quotient

$$\begin{array}{ccc} \mathrm{GL}(2,\mathbb{R}) \times \mathcal{T}(k_1,..,k_r) & \to & \mathcal{T}(k_1,..,k_r) \\ (G,[\xi]) & \mapsto & [G\xi]. \end{array} \tag{2.22}$$

Moreover the action of $\mathrm{GL}(2,\mathbb{R})$ and commutes with the action of $\mathrm{Mod}(g,r)$ defined at equation (2.21).

For any $\xi \in \mathtt{Transl}(M,\Sigma,k)$ the corresponding holomorphic 1-form w_ξ induces an area form $w_\xi \wedge \overline{w}_\xi$ on $M \setminus \Sigma$. The area of a translation structure ξ is therefore given by

$$\mathrm{Area}(\xi) := \int_{M\setminus\Sigma} w_\xi \wedge \overline{w}_\xi. \tag{2.23}$$

The integral in equation (2.23) is obviously invariant under isotopy (that is under the action of $\mathtt{Diff}_0^+(M,\Sigma)$), moreover it is a real analytic function of the *periods* of the structure ξ, that is the entries of the complex vector $\Theta([\xi])$, where Θ is the period map defined by equation (2.19)). We get a real analytic function on the Teichmüller space, that we call the *area function*:

$$\begin{array}{ccc} \mathrm{Area} : \mathcal{T}(k_1,..,k_r) & \to & \mathbb{R}_+ \\ \xi & \mapsto & \mathrm{Area}(\xi). \end{array} \tag{2.24}$$

Since the map $[\xi] \mapsto \mathrm{Area}([\xi])$ is real analytic, the set of area one translation structures

$$\mathcal{T}^{(1)}(k_1,..,k_r) := \{[\xi] \in \mathcal{T}^{(1)}(k_1,..,k_r); \mathrm{Area}([\xi]) = 1\} \tag{2.25}$$

is a codimension one embedded sub-manifold of $\mathcal{T}(k_1,..,k_r)$.

Let us consider the standard volume form $d\mathrm{Leb}$ on $\mathbb{C}^d$ normalized in order to give co-volume one to the integer lattice $(\mathbb{Z}\oplus i\mathbb{Z})^d$. Using the period map Θ defined in equation (2.19) we pull back it and we get a volume form

$$m := \Theta^* d\mathrm{Leb} \tag{2.26}$$

on the stratum $\mathcal{T}(k_1,..,k_r)$. An element $f \in \mathtt{Diff}^+(M,\Sigma)$ induces an isomorphism in the relative homology $H_1(M,\Sigma,\mathbb{Z})$, that is an element of $\mathrm{SL(d},\mathbb{Z})$, which preserves the standard volume form $d\mathrm{Leb}$ on $H^1(M,\Sigma,\mathbb{C})$. The action of $\mathrm{Mod}(g,r)$ therefore preserve the volume form m. $\mathrm{SL}(2,\mathbb{R})$ acts on $\mathcal{T}(k_1,..,k_r)$ as subgroup of $\mathrm{GL}(2,\mathbb{R})$. Looking at the effect of the action on the space of periods, that is on $H^1(M,\Sigma,\mathbb{C})$, we get that $\mathrm{SL}(2,\mathbb{R})$ too preserves the form m. The subgroup of $\mathrm{GL}(2,\mathbb{R})$ of matrices $\begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix}$ with $r > 0$ determines a diffeomorphism

$$\begin{array}{ccc} (0,+\infty) \times \mathcal{T}^{(1)}(k_1,..,k_r) & \to & \mathcal{T}(k_1,..,k_r) \\ (r,[\xi]) & \mapsto & [r\xi] \end{array}$$

that induces a decomposition

$$m = dr \wedge m^{(1)} \tag{2.27}$$

where the form $m^{(1)}$ in equation (2.27) is a positive volume form on $\mathcal{T}^{(1)}(k_1,..,k_r)$. It is easy to check that the area function defined by equation (2.24) is invariant both for the action of $\mathrm{Mod}(g,r)$ and of $\mathrm{SL}(2,\mathbb{R})$. Thus the area one hyper-surface $\mathcal{T}^{(1)}(k_1,..,k_r)$ is preserved both by $\mathrm{SL}(2,\mathbb{R})$ and $\mathrm{Mod}(g,r)$ and the two groups act on $\mathcal{T}^{(1)}(k_1,..,k_r)$ by restriction of their actions on $\mathcal{T}(k_1,..,k_r)$. Moreover $\mathrm{SL}(2,\mathbb{R})$ and $\mathrm{Mod}(g,r)$ preserve the volume form $m^{(1)}$ on $\mathcal{T}(1)(k_1,..,k_r)$.

2.3.4. Moduli space and Teichmüller flow. Given two positive integers g and r and r non-negative integers $k_1,..,k_r$ satisfying equation (2.15), the *stratum* of the *moduli space* of translation surfaces with a cone singularities of orders $2(k_1+1)\pi,...,2(k_r+1)\pi$ is

$$\mathcal{H}(k_1,..,k_r) := \mathcal{T}(k_1,..,k_r)/\mathrm{Mod}(g,r). \tag{2.28}$$

We call X the general element of $\mathcal{H}(k_1,..,k_r)$. Once we choose a pair (M,Σ) as above such an X correspond to the datum (S,w) of a Riemann surface S of genus g and an holomorphic 1-form w on S with a zeroes of orders $k_1...k_k$. Since the group $\mathrm{Mod}(g,r)$ acts with non-trivial stabilizer on $\mathcal{T}(k_1,..,k_r)$ the moduli space is not a manifold but just an orbifold.

We may also consider the moduli space of area one translation surface. The stratum corresponding to the prescribed order of singularities $k_1,..,k_r$ is

$$\mathcal{H}^{(1)}(k_1,..,k_r) := \mathcal{T}^{(1)}(k_1,..,k_r)/\mathrm{Mod}(g,r). \tag{2.29}$$

Sometimes when the base topological surface M and its subset Σ are specified we will also use the notation $\mathcal{H}(M,\Sigma,k)$ for $\mathcal{H}(k_1,..,k_r)$ and $\mathcal{H}^{(1)}(M,\Sigma,k)$ for $\mathcal{H}^{(1)}(k_1,..,k_r)$.

The volume forms m and $m^{(1)}$, defined respectively on $\mathcal{T}(k_1,..,k_r)$ by equation (2.26) and on $\mathcal{T}^{(1)}(k_1,..,k_r)$ by equation (2.27), are invariant for the action of $\mathrm{Mod}(g,r)$. Therefore m and $m^{(1)}$ define two measures ν and $\nu^{(1)}$ respectively on $\mathcal{T}(k_1,..,k_r)$ and on $\mathcal{T}^{(1)}(k_1,..,k_r)$. These two measures have smooth density and are invariant under the action of the mapping class group. The natural projections

$$\mathcal{T}(k_1,..,k_r) \to \mathcal{H}(k_1,..,k_r) \ \texttt{and} \ \mathcal{T}^{(1)}(k_1,..,k_r) \to \mathcal{H}^{(1)}(k_1,..,k_r)$$

induces two smooth measures μ and $\mu^{(1)}$ respectively on $\mathcal{H}(k_1,..,k_r)$ and on $\mathcal{H}^{(1)}(k_1,..,k_r)$. Moreover ν and $\nu^{(1)}$ are invariant under $\mathrm{SL}(2,\mathbb{R})$ (since the volume forms m and $m^{(1)}$ are invariant) therefore $\mathrm{SL}(2,\mathbb{R})$ preserves μ and $\mu^{(1)}$.

The *Teichmüller flow* $\mathcal{F}_t$ on $\mathcal{H}^{(1)}(k_1,..,k_r)$ is the action of the diagonal subgroup of $\mathrm{SL}(2,\mathbb{R})$, that if for any $t \in \mathbb{R}$:

$$\mathcal{F}_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}. \tag{2.30}$$

The following one is the basic result in Teichmüller dynamics (see [**Ma1**] and [**Ve**]).

THEOREM 2.3.6. *The smooth measure $\mu^{(1)}$ defined above is invariant for the Teichmüller flow, it gives to $\mathcal{H}^{(1)}(k_1,..,k_r)$ finite volume and its restriction on any connected component is ergodic.*

2.4. Zippered rectangles and Veech space

In this paragraph we describe a fundamental construction given by Veech (see [**Ve**]). We follow the presentation appearing in [**Y1**] and [**M,M,Y**]. Let $\mathcal{R}$ be a Rauzy class and $\pi = (\pi^t, \pi^b) \in \mathcal{R}$. Let $\Theta_\pi \subset \mathbb{R}^{\mathcal{A}}$ be the set of all τ such that

$$\sum_{\pi^t(\alpha)\le k} \tau_\alpha > 0 \texttt{ and } \sum_{\pi^t(\beta)\le k} \tau_\beta < 0 \texttt{ for all } 1 \le k \le d-1. \tag{2.31}$$

Θ_π is an open convex polyhedral cone. It is not empty, since the vector τ with coordinates $\tau_\alpha := \pi^b(\alpha) - \pi^t(\alpha)$ belongs to Θ_π. We say that the $\tau \in \Theta_\pi$ are the *suspension data* for π.

2.4.1. The Veech construction. From the data (π, λ, τ) it is possible to define a translation surface $X = X(\pi, \lambda, \tau)$ in some $\mathcal{H}(k_1, .., k_r)$, where the orders of singularities $k_1, .., k_r$ depend only on the Rauzy class $\mathcal{R}$ of π. The construction is due to Veech and is known as the *zippered rectangles construction.* For any length and suspension data $(\lambda, \tau) \in \Delta_\pi \times \Theta_\pi$ we define the complex vector $\zeta = \lambda + i\tau \in \mathcal{R}^d$. Then for any $\alpha \in \mathcal{A}$ we define two complex numbers

$$\begin{aligned} \xi^t_\alpha &:= \textstyle\sum_{\pi^t(x)<\pi^t(\alpha)} \zeta_x \\ \xi^b_\alpha &:= \textstyle\sum_{\pi^b(x)<\pi^b(\alpha)} \zeta_x \end{aligned} \tag{2.32}$$

Let T be the i.e.t. defined by the data (π, λ). Note that, if u^t_α and u^b_β denote the singularities for T and T^{-1} (introduced in equation (1.2)), then $u^t_\alpha = \mathrm{Re}(\xi^t_\alpha)$ and $u^b_\alpha = \mathrm{Re}(\xi^b_\alpha)$ for all $\alpha \in \mathcal{A}$. We see that the condition in equation (2.31) means that

$$\mathrm{Im}(\xi^t_\alpha) > 0 \texttt{ if } \pi^t(\alpha) > 1 \texttt{ and } \mathrm{Im}(\xi^b_\alpha) < 0 \texttt{ if } \pi^b(\alpha) > 1$$

In particular we can define a vector $h = h(\pi, \tau) \in \mathbb{R}^{\mathcal{A}}_+$ setting for any $\alpha \in \mathcal{A}$:

$$h_\alpha := \Im(\xi^t_\alpha) - \Im(\xi^b_\alpha). \tag{2.33}$$

We also define the *translation vector* $\theta \in \mathbb{R}^d$ by $\theta_\alpha := \xi^b_\alpha - \xi^t_\alpha$. We have $\theta = \delta - ih$ for some $\delta \in \mathbb{R}^{\mathcal{A}}$. We form $2d$ open rectangles in the complex plane defining for any $\alpha \in \mathcal{A}$:

$$\begin{aligned} R^t_\alpha &:= (u^t_\alpha, u^t_\alpha + \lambda_\alpha) \times (0, h_\alpha) \\ R^b_\alpha &:= (u^b_\alpha, u^b_\alpha + \lambda_\alpha) \times (-h_\alpha, 0). \end{aligned} \tag{2.34}$$

In order to get a surface we past together these rectangles. The identification arr the following

(1) For each α the rectangle R^t_α is equivalent to the rectangle R^b_α via the translation by the complex number θ_α.
(2) For each α with $\pi^t(\alpha) > 1$ we consider α' with $\pi^t(\alpha) = \pi^t(\alpha') + 1$ and we paste together R^t_α and $R^t_{\alpha'}$ along the common vertical open segment in their boundaries that connects the point $u^t_\alpha + i0$ to the point ξ^t_α.
(3) For each β with $\pi^b(\beta) > 1$ we consider β' with $\pi^b(\beta) = \pi^t(\beta') + 1$ and we paste together R^b_β and $R^b_{\beta'}$ along the common vertical open segment in their boundaries that connects the point $u^b_\beta + i0$ to the point ξ^b_β.
(4) We paste all the rectangles R^t_α to the horizontal open segment $(0, \sum_{\alpha\in\mathcal{A}} \lambda_\alpha)$ along their lower horizontal boundary segment $(u_\alpha, u_\alpha + \lambda_\alpha) \times \{0\}$
(5) We paste all the rectangles R^b_α to the horizontal open segment $(0, \sum_{\alpha\in\mathcal{A}} \lambda_\alpha)$ along their upper horizontal boundary segment $(u_\alpha, u_\alpha + \lambda_\alpha) \times \{0\}$.

(6) Finally we add the origin 0 of $\mathbb{C}$, the points ξ^t_α with $\pi^t(\alpha) > 1$, the points ξ^b_β with $\pi^b(\beta) > 1$ and the point $\xi^* := \sum_{\alpha\in\mathcal{A}} \zeta_\alpha$

DEFINITION 2.4.1. For any combinatorial datum π over the alphabet $\mathcal{A}$ and for any pair of length-suspension data (λ, τ) for π we call $X(\pi, \lambda, \tau)$ the translation surface that is obtained following the procedure above, which is also known as the *zippered rectangles Veech's construction.*

We recall from paragraph 2.1.1 the definition of the two letters α_t and α_b in $\mathcal{A}$ such that $\pi^t(\alpha_t) = \pi^b(\alpha_b) = d$. We also introduce the letters α'_t and α'_b in $\mathcal{A}$ such that $\pi^t(\alpha'_t) = \pi^b(\alpha'_b) = 1$. We introduce two symbols R and L and we define the set of pairs

$$\{(\alpha, R), (\alpha, L)\,;\, \alpha \in \mathcal{A}\}\,.$$

On this set we define the equivalence relation $\sim$ given by the two identifications $(\alpha'_t, L) \sim (\alpha'_b, L)$ and $(\alpha_t, R) \sim (\alpha_b, R)$ and we consider the set $\tilde{\mathcal{A}}$ of equivalence classes. We consider the permutation σ of the elements in $\tilde{\mathcal{A}}$ given by:

$$\begin{aligned} \sigma(\alpha, L) &:= (\beta_0, R) \text{ such that } \pi^t(\beta_0) + 1 = \pi^t(\alpha) \\ \sigma(\beta, R) &:= (\alpha_0, L) \text{ such that } \pi^b(\alpha_0) = \pi^b(\beta) + 1. \end{aligned} \tag{2.35}$$

The permutation σ can be interpreted as follows. For any $\alpha \in \mathcal{A}$ such that $\pi^t(\alpha) > 1$ we associate to the symbol (α, L) the right part of a neighborhood of ξ^t_α in R^t_α, that we call U^t_α. Similarly for any $\beta \in \mathcal{A}$ such that $\pi^b(\beta) > 1$ we associate to the symbol (β, R) the left part of a neighborhood of $\xi^b_\beta + \zeta_\beta$ in R^t_β, that we call U^b_β. Now let p_i be any point in Σ and let us turn in counterclockwise sense around p_i in the translation surface $X(\pi, \lambda, \tau)$. It is easy to check that the half neighborhoods U^t_α and U^b_β that we meet in this way correspond to the elements (α, L) and (β, R) of some cycle of σ. If k_i is the order of the zero of w at p_i then the length of the corresponding cycle is $2(k_i + 1)$. It follows that if $(2k_1 + 2)..(2k_r + 2)$ is the decomposition in cycles of σ then $X(\pi, \lambda, \tau)$ is in $\mathcal{H}(k_1, .., k_r)$. It is also easy to check that σ is an invariant of $\mathcal{R}$.

2.4.2. Marked translation surfaces. Let us consider an i.e.t. $T : I \to I$ defined by the data π, λ. Let us also consider a suspension datum τ for π and consider the translation surface $X = X(\pi, \lambda, \tau)$ obtained with the Veech construction in paragraph 2.4.1. For such X a point $p_i \in \Sigma$ and a horizontal separatrix S in p_i are naturally marked, they correspond to the left endpoint of I and to the separatrix that contains the interval.

Let's fix a point $p_i \in \Sigma$. We introduce the set $\texttt{Transl}_i(M, \Sigma, k)$ of pairs (ξ, S), where ξ is a translation structure in $\texttt{Transl}(M, \Sigma, k)$ and S is an horizontal separatrix starting at p_i. We call such a datum a *marked translation surface* (at the point p_i). The group $\texttt{Diff}^+(M, \Sigma)$ defined in paragraph 2.3.2 acts on $\texttt{Transl}_i(M, \Sigma, k)$ by

$$\xi, S \mapsto f_*\xi, f(S) \tag{2.36}$$

where $f_*\xi$ is defined as in equation (2.16) and $f(S)$ is just the image of the separatrix S under f. As before we define the Teichmüller space of translation surfaces marked in p_i considering the action of isotopies:

$$\widetilde{\mathcal{T}}_i(M, \Sigma, k) := \texttt{Transl}_i(M, \Sigma, k)/\texttt{Diff}^+_0(M, \Sigma). \tag{2.37}$$

When there is no ambiguity on the choice of the point $p_i \in \Sigma$ we simply write $\widetilde{\mathcal{T}}(M, \Sigma, k)$ instead of $\widetilde{\mathcal{T}}_i(M, \Sigma, k)$.

PROPOSITION 2.4.2. *For any choice of a point $p_i \in \Sigma$ the space $\widetilde{\mathcal{T}}_i(M,\Sigma,k)$ has a structure of complex manifold of complex dimension $2g+r-1$. Moreover the natural projection*

$$\begin{array}{ccc} \widetilde{\mathcal{T}}_i(M,\Sigma,k) & \to & \mathcal{T}(M,\Sigma,k) \\ (\xi,S) & \mapsto & \xi \end{array} \tag{2.38}$$

is a covering map of degree k_i+1, where k_i is the order of the zero in p_i.

Proof: See [**Y3**]. □

The modular group $\mathrm{Mod}(M,\Sigma)$ naturally acts on $\widetilde{\mathcal{T}}_i(M,\Sigma,k)$ as it does on $\mathcal{T}(M,\Sigma,k)$. The quotient space with respect to this action gives the moduli space of marked translation surfaces:

$$\widetilde{\mathcal{H}}_i(M,\Sigma,k) := \widetilde{\mathcal{T}}_i(M,\Sigma,k)/\mathrm{Mod}(M,\Sigma). \tag{2.39}$$

When we are not interested on the underlying topological pair (M,Σ) we also use the notation $\widetilde{\mathcal{H}}_i(k_1,..,k_k)$. The covering map in equation (2.38) induces a covering map $\widetilde{\mathcal{H}}_i(k_1,..,k_r) \to \mathcal{H}(k_1,..,k_r)$. The pull-back of the measures μ and $\mu^{(1)}$ introduced in paragraph 2.3.4 defines two measures, that we still denote μ and $\mu^{(1)}$, in the lebesgue class respectively of $\widetilde{\mathcal{H}}_i(k_1,..,k_r)$ and the hypersurface $\widetilde{\mathcal{H}}_i^{(1)}(k_1,..,k_r)$ of area one marked translation surfaces. The group $\mathrm{SL}(2,\mathbb{R})$ acts on $\widetilde{\mathcal{H}}_i(k_1,..,k_r)$ as it does on $\mathcal{H}(k_1,..,k_r)$ (see paragraph 2.3.3), in particular the Teichmüller flow $\mathcal{F}_t$ is defined. The area of a marked translation surface is obviously invariant under $\mathcal{F}_t$, therefore we have an action on the hyper-surface $\widetilde{\mathcal{H}}_i^{(1)}(k_1,..,k_r)$ of area one marked translation surfaces.

2.4.3. Extension of the induction and the Veech space. Let us consider a Rauzy class $\mathcal{R}$ over the alphabet $\mathcal{A}$. For any $\pi \in \mathcal{R}$ recall the definition of the open cone $\Theta_\pi \subset \mathbb{R}^{\mathcal{A}}$ given by equation (2.31) and let us define

$$\hat{\Delta}(\mathcal{R}) := \bigsqcup_{\pi\in\mathcal{R}} \Delta_\pi \times \Theta_\pi.$$

If $\gamma \in \Pi(\mathcal{R})$ is a path starting at π and ending in π', let Θ_γ be the open sub-cone of $\Theta_{\pi'}$ defined by the condition

$${}^tB_\gamma \Theta_\gamma = \Theta_\pi.$$

If γ is a top arrow ending at π' we have an invertible map

$$\begin{array}{cccc} \hat{Q}_\gamma & : \Delta_\gamma \times \Theta_\pi & \to & \Delta_{\pi'} \times \Theta_\gamma \\ & (\pi,\lambda,\tau) & \mapsto & (Q(\pi,\lambda), {}^tB_\gamma^{-1}\tau). \end{array} \tag{2.40}$$

Putting together the maps $\hat{Q}_\gamma$ defined by equation (2.40) for every arrow γ, we get a map $\hat{Q} : \hat{\Delta}(\mathcal{R}) \to \hat{\Delta}(\mathcal{R})$ which is defined and invertible almost everywhere. In particular its domain is the union of the sets $\Delta_\gamma \times \Theta_\pi$, where π is any element in $\mathcal{R}$ and γ is any arrow starting at π. The image of $\hat{Q}$ is the union of all the sets $\Delta_{\pi'} \times \Theta_\gamma$, where π' is any element in $\mathcal{R}$ and γ is any arrow ending in π'. The map $\hat{Q}$ is a *skew-product map* over the Rauzy-Veech algorithm Q, that is for any $(\pi,\lambda,\tau) \in \hat{\Delta}(\mathcal{R})$ the matrix ${}^tB_\gamma^{-1}$ in equation (2.40) depend just from (π,λ) and not from τ.

The action of $\hat{Q}$ on $\hat{\Delta}(\mathcal{R})$ admits a nice fundamental domain. Let $\phi(\pi,\lambda,\tau) := \|\lambda\|$ and let $\mathcal{V}(\mathcal{R})$ be the set of $(\pi,\lambda,\tau) \in \hat{\Delta}(\mathcal{R})$ such that either

(1) $\hat{Q}(\pi,\lambda,\tau)$ is defined and $\phi(\hat{Q}(\pi,\lambda,\tau)) < 1 \leq \phi(\pi,\lambda,\tau)$,
(2) $\hat{Q}(\pi,\lambda,\tau)$ is not defined and $\phi(\pi,\lambda,\tau) \geq 1$,
(3) $\hat{Q}(\pi,\lambda,\tau)^{-1}$ is not defined and $\phi(\pi,\lambda,\tau) < 1$.

It is evident that each orbit of $\hat{Q}$ intersects $\mathcal{V}(\mathcal{R})$ in exactly one point, therefore the latter is a fundamental domain for the action of $\hat{Q}$. We call $\mathcal{V}(\mathcal{R})$ the *Veech space*. There is a useful alternative construction of the same fundamental domain, that we describe here. If $T = (\pi,\lambda)$ is an i.e.t. in $\Delta(\mathcal{R})$ that satisfies $\lambda_{\alpha_t} \neq \lambda_{\alpha_t}$ we call $\alpha_{lose}(T)$ the letter in $\{\alpha_b, \alpha_t\}$ that loses in T. For any element π of $\mathcal{R}$ we define the subset $\mho_\pi$ of Δ_π by

$$\mho_\pi := \{T \in \Delta_\pi;\ \sum_{\alpha\neq\alpha_{lose}(T)} \lambda_\alpha \leq 1 < \sum_{\alpha\in\mathcal{A}} \lambda_\alpha\}.$$

Then it is easy to check that we have:

$$\mathcal{V}(\mathcal{R}) := \bigsqcup_{\pi\in\mathcal{R}} \mho_\pi \times \Theta_\pi. \tag{2.41}$$

2.4.4. Representability. Let us consider any Rauzy class $\mathcal{R}$. During all this paragraph we fix an element π of $\mathcal{R}$.

We define two standard length and suspension data $\lambda^\pi \in \Delta_\pi$ and $\tau^\pi \in \Theta_\pi$ setting for any $\alpha \in \mathcal{A}$:

$$\lambda^\pi_\alpha := 1 \ \texttt{and}\ \tau^\pi_\alpha := \pi^b(\alpha) - \pi^t(\alpha). \tag{2.42}$$

The Veech construction in paragraph 2.4.1, applied to the data $(\pi,\lambda^\pi,\tau^\pi)$, gives a marked translation surface. Let M^π and Σ^π be respectively the topological surface supporting the conformal structure and the set of zeros of the one form associated to the data $(\pi,\lambda^\pi,\tau^\pi)$. Let also denote k^π the integer vector that gives the order of the zeros of the holomorphic one form. The entries of k^π are the length of the cicles of the permutation σ defined by equation (2.35) in paragraph 2.4.1. Since the decomposition in cycles of σ is an invariant for Rauzy classes, the vector k^π too is an invariant, and we can denote it simply k.

For a pair of length-suspension data (λ,τ) in $\Delta_\pi \times \Theta_\pi$ the translation surface $X(\pi,\lambda,\tau)$ given by the Veech construction corresponds to a marked translation structure on the standard topological data (M^π,Σ^π,k), therefore we can introduce an application

$$\begin{array}{rccc} i_\pi : & \Theta_\pi \times \Delta_\pi & \to & \widetilde{\mathcal{T}}(M^\pi,\Sigma^\pi,k) \\ & (\lambda,\tau) & \mapsto & X(\pi,\lambda,\tau). \end{array} \tag{2.43}$$

which is an embedding onto an open subset of $\widetilde{\mathcal{T}}(M^\pi,\Sigma^\pi,k)$.

We consider an arrow $\gamma : \pi \to \pi'$ in the Rauzy diagram starting at the fixed vertex π. For such a γ let's take any (λ,τ) in $\Delta_\gamma \times \Theta_\pi$ and apply the extended Rauzy induction $\hat{Q}_\gamma$ to (π,λ,τ). In terms of translation structures this means that we have a bi-holomorphic map between the marked translation surfaces $X(\pi,\lambda,\tau)$ and $X(\pi',\lambda',\tau')$, where $(\pi',\lambda',\tau') = \hat{Q}_\gamma(\pi,\lambda,\tau)$. In particular $\hat{Q}_\gamma$ defines a diffemorphism between the underlying topological pairs (M^π,Σ^π) and $(M^{\pi'},\Sigma^{\pi'})$. We denote this diffeomorphism by

$$\hat{Q}_\gamma : (M^\pi,\Sigma^\pi) \to (M^{\pi'},\Sigma^{\pi'}).$$

The map $(\xi, S) \mapsto \hat{Q}^*_\gamma(\xi, S)$, assigning to any marked translation structure in $\widetilde{\mathcal{T}}(M^\pi, \Sigma^\pi, k)$ its push-forward under $\hat{Q}_\gamma$, gives an analytic bijection between Teichmüller spaces:

$$\hat{Q}^*_\gamma : \widetilde{\mathcal{T}}(M^\pi, \Sigma^\pi, k) \to \widetilde{\mathcal{T}}(M^{\pi'}, \Sigma^{\pi'}, k). \tag{2.44}$$

Let us consider all the non-oriented Rauzy paths $\gamma : \pi \to \pi'$, where π is still the fixed element and π' varies in $\mathcal{R}$. The definition of $\hat{Q}^*_\gamma$ in equation (2.44) can be extended to any such non-oriented paths γ by concatenation. For any $\pi' \in \mathcal{R}$ the sets $i_{\pi'}(\Delta_{\pi'} \times \Theta_{\pi'})$ are open embedded sets in $\widetilde{\mathcal{T}}(M^{\pi'}, \Sigma^{\pi'}, k)$, thus for any $\gamma : \pi \to \pi'$ the set $\hat{Q}^{*-1}_\gamma \circ i_{\pi'}(\Delta_{\pi'} \times \Theta_{\pi'})$ is an open embedded set in $\widetilde{\mathcal{T}}(M^\pi, \Sigma^\pi, k)$ that intersects $i_\pi(\Delta_\pi \times \Theta_\pi)$. We paste together all these sets to get a bigger open set:

$$\mathcal{U}_\pi := \bigcup_{\gamma:\pi\mapsto\pi'} \hat{Q}^{*-1}_\gamma \circ i_{\pi'}(\Delta_{\pi'} \times \Theta_{\pi'}). \tag{2.45}$$

The stratum $\widetilde{\mathcal{H}}(M^\pi, \Sigma^\pi, k)$ is obtained taking the quotient by the action of the mapping class group $\mathrm{Mod}(M^\pi, \Sigma^\pi)$ on $\widetilde{\mathcal{T}}(M^\pi, \Sigma^\pi, k)$ and it is identified with some stratum $\widetilde{\mathcal{H}}_i(k_1, .., k_r)$ for some values $k_1, .., k_r$ (depending from the Rauzy class of π). We call *representable marked translation surfaces* the marked translation surfaces that can be obtained with the Veech construction of paragraph 2.4.1.

Lemma 2.4.3. *Let $\widetilde{\mathcal{H}}_i(k_1, .., k_r)$ be any stratum of the moduli space of marked translation surfaces with genus g and let (X, S) be any of its elements. Then either X has both vertical and horizontal saddle connections, or the following is true: there exists an admissible combinatorial datum π with $2g + r - 1$ letters and length-suspension data λ and τ for π such that X coincides with the translation surface $X(\pi, \lambda, \tau)$ obtained with the Veech construction of paragraph 2.4.1 and the horizontal separatrix S coincides with the positive real line in $X(\pi, \lambda, \tau)$.*

Proof: See [**Y3**]. □

Consider the natural projection $\Pi : \widetilde{\mathcal{T}}(M^\pi, \Sigma^\pi, k) \to \widetilde{\mathcal{H}}(M^\pi, \Sigma^\pi, k)$ to the quotient space. Since the open set $\mathcal{U}_\pi$ introduced in equation (2.45) is connected, then its image $\Pi(\mathcal{U}_\pi)$ under the map Π is also connected, therefore it is contained in just one connected component of $\widetilde{\mathcal{H}}(M^\pi, \Sigma^\pi, k)$. Let us call $\mathcal{C}_\pi$ this connected component. Lemma 2.4.3 has the following consequence.

Corollary 2.4.4. *Any marked translation structure (X, S) in $\mathcal{C}_\pi \setminus \Pi(\mathcal{U}_\pi)$ must have both vertical and horizontal connections, therefore the set of such (X, S) has codimension 2 in $\mathcal{C}_\pi$. In particular $\Pi(\mathcal{U}_\pi)$ has full measure in $\mathcal{C}_\pi$. Moreover any stratum $\widetilde{\mathcal{H}}_i(k_1, .., k_r)$ is identified with some representable stratum $\widetilde{\mathcal{H}}(M^\pi, \Sigma^\pi, k)$ and therefore almost any marked translation structure is representable.*

Let us consider any π' in the same Rauzy class $\mathcal{R}$ of π, any path $\gamma : \pi \to \pi'$ in the Rauzy diagram and the associated map $\hat{Q}^*_\gamma$ as in equation (2.44). Since $\mathcal{U}_\pi \cap \hat{Q}^{*-1}_\gamma(\mathcal{U}_{\pi'}) \neq \emptyset$ then the two sets belong to the same connected component $\mathcal{C}_\pi$ of $\widetilde{\mathcal{H}}(M^\pi, \Sigma^\pi, k)$, moreover corollary 2.4.4 implies that $\mathcal{U}_\pi = \hat{Q}^{*-1}_\gamma(\mathcal{U}_{\pi'}) = \mathcal{C}_\pi$ mod 0. It follows that we can forget the dependence from π and denote $\mathcal{C}_\pi$ simply $\mathcal{C}(\mathcal{R})$ (everything is meant modulo identifications between $\widetilde{\mathcal{H}}(M^\pi, \Sigma^\pi, k)$ and $\widetilde{\mathcal{H}}(M^{\pi'}, \Sigma^{\pi'}, k)$ with π and π' in the same $\mathcal{R}$).

DEFINITION 2.4.5. For any stratum $\widetilde{\mathcal{H}}_i(k_1,..,k_r)$ of the moduli space of translation surfaces marked at the singular point p_i we consider any alphabet $\mathcal{A}$ with $d = 2g + r - 1$ letters and define the set $\mathfrak{S}_i(k_1,..,k_r)$ of those admissible combinatorial data π over $\mathcal{A}$ such that for any pair of length-suspension data (λ, τ) in $\Delta_\pi \times \Theta_\pi$ the surface $X(\pi, \lambda, \tau)$ given by the Veech construction coincides with a translation surface (X, S_i) in $\widetilde{\mathcal{H}}_i(k_1,..,k_r)$.

The set $\mathfrak{S}_i(k_1,..,k_r)$ introduced in definition 2.4.5 is union of a finite number of Rauzy classes. Kontsevich and Zorich classified the connected components of strata in the moduli space and proved that such number is always at most 3 (see [**KZ**]). Let $\mathcal{C}$ be any connected component of $\widetilde{\mathcal{H}}_i(k_1,..,k_r)$. Since almost any element in $\mathcal{C}$ is representable via the Veech construction of paragraph 2.4.1 with some combinatorial datum $\pi \in \mathfrak{S}_i(k_1,..,k_r)$, then $\mathcal{C}$ is identified with some connected component $\mathcal{C}(\mathcal{R})$ of $\widetilde{\mathcal{H}}(M^\pi, \Sigma^\pi, k)$. The Veech construction therefore establish a surjective application

$$\mathcal{R} \mapsto \mathcal{C} = \mathcal{C}(\mathcal{R}) \tag{2.46}$$

that assigns to any Rauzy class $\mathcal{R} \subset \mathfrak{S}_i(k_1,..,k_r)$ a connected component $\mathcal{C}$ of the stratum $\widetilde{\mathcal{H}}_i(k_1,..,k_r)$. The following lemma holds.

LEMMA 2.4.6. *Let π and π' two admissible combinatorial data on the same alphabet $\mathcal{A}$. Let $g : (M^\pi, \Sigma^\pi) \to (M^{\pi'}, \Sigma^{\pi'})$ an homeomorphism of pairs such that*

$$g(\mathcal{U}_\pi) \cap \mathcal{U}_{\pi'} \neq \emptyset. \tag{2.47}$$

Then π and π' belong to the same Rauzy class $\mathcal{R}$ and there exist a non-oriented Rauzy path $\gamma : \pi \to \pi'$ such that $g = \hat{Q}^*_\gamma$.

Proof: See [**Y3**]. □

Lemma 2.4.6 has the following consequence.

COROLLARY 2.4.7. *The application $\mathcal{R} \mapsto \mathcal{C} = \mathcal{C}(\mathcal{R})$ in equation (2.46) is a bijection, that is there is a ono-to-one correspondence between Rauzy classes and connected component of the moduli space of translation surfaces marked at some point.*

For any $\pi \in \mathfrak{S}_i(k_1,..,k_r)$ recall the subset $\mho_\pi$ of Δ_π defined at the end of paragraph 2.4.3. We consider the restriction to $\mho_\pi \times \Theta_\pi$ of the map i_π defined by equation (2.43). Then we compose $i_\pi|_{\mho_\pi \times \Theta_\pi}$ with the projection Π on the moduli space. We get an homeomorphism

$$\begin{array}{rccc} \mathcal{I}_\pi : & \mho_\pi \times \Theta_\pi & \to & \widetilde{\mathcal{H}}_i(k_1,..,k_r) \\ & (\lambda, \tau) & \mapsto & X(\pi, \lambda, \tau) \end{array} \tag{2.48}$$

which is surjective onto an open subset of $\widetilde{\mathcal{H}}_i(k_1,..,k_r)$. Let $\mathcal{R}$ be one of those Rauzy classes that compose $\mathfrak{S}_i(k_1,..,k_r)$ and recall the definition of the map $\hat{Q} : \hat{\Delta}(\mathcal{R}) \to \hat{\Delta}(\mathcal{R})$ and of its fundamental domain $\mathcal{V}(\mathcal{R})$ given in paragraph 2.4.3. For any $(\pi, \lambda, \tau) \in \hat{\Delta}(\mathcal{R})$ in the domain of $\hat{Q}$ the translation surfaces corresponding to (π, λ, τ) and $\hat{Q}(\pi, \lambda, \tau)$ are obtained by appropriate cutting and pasting, therefore they correspond to the same element in the moduli space. Let us recall equation (2.41) in paragraph 2.4.3. Corollary 2.4.4, together with the identification between $\widetilde{\mathcal{H}}_i(k_1,..,k_r)$ and $\widetilde{\mathcal{H}}(M^\pi, \Sigma^\pi, k)$, implies the following corollary.

Corollary 2.4.8. *For any stratum $\widetilde{\mathcal{H}}_i(k_1, .., k_r)$ of the moduli space of translation surfaces marked at some singular point p_i and for the set $\mathfrak{S}_i(k_1, .., k_r)$ defined above we have*

$$\widetilde{\mathcal{H}}_i(k_1, .., k_r) = \bigsqcup_{\pi \in \mathfrak{S}_i(k_1, .., k_r)} \mathcal{I}_\pi(\mho_\pi \times \Theta_\pi) \mod 0, \tag{2.49}$$

where the union is disjoint.

CHAPTER 3

Preliminary results.

3.1. Configurations of saddle connections.

3.1.1. Marking all the separatrices. We consider the set $\mathtt{Transl}^{tot}(M,\Sigma,k)$ whose elements are the data $(\xi, S_1, .., S_r)$, where ξ is a translation structure in $\mathtt{Transl}(M,\Sigma,k)$ and for any singular point $p_i \in \Sigma$ S_i is an horizontal separatrix starting at p_i. We call such a datum a *totally marked translation surface*. The group $\mathtt{Diff}^+(M,\Sigma)$ defined in paragraph 2.3.2 acts on $\mathtt{Transl}^{tot}(M,\Sigma,k)$ by

$$(\xi, S_1, .., S_r) \mapsto (f_*\xi, f(S_1), .., f(S_r)) \tag{3.1}$$

where $f_*\xi$ is defined as in equation (2.16) in paragraph 2.3.2 and for any $i \in \{1, .., r\}$ the separatrix $f(S_i)$ is the image of the separatrix S_i under f. The Teichmüller space of totally marked translation surfaces, that we denote $\widehat{\mathcal{T}}(M,\Sigma,k)$, is obtained as before taking the quotient by the action of isotopies:

$$\widehat{\mathcal{T}}(M,\Sigma,k) := \mathtt{Transl}^{tot}(M,\Sigma,k)/\mathtt{Diff}_0^+(M,\Sigma). \tag{3.2}$$

Proposition 3.1.1. *The space $\widehat{\mathcal{T}}(M,\Sigma,k)$ is a complex non-compact manifold of complex dimension $2g+r-1$. Moreover the natural projection*

$$\begin{array}{ccc} \widehat{\mathcal{T}}(M,\Sigma,k) & \rightarrow & \mathcal{T}(M,\Sigma,k) \\ (\xi, S_1, ..., S_r) & \mapsto & \xi \end{array} \tag{3.3}$$

is a covering map of degree $\Pi_{i=1..r}(k_i+1)$.

Proof: The proof is the same as the one in proposition 2.4.2 in paragraph 2.4.2 for the Teichmüller spaces $\widetilde{\mathcal{T}}_i(M,\Sigma,k)$ of translation surfaces marked at some point $p_i \in \Sigma$. □

The action in equation (3.1) can be considered modulo elements in $\mathtt{Diff}_0^+(M,\Sigma)$, in this way we get an action of the group $\mathrm{Mod}(M,\Sigma)$ (defined by equation (2.20)) in paragraph 2.3.3 on the space $\widehat{\mathcal{T}}(M,\Sigma,k)$. The quotient

$$\widehat{\mathcal{H}}(M,\Sigma,k) := \widehat{\mathcal{T}}(M,\Sigma,k)/\mathrm{Mod}(M,\Sigma) \tag{3.4}$$

is called the *moduli space of totally marked translation surfaces*. Let us denote with $\hat{X}$ its elements, that is any $\hat{X}$ is the datum $(X, S_1, .., S_r)$ of a translation surface in $\mathcal{H}(M,\Sigma,k)$ plus a choice of an horizontal separatrix S_i starting at p_i for any $p_i \in \Sigma$. The projection in equation (3.3) passes to the quotient and give a projection

$$\begin{array}{ccc} \widehat{\mathcal{H}}(M,\Sigma,k) & \rightarrow & \mathcal{H}(M,\Sigma,k) \\ (X, S_1, ..., S_r) & \mapsto & X \end{array} \tag{3.5}$$

which is a covering map of degree $\Pi_{i=1..r}(k_i+1)$. Let $\hat{X} = (X, S_1, .., S_r)$ be any totally marked translation surface in $\widehat{\mathcal{H}}(M,\Sigma,k)$. The vertical separatrices starting and ending at singular points of $\hat{X}$ can be labelled as follows. For any $p_i \in \Sigma$

and any $l \in \{0,..,k_i\}$ we call $V^{start}_{p_i,l}(\hat{X})$ and $V^{end}_{p_i,l}(\hat{X})$ the two vertical separatrices respectively starting and ending at p_i such that

$$\begin{array}{c} \text{angle}(V^{start}_{p_i,l}, S_i) = 2\pi(l+\frac{1}{4}) \\ \text{angle}(V^{end}_{p_i,l}, S_i) = 2\pi(l+\frac{3}{4}). \end{array} \tag{3.6}$$

3.1.2. Configurations of saddle connections. Let us consider any $\hat{X} = (X, S_1, .., S_r)$ in $\widehat{\mathcal{H}}(k_1, .., k_r)$. If γ is a saddle connection for X that starts at $p_j \in \Sigma$ then there exists an only integer $m \in \{0,..,k_j\}$ such that the vertical separatrix $V^{start}_{p_j,m}$ defined in equation (3.6) satisfies $-\pi \leq \text{angle}(V^{start}_{p_j,m}, \gamma) < \pi$. Similarly, if p_i is the ending point of γ then there exists an only integer $l \in \{0,..,k_i\}$ such that the vertical separatrix $V^{end}_{p_i,l}$ defined in equation (3.6) satisfies $-\pi \leq \text{angle}(\gamma, V^{end}_{p_i,l}) < \pi$. This motivates the following definition.

Definition 3.1.2. Let $\hat{X}$ be any element in $\widehat{\mathcal{H}}(M,\Sigma,k)$. For any pair of singular points $p_i, p_j \in \Sigma$ and any pair of integers l, m with $l \in \{0,..,k_i\}$ and $m \in \{0,..,k_j\}$ we define the set $\mathcal{C}^{(p_i,p_j,l,m)}(\hat{X})$ as the set of saddle connections γ for the translation surface X that start at p_i, end in p_j and such that

$$\begin{array}{c} -\pi \leq \text{angle}(\gamma, V^{start}_{p_j,m}) < \pi \\ -\pi \leq \text{angle}(\gamma, V^{end}_{p_i,l}) < \pi. \end{array} \tag{3.7}$$

Lemma 3.1.3. *Let $\hat{X}$ be any element in $\widehat{\mathcal{H}}(M,\Sigma,k)$. For any datum (p_i, p_j, l, m) as in definition 3.1.2 let us fix any saddle connection $\gamma \in \mathcal{C}^{(p_i,p_j,l,m)}(\hat{X})$. Then for any $t \in \mathbb{R}$ the smooth curve $\gamma : [0,T] \to M$ associated to γ is still a saddle connection for $\mathcal{F}_t\hat{X}$ and moreover it belongs to $\mathcal{C}^{(p_i,p_j,l,m)}(\mathcal{F}_t\hat{X})$.*

Proof: Let us consider any $\gamma \in \mathcal{C}^{(p_i,p_j,l,m)}(\hat{X})$. It is a geodesic for the flat metric of X and as a smooth curve $\gamma : [0,T] \to M$ it satisfies $\gamma^{-1}(\Sigma) = \{0, T\}$. This last condition does not depend from the translation structure, therefore keeps true for the translation structure $\mathcal{F}_t(X)$. Moreover the action of $\mathcal{F}_t$ is affine in charts and sends straight lines into straight lines, therefore for any $t \in \mathbb{R}$ the curve γ is a geodesic for the flat metric of $\mathcal{F}_tX$ and therefore is a saddle connection. The Teichmüller flow $\mathcal{F}_t$ does not acts by conformal maps, therefore it does not preserve angles, anyway if we fix any saddle connection γ and a vertical separatrix V starting (or ending) at the same endpoint in Σ, then for any $t \in \mathbb{R}$ we have

$$\tan \text{angle}_{X,t}(\gamma, V) = e^{2t} \tan \text{angle}_{X,0}(\gamma, V),$$

which means that the configuration $\mathcal{C}^{(p_i,p_j,l,m)}(\hat{X})$ of γ is preserved by $\mathcal{F}_t$. The lemma is proved. □

3.1.3. Relation with the Veech construction. Let us fix any point $p_i \in \Sigma$ and consider the moduli space $\widetilde{\mathcal{H}}_i(M,\Sigma,k)$ of translation surfaces marked in p_i. The projection in equation (3.5) in paragraph 3.1.1 can be factorized into two natural projections:

$$\begin{array}{ccccc} \widehat{\mathcal{H}}(M,\Sigma,k) & \to & \widetilde{\mathcal{H}}_i(M,\Sigma,k) & \to & \mathcal{H}(M,\Sigma,k) \\ (X, S_1, ..., S_r) & \mapsto & (X, S_i) & \mapsto & X. \end{array} \tag{3.8}$$

The projections in equation (3.8) are two covering maps, the one on the left has degree $\Pi_{j \neq i}(k_j + 1)$ and the one on the right has degree $k_i + 1$. Let us call Proj_i

the projection on the left hand side of equation (3.8), that is we write

$$(X, S_i) = \mathrm{Proj}_i(X, S_1, .., S_r).$$

Let us consider the set $\mathfrak{S}_i(k_1, .., k_r)$ associated to $\widetilde{\mathcal{H}}_i(M, \Sigma, k)$ in paragraph 2.4.4 of the background and consider any $\pi \in \mathfrak{S}_i(k_1, .., k_r)$. Equation (2.48) of the same paragraph defines an homeomorphism

$$\mathcal{I}_\pi : \mho_\pi \times \Theta_\pi \to \widetilde{\mathcal{H}}_i(k_1, .., k_r).$$

For any $\pi \in \mathfrak{S}_i(k_1, .., k_r)$ let us consider a fixed pair $(\lambda_\pi^{st}, \tau_\pi^{st})$ of length suspension data in $\mho_\pi \times \Theta_\pi$ and consider the associated translation surface $X_\pi^{st} := \mathcal{I}_\pi(\pi, \lambda_\pi^{st}, \tau_\pi^{st})$. Any choice of a pre-image $\hat{X}_\pi^{st} \in \widehat{\mathcal{H}}(M, \Sigma, k)$ of X_π^{st} under Proj_i determines an unique lift

$$\mathcal{I}_{\pi, \hat{X}_\pi^{st}} : \mho_\pi \times \Theta_\pi \to \widehat{\mathcal{H}}(k_1, .., k_r) \tag{3.9}$$

of the homeomorphism $\mathcal{I}_\pi$. Corollary 2.4.4 implies the following corollary.

COROLLARY 3.1.4. *For any stratum $\widetilde{\mathcal{H}}_i(k_1, .., k_r)$ of the moduli space of translation surfaces marked at some singular point p_i we have*

$$\widehat{\mathcal{H}}(k_1, .., k_r) = \bigsqcup_{\pi \in \mathfrak{S}_i(k_1, .., k_s)} \ \bigsqcup_{\hat{X}_\pi^{st} \in \mathrm{Proj}_i^{-1}\{X_\pi^{st}\}} \mathcal{I}_{\pi, \hat{X}_\pi^{st}}(\mho_\pi \times \Theta_\pi) \mod 0. \tag{3.10}$$

Recall the zippered rectangles construction in paragraph 2.4.1. For any admissible combinatorial datum π over the alphabet $\mathcal{A}$ and for any $(\lambda, \tau) \in \Delta_\pi \times \Theta_\pi$, the complex numbers $\zeta_\alpha = \lambda_\alpha + i\tau_\alpha$ define $2d$ curves in $\mathbb{C}$, two for each letter $\alpha \in \mathcal{A}$:

$$\hat{\zeta}_\alpha^t : (0, 1) \to \mathbb{C}; t \mapsto \xi_\alpha^t + t\zeta_\alpha$$
$$\hat{\zeta}_\alpha^b : (0, 1) \to \mathbb{C}; t \mapsto \xi_\alpha^b + t\zeta_\alpha.$$

LEMMA 3.1.5. *For any $\alpha \in \mathcal{A}$ either $\hat{\zeta}_\alpha^t$ or $\hat{\zeta}_\alpha^t$ projects to a saddle connection ζ_α for the translation surface $X(\pi, \lambda, \tau)$.*

Note: We denote with ζ_α both the complex number $\lambda_\alpha + i\tau_\alpha$ and the associated saddle connection, when any confusion risk to arise we specify if we are talking about the former or the latter.

Proof: For any $\alpha \in \mathcal{A}$ the Veech construction associates to the triple π, λ, τ a pair of open rectangles R_α^t and R_α^b in the complex plane, moreover the identifications that define the translation surface $X(\pi, \lambda, \tau)$ give an embedding of each one of these rectangles onto a well defined open rectangle $R_\alpha \subset X(\pi, \lambda, \tau)$. It follows that in order to prove the statement it is enough to prove that for any $\alpha \in \mathcal{A}$ either $\hat{\zeta}_\alpha^t$ is contained in the interior of R_α^t or $\hat{\zeta}_\alpha^b$ is contained in the interior of R_α^b. The height h_α is defined by $h_\alpha = \Im(\xi_\alpha^t) - \Im(\xi_\alpha^b)$ and we have that the lower horizontal side of R_α^t is on the real axis and the upper one is at eight h_α, conversely the upper horizontal side of R_α^b is on the real axis and the lower one is at eight $-h_\alpha$. We first observe that for any $\alpha \in \mathcal{A}$ we have $\Im(\xi_\alpha^t) \le h_\alpha$ and $\Im(\xi_\alpha^t) \ge -h_\alpha$. Since π is admissible for any $\alpha \in \mathcal{A}$ there are two cases: either $\pi^b(\alpha) < d$ or $\pi^t(\alpha) < d$.

- If $\pi^b(\alpha) < d$ then the suspension condition (equation (2.34) in paragraph 2.4.1) implies that $\Im(\xi_\alpha^b + \zeta_\alpha) < 0$ and therefore

$$h_\alpha = \sum_{\pi^t(x) \le \pi^t(\alpha)} \tau_x - \sum_{\pi^b(y) \le \pi^b(\alpha)} \tau_y = \Im(\xi_\alpha^t + \zeta_\alpha) - \Im(\xi_\alpha^b + \zeta_\alpha) > \Im(\xi_\alpha^t + \zeta_\alpha).$$

  It follows that $\hat{\zeta}_\alpha^t$ is contained in the interior of R_α^t.

- If If $\pi^t(\alpha) < d$ then then the suspension condition (equation (2.34) in paragraph 2.4.1) implies that $\Im(\xi^t_\alpha + \zeta_\alpha) > 0$ and therefore

$$-h_\alpha = -\sum_{\pi^t(x)\le\pi^t(\alpha)} \tau_x + \sum_{\pi^b(y)\le\pi^b(\alpha)} \tau_y = -\Im(\xi^t_\alpha + \zeta_\alpha) + \Im(\xi^b_\alpha + \zeta_\alpha) < \Im(\xi^b_\alpha + \zeta_\alpha).$$

Then it follows that $\hat{\zeta}^b_\alpha$ is contained in the interior of R^b_α.

The lemma is proved. □

Let us consider any $\pi \in \mathfrak{S}_i(k_1, .., k_r)$ and a pair (λ, τ) of length-suspension data for π. Let $X(\pi, \lambda, \tau)$ the translation surface associated to the data (π, λ, τ) by the zippered rectangles construction in paragraph 2.4.1. For any $\alpha \in \mathcal{A}$ we consider the saddle connection ζ_α for $X(\pi, \lambda, \tau)$ given by lemma 3.1.5 and the point $p(\alpha) \in \Sigma$ where ζ_α start from.

For any α such that $\pi^t(\alpha) > 1$ we call V^{end}_α the only vertical separatrix ending at the point $p(\alpha)$ where ζ_α starts such that

$$\text{angle}(V^{end}_\alpha, \zeta_\alpha) < \pi. \tag{3.11}$$

Similarly, for any β such that $\pi^b(\beta) > 1$ we call V^{start}_β the only vertical separatrix ending at the point $p(\beta)$ where ζ_β starts such that

$$\text{angle}(V^{start}_\beta, \zeta_\beta) < \pi. \tag{3.12}$$

Definition 3.1.6. Let π be an admissible datum over the alphabet $\mathcal{A}$ and let α, β letters such that $\pi^t(\alpha) > 1$ and $\pi^b(\beta) > 1$. For any pair of length-suspension data (λ, τ) for π we define the set $\mathcal{C}^{(\beta,\alpha)}(\pi, \lambda, \tau)$ of those saddle connections γ for the translation surface $X(\pi, \lambda, \tau)$ that start at the same point where V^{start}_β starts, end in the same point where V^{end}_α ends, and such that

$$\begin{aligned} -\pi &\le \text{angle}(\gamma, V^{start}_\beta) < \pi \\ -\pi &\le \text{angle}(\gamma, V^{end}_\alpha) < \pi. \end{aligned} \tag{3.13}$$

For any admissible combinatorial datum over the alphabet $\mathcal{A}$ we introduce the subset $\mathcal{A}^t_\pi \subset \mathcal{A}$ of those $\alpha \in \mathcal{A}$ such that $\pi^t(\alpha) > 1$ and the subset $\mathcal{A}^b_\pi \subset \mathcal{A}$ of those $\beta \in \mathcal{A}$ such that $\pi^b(\beta) > 1$.

Lemma 3.1.7. *Let $\widetilde{\mathcal{H}}_i(k_1, .., k_r)$ be any stratum of the moduli space of translation surfaces marked at some singular point $p_i \in \Sigma$. Let $\pi \in \mathfrak{S}_i(k_1, .., k_r)$ and let $\hat{\mathcal{I}}_\pi$ be any lift to $\widehat{\mathcal{H}}(k_1, .., k_r)$ of the map $\mathcal{I}_\pi$ as in equation (3.9). Then there exist a bijection from the set of the data (p_i, p_j, l, m), where p_i, p_j is any pair of points in Σ and l, m are indexes respectively in $\{0, .., k_i\}$ and $\{0, .., k_j\}$, and the set of pairs of letters (β, α) with $\pi^t(\alpha) > 1$ and $\pi^b(\beta) > 1$. The bijection is such that for any $(\lambda, \tau) \in \mho_\pi \times \Theta_\pi$, if $\hat{X} = \hat{\mathcal{I}}_\pi(\lambda, \tau)$ and if (β, α) is the pair associated to (p_i, p_j, l, m), then the set $\mathcal{C}^{(\beta,\alpha)}(\pi, \lambda, \tau)$ coincides with the configuration $\mathcal{C}^{(p_i,p_j,l,m)}(\hat{X})$.*

Proof: Let $\hat{X} = (X, S_1, .., S_r)$ be a totally marked translation surface in the image of $\hat{\mathcal{I}}_\pi$ and consider the data λ, τ such that $\hat{X} = \hat{\mathcal{I}}_\pi(\lambda, \tau)$. We define r letters $\alpha_i = \alpha_i(\pi, \lambda, \tau)$ such that for any $i = 1..r$ we have that S_i starts in $R^t_{\alpha_i}$. Let us consider the permutation σ introduced in equation (2.35) in paragraph 2.4.1. The second iterated σ^2 of σ induces two permutations respectively of $\mathcal{A}^t$ and $\mathcal{A}^b$. The permutation σ^2 has a cycle decomposition with cycles of lengths $k_1 + 1, .., k_r + 1$, moreover all the letter α_i defined above are in different cycles. Let us consider any

$i \in \{1,..,r\}$ and any letter $\alpha \in \mathcal{A}^t$ in the σ^2 orbit of α_i. If V_α^{end} is the vertical separatrix for $X(\pi,\lambda,\tau)$ defined according to equation (3.11) then the unique integer $l \in \{0,..,k_i\}$ such that $\alpha = \sigma^{2l}(\alpha_i)$ satisfies

$$\text{angle}(V_\alpha^{end}, S_i) = 2\pi(l + \frac{3}{4}).$$

Similarly for any other $j \in \{1,..,r\}$ and any letter $\beta \in \mathcal{A}^b$ in the σ^2 orbit of α_j, if V_β^{start} is the vertical separatrix for $X(\pi,\lambda,\tau)$ defined according to equation (3.11) then the unique integer $m \in \{0,..,k_m\}$ such that $\beta = \sigma^{2l}(\alpha_j)$ satisfies

$$\text{angle}(V_\beta^{start}, S_j) = 2\pi(l + \frac{1}{4}).$$

Recalling equation (3.6) in paragraph 3.1.1, modulo the identification $\hat{X} = \hat{\mathcal{I}}_\pi(\lambda,\tau)$, we have $V_\alpha^{end} = V_{p_i,l}^{end}$ and $V_\alpha^{end} = V_{p_i,l}^{end}$, therefore the set $\mathcal{C}^{(\beta,\alpha)}(\pi,\lambda,\tau)$ is identified with the configuration $\mathcal{C}^{(p_i,p_j,l,m)}(\hat{X})$. The lemma is proved. □

3.2. On the suspension.

3.2.1. Combinatorially defined homology classes. Let π be any admissible combinatorial datum over the alphabet $\mathcal{A}$. For any (λ,τ) in $\Delta_\pi \times \Theta_\pi$ and any $\alpha \in \mathcal{A}$ consider saddle connection ζ_α for $X(\pi,\lambda,\tau)$ defined by lemma 3.1.5 in paragraph 3.1.3. In particular any ζ_α defines a relative homology class $\tilde{\zeta}_\alpha \in H_1(M,\Sigma,\mathbb{Z})$. The formulae that define $\xi_\alpha^t := \sum_{\pi^t(x)<\pi^t(\alpha)} \zeta_x, \xi_\alpha^b := \sum_{\pi^b(x)<\pi^b(\alpha)} \zeta_x$ and $\theta_\alpha := \xi_\alpha^b - \xi_\alpha^t$ for $\alpha \in \mathcal{A}$ may be extended formally on relative (or absolute) homology classes

$$\tilde{\xi}_\alpha^t := \sum_{\pi^t(x)<\pi^t(\alpha)} \tilde{\zeta}_x \in H_1(M,\Sigma,\mathbb{Z})$$

$$\tilde{\xi}_\alpha^b := \sum_{\pi^b(x)<\pi^b(\alpha)} \tilde{\zeta}_x \in H_1(M,\Sigma,\mathbb{Z})$$

$$\tilde{\theta}_\alpha := \tilde{\xi}_\alpha^b - \tilde{\xi}_\alpha^t \in H_1(M,\mathbb{Z}).$$

All the homology classes appearing here depend only on the combinatorial datum $\pi \in \mathcal{R}$. For any pair of length-suspension data $(\lambda,\tau) \in \Delta_\pi \times \Theta_\pi$ each one of the homology classes above has a representant which is concatenation of saddle connections for the flat structure $X = (\pi,\lambda,\tau)$, anyway this representant is no more a saddle connection since it contains in its interior singular points in Σ.

For any admissible combinatorial datum π over the alphabet $\mathcal{A}$ and any $\lambda \in \Delta_\pi$ let us consider the i.e.t. $T : I \to I$ corresponding to the data (π,λ). We define a piecewise constant map $\tilde{\theta}_T : I \to H_1(M,\mathbb{Z})$ by the formula

$$\tilde{\theta}_T(x) = \tilde{\theta}_\alpha \texttt{ iff } x \in I_\alpha^t.$$

Let us consider the Birkhoff sum $S_n\tilde{\theta}_T$ over the map T of the function $\tilde{\theta}_T$. We fix a pair of letters $(\beta,\alpha) \in \mathcal{A}^2$ with $\pi^t(\alpha), \pi^b(\beta) > 1$. If T has no connection we can iterate T on u_β^b infinitely many times and we get a sequence of elements in the relative homology $H_1(M,\Sigma,\mathbb{Z})$ defined by

$$\tilde{\gamma}_{\beta,\alpha,n,T} := \tilde{\xi}_\alpha^t - \tilde{\xi}_\beta^b - S_n\tilde{\theta}_T(u_\beta^b). \tag{3.14}$$

3.2.2. Combinatorially defined saddle connections. For any admissible combinatorial datum π over the alphabet $\mathcal{A}$ and any pair of length-suspension data (λ,τ) in $\Delta_\pi \times \Theta_\pi$ the abelian differential w_X associated to the flat structure $X = X(\pi,\lambda,\tau)$ defines an element in $\hom(H_1(M,\Sigma,\mathbb{Z}),\mathbb{C})$. Let us recall the set $\mathcal{C}^{(\beta,\alpha)}(\pi,\lambda,\tau)$ of saddle connections for $X(\pi,\lambda,\tau)$ defined by equation (3.13) in paragraph 3.1.3.

LEMMA 3.2.1. *Let π be an admissible combinatorial datum over the alphabet $\mathcal{A}$ and let $T = (\pi,\lambda) \in \Delta_\pi$ be an i.e.t. without connections. Let us consider any suspension datum τ for π and the associated translation surface $X = X(\pi,\lambda,\tau)$. For any pair of letters $(\beta,\alpha) \in \mathcal{A}^2$ with $\pi^t(\alpha),\pi^b(\beta) > 1$ and for any triple (β,α,n) reduced for T there exists a saddle connection $\gamma_{\beta,\alpha,n,X}$ in $\mathcal{C}^{(\beta,\alpha)}(\pi,\lambda,\tau)$ for $X(\pi,\lambda,\tau)$ such that for the homology class $\tilde{\gamma}_{\beta,\alpha,n,T}$ defined by equation (3.14) we have*

$$\int_{\gamma_{\beta,\alpha,n,X}} w_X = w_X(\tilde{\gamma}_{\beta,\alpha,n,T}). \tag{3.15}$$

Proof: Let $T : I \to I$ be the i.e.t. defined by the data (π,λ) and let $\tilde{\theta}_T : I \to H_1(M,\mathbb{Z})$ be the piecewise constant map defined in paragraph 3.2.1. Let τ be any suspension datum for π and let w_X be the element in $\hom(H_1(M,\Sigma,\mathbb{Z}),\mathbb{C})$ associated to $X(\pi,\lambda,\tau)$. In terms of the notation of paragraph 2.4.1 we have the identities

$$w_X(\tilde{\xi}_\alpha^{t/b}) = \xi_\alpha^{t/b} \quad \texttt{and} \quad w_X(\tilde{\theta}_\alpha) = \theta_\alpha$$

for any $\alpha \in \mathcal{A}$. In particular we can define a piecewise constant map $\theta_X : I \to \mathbb{C}$ by

$$\theta_X(x) := w_X(\tilde{\theta}_T(x)).$$

We fix a cartesian frame of reference on $\mathbb{C}$ by choosing as origin the left endpoint of the interval I and as positive real half-line the half-line starting from the origin and containing the interval I. Let (β,α,n) be a reduced triple for T and let us consider the relative homology class $\tilde{\gamma}_{\beta,\alpha,n,T}$ defined in equation (3.14). Observe that we have

$$w_X(\tilde{\gamma}_{\beta,\alpha,n,T}) = \xi_\alpha^t - \xi_\beta^b - S_n\theta_X(u_\beta^b).$$

We consider the path $\hat{\gamma}_{\beta,\alpha,n,X} : (0,1) \to \mathbb{C}$ defined by

$$\hat{\gamma}_{\beta,\alpha,n,X}(t) := \xi_\beta^b + S_n\theta_X(u_\beta^b) + t(\xi_\alpha^t - \xi_\beta^b - S_n\theta_X(u_\beta^b)),$$

Our aim is to project $\hat{\gamma}_{\beta,\alpha,n,X}$ onto a saddle connection in X. In order to construct the immersion we define the open rectangle $R(\beta,\alpha,n,X)$ in $\mathbb{C}$ with sides parallel to the vertical and horizontal directions and vertices in the points $\xi_\beta^b + S_n\theta_T(u_\beta^b)$ and ξ_α^t (that is the rectangle that has $\hat{\gamma}_{\beta,\alpha,n,X}$ as diagonal). For any $i \in \mathbb{N}$ and any $x \in I$ such that T^{-1} may be iterated $i-1$ times on x we define

$$S_i^-\theta_X(x) := \theta_X(x) + .. + \theta_X(T^{-(i-1)}x).$$

For any $i \in \{0,..,n\}$ the iterated $T^{n-1}u_\beta^b$ is in the domain of $S_i^-\theta_T^-$ and for these i we have $S_n\theta_X(u_\beta^b) - S_i^-\theta_X(T^{n-1}u_\beta^b)) = S_{n-i}\theta_X(u_\beta^b)$. For $i = 0,..,n+1$ we define the rectangles $R_i \subset \mathbb{C}$ by

for $i = 0$**:** $R_0 := I(\beta,\alpha,n) \times [0, \Im(\xi_\alpha^t))$,
for $i \in \{1,..,n\}$**:** $R_i := I(\beta,\alpha,n) \times \Big[\Im(S_i^-\theta_X(T^{n-1}u_\beta^b)), \Im(S_{i-1}\theta_X(T^{n-1}u_\beta^b))\Big)$,
for $i = n+1$**:** $R_{n+1} := I(\beta,\alpha,n) \times \Big[\Im(S_n^-\theta_X(T^{n-1}u_\beta^b) + \xi_\beta^b), \Im(S_n^-\theta_X(T^{n-1}u_\beta^b))\Big)$.

We have $R(\beta,\alpha,n,X)=\bigsqcup_{i=0,..,n+1}R_i$. For any $i\in\{0,..,n\}$ we have

$$(R_i-S_i^-\theta_X(T^{n-1}u_\beta^b))\cap\mathbb{R}=T^{-i}I(\beta,\alpha,n).$$

Since the triple (β,α,n) is reduced we have that $T^{-i}I(\beta,\alpha,n)$ never contains the points $u_{\alpha'}^t$ and $u_{\beta'}^b$ for any $\alpha',\beta'\in\mathcal{A}$ and any $i\in\{0,..,n\}$. The relations $u_{\alpha'}^t=\Re(\xi_{\alpha'}^t)$ and $u_{\beta'}^b=\Re(\xi_{\beta'}^b)$ therefore imply that the points $\xi_{\alpha'}^t$ and $\xi_{\beta'}^b$ with $\alpha',\beta'\in\mathcal{A}$ are not contained in any of the rectangles $R_i-S_i^-\theta_X(T^{n-1}u_\beta^b)$ for any $i\in\{0,..,n\}$. We introduce the pair of piecewise constant functions $\alpha^t:I\to\mathcal{A}$ and $\alpha^b:I\to\mathcal{A}$ defined respectively by $\alpha^t(x)=\delta$ iff $x\in I_\delta^t$ and $\alpha^b(x)=\delta$ iff $x\in I_\delta^b$. The previous relation implies that for any $i\in\{0,..,n\}$ we have

$$R_i-S_i^-\theta_X(T^{n-1}u_\beta^b)\subset R^t_{\alpha^t(T^{n-i}u_\beta^b)}.$$

For any $\alpha\in\mathcal{A}$ recall the rectangles R_α^t and R_β^b associated to the data (π,λ,τ) by the zippered rectangles construction in paragraph 2.4.1 in the background. Let us denote with $\rho:\bigsqcup_{\alpha,\beta,\in\mathcal{A}}R_\alpha^t,R_\beta^b\to X$ the projection from open subset of $\mathbb{C}$ union of these rectangles into the translation surface $X(\pi,\lambda,\tau)$. For any $i\in\{0,..,n\}$ and the corresponding rectangle R_i in $\mathbb{C}$ defined above we have an isometric embedding $f_i:R_i\hookrightarrow X(\pi,\lambda,\tau)$ defined by

$$f_i(z):=\rho(z-S_i^-\theta_X(T^{n-1}u_\beta^b)).$$

We also have $T^{-n}\subset I(\beta,\alpha,n)\subset I_\beta^b$, therefore $R_{n+1}-S_n^-\theta_X(T^{n-1}u_\beta^b)-\theta_\beta\subset R_\beta^t$ and we have an other isometric embedding $f_{n+1}:R_{n+1}\hookrightarrow X(\pi,\lambda,\tau)$ defined by

$$f_i(z):=\rho(z-S_n^-\theta_X(T^{n-1}u_\beta^b)-\theta_\beta).$$

For any $i\in\{0,..,n\}$ and for any $z\in R(\beta,\alpha,n,X)$ with $\Im(z)=S_i\theta(u_\beta^b)$ we have $f_i(z)=\lim_{x\to z}f_{i+1}(x)$, therefore the embedding f_i past together to a local isometry

$$f:R(\beta,\alpha,n,X)\to X(\pi,\lambda,\tau),$$

(that may not be injective). We define the curve $\gamma_{\beta,\alpha,n,X}:=f\circ\hat{\gamma}_{\beta,\alpha,n,X}$. Since it satisfies the relations

$$\lim_{t\to0}\gamma_{\beta,\alpha,n,X}(t)=\rho(\lim_{t\to0}\hat{\gamma}_{\beta,\alpha,n,X}-S_n\theta_X(u_\beta^b))=\rho(\xi_\beta^b)$$

$$\lim_{t\to1}\gamma_{\beta,\alpha,n,X}(t)=\rho(\lim_{t\to1}\hat{\gamma}_{\beta,\alpha,n,X})=\rho(\xi_\alpha^t),$$

extended continuously on $[0,1]$, the curve $\gamma_{\beta,\alpha,n,X}$ is a saddle connection. It is also evident that $\gamma_{\beta,\alpha,n,X}\in\mathcal{C}^{(\beta,\alpha)}(\pi,\lambda,\tau)$. Finally, since f is a local isometry, the holonomy of $\gamma_{\beta,\alpha,n,X}$ is the same of the holonomy of $\hat{\gamma}_{\beta,\alpha,n,X}$, that is $\xi_\alpha^t-\xi_\beta^b-S_n\theta_T(u_\beta^b)$, therefore the lemma is proved. □

Given an i.e.t. $T=(\pi,\lambda)$, a suspension datum τ for π and the associated translation structure $X=(\pi,\lambda,\tau)$, recall the vector h_X defined by equation (2.33) in paragraph 2.4.1. It can be seen as a piecewise constant function $h_X:I\to\mathbb{R}_+$ defined by $h_X(x)=h_\delta$ iff $x\in I_\delta^t$. We denote with S_nh_X the Birkhoff sum of the function h_X.

LEMMA 3.2.2. *Let π be an admissible combinatorial datum over the alphabet $\mathcal{A}$ and let $T=(\pi,\lambda)\in\Delta_\pi$ be an i.e.t. without connections. Let us consider any suspension datum τ for π and the associated translation surface $X=X(\pi,\lambda,\tau)$. Let $(\beta,\alpha)\in\mathcal{A}^2$ be any pair of letters with $\pi^t(\alpha),\pi^b(\beta)>1$. For any $\epsilon>0$ and* *any triple (β,α,n) reduced for T with n big enough the associated saddle connection $\gamma_{\beta,\alpha,n,X}$ given by lemma3.2.1 satisfies the following two estimates:*

$$(1-\epsilon)|\gamma_{\beta,\alpha,n,X}| \leq S_n h_X(u_\beta^b) \leq |\gamma_{\beta,\alpha,n,X}| \tag{3.16}$$

$$(1-\epsilon)\frac{|T^n u_\beta^b - u_\alpha^t|}{|\gamma_{\beta,\alpha,n,X}|} \leq |\tan \text{angle}(\gamma_{\beta,\alpha,n,X},\partial_y)| \leq (1+\epsilon)\frac{|T^n u_\beta^b - u_\alpha^t|}{|\gamma_{\beta,\alpha,n,X}|}. \tag{3.17}$$

Proof: Let $\gamma_{\beta,\alpha,n,X}$ be the curve given by lemma 3.2.1. We have

$$\int_{\gamma_{\beta,\alpha,n,X}} w_X = \xi_\alpha^t - \xi_\beta^b - S_n\theta_X(u_\beta^b)$$
$$\Re(\xi_\alpha^t - \xi_\beta^b - S_n\theta_X(u_\beta^b)) = T^n u_\beta^b - u_\alpha^t$$
$$\Im(\xi_\alpha^t - \xi_\beta^b - S_n\theta_X(u_\beta^b)) = S_n h_X(u_\beta^b) + \Im\xi_\alpha^t - \Im\xi_\beta^b.$$

We observe that $|T^n u_\beta^b - u_\alpha^t| \leq \|\lambda\|$ for all $n \in \mathbb{N}$ and on the other hand $S_n h_X(u_\beta^b) \geq n \min_{\alpha' \in \mathcal{A}} h_\alpha$ and $\min_{\alpha' \in \mathcal{A}} h_\alpha$ is strictly positive. Therefore for any $\epsilon > 0$ and any $n \in \mathbb{N}$ big enough equation (3.16) holds. We observe that

$$\tan(\text{angle}(\gamma_{\beta,\alpha,n,X},\partial_y)) = \frac{T^n u_\beta^b - u_\alpha^t}{S_n h_X(u_\beta^b) + \Im\xi_\alpha^t - \Im\xi_\beta^b}$$

therefore using equation (3.16) we get that for any $\epsilon > 0$ and any $n \in \mathbb{N}$ big enough equation (3.17) holds. The lemma is proved. □

For any bounded function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ such that $t\varphi(t)$ is decreasing monotone and for any (small) $\epsilon > 0$ let us introduce the function $\varphi_\epsilon : \mathbb{R}_+ \to \mathbb{R}_+$ defined by

$$\varphi_\epsilon(t) := (1-\epsilon)\varphi((1-\epsilon)t). \tag{3.18}$$

PROPOSITION 3.2.3. *Let π be an admissible combinatorial datum over the alphabet $\mathcal{A}$ and let $T = (\pi,\lambda) \in \Delta_\pi$ be an uniquely ergodic i.e.t. without connections. Let (β,α) be a pair in $\mathcal{A}^2$ with $\pi^t(\alpha) > 1$ and $\pi^b(\beta) > 1$ and suppose that there exist infinitely many triples (β,α,n) reduced for T that are solutions of equation (1.3) with respect to φ. Then for any suspension $X(\pi,\lambda,\tau)$ of T with* Area $= 1$ *and any $\epsilon > 0$ there are infinitely many saddle connections γ for $X(\pi,\lambda,\tau)$ in configuration $\mathcal{C}^{(\beta,\alpha)}(\pi,\lambda,\tau)$ that are solutions of equation (1.4) with respect to φ_ϵ, that is*

$$|\tan \text{angle}(\gamma,\partial_y)| < \frac{\varphi_\epsilon(|\gamma|)}{|\gamma|}.$$

Proof: Let us suppose that we have infinite reduced triples (β,α,n) for T that are solution of equation (1.3) with respect to the function φ. Let us consider any suspension datum τ for π such that $\text{Area}(X(\pi,\lambda,\tau)) = 1$. To any triple (β,α,n) as before lemma 3.2.1 associates a saddle connection $\gamma = \gamma_{\beta,\alpha,n,X}$ in the set $\mathcal{C}^{(\beta,\alpha)}(\pi,\lambda,\tau)$. Equation (3.17) of lemma 3.2.2 implies that for any $\epsilon > 0$ and for all n big enough (and such that the triple (β,α,n) is reduced for T) we have

$$|\tan \text{angle}(\gamma_{\beta,\alpha,n,X},\partial_y)| \leq (1+\epsilon)\frac{\varphi(n)}{|\gamma_{\beta,\alpha,n,X}|}.$$

Since T is uniquely ergodic, for $n \to \infty$ the Birkoff average $\frac{1}{n}S_n h_X(u_\beta^b)$ converges to $\int_I h_X = \texttt{Area}(X) = 1$, therefore equation (3.16) of lemma 3.2.2 implies that for any $\epsilon > 0$ and any $\in \mathbb{N}$ big enough we have

$$(1-\epsilon)|\gamma_{\beta,\alpha,n,X}| < n < (1+\epsilon)|\gamma_{\beta,\alpha,n,X}|.$$

Since $t\varphi(t)$ is decreasing monotone, for any $\epsilon > 0$ and any $\in \mathbb{N}$ big enough we have $n\varphi(n) \leq (1-\epsilon)|\gamma_{\beta,\alpha,n,X}|\varphi((1-\epsilon)|\gamma_{\beta,\alpha,n,X}|)$, that is

$$|\tan \text{angle}(\gamma_{\beta,\alpha,n,X}, \partial_y)| \leq (1+\epsilon)\frac{\varphi((1-\epsilon)|\gamma_{\beta,\alpha,n,X}|)}{|\gamma_{\beta,\alpha,n,X}|} \leq \frac{\varphi_\epsilon(|\gamma_{\beta,\alpha,n,X}|)}{|\gamma_{\beta,\alpha,n,X}|}$$

since $(1+\epsilon)(1-\epsilon) \leq 1$. The proposition is proved. □

3.3. Normalization of length and area.

3.3.1. Normalization of length. We want to take advantage of the Rauzy-Veech and Zorich algorithms. Since they have interesting recurrence properties just at projective level we need to relate part b) of theorem 1.1.6 to an equivalent formulation (proposition 3.3.1) on the subset $\Delta^{(1)}(\mathcal{R})$ of length one i.e.t..

PROPOSITION 3.3.1. *Let us consider a sequence $\varphi : \mathbb{N} \to \mathbb{R}_+$ such that $n\varphi(n)$ is decreasing monotone. Let us fix any admissible combinatorial datum π over an alphabet $\mathcal{A}$. Theorem 1.1.6 is equivalent to the following dichotomy:*

- **a:** *If $\sum_{n\in\mathbb{N}} \varphi(n) < \infty$ then almost any i.e.t. $T \in \Delta_\pi^{(1)}$ is* modφ*-Diophantine.*
- **b:** *If $\sum_{n\in\mathbb{N}} \varphi(n) = \infty$ then almost any i.e.t. $T \in \Delta_\pi^{(1)}$ is* modφ*-Liouville.*

Proof: For any $s \in \mathbb{R}_+$ and any admissible π let us define the subsets $\Delta_\pi^{(\leq s)}$ and $\Delta_\pi^{(\geq s)}$ of the i.e.t. $T = (\pi, \lambda)$ in Δ_π such that respectively $\|\lambda\| \leq s$ and $\|\lambda\| \geq s$. For any $(\pi,\lambda) \in \Delta_\pi$ and any $t \in \mathbb{R}_+$ let us introduce the i.e.t. $\tilde{T} := (\pi, t\lambda)$. For any pair of letters (β,α) such that $\pi^t(\alpha), \pi^b(\beta) > 1$ we denote as usual with u_β^b and u_α^t the bottom and top singularities of T and with $\tilde{u}_\beta^b$ and $\tilde{u}_\alpha^t$ the bottom and top singularities of $\tilde{T}$. For any $n \in \mathbb{N}$ we have

$$|\tilde{T}^n \tilde{u}_\beta^b - \tilde{u}_\alpha^t| = t|T^n u_\beta^b - u_\alpha^t| \tag{3.19}$$

therefore the triple (β,α,n) is a reduced solution for T of $|T^n u_\beta^b - u_\alpha^t| < \varphi(n)$ if and only if it is a reduced solution for $\tilde{T}$ of $|\tilde{T}^n \tilde{u}_\beta^b - \tilde{u}_\alpha^t| < t\varphi(n)$. We also note that for any π we have an homeomorphism

$$\begin{array}{ccc} \mathbb{R}_+ \times \Delta_\pi^{(1)} & \to & \Delta_\pi \\ (t,(\pi,\lambda)) & \mapsto & (\pi, t\lambda), \end{array}$$

and the Lebesgue measure on Δ_π is equivalent to the product of the Lebesgue measures on the two factors.

Let us suppose that statement a) of proposition 3.3.1 is true. We observe that if $(\pi,\lambda) \in \Delta_\pi^{(1)}$ is modφ-Diophantine then also any $(\pi, t\lambda)$ is modφ-Diophantine for any $t \in [1,\infty)$, therefore by our hypothesis almost any $(\pi,\lambda) \in \Delta_\pi^{(\geq 1)}$ is modφ-Diophantine for any φ such that $n\varphi(n)$ is decreasing monotone and such that $\sum_{n\in\mathbb{N}} \varphi(n) < \infty$. Let us consider such a φ and for any fixed $s \in (0,1)$ let us consider the sequence $\tilde{\varphi} := s\varphi$. We still have $\sum_{n\in\mathbb{N}} \tilde{\varphi}(n) < \infty$, therefore almost any $T = (\pi,\lambda) \in \Delta_\pi^{(\geq 1)}$ is mod $\tilde{\varphi}$-Diophantine. Let us consider such a T: for any pair of letters (β,α) such that $\pi^t(\alpha), \pi^b(\beta) > 1$ there exist just a finite number of reduced triples (β,α,n) solution of $|T^n u_\beta^b - u_\alpha^t| < \tilde{\varphi}(n) = s\varphi(n)$, therefore equation (3.19) implies that for $\tilde{T} := (\pi, s^{-1}\lambda)$ there exist just a finite number of reduced triples (β,α,n) that are solutions of $|\tilde{T}^n \tilde{u}_\beta^b - \tilde{u}_\alpha^t| < \varphi(n)$, that is $\tilde{T}$ is modφ-Diophantine. Since such $\tilde{T}$ is generic we get that almost any $(\pi,\lambda) \in \Delta_\pi^{(\geq s^{-1})}$

is modφ-Diophantine. Since s is any arbitrary real number in $(0,1)$ and we get part a) of theorem 1.1.6.

Let us now suppose that statement b) of proposition 3.3.1 is true. We observe that if $(\pi,\lambda)\in\Delta_\pi^{(1)}$ is modφ-Liouville then also any $(\pi,t\lambda)$ is modφ-Liouville for any $t\in(0,1]$, therefore by our hypothesis almost any $(\pi,\lambda)\in\Delta_\pi^{(\leq 1)}$ is modφ-Liouville for any φ such that $n\varphi(n)$ is decreasing monotone and such that $\sum_{n=1}^{\infty}\varphi(n)=\infty$. Let us consider such a φ and for any fixed $s>1$ and let us define the sequence $\tilde\varphi:=s^{-1}\varphi$. We still have $\sum_{n\in\mathbb{N}}\tilde\varphi(n)=\infty$, therefore almost any $T=(\pi,\lambda)\in\Delta_\pi^{(\leq 1)}$ is mod $\tilde\varphi$-Liouville. Let us consider such a T: for any pair of letters (β,α) such that $\pi^t(\alpha),\pi^b(\beta)>1$ there exist infinite reduced triples (β,α,n) solution of $|T^n u_\beta^b-u_\alpha^t|<\tilde\varphi(n)=s^{-1}\varphi(n)$, therefore equation (3.19) implies that for $\tilde T:=(\pi,s\lambda)$ there exist infinite reduced triples (β,α,n) that are solutions of $|\tilde T^n\tilde u_\beta^b-\tilde u_\alpha^t|<\varphi(n)$, that is $\tilde T$ is modφ-Diophantine. Since such $\tilde T$ is generic we get that almost any $(\pi,\lambda)\in\Delta_\pi^{(\geq s)}$ is modφ-Diophantine. Since r is arbitrary we get part a) of theorem 1.1.6. The proposition is proved. □

3.3.2. Normalization of area. Proposition 3.2.3) introduces a normalization of area, therefore we need an equivalent formulation of theorem 1.2.3 for strata $\widehat{\mathcal{H}}^{(1)}(k_1,..,k_r)$ of area one totally marked translation surfaces. The proof has evident analogies with the one of proposition 3.3.1.

Proposition 3.3.2. *Let us consider $\varphi:\mathbb{R}_+\to\mathbb{R}_+$ such that $t\varphi(t)$ is a decreasing function and fix a stratum $\widehat{\mathcal{H}}(k_1,..,k_r)$. Theorem 1.2.3 is equivalent to the following dichotomy:*

- **a:** *If $\int_{\mathbb{R}_+}\varphi(t)dt<\infty$ then almost any $X\in\widehat{\mathcal{H}}^{(1)}(k_1,..,k_r)$ is* modφ*-Diophantine.*
- **b:** *If $\int_{\mathbb{R}_+}\varphi(t)dt=\infty$ then almost any $X\in\widehat{\mathcal{H}}^{(1)}(k_1,..,k_r)$ is* modφ*-Liouville.*

Proof: For any $s\in\mathbb{R}_+$ let $\widehat{\mathcal{H}}^{(\leq s)}(k_1,..,k_r)$ and $\widehat{\mathcal{H}}^{(\geq s)}(k_1,..,k_r)$ be the sets of totally marked translation surfaces $\hat X=(X,S_1,..,S_r)$ such that respectively $\mathrm{Area}(X)\leq s$ and $\mathrm{Area}(X)\geq s$. For any $\hat X\in\widehat{\mathcal{H}}(k_1,..,k_r)$ and any $t\in\mathbb{R}_+$ let us consider the flat structure $t\cdot X$, where the multiplication is given by the action of $\mathrm{GL}(2,\mathbb{R})$, and the totally marked translation surface $t\cdot\hat X:=(t\cdot X,S_1,..,S_r)$. It is evident that γ is saddle connection for X if and only if it is a saddle connection for $t\cdot X$. For any $t\in\mathbb{R}_+$ we denote with $|\gamma|_{t\cdot X}$ its length with respect to the flat structure $t\cdot X$. The angle $\mathrm{angle}(\gamma,\partial_y)$ with the vertical directions is evidently the same for X and $t\cdot X$. For any γ we have

$$\frac{\varphi(|\gamma|_{t\cdot X})}{|\gamma|_{t\cdot X}}=\frac{\varphi(t|\gamma|_X)}{t|\gamma|_X}. \tag{3.20}$$

It follows that γ is a solution for $t\cdot X$ of $|\tan\mathrm{angle}(\gamma,\partial_y)|\leq\frac{\varphi(|\gamma|_{t\cdot X})}{|\gamma|_{t\cdot X}}$ if and only if it is a solution for X of $|\tan\mathrm{angle}(\gamma,\partial_y)|\leq\frac{\varphi(t|\gamma|_X)}{t|\gamma|_X}$. We also note that we have an homeomorphism

$$\begin{array}{ccc}\mathbb{R}_+\times\widehat{\mathcal{H}}^{(1)}(k_1,..,k_r) & \to & \widehat{\mathcal{H}}(k_1,..,k_r)\\ (t,X) & \mapsto & t\cdot X\end{array}$$

and the Lebesgue measure on $\widehat{\mathcal{H}}(k_1,..,k_r)$ is equivalent to the product of the Lebesgue measures on the two factors.

Let us suppose that statement a) of proposition 3.3.2 is true. We observe that if $\hat{X} \in \widehat{\mathcal{H}}^{(1)}(k_1,..,k_r)$ is modφ-Diophantine then also any $t \cdot \hat{X}$ is modφ-Diophantine for any $t \in [1,\infty)$. It follows by our hypothesis that almost any $\hat{X} \in \widehat{\mathcal{H}}^{(\geq 1)}(k_1,..,k_r)$ is modφ-Diophantine for any φ such that $t\varphi(t)$ is decreasing monotone and such that $\int_0^{+\infty} \varphi(t) < \infty$. Let us consider such a φ and for any fixed $s \in (0,1)$ let us put $\tilde{\varphi}(x) := \varphi(sx)/s$. We still have $\int_0^{+\infty} \tilde{\varphi}(t) < \infty$, therefore almost any $\hat{X} \in \widehat{\mathcal{H}}^{(\geq 1)}(k_1,..,k_r)$ is mod $\tilde{\varphi}$-Diophantine. For any such $\hat{X}$ there exist just finitely many saddle connections γ that are solutions of $|\tan \text{angle}(\gamma, \partial_y)| \leq \tilde{\varphi}(|\gamma|_X)/|\gamma|_X$, therefore equation (3.20) implies that for $s \cdot \hat{X}$ there exist just finitely many solutions of $|\tan \text{angle}(\gamma, \partial_y)| \leq \varphi(|\gamma|_{s \cdot X})/|\gamma|_{s \cdot X}$, that is $s \cdot \hat{X}$ is modφ-Diophantine. Since such $\hat{X}$ is generic we get that almost any $\hat{X} \in \widehat{\mathcal{H}}^{(\geq s^2)}(k_1,..,k_r)$ is modφ-Diophantine. Since s is arbitrary we get part a) of theorem 1.2.3.

Let us now suppose that statement b) of proposition 3.3.2 is true. We observe that if $\hat{X} \in \widehat{\mathcal{H}}^{(1)}(k_1,..,k_r)$ is modφ-Liouville then also any $t \cdot \hat{X}$ is modφ-Liouville for any $t \in (0,1]$, therefore by our hypothesis almost any $\hat{X} \in \widehat{\mathcal{H}}^{(\leq 1)}(k_1,..,k_s)$ is modφ-Liouville for any φ such that $t\varphi(t)$ is decreasing monotone and such that $\int_{\mathbb{R}_+} \varphi(t) = +\infty$. Let us consider any such φ and for any fixed $s > 1$ let us put $\tilde{\varphi}(x) := \varphi(sx)/s$. We still have $\int_0^{+\infty} \tilde{\varphi}(t) = +\infty$, therefore almost any $\hat{X} \in \widehat{\mathcal{H}}^{(\leq 1)}(k_1,..,k_r)$ is mod $\tilde{\varphi}$-Liouville. Let us consider any such $\hat{X}$: any configuration $\mathcal{C}^{(p_i,p_j,l,m)}(\hat{X})$ contains infinitely many saddle connections γ that are solutions of $\text{angle}(\gamma, \partial_y) \leq \tilde{\varphi}(|\gamma|_X)/|\gamma|_X$. Let us consider the corresponding configuration $\mathcal{C}^{(p_i,p_j,l,m)}(s \cdot \hat{X})$ for $s \cdot \hat{X}$. Equation (3.20) implies that $\mathcal{C}^{(p_i,p_j,l,m)}(s \cdot \hat{X})$ contains infinitely many saddle connections γ that are solutions of $|\tan \text{angle}(\gamma, \partial_y)| \leq \varphi(|\gamma|_{s \cdot X})/|\gamma|_{s \cdot X}$ and this is true for any configuration, that is $s \cdot \hat{X}$ is modφ-Liouville. Since such $\hat{X}$ is generic we get that almost any $\hat{X} \in \widehat{\mathcal{H}}^{(\leq s^2)}(k_1,..,k_r)$ is modφ-Liouville. Since s is any real number in $(1,+\infty)$ we get part b) of theorem 1.2.3. The proposition is proved. $\square$

3.4. Proof of the strong statement in the convergent case.

As we said in the introduction proposition 1.2.4 is a strong version of part **a** of theorem 1.2.3. In this paragraph we prove the proposition (and therefore part **a** of the theorem too). We recall the statement.

Let us consider a positive function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\int_0^{\infty} \varphi(t)dt < \infty$. Then for any $\hat{X}$ in $\widehat{\mathcal{H}}(k_1,..,k_r)$, almost any $\hat{X}_\theta \in \mathrm{SO}(2,\mathbb{R})X$ is modφ-Diophantine.

Proof: In the proof we don't need any information about configurations of saddle connections, therefore the proposition can be proved directly for translations surfaces (without any marking of horizontal saddle connections at singularities). Let us therefore fix a stratum $\mathcal{H}(k_1,..,k_r)$ of the moduli space of abelian differentials and $X \in \mathcal{H}(k_1,..,k_r)$, then let us consider the orbit $\mathrm{SO}(2,\mathbb{R})X$ of X under the action of $\mathrm{SO}(2,\mathbb{R})$. It is an immersed circle in $\mathcal{H}(k_1,..,k_r)$ (the immersion may fail to be injective at ramification points). We have a parametrization $[0,2\pi) \to \mathrm{SO}(2,\mathbb{R})X; \theta \mapsto X_\theta$, where $X_\theta := R_\theta X$ and R_θ denotes the rotation of angle θ in $\mathbb{R}^2$ (the multiplication is induced by the action of $\mathrm{SL}(2,\mathbb{R})$). Let us define the set $\mathcal{S}$ of all the saddle connection for the flat structure X, it is a countable set. For any $\theta \in [0,2\pi)$ the curve γ is a saddle connection for X if and only if is a saddle connection for X_θ. For any θ the flat metric determined by X_θ is the same as the flat metric determined by X, therefore the length of a saddle connection γ does not change passing from X to X_θ and we denote it as $|\gamma|$. For any $\gamma \in \mathcal{S}$ we define the set

$$I_\gamma := \{\theta \in [0,2\pi); |\tan \text{angle}(\gamma, R_\theta \partial_y)| \leq \frac{\varphi(|\gamma|)}{|\gamma|}\},$$

where $R_\theta \partial_y$ denotes the constant vertical vector field associated to the rotated flat structure X_θ. The set $I(\gamma)$ is the set of θ such that the flat structure X_θ has γ as solution of equation (1.4). For any $\gamma \in \mathcal{S}$ we have $\text{Leb}(I(\gamma)) = 2\varphi(|\gamma|)/|\gamma|$. It is sufficient to prove that $\sum_{\gamma \in \mathcal{S}} \text{Leb}(I_\gamma) < \infty$, then with the classical Borel-Cantelli argument we conclude that almost any $\theta \in [0,2\pi)$ is contained into a finite number of sets $I(\gamma)$ for $\gamma \in \mathcal{S}$. We re-order our sum as:

$$\sum_{k \in \mathbb{N}} \left(\sum_{\gamma \in \mathcal{S}, 2^{k-1} < |\gamma| \leq 2^k} \text{Leb}(I_\gamma) \right).$$

In [**Ma2**] it is proved that for any $Y \in \mathcal{H}(k_1,..,k_r)$ the number $N(Y,L)$ of saddle connections γ for Y of length $|\gamma| \leq L$ has quadratic growth with L, that is there are two positive constants $c < C$ such that for any L big enough we have

$$cL^2 \leq N(Y,L) \leq CL^2.$$

This implies (recalling that $t\varphi(t)$ is decreasing monotone and therefore $\varphi(t)$ too)

$$\sum_{\gamma \in \mathcal{S}, 2^{k-1} < |\gamma| \leq 2^k} \text{Leb}(I_\gamma) \leq C 2^{2k} \frac{\varphi(2^{k-1})}{2^{k-1}}$$

and $\sum_{\gamma \in \mathcal{S}} \text{Leb}(I_\gamma) \leq 8C \sum_{k \in \mathbb{N}} 2^k \varphi(2^k) < \infty$. Proposition 1.2.4 is proved. □

3.5. Proofs of non-direct results in arbitrary genus.

In this paragraph we assume part b) of theorem 1.1.6 and together with proposition 3.2.3 we show theorems 1.1.6 and 1.2.3. We first develop some useful machinery.

Let us fix a singular point p_i and consider a stratum $\widetilde{\mathcal{H}}_i(k_1,..,k_r)$ of the moduli space of translation surfaces marked at p_i. Recall that we have a covering map of degree $\Pi_{j \neq i}(k_j+1)$ from the moduli space $\widehat{\mathcal{H}}(k_1,..,k_r)$ of totally marked translation surfaces:

$$\text{Proj}_i : \widehat{\mathcal{H}}(k_1,..,k_r) \to \widetilde{\mathcal{H}}_i(k_1,..,k_r)$$

Recall the set $\mathfrak{S}_i(k_1,..,k_r)$ introduced in definition 2.4.5 in paragraph 2.4.4 of the background. For any $\pi \in \mathfrak{S}_i(k_1,..,k_r)$ we have a map $\mathcal{I}_\pi : \mho_\pi \times \Theta_\pi \to \widetilde{\mathcal{H}}_i(k_1,..,k_r)$, defined in equation (2.48) of paragraph 2.4.4, which is an homeomorphism onto an open subset of $\widetilde{\mathcal{H}}_i(k_1,..,k_r)$. For any marked translation surface $X^* \in \mathcal{I}_\pi(\mho_\pi \times \Theta_\pi)$ a choice of a pre-image $\hat{X}^*$ in $\widehat{\mathcal{H}}(k_1,..,k_r)$ determines uniquely a lift $\mathcal{I}_{\pi,\hat{X}^*}$ of $\mathcal{I}_\pi$ to the moduli space of totally marked translation surfaces (see paragraph 3.1.3). The map $\mathcal{I}_{\pi,\hat{X}^*}$ extends continuously to a map on $\Delta_\pi \times \Theta_\pi$, that we still call $\mathcal{I}_{\pi,\hat{X}^*}$. Such a map will be no more injective. Let us denote with

$$\widehat{\mathcal{I}}_\pi : \Delta_\pi \times \Theta_\pi \to \widehat{\mathcal{H}}(k_1,..,k_r) \tag{3.21}$$

the general map obtained in this way. For any combinatorial datum $\pi \in \mathfrak{S}_i(k_1,..,k_r)$ we define the hyperboloid $H_\pi^{(1)} \subset \Delta_\pi \times \Theta_\pi$ by

$$H_\pi^{(1)} := \{(\lambda,\tau) \in \Delta_\pi \times \Theta_\pi; \langle \lambda, \Omega_\pi \tau \rangle = 1\}. \tag{3.22}$$

For any $(\pi,\lambda) \in \Delta_\pi$ we also introduce the set $\Theta_{(\pi,\lambda)}^{(1)} := \{\tau \in \Theta_\pi; \langle \lambda, \Omega_\pi \tau \rangle = 1\}$. We have a decomposition $H_\pi^{(1)} = \bigsqcup_{(\pi,\lambda)\in\Delta_\pi} \Theta_{(\pi,\lambda)}^{(1)}$. Moreover, since for any $(\pi,\lambda) \in \Delta_\pi$ the fiber $\Theta_{(\pi,\lambda)}^{(1)}$ is diffeomorphic to $\mathbb{P}\Theta_\pi$, the hyperboloid $H_\pi^{(1)}$ is diffeomorphic to the product $\Delta_\pi \times \mathbb{P}\Theta_\pi$. Let us denote Leb_d and Leb_{d-1} the lebesgue measure respectively on Δ_π and on $\mathbb{P}\Theta_\pi$. By Fubini's theorem the lebesgue measure Leb_{2d-1} on $H_\pi^{(1)}$ is equivalent to the product measure on $\Delta_\pi \times \mathbb{P}\Theta_\pi$.

The restriction to the hyperboloid $H_\pi^{(1)}$ of any map $\widehat{\mathcal{I}}_\pi$ as in equation (3.21) gives a continuous map

$$\widehat{\mathcal{I}}_\pi|_{H_\pi^{(1)}} : H_\pi^{(1)} \to \widehat{\mathcal{H}}^{(1)}(k_1,..,k_r)$$

onto an open subset of the hyper-surface or area one totally marked translation surfaces. Moreover the invariant volume $\mu^{(1)}$ on $\hat{\mathcal{H}}(k_1,..,k_r)$, restricted to $\widehat{\mathcal{I}}_\pi(H_\pi^{(1)})$, is equivalent to the push-forward $\widehat{\mathcal{I}}_{\pi*}\mathrm{Leb}_{2d-1}$ of the lebesgue measure on $H_\pi^{(1)}$. Corollary 3.1.4 implies that

$$\widehat{\mathcal{H}}^{(1)}(k_1,..,k_r) = \bigcup_{\pi\in\mathfrak{S}_i(k_1,..,k_r)} \bigcup_{\mathrm{Proj}_i\circ\widehat{\mathcal{I}}_\pi = \mathcal{I}_\pi} \widehat{\mathcal{I}}_\pi(H_\pi^{(1)}) \mod 0. \tag{3.23}$$

where in the formula for any $\pi \in \mathfrak{S}_i(k_1,..,k_r)$ we take the union over all the lifts $\widehat{\mathcal{I}}_\pi$ of $\mathcal{I}_\pi$ (the union will be no more disjoint).

3.5.1. Convergent case, proof of part a) of theorem 1.2.3. The proof is a trivial consequence of proposition 1.2.4, just observe that the foliation induced by the action of $\mathrm{SO}(2,\mathbb{R})$ on $\widehat{\mathcal{H}}(k_1,..,k_r)$ is smooth, therefore the lebesgue measure in $\widehat{\mathcal{H}}(k_1,..,k_r)$ is equivalent to the product of the lebesgue measure on the leaves and the transversal measure. Observe that the area one hyper-surface $\widehat{\mathcal{H}}^{(1)}(k_1,..,k_r)$ is $\mathrm{SO}(2,\mathbb{R})$-invariant, therefore the theorem holds also for area one translation surfaces.

3.5.2. Convergent case, proof of part a) of theorem 1.1.6. Let us consider a positive sequence $\varphi : \mathbb{N} \to \mathbb{R}_+$ such that $n\varphi(n)$ is decreasing monotone. Let us take any admissible combinatorial datum π over the alphabet $\mathcal{A}$. In this paragraph we prove that if $\sum_{n=1}^\infty \varphi(n) < +\infty$, then almost any $T \in \Delta_\pi$ is modφ-diophantine.

If the claim is not true, then there exist two letters β, α such that $\pi^t(\alpha) > 1$ and $\pi^b(\beta) > 1$ and a positive measure set $\mathcal{D} \subset \Delta_\pi$ such that for all $T \in \mathcal{D}$ there exist infinitely many triples (β,α,n) reduced for T that are solutions of equation (1.3), that is such that

$$|T^n u_\beta^b - u_\alpha^t| \leq \varphi(n).$$

By the celebrated result of Masur and Veech (see [**Ma1**] and [**Ve**]) almost any i.e.t. is uniquely ergodic, therefore we can assume without losing in generality that any T in the subset $\mathcal{D}$ is also uniquely ergodic.

The sequence φ may be extended to a bounded positive bounded function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ such that $t\varphi(t)$ is decreasing monotone and $\int_0^\infty \varphi(t)dt < +\infty$. If $T = (\pi, \lambda)$ is any element in the set $\mathcal{D}$ introduced above, for any $\tau \in \Theta^{(1)}_{(\pi,\lambda)}$ the translation surface $X(\pi, \lambda, \tau)$ has area one and uniquely ergodic vertical flow ∂_y. In particular proposition 3.2.3 applies and we get that for any $\epsilon > 0$ there are infinitely many saddle connections γ for $X(\pi, \lambda, \tau)$ that are solutions of

$$|\tan \mathrm{angle}(\gamma, \partial_y)| \leq \frac{\varphi_\epsilon(|\gamma|)}{|\gamma|},$$

where $\varphi_\epsilon(t) = (1 - \epsilon)\varphi((1 + \epsilon)t)$. We introduce the set $\widehat{\mathcal{D}} := \bigcup_{(\pi,\lambda)\in\mathcal{D}} \Theta^{(1)}_{(\pi,\lambda)}$. Since $\mathrm{Leb}_d(\mathcal{D}) > 0$ and the lebesgue measure Leb_{2d-1} on the hyperboloid $H^{(1)}_\pi$ is equivalent to the product $\mathrm{Leb}_d \times \mathrm{Leb}_{d-1}$ on $\Delta_\pi \times \mathbb{P}\Theta_\pi$, then $\mathrm{Leb}_{2d-1}(\hat{\mathcal{D}}) > 0$. Now we consider any map $\widehat{\mathcal{I}}_\pi : H^{(1)}_\pi \to \widehat{\mathcal{H}}^{(1)}(k_1, .., k_r)$ that appears in the union in equation (3.23). The invariant volume $\mu^{(1)}$ restricted to the image of this map is equivalent to $\widehat{\mathcal{I}}_{\pi*}\mathrm{Leb}_{2d-1}$, thus the set $\widehat{\mathcal{I}}_\pi(\hat{\mathcal{D}})$ is a positive measure set in $\widehat{\mathcal{H}}^{(1)}(k_1, .., k_r)$ of area one totally marked translation surfaces that are not φ_ϵ-Liouville. Since $\int_0^\infty \varphi(t)dt < +\infty$ implies that also $\int_0^\infty \varphi_\epsilon(t)dt < +\infty$ we get an absurd. Part a) of theorem 1.1.6 is proved.

3.5.3. Divergent case: reduction to theorem 1.1.6. In this paragraph we show that part b) of theorem 1.1.6 implies part b) of theorem 1.2.3. We first show how part b) of theorem 1.1.6 implies the following proposition.

PROPOSITION 3.5.1. *Let us fix any pair of points $p_i, p_j \in \Sigma$ and any pair of indexes l, m with $l \in \{0, .., k_i\}$ and $m \in \{0, .., k_j\}$. If $\int_0^\infty \varphi(t)dt = +\infty$ then for almost any $\hat{X} \in \widehat{\mathcal{H}}^{(1)}(k_1, .., k_r)$ the configuration $\mathcal{C}^{(p_i,p_j,l,m)}(\hat{X})$ contains infinitely many saddle connections γ which are solution of equation (1.4), that is*

$$|\tan \mathrm{angle}(\gamma, \partial_y)| \leq \frac{\varphi(|\gamma|)}{|\gamma|}.$$

Proof: We recall that from corollary 3.1.4 in paragraph 3.1.3 we have that

$$\widehat{\mathcal{H}}^{(1)}(k_1, .., k_r) = \bigsqcup_{\pi\in\mathfrak{S}_i(k_1,..,k_r)} \bigsqcup_{\mathrm{Proj}_i\circ\widehat{\mathcal{I}}_\pi=\mathcal{I}_\pi} \widehat{\mathcal{I}}_\pi(H^{(1)}_\pi \cap (\mho_\pi \times \Theta_\pi)) \mod 0.$$

where in the formula for any $\pi \in \mathfrak{S}_i(k_1, .., k_r)$ we take the union over all the lifts $\widehat{\mathcal{I}}_\pi$ of $\mathcal{I}_\pi$ and the hyperboloid $H^{(1)}_\pi$ is defined in equation (3.22). The union is disjoint since any $\widehat{\mathcal{I}}_\pi$ is restricted to $\mho_\pi \times \Theta_\pi$. It is enough to prove the statement on each element of the union.

Let us fix any pair of points $p_i, p_j \in \Sigma$ and any pair of indexes l, m with $l \in \{0, .., k_i\}$ and $m \in \{0, .., k_j\}$. Lemma 3.1.7 implies that there exist a pair of letters (β, α) (uniquely determined) with $\pi^t(\alpha) > 1$ and $\pi^b(\beta) > 1$ such that for any $(\lambda, \tau) \in \mho_\pi \times \Theta_\pi$ the map $\widehat{\mathcal{I}}_\pi$ induces a bijection between the set $\mathcal{C}^{(\beta,\alpha)}(\pi, \lambda, \tau)$ of saddle connections γ for the marked translation surface $X(\pi, \lambda, \tau)$ and the configuration $\mathcal{C}^{(p_i,p_j,l,m)}(\hat{X})$ for $\hat{X} = \widehat{\mathcal{I}}_\pi(\lambda, \tau)$.

Let us fix any $\epsilon > 0$ and consider a bounded positive function $\varphi' : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\varphi'_\epsilon = \varphi$, where $\varphi'_\epsilon(t) = (1 - \epsilon)\varphi'((1 + \epsilon)t)$. It is easy to check that $t\varphi'(t)$ is decreasing monotone and $\int_0^\infty \varphi'(t)dt = \infty$. Part b) of theorem therefore implies that there exists a full measure subset $\mathcal{D}_\pi$ of $\mho_\pi$ such that for almost any

$T = (\pi, \lambda)$ in $\mathcal{D}_\pi$ there exists infinite triples (β, α, n) reduced for T that are solutions of equation (1.3) with respect to φ', that is $|T^n u^b_\beta - u^t_\alpha| \leq \varphi'(n)$. By the theorem of Masur and Veech (see [**Ma1**] and [**Ve**]) we can suppose without losing in generality that any T in $\mathcal{D}_\pi$ also uniquely ergodic. Proposition 3.2.3 in paragraph 3.2 implies that for any $T = (\pi, \lambda) \in \mathcal{D}_\pi$ and for any suspension datum $\tau \in \Theta^{(1)}_{(\pi,\lambda)}$ the area one marked translation surface $X(\pi, \lambda, \tau)$ has infinitely many saddle connections γ in $\mathcal{C}^{(\beta,\alpha)}(\pi, \lambda, \tau)$ that are solutions of equation (1.4), that is

$$\tan \text{angle}(\gamma, \partial_y) \leq \frac{\varphi(|\gamma|)}{|\gamma|}.$$

With the identification $\hat{X} = \widehat{\mathcal{I}}_\pi(\pi, \lambda, \tau)$ we have that these saddle connections correspond to infinite elements in configuration $\mathcal{C}^{(p_i,p_j,l,m)}(\hat{X})$ for $\hat{X}$ that are solutions of the same equation. We consider the set $\hat{\mathcal{D}}_\pi := \bigcup_{(\pi,\lambda)\in\mathcal{D}_\pi} \Theta^{(1)}_{(\pi,\lambda)}$. Since $\mathcal{D}_\pi$ has full measure in $\mho_\pi$ and since the lebesgue measure Leb_{2d-1} on the hyperboloid $H^{(1)}_\pi$ is equivalent to the product $\text{Leb}_d \times \text{Leb}_{d-1}$ on $\Delta_\pi \times \mathbb{P}\Theta_\pi$, then $\hat{\mathcal{D}}_\pi$ has full measure in $(\mho_\pi \times \Theta_\pi) \cap H^{(1)}_\pi$. It follows that $\widehat{\mathcal{I}}_\pi(\hat{\mathcal{D}})_\pi$ as full $\mu^{(1)}$-measure in $\widehat{\mathcal{I}}_\pi(H^{(1)}_\pi \cap (\mho_\pi \times \Theta_\pi))$. The same argument works for any lift $\widehat{\mathcal{I}}_\pi$ in equation (3.21), therefore the proposition is proved. □

Proposition 3.5.1 together with proposition 3.3.2 implies part b) of theorem 1.2.3.

3.6. Generalization of the logarithmic law.

3.6.1. Preliminary facts. In this paragraph we prove some useful estimations about saddle connections. In what we do we don't use any information about the configuration they belong to, therefore in all the paragraph we will work with simple translation surfaces without any marking of horizontal separatrix at the singularities. Let us fix a stratum $\mathcal{H}(k_1, .., k_r)$ in the moduli space of translation surfaces and consider the action on it of Teichmüller flow

$$\mathcal{F}_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}.$$

Let us consider any $X \in \mathcal{H}(k_1, .., k_r)$ and a saddle connection γ for X. As we have seen in the prove of lemma 3.1.3 the curve γ is a saddle connection for $\mathcal{F}_t X$ for any $t \in \mathbb{R}$. We recall that for any $t \in \mathbb{R}$ we denote with $|\gamma|_{X,t}$ the length of γ with respect to the flat metric of $\mathcal{F}_t X$ and with $\text{angle}_{X,t}(\gamma, \partial_y)$ the angle between γ and the vertical direction ∂_y in the translation structure $\mathcal{F}_t X$.

Lemma 3.6.1. *Let us fix any $X \in \mathcal{H}(k_1, .., k_r)$ and any saddle connection γ for X. Let us denote $\tau := \tau(X, \gamma) \in \mathbb{R}$ the instant defined by*

$$|\gamma|_{X,\tau} := \min\{|\gamma|_{X,t}; t \in \mathbb{R}\}. \quad (3.24)$$

Then $\tau(X, \gamma)$ satisfies

$$e^{2\tau(X,\gamma)} |\tan \text{angle}_{X,0}(\gamma, \partial_y)| = 1 \quad (3.25)$$

Proof: For any translation surface X and any saddle connection γ for X we have

$$\tan \text{angle}_{X,t}(\gamma, \partial_y) = e^{2t} \tan \text{angle}_{X,0}(\gamma, \partial_y),$$

We observe that for any fixed γ which is a saddle connection for X, the length $|\gamma|_{X,t}$ is minimal when $\text{angle}_{X,t}(\gamma, \partial_y) = \pi/4$. The lemma is proved. □

LEMMA 3.6.2. *If ϵ is any fixed positive real number, then for almost any X in $\mathcal{H}(k_1,..,k_r)$ and for any saddle connection γ for X with initial length $|\gamma| = |\gamma|_{X,0}$ big enough, the instant $\tau(X,\gamma)$ associated to γ by equation (3.24) satisfy:*

$$\tau(X,\gamma) \leq (1+\epsilon)\log|\gamma|. \tag{3.26}$$

Proof: Let us fix any $\epsilon > 0$. Since the function $t \mapsto t^{-(1+2\epsilon)}$ has convergent tail, part a) of theorem 1.2.3 implies that for almost any $X \in \mathcal{H}(k_1,..,k_r)$ there are just finitely many saddle connections γ for X such that

$$|\tan \mathrm{angle}(\gamma,\partial_y)| \leq |\gamma|^{-2(1+\epsilon)}.$$

On the other hand the instant $\tau(X,\gamma)$ associated to γ by equation (3.24) satisfies equation (3.25), therefore we have

$$\tau(X,\gamma) = -\frac{1}{2}\log|\tan \mathrm{angle}(\gamma,\partial_y)| \leq (1+\epsilon)\log|\gamma|.$$

□

LEMMA 3.6.3. *Let X be any translation surface and γ any saddle connection for X. If for some instant $t \geq 0$ we have $|\gamma|_{X,t} < 1$ then*

$$t \geq \log|\gamma|_{X,0}. \tag{3.27}$$

Proof: The quantity

$$|\gamma|^2_{X,t}\frac{\tan \mathrm{angle}_{X,t}(\gamma,\partial_y)}{1+\tan^2 \mathrm{angle}_{X,t}(\gamma,\partial_y)}$$

is constant in t, therefore putting it equal to its value for $t = 0$ we get

$$\tan \mathrm{angle}_{X,0}(\gamma,\partial_y) = \frac{|\gamma|^2_{X,t}}{|\gamma|^2_{X,0}}\left(\frac{1+\tan^2 \mathrm{angle}_{X,0}(\gamma,\partial_y)}{1+\tan^2 \mathrm{angle}_{X,t}(\gamma,\partial_y)}\right)\tan \mathrm{angle}_{X,t}(\gamma,\partial_y). \tag{3.28}$$

We recall that $\tan \mathrm{angle}_{X,t}(\gamma,\partial_y) = e^{2t}\tan \mathrm{angle}_{X,0}(\gamma,\partial_y)$ and that for $t \geq 0$ we have $\mathrm{angle}_{X,t}(\gamma,\partial_y) \geq \mathrm{angle}_{X,0}(\gamma,\partial_y)$, therefore it follows that

$$e^{-2t} \leq \frac{|\gamma|^2_{X,t}}{|\gamma|^2_{X,0}}$$

and taking the logarithm we get $t \geq \log|\gamma|_{X,0} - \log|\gamma|_{X,t}$. Since by assumption we have $|\gamma|_{X,t} \leq 1$ then the lemma follows. □

LEMMA 3.6.4. *Let X be any element in $\mathcal{H}(k_1,..,k_r)$. Let γ_n with $n \in \mathbb{N}$ be a family of saddle connections for X such that*

$$\mathrm{angle}_{X,0}(\gamma_n,\partial_y) \to 0 \tag{3.29}$$

as $n \to \infty$. If $\tau_n \in \mathbb{R}$ is the instant associated to γ_n by equation (3.24) for any $n \in \mathbb{N}$, then we have

$$\lim_{n\to\infty}\frac{e^{2\tau_n}|\gamma_n|^2_{X,\tau_n}}{|\gamma_n|^2_{X,0}} = 2. \tag{3.30}$$

Proof: For any $n \in \mathbb{N}$ we call $\theta_n := \text{angle}_{X,0}(\gamma_n, \partial_y)$. Any saddle connection γ_n is the diagonal of some open rectangle R_n immersed into X with sides parallel to the vertical and horizontal directions. We have

$$\text{Area}(R_n) = |\gamma_n|^2 \cos\theta_n |\sin\theta_n| = |\gamma_n|^2 \frac{|\tan\theta_n|}{1+\tan^2\theta_n}.$$

Equation (3.29) says that $\tan\theta_n \to 0$ as $n \to \infty$ and therefore

$$\frac{\text{Area}(R_n)}{|\gamma_n|^2 |\tan\theta_n|} \to 1$$

as $n \to \infty$. Equation (3.25) in lemma 3.6.1 implies that for any $n \in \mathbb{N}$ and for the instants τ_n defined in equation (3.25) we have $|\tan\theta_n| = e^{-2\tau_n}$, therefore

$$\text{Area}(R_n)|\gamma_n|^{-2} e^{2\tau_n} \to 1$$

as $n \to \infty$. Finally we observe that $\text{Area}(R_n)$ is invariant under the Teichmüller flow. In particular for any $n \in \mathbb{N}$, at the instant τ_n associated to γ_n by equation (3.25) we have $|\gamma_n|^2_{X,\tau_n} = 2\text{Area}(R_n)$ and the lemma follows. □

3.6.2. Proof of theorem 1.3.1. Here we prove theorem 1.3.1. We recall the statement:

If $\int_0^\infty \psi(t)dt < +\infty$ *then for almost any* $\hat{X} \in \widehat{\mathcal{H}}^{(1)}(k_1, .., k_r)$ *the condition in equation (1.6) holds, that is we have*

$$\lim_{t\to\infty} \frac{\texttt{Sys}(\mathcal{F}_t\hat{X})}{\sqrt{\psi(t)}} = \infty.$$

If $\int_0^\infty \psi(t)dt = +\infty$ *then for any pair of points* $p_i, p_j \in \Sigma$*, any* $l \in \{0, .., k_i\}$*, any* $m \in \{0, .., k_j\}$ *and for almost any* $\hat{X} \in \widehat{\mathcal{H}}^{(1)}(k_1, .., k_r)$ *the condition in equation (1.7) holds, that is we have:*

$$\liminf_{t\to\infty} \frac{\texttt{Sys}^{(p_i,p_j,l,m)}(\mathcal{F}_t\hat{X})}{\sqrt{\psi(t)}} = 0.$$

Before starting the proof we recall that in lemma 3.1.3 in paragraph 3.1.2 we proved that for $\hat{X} \in \widehat{\mathcal{H}}^{(1)}(k_1, .., k_r)$ the configurations $\mathcal{C}^{(p_i,p_j,l,m)}(\hat{X})$ are invariant for the Teichmüller flow $\mathcal{F}_t$. The following lemma is useful.

Lemma 3.6.5. *Let* $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ *be a positive function such that* $t\varphi(t)$ *is decreasing monotone and associate to it the function* $\hat{\varphi} : \mathbb{R} \to \mathbb{R}_+$ *defined by*

$$\hat{\varphi}(s) := e^s \varphi(e^s).$$

Then $\hat{\varphi}$ *is decreasing monotone on* $\mathbb{R}$*. On the other hand, for any decreasing monotone positive function* $\psi : \mathbb{R} \to \mathbb{R}_+$ *there exists a unique positive function* $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ *with* $t\varphi(t)$ *decreasing monotone such that* $\psi = \hat{\varphi}$*. Such* φ *is given by*

$$\varphi(t) = \psi(\log t)/t.$$

Finally we have

$$\int_0^\infty \psi(t)dt = \int_1^\infty \varphi(s)ds. \tag{3.31}$$

Proof: The proof is a simple exercise in calculus. □

Proof: (Of theorem 1.3.1)

We first consider the case $\int_0^\infty \psi(t)dt < +\infty$. We show that for almost any $\hat{X} \in \widehat{\mathcal{H}}^{(1)}(k_1, .., k_r)$ we have

$$\liminf_{t\to\infty} \frac{\mathtt{Sys}(\mathcal{F}_t\hat{X})}{\sqrt{\psi(t)}} > 1. \tag{3.32}$$

Before proving the claim we observe that for any positive constant $C > 1$ the function $C^2\psi$ still has finite integral, thus we can substitute ψ with $C^2\psi$ in the preceding result and get that the the lim inf above is greater than C. We conclude that the lim inf in infinite, therefore the limit exists and is infinite too and the first part of the theorem follows from the claim. Now we prove the claim. Let us suppose that there exists a positive measure subset $\mathcal{S}$ of $\widehat{\mathcal{H}}^{(1)}(k_1, .., k_r)$ such that for any $\hat{X} \in \mathcal{S}$, in equation (3.32) we have

$$\liminf_{t\to\infty} \frac{\mathtt{Sys}(\mathcal{F}_tX)}{\sqrt{\psi(t)}} \leq 1.$$

Let us fix any $\hat{X} \in \mathcal{S}$. There exists a sequence of instants $t_1 < t_2 < .. < t_n < ...$ with $t_n \to +\infty$ and such that $\mathtt{Sys}(\mathcal{F}_{t_n}X) \leq \sqrt{\psi(t_n)}$. Let γ_n be a sequence of saddle connections for the fixed X such that for every $n \in \mathbb{N}$ we have $\mathtt{Sys}(\mathcal{F}_{t_n}X) = |\gamma_n|_{X,t_n}$. On any translation surface there are just finitely many saddle connections with length smaller than some fixed bound, thus the initial length $|\gamma_n|_{X,0}$ of the saddle connection γ_n satisfies $|\gamma_n|_{X,0} \to \infty$ as $n \to \infty$ and therefore $\text{angle}_{X,0}(\gamma_n, \partial_y) \to 0$ as $n \to \infty$. Let us fix any $\epsilon > 0$. Equation (3.28) in lemma 3.6.3 implies that for any n big enough we have

$$|\tan \text{angle}_{X,0}(\gamma_n, \partial_y)| \leq (1+\epsilon)\frac{|\gamma_n|^2_{X,t_n}}{|\gamma_n|^2_{X,0}}.$$

We recall that by definition $|\gamma_n|_{X,t_n} = \mathtt{Sys}(\mathcal{F}_{t_n}X) \leq \sqrt{\psi(t_n)}$. In particular, since $\psi(t) \to 0$ as $t \to \infty$ (the integral is finite), lemma 3.6.3 implies that $t_n \geq \log |\gamma_n|_{X,0}$ and since ψ is decreasing monotone we get

$$|\tan \text{angle}_{X,0}(\gamma_n, \partial_y)| \leq (1+\epsilon)\frac{\psi(\log |\gamma_n|_{X,0})}{|\gamma_n|^2_{X,0}}.$$

Now we consider the function $\varphi(s) = (1+\epsilon)\psi(\log s)/s$. Lemma 3.6.5 implies that $\int_0^\infty \varphi(s)ds < +\infty$, but on the other hand the last condition implies that for any $X \in \mathcal{S}$ and for all n big enough the saddle connection γ_n for X defined above satisfies

$$|\tan \text{angle}(\gamma_n, \partial_y)| < \frac{\varphi(|\gamma_n|)}{|\gamma_n|}.$$

Part a) of theorem 1.2.3 implies that we have an absurd, therefore the claim is proved.

Now we consider the case $\int_0^\infty \psi(t)dt = +\infty$. Let us fix any pair of points $p_i, p_j \in \Sigma$ and any pair of indexes l, m with $l \in \{0, .., k_i\}$ and $m \in \{0, .., k_j\}$. We show that for almost any $\hat{X} \in \widehat{\mathcal{H}}^{(1)}(k_1, .., k_r)$ we have

$$\liminf_{t \to \infty} \frac{\mathtt{Sys}^{(p_i, p_j, l, m)}(\mathcal{F}_t \hat{X})}{\sqrt{\psi(t)}} < 1. \tag{3.33}$$

The second part of the theorem follows from the claim, just observe that for any positive constant $\epsilon > 0$ the function $\epsilon^2 \psi$ still has divergent integral and therefore substituting ψ with $\epsilon^2 \psi$ we get that the lim inf above is less than ϵ. Here we show the claim. Let us consider the function $\varphi(t) := \psi(\log t)/t$. Lemma 3.6.5 implies that we have $\int_0^\infty \varphi(t) dt = \infty$, therefore part b) of theorem 1.2.3 applies. We have that there exists a full lebesgue measure set $\mathcal{S}$ in $\widehat{\mathcal{H}}^{(1)}(k_1, .., k_r)$ such that for any $\hat{X} \in \mathcal{S}$ the configuration $\mathcal{C}^{(p_i, p_j, l, m)}(\hat{X})$ contains infinitely many saddle connections γ_n such that

$$|\tan \text{angle}_{X,0}(\gamma_n, \partial_y)| \leq \frac{\varphi(|\gamma_n|_{X,0})}{|\gamma_n|_{X,0}}.$$

For any such γ_n let $\tau_n = \tau_n(\hat{X}, \gamma_n)$ be the instant associated to γ_n by equation (3.24) in lemma 3.6.1. Let us fix any $\epsilon > 0$. Since $\text{angle}_{X,0}(\gamma_n, \partial_y) \to 0$ lemma 3.6.4 implies that

$$|\gamma_n|^2_{X,\tau_n} < 2(1 + \epsilon)|\gamma_n|^2_{X,0} e^{-2\tau_n},$$

therefore equation (3.25) implies that

$$|\gamma_n|^2_{X,\tau_n} < 2(1 + \epsilon)|\gamma_n|_{X,0} \varphi(|\gamma_n|_{X,0}).$$

Lemma 3.6.2 implies that $e^{\tau_n/(1+\epsilon)} \leq |\gamma_n|_{X,0}$ for almost any $\hat{X} \in \widehat{\mathcal{H}}^{(1)}(k_1, .., k_r)$ and of course we do not lose in generality if we assume that our fixed $\hat{X} \in \mathcal{S}$ satisfies this supplementary condition. Since $t\varphi(t)$ is decreasing monotone we have

$$|\gamma_n|_{X,0} \varphi(|\gamma_n|_{X,0}) \leq e^{\tau_n/(1+\epsilon)} \varphi(e^{\tau_n/(1+\epsilon)})$$

and therefore recalling that for any t we have $\psi(t) = e^t \varphi(e^t)$ we get

$$|\gamma_n|^2_{X,\tau_n} < 2(1 + \epsilon) \psi(\frac{\tau_n}{1 + \epsilon}).$$

The condition on ψ is of course equivalent to $2(1 + \epsilon) \int_0^\infty \psi(t/(1 + \epsilon)) dt = \infty$, therefore modulo changing the initial function $\psi(t)$ with $2(1+\epsilon)\psi(t/(1+\epsilon))$, we get $|\gamma_n|^2_{X,\tau_n} < \psi(\tau_n)$ for any n. Since the Teichmüller flow preserves the configurations acting on $\widehat{\mathcal{H}}^{(1)}(k_1, .., k_r)$ we have that for any τ_n the saddle connection γ_n is in $\mathcal{C}^{(p_i, p_j, l, m)}(\mathcal{F}_{\tau_n} \hat{X})$, therefore

$$\mathtt{Sys}^{(p_i, p_j, l, m)}(\mathcal{F}_{\tau_n} \hat{X}) \leq |\gamma_n|_{X,\tau_n} < \sqrt{\psi(\tau_n)}.$$

Finally, it is evident that $\tau_n \to \infty$ therefore the statement follows. $\square$

CHAPTER 4

Divergent Case, arbitrary genus

Let us fix any Rauzy class $\mathcal{R}$ on an alphabet $\mathcal{A}$ with $d \geq 2$ letters. For any $\pi \in \mathcal{R}$ we consider the pairs of letters (β, α) such that $\pi^b(\beta) > 1$ and $\pi^t(\alpha) > 1$. It is easy to note that for π varying in $\mathcal{R}$ the first letter respectively in the top line and in the bottom line of π are always the same, therefore the set of pairs (β, α) that satisfy the property above depends only from the Rauzy class $\mathcal{R}$ and not from the combinatorial datum. In what follows we keep writing the condition above as $\pi^b(\beta) > 1$ and $\pi^t(\alpha) > 1$, even when just a Rauzy class is fixed but no combinatorial datum $\pi \in \mathcal{R}$ is specified. This chapter is entirely devoted to the proof of the following:

Proposition 4.0.6. *Let us consider a positive sequence $\varphi : \mathbb{N} \to \mathbb{R}_+$ such that $n\varphi(n)$ is monotone decreasing and $\sum_{n=1}^{\infty} \varphi(n) = \infty$. Then for and any pair of letters (β, α) such that $\pi^b(\beta) > 1$ and $\pi^t(\alpha) > 1$ and for almost any i.e.t. $T \in \Delta^{(1)}(\mathcal{R})$ there exist infinitely many triples (β, α, n) reduced for T that are solution of equation (1.3), that is such that*

$$\left|T^n(u^b_\beta) - u^t_\alpha\right| < \varphi(n).$$

Proposition 3.3.1 in paragraph 3.3.1 implies that part **b)** of theorem 1.1.6 then follows.

4.1. Reduction to a shrinking target property.

In this section we consider pairs of letters $(\beta, \alpha) \in \mathcal{A}^2$ with $\pi^t(\alpha), \pi^b(\beta) > 1$ for all $\pi \in \mathcal{R}$ and we introduce two properties of these pairs (properties **A** or **B** in definition 4.1.1). In paragraph 4.2 we prove a combinatorial proposition (proposition 4.2.1) on Rauzy classes which implies that any pair (β, α) as above satisfies at least one of the two properties. In this paragraph, for a pair of letters (β, α) that satisfies one of the two properties, we give a sufficient condition to get the result in proposition 4.0.6 for the pair (β, α). The sufficient condition that we give is stated in proposition 4.1.12 and is a *shrinking target property* for the Rauzy-Veech algorithm.

Definition 4.1.1. Let $(\beta, \alpha) \in \mathcal{A}^2$ be a pair of letters with $\pi^t(\alpha), \pi^b(\beta) > 1$ for all $\pi \in \mathcal{R}$.

- (β, α) has *property A* if there exists some $\pi = \pi(\beta, \alpha) \in \mathcal{R}$ such that

$$\pi^t(\alpha) = \pi^b(\beta) = d \tag{4.1}$$

  that is we have

$$\pi = \begin{pmatrix} X & \dots & \alpha \\ Y & \dots & \beta \end{pmatrix}.$$

- (β, α) has *property B* if there exists some $\pi = \pi(\beta, \alpha) \in \mathcal{R}$ and some letter $V \in \mathcal{A}$ such that

$$\begin{aligned} &\{x \in \mathcal{A}\,;\, \pi^t(x) < \pi^t(\alpha)\} \cup \{V\} = \{y \in \mathcal{A}\,;\, \pi^b(y) < \pi^b(\beta)\} \\ &\pi^t(V) = \pi^b(\alpha) = d. \end{aligned} \tag{4.2}$$

When (β, α) has *property B*, for an element $\pi \in \mathcal{R}$ that satisfies equation (4.16), let us introduce the letter $L \in \mathcal{A}$ such that $\pi^b(L) = \pi^b(\beta) - 1$ and $\pi^t(L) < \pi^t(\alpha)$ (observe that it has to be $L \neq V$ since π is irreducible). We have

$$\pi = \begin{pmatrix} \dots & L & \dots & \alpha & \dots & & V \\ \dots & V & \dots & L & \beta & \dots & \alpha \end{pmatrix}.$$

4.1.1. Two fundamental lemmas. Here we look at the non-normalized version of the Rauzy-Veech algorithm:

$$Q : \Delta(\mathcal{R}) \to \Delta(\mathcal{R})$$

For any $T = (\pi, \lambda) \in \Delta(\mathcal{R})$ and for any $k \in \mathbb{N}$ we write $Q^k(T) = (\pi^{(k)}, \lambda^{(k)})$ for its k-th iterated. If T undergoes k iterations of Q it generates a path in the Rauzy diagram of length k, that we denote $\gamma(k)$.

Lemma 4.1.2. *If the pair (β, α) with $\pi^t(\alpha), \pi^b(\beta) > 1$ has property A (that is equation (4.1) holds) then there exists a finite path η in the Rauzy diagram such that the following holds. The element $\pi(\beta, \alpha)$ that satisfies equation (4.1) is in third to last position in η. Moreover for any Rauzy path $\gamma(k)$ ending with η and any $T = (\pi, \lambda) \in \Delta_{\gamma(k)}$ we have an integer $n = n(\gamma(k))$ with $n \leq \|q^{\gamma(k)}\|$ such that*

$$\lambda^{(k)}_\alpha = |T^n u^b_\beta - u^t_\alpha| \tag{4.3}$$

and the triple (n, β, α) is reduced for T.

Proof: We first recall a general fact. For any i.e.t. $T : I \to I$ without connections let us call $\nu = \nu(T)$ the half-infinite Rauzy path generated by T and let us consider the finite subpath $\nu(i)$ of $\nu(T)$ truncated at time i. To simplify notation we write $q^{(i)}$ instead of $q^{\nu(i)}$, where for any finite Rauzy path γ the vector q^γ is defined in paragraph 2.1.2. Let us call $T^{(i)} : I^{(i)} \to I^{(i)}$ the corresponding i.e.t. and $(\pi^{(i)}, \lambda^{(i)})$ the corresponding combinatorial and length data (not normalized). For any $x \in \mathcal{A}$ we also define $u^{(i),t}_x$ and $u^{(i),b}_x$ respectively the singularity for $T^{(i)}$ at position $\pi^{(i),t}(x)$ and the singularity for $T^{(i)-1}$ at position $\pi^{(i),b}(x)$. For any pair of letters $\beta, \alpha \in \mathcal{A}^2$ there exist two nonnegative integers $l(i, \beta)$ and $h(i, \alpha)$ that depends only from $\nu(i)$ such that:

$$u^{(i),b}_\beta = T^{l(i,\beta)} u^b_\beta, \;\; u^{(i),t}_\alpha = T^{-h(i,\alpha)} u^t_\alpha.$$

For any $i \in \mathbb{N}$ and any $x \in \mathcal{A}$ we also define $I^{(i),t}_x$ as the interval in the domain of $T^{(i)}$ at position $\pi^{(i),t}(x)$ and $I^{(i),b}_x$ as the interval in the domain of $(T^{(i)})^{-1}$ at position $\pi^{(i),b}(x)$. It is easy to see that the interval $I^{(i),t}_x$ undergoes exactly $q^{(i)}_x = l(i, x) + h(i, x) + 1$ iterations under the map $T = T^{(0)}$ before coming back to $I^{(i)}$, therefore it is clear that for any pair $\alpha, \beta \in \mathcal{A}$ we have:

$$l(i, \beta) + h(i, \alpha) \leq \|q^{\nu(i)}\|. \tag{4.4}$$

Since the pair (β, α) has property A then there exists an element $\pi = \pi(\beta, \alpha) \in \mathcal{R}$ such that $\pi^t(\alpha) = \pi^b(\beta) = d$. Let us consider the top arrow $\gamma^t_\alpha : \pi \to \pi_{int}$ with winner α starting from π and the bottom arrow $\gamma^b_W : \pi_{int} \to \pi_{end}$ starting from π_{int} with winner the last letter W in the bottom line of π_{int}. These two arrow can be concatenated. Let us consider any Rauzy path $\eta : \pi_{start} :\to \pi_{end}$ ending with their concatenation $\gamma^t_\alpha \gamma^b_W$. ($\pi(\beta,\alpha)$ is therefore in third to last position in η).

Let us consider a length k path $\gamma(k)$ in the Rauzy diagram as in the hypothesis. Since $\gamma(k)$ ends with η we have $(\pi^{(k-2)}, \lambda^{(k-2)}) \in \Delta_{\gamma^t_\alpha \gamma^b_W} \subset \Delta_{\pi(\beta,\alpha)}$, therefore the next step $(\pi^{(k-2)}, \lambda^{(k-2)}) \mapsto (\pi^{(k-1)}, \lambda^{(k-1)})$ of the algorithm Q is given by the top arrow γ^t_α and we get $\lambda^{k-1}_\alpha = |u^{(k-2),b}_\beta - u^{(k-2),t}_\alpha|$, that is

$$\lambda^{(k-1)}_\alpha = |T^{l(k-2,\beta)} u^b_\beta - T^{-h(k-2,\alpha)} u^t_\alpha|.$$

For the same reason we have $u^{(k-2),b}_\beta \in I^{(k-2),t}_\alpha$, therefore

$$|u^{(k-2),b}_\beta - u^{(k-2),t}_\alpha| = |T^i u^{(k-2),b}_\beta - T^i u^{(k-2),t}_\alpha|$$

for all $i \in \{0, .., q^{(k-2)}_\alpha\}$ (we recall that $q^{(k-2)}_\alpha$ is the first return time into $I^{(k-2)}$ of the interval $I^{(k-2),t}_\alpha$ under iterations of T). Since $h(k-2,\alpha) < q^{(k-2)}_\alpha$, if we associate to the path $\gamma(k)$ the integer $n = n(\gamma(k-2)) := l(k-2,\beta) + h(k-2,\alpha)$ we have $\lambda^{(k-1)}_\alpha = |T^n u^b_\beta - u^t_\alpha|$. Since γ^b_W is the last arrow in η, and therefore in $\gamma(k)$, and the loser in the arrow γ^b_W is the letter α, then the length $\lambda^{(k-1)}_\alpha$ keeps unchanged at this step and we get

$$\lambda^{(k)}_\alpha = |T^n u^b_\beta - u^t_\alpha|.$$

Thanks to equation (4.4) we have $n(\gamma(k)) \le \|q^{\gamma(k-2)}\|$. Finally since $(\pi^{(k-2)}, \lambda^{(k-2)}) \in \Delta_{\gamma^b_W} \subset \Delta_{\pi_{int}}$ then $u^{(k-2),b}_\beta \in I^{(k-2),t}_\alpha$ and $u^{(k-2),t}_\alpha \in I^{(k-2),b}_W$, that implies that the interval $(u^{(k-2),t}_\alpha, u^{(k-2),b}_\beta)$ may be iterated $l(k-2,\beta)$ times in the past and $h(k-2,\alpha)$ times in the future without meeting any other singularity both for T and T^{-1}, that is $(n = l(k-2,\beta) + h(k-2,\alpha), \beta, \alpha)$ is a reduced triple for T. The lemma is proved. □

LEMMA 4.1.3. *If the pair (β,α) with $\pi^t(\alpha), \pi^b(\beta) > 1$ has property B (that is equation (4.2) holds) then there exists a finite path η in the Rauzy diagram such that the following holds. The element $\pi(\beta,\alpha)$ that satisfies equation (4.2) is in second to last position in η and the last arrow of η is of type bottom with winner α. Moreover for any Rauzy path $\gamma(k)$ ending with η and any $T = (\pi,\lambda) \in \Delta_{\gamma(k)}$ we have an integer $n = n(\gamma(k))$ with $n \le \|q^{\gamma(k)}\|$ such that*

$$\lambda^{(k)}_V = |T^n u^b_\beta - u^t_\alpha|. \tag{4.5}$$

Furthermore if $\lambda^{(k)}_V < \lambda^{(k)}_L$ then the triple (n,β,α) is reduced for T (the letters V and L are the ones defined by equation (4.16)).

Proof: Since the pair (β,α) has property B then there exists an element $\pi = \pi(\beta,\alpha) \in \mathcal{R}$ such that

$$\{x \in \mathcal{A}\,;\, \pi^t(x) < \pi^t(\alpha)\} \cup \{V\} = \{y \in \mathcal{A}\,;\, \pi^b_y < \pi^b(\beta)\}$$
$$\pi^t(V) = \pi^b(\alpha) = d.$$

Let us consider the bottom arrow $\gamma^b_\alpha : \pi \to \pi_{end}$ with winner α starting from π and pick as $\eta : \pi_{start} \to \pi_{end}$ any Rauzy path ending with γ^b_α (thus $\pi(\beta,\alpha)$ is in second to last position in η).

Since $(\pi^{(k-1)}, \lambda^{(k-1)}) \in \Delta_\pi$ then the combinatory of π implies that $\lambda_V^{(k-1)} = |u_\beta^{(k-1),b} - u_\alpha^{(k-1),t}|$ (the notation is the same as in lemma 4.1.2), that is

$$\lambda_V^{(k-1)} = |T^{l(k-1,\beta)} u_\beta^b - T^{-h(k-1,\alpha)} u_\alpha^t|.$$

Since in fact $(\pi^{(k-1)}, \lambda^{(k-1)}) \in \Delta_{\gamma_\alpha^b} \subset \Delta_\pi$ then $\lambda_V^{(k-1)} < \lambda_\alpha^{(k-1)}$, that is $u_\beta^{(k-1),b} \in I_\alpha^{(k-1),t}$, therefore

$$|T^i u_\beta^{(k-1),b} - T^i u_\alpha^{(k-1),t}| = |u_\beta^{(k-1),b} - u_\alpha^{(k-1),t}|$$

for all $i \in \{0, .., q_\alpha^{(k-1)}\}$, where $q_\alpha^{(k-1)}$ is the first return time to $I^{(k-1)}$ of the interval $I_\alpha^{(k-1),t}$ under iterations of T. Since $h(k-1,\alpha) < q_\alpha^{(k-1)}$, if we associate to the path $\gamma(k)$ the integer $n = n(\gamma(k-1)) := l(k-1,\beta) + h(k-1,\alpha)$ we have $\lambda_\alpha^{(k-1)} = |T^n u_\beta^b - u_\alpha^t|$. Since γ_α^b is the last arrow in η, and therefore in $\gamma(k)$, and the loser of γ_α^b is the letter V, then the length $\lambda_V^{(k-1)}$ keeps unchanged at this step and we get

$$\lambda_V^{(k)} = |T^n u_\beta^b - u_\alpha^t|.$$

Thanks to equation (4.4) we have $n(\gamma(k)) \leq \|q^{\gamma(k-2)}\|$ and the first part of the lemma is proved. To prove the second part just observe that $\lambda_V^{(k)} < \lambda_L^{(k)}$ is equivalent to $\lambda_V^{(k-1)} < \lambda_L^{(k-1)}$ (since the last arrow in $\gamma(k)$ is γ_α^b) and this condition implies that $u_\alpha^{(k-1),t} \in I_L^{(k-1),b}$. Together with $u_\beta^{(k-1),b} \in I_\alpha^{(k-1),t}$ the former condition implies that the interval $(u_\alpha^{(k-1),t}, u_\beta^{(k-1),b})$ may be iterated $l(k-1,\beta)$ times in the past and $h(k-1,\alpha)$ times in the future without meeting any other singularity both for T and T^{-1}, that is $(n = l(k-1,\beta) + h(k-1,\alpha), \beta, \alpha)$ is a reduced triple for T. The lemma is proved. □

4.1.2. First return to a pre-reference path. From now on we will always consider the normalized version of the Rauzy-Veech algorithm, that we denote:

$$\mathbb{P}Q : \Delta^{(1)}(\mathcal{R}) \to \Delta^{(1)}(\mathcal{R}).$$

Let us fix any Rauzy path $\eta : \pi_{start} \to \pi_{end}$ starting at π_{start} and ending at π_{end} and consider the sub-simplex $\Delta_\eta^{(1)}$ of $\Delta_{\pi_{start}}^{(1)}$. According to the discussion at the end of paragraph 2.1.4, the first return map $R_\eta : \Delta_\eta^{(1)} \to \Delta_\eta^{(1)}$ of the Rauzy-Veech algorithm to the sub-simplex $\Delta_\eta^{(1)}$ is defined almost everywhere. The connected components of the domain of R_η are exactly the sub-simplices $\Delta_\gamma^{(1)}$ of $\Delta_\eta^{(1)}$ with $\gamma \in \Gamma_\eta$, where Γ_η is the set of Rauzy paths $\gamma : \pi_{start} \to \pi_{end}$ that admit two decompositions

$$\gamma = \gamma'\eta \texttt{ and } \gamma = \eta\gamma'' \texttt{ for } \gamma', \gamma'' \in \Pi(\mathcal{R})$$

and that are minimal with this property with respect to the ordering $\prec$ defined by equation (2.8) in paragraph 2.1.4. We also recall that for any such $\gamma \in \Gamma_\eta$, written according to the decomposition $\gamma = \gamma'\eta$ as above, and for any $T = (\pi_{start}, \lambda) \in \Delta_\gamma^{(1)}$ we have

$$R_\eta(T) = (\pi_{start}, \frac{{}^tB_{\gamma'}^{-1}\lambda}{\|{}^tB_{\gamma'}^{-1}\lambda\|}).$$

Definition 4.1.4. We say that a finite Rauzy path η is *neat* if any time that we can write $\eta = \eta_1\eta_2 = \eta_3\eta_1$ either $\eta = \eta_1$ or η_1 is trivial.

LEMMA 4.1.5. *The Rauzy path η is neat if and only if any $\gamma \in \Gamma_\eta$ is of the form*

$$\gamma = \eta\gamma_0\eta \tag{4.6}$$

where γ_0 does not contain η.

Proof: Let us first suppose that η is not neat, which means that there exist three non trivial paths η_1, η_2, η_3 such that $\eta = \eta_1\eta_2 = \eta_2\eta_3$. The concatenation $\eta_1\eta_2\eta_3$ begins with η, then for $\lambda \in \Delta^{(1)}_{\eta_1\eta_2\eta_3}$ we have ${}^tB^{-1}_{\eta_3}\lambda/\|{}^tB^{-1}_{\eta_3}\lambda\| \in \Delta^{(1)}_\eta$. Without loosing in generality η_3 may be taken minimal, therefore $\lambda \mapsto^t B^{-1}_{\eta_3}\lambda/\|{}^tB^{-1}_{\eta_3}\lambda\|$ is a branch of the first return map R_η to $\Delta^{(1)}_\eta$ and $\eta_1\eta_2\eta_3 \in \Gamma_\eta$, which is absurd.

On the other hand for any η, any path in Γ_η begins and ends with η, therefore if there are paths different from those in equation (4.6) then η is not neat. The lemma is proved. □

DEFINITION 4.1.6. Let (β, α) be a pair with $\pi^t(\alpha), \pi^b(\beta) > 1$ satisfying properties A or B. A *pre-reference path* for the pair (β, α) is a neat Rauzy path $\eta : \pi_{start} \to \pi_{end}$ (see definition 4.1.4) chosen according to lemma 4.1.2 (if the pair (β, α) has property A) or according to lemma 4.1.3 (if the pair (β, α) has property B).

REMARK 4.1.7. Lemmas 4.1.2 and 4.1.3 just specify the ending part (the last arrow or the last two) of the path η that they provide, whereas they leave complete freedom in the choice of its beginning. This make possible to have a neat path η compatible with the prescription given in both the two lemmas. Later on (definition 4.7.5) we will need η to satisfy more properties and we will fix more restrictions on its beginning.

Let us fix a pair (β, α) with $\pi^t(\alpha), \pi^b(\beta) > 1$ for all $\pi \in \mathcal{R}$ and suppose that it satisfies property A or B. Then let us consider a pre-reference path $\eta : \pi_{start} \to \pi_{end}$ for the pair (β, α) as in definition 4.1.6.

Let us consider the sub-simplex $\Delta^{(1)}_\eta$ of $\Delta^{(1)}_{\pi_{start}}$ and the first entering map of the Rauzy algorithm into $\Delta^{(1)}_\eta$, that is the map $\tilde{R}_\eta : \Delta^{(1)}(\mathcal{R}) \to \Delta^{(1)}_\eta$ defined by $\tilde{R}_\eta(\pi, \lambda) := \mathbb{P}Q^{n(\pi,\lambda)}(\pi, \lambda)$, where

$$n(\pi, \lambda) := \min\{k \in \mathbb{N}^*; \mathbb{P}Q^k(\pi, \lambda) \in \Delta^{(1)}_\eta\}.$$

The map $\tilde{R}_\eta$ is defined almost everywhere on the whole set $\Delta^{(1)}(\mathcal{R})$ and has image $\Delta^{(1)}_\eta$. Restricted to the sub-simplex $\Delta^{(1)}_\eta \subset \Delta^{(1)}_{\pi_{start}}$ it coincides with the first return map R_η defined in paragraph 2.1.4. Let Γ^η be the set of Rauzy paths $\gamma : \pi \to \pi_{end}$ starting at any $\pi \in \mathcal{R}$ and ending in π_{end} which end with η and are minimal with this property with respect to the ordering $\prec$ introduced in equation (2.8) in paragraph 2.1.4. In other words the elements of Γ^η are the paths γ such that there exists a sub-path $\gamma' \in \Pi(\mathcal{R})$ such that we can write

$$\gamma = \gamma'\eta \tag{4.7}$$

and no proper sub-path ν with $\nu \prec \gamma$ has the decomposition in equation (4.7). The connected components of the domain of $\tilde{R}_\eta$ are exactly the simplices $\Delta^{(1)}_\gamma$ in

$\Delta^{(1)}(\mathcal{R})$, with $\gamma \in \Gamma^\eta$. For any such γ, written according to the decomposition $\gamma = \gamma'\eta$ as above, and for any $T = (\pi, \lambda) \in \Delta^{(1)}_\gamma$ we have

$$\tilde{R}_\eta(\pi, \lambda) = (\pi_{start}, \frac{{}^tB^{-1}_{\gamma'}\lambda}{\|{}^tB^{-1}_{\gamma'}\lambda\|}). \tag{4.8}$$

We consider the homeomorphism $\mathbb{P}Q_\eta : \Delta^{(1)}_\eta \to \Delta^{(1)}_{\pi_{end}}$ defined by

$$\mathbb{P}Q_\eta(\pi_{start}, \lambda) = (\pi_{end}, \frac{{}^tB^{-1}_{\eta}\lambda}{\|{}^tB^{-1}_{\eta}\lambda\|}).$$

Definition 4.1.8. Let (β, α) be a pair of letters satisfying property A or B and let η be a pre-reference path associated to (β, α) as in definition 4.1.4. We introduce the map $\mathcal{F}_\eta : \Delta^{(1)}(\mathcal{R}) \to \Delta^{(1)}_{\pi_{end}}$ defined by

$$\mathcal{F}_\eta(\pi, \lambda) := \mathbb{P}Q_\eta \circ \tilde{R}_\eta(\pi, \lambda). \tag{4.9}$$

The connected components of the domain of $\mathcal{F}_\eta$ are the same as those of the map $\tilde{R}_\eta$, that is they are the simplices $\Delta^{(1)}_\gamma$ with $\gamma \in \Gamma^\eta$. For any path $\gamma : \pi \to \pi_{end}$ in Γ^η we have

$$(\pi, \lambda) \in \Delta^{(1)}_\gamma \Leftrightarrow \mathcal{F}_\eta(\pi, \lambda) = (\pi_{end}, \frac{{}^tB^{-1}_{\gamma}\lambda}{\|{}^tB^{-1}_{\gamma}\lambda\|}).$$

Lemma 4.1.9. *For any $k \in \mathbb{N}$ the k-th iterated of the map $\mathcal{F}_\eta$ introduced in definition 4.1.8 satisfy:*

$$\mathcal{F}^k_\eta = \mathbb{P}Q_\eta \circ \tilde{R}^k_\eta. \tag{4.10}$$

Proof: We recall that the pre-refernece path η is neat by definition. For $k = 1$ the statement follows trivially from the definition of the map $\mathcal{F}_\eta$ (definition 4.1.8). Let us consider the first return map $R_\eta : \Delta^{(1)}_\eta \to \Delta^{(1)}_\eta$ of the Rauzy-Veech algorithm to the sub-simplex $\Delta^{(1)}_\eta$. For any $k > 1$ we have $\tilde{R}^k_\eta = R^{k-1}_\eta \circ \tilde{R}_\eta$. Let us call $N = N(\eta)$ the number of Rauzy arrows in η. Since η is neat, equation (4.6) in lemma 4.1.5 implies that for any $(\pi_{start}, \lambda) \in \Delta^{(1)}_\eta$ and any $i \in \{1, .., N-1\}$ we have that $\mathbb{P}Q^i(\pi_{start}, \lambda)$ is not in $\Delta^{(1)}_\eta$, therefore

$$R_\eta = \tilde{R}_\eta \circ \mathbb{P}Q_\eta.$$

This implies that for any $k > 1$ we have

$$\tilde{R}^k_\eta = R^{k-1}_\eta \circ \tilde{R}_\eta = \tilde{R}^{k-1}_\eta \circ \mathbb{P}Q_\eta \circ \tilde{R}_\eta = \tilde{R}^{k-1}_\eta \circ \mathcal{F}_\eta.$$

It follows that $\mathbb{P}Q_\eta \circ \tilde{R}^k_\eta = \mathbb{P}Q_\eta \circ \tilde{R}^{k-1}_\eta \circ \mathcal{F}_\eta$ and therefore equation (4.10) follows by induction over k. The lemma is proved. □

For any $k \in \mathbb{N}$ let us introduce the set $\Gamma^{(k),\eta}$ of those finite paths $\gamma : \pi \to \pi_{end}$ starting at any $\pi \in \mathcal{R}$ and ending in π_{end} that contain η exactly k times and are minimal with this property with respect to the ordering $\prec$ defined in equation (2.8) in paragraph 2.1.4 (we observe that by minimality all these paths end with η). The connected components of the k-th iterated $\mathcal{F}^k_\eta$ of $\mathcal{F}_\eta$ are exactly the simplices $\Delta^{(1)}_{\gamma_k}$ with $\gamma_k \in \Gamma^{(k),\eta}$. For any path $\gamma_k : \pi \to \pi_{end}$ in $\Gamma^{(k),\eta}$ we have

$$(\pi, \lambda) \in \Delta^{(1)}_{\gamma_k} \Leftrightarrow \mathcal{F}^k_\eta(\pi, \lambda) = (\pi_{end}, \frac{{}^tB_{\gamma_k}\lambda}{\|{}^tB_{\gamma_k}\lambda\|}).$$

Since $\mathcal{F}_\eta^{(k)}$ is defined almost everywhere, for all $k \in \mathbb{N}$ we have

$$\Delta^{(1)}(\mathcal{R}) = \bigsqcup_{\gamma_k \in \Gamma^{(k),\eta}} \Delta^{(1)}_{\gamma_k} \mod 0.$$

For any $(\pi,\lambda) \in \bigsqcup_{\gamma_k \in \Gamma^{(k),\eta}} \Delta^{(1)}_{\gamma_k}$ we denote with $\gamma_k(\pi,\lambda)$ the element in $\Gamma^{(k),\eta}$ such that $(\pi,\lambda) \in \Delta^{(1)}_{\gamma_k(\pi,\lambda)}$. We also denote with $r_k = r_k(\pi,\lambda)$ the Rauzy time of the path $\gamma_k(\pi,\lambda)$, that is the number of simple Rauzy arrows that compose $\gamma_k(\pi,\lambda)$. With this notation the k-th iterated of the map $\mathcal{F}_\eta$ (defined by equation (4.9)) on the element (π,λ) can be written as

$$\mathcal{F}_\eta^k(\pi,\lambda) = (\pi_{end}, \hat{\lambda}^{(r_k)}).$$

Lemma 4.1.9 has the following corollary:

Corollary 4.1.10. *For any $k \in \mathbb{N}$ and any (π,λ) in the domain of $\mathcal{F}_\eta^k$ we have*

$$\sharp\{i \in \mathbb{N}^*; 1 \le i < r_k(\pi,\lambda) \text{ and } \mathbb{P}Q^i(\pi,\lambda) \in \Delta^{(1)}_\eta\} = k. \tag{4.11}$$

Let us fix any other finite path ν in the Rauzy diagram starting at some element $\pi \in \mathcal{R}$. Let us denote $\Gamma_\nu^{(k),\eta}$ the set of those finite paths $\gamma_k : \pi \to \pi_{end}$ that start with ν, contain η exactly k times and are minimal with this property with respect to the ordering $\prec$ defined in equation (2.8) in paragraph 2.1.4 (we stress on the fact that by minimality all these paths end with η). The connected components of the restriction $\mathcal{F}_\eta^k|_{\Delta^{(1)}_\nu}$ of $\mathcal{F}_\eta^k$ to $\Delta^{(1)}_\nu$ are exactly the simplices $\Delta^{(1)}_{\gamma_k}$ with $\gamma_k \in \Gamma_\nu^{(k),\eta}$.

Lemma 4.1.11. *If the fixed path ν does not contain η, then for any $k \in \mathbb{N}$ the set $\Gamma_\nu^{(k),\eta}$ contains more than one element and we have a non-trivial partition*

$$\Delta^{(1)}_\nu = \bigsqcup_{\gamma_k \in \Gamma_\nu^{(k),\eta}} \Delta^{(1)}_{\gamma_k} \mod 0. \tag{4.12}$$

Proof: The proof is just a rephrasing of the definition of first entering map. □

Here we formulate our first sufficient criterion to get proposition 4.0.6. It is formulated as a (non-uniform) shrinking target property for the map $\mathcal{F}_\eta$ introduced in definition 4.1.8.

Proposition 4.1.12. *If the pair (β,α) has property A and η is given by lemma 4.1.2 then for any $T = (\pi,\lambda) \in \Delta^{(1)}(\mathcal{R})$, in order to have infinitely many triples (β,α,n) that are reduced for T and solutions of equation (1.3), it is sufficient to find infinitely many solutions $k \in \mathbb{N}$ of*

$$\hat{\lambda}^{(r_k)}_\alpha \le \frac{1}{\|\lambda^{(r_k)}\|}\varphi(\|q^{\gamma_k(\pi,\lambda)}\|). \tag{4.13}$$

If the pair (β,α) has property B and η is given by lemma 4.1.3 then for any $T = (\pi,\lambda) \in \Delta^{(1)}(\mathcal{R})$, in order to find infinitely many triples (β,α,n) that are reduced for T and solutions of equation (1.3), it is sufficient to find infinitely many solutions of

$$\hat{\lambda}^{(r_k)}_V \le \min\{\hat{\lambda}^{(r_k)}_L, \frac{1}{\|\lambda^{(r_k)}\|}\varphi(\|q^{\gamma_k(\pi,\lambda)}\|)\}. \tag{4.14}$$

Proof: Just observe that paths in $\Gamma^{(k),\eta}$ always end with the fixed pre-reference path η, therefore lemmas 4.1.2 or 4.1.3 apply. For any any $(\pi,\lambda)\in\Delta^{(1)}(\mathcal{R})$ and any $k\in\mathbb{N}$ the integer $n(\gamma_k(\pi,\lambda))$ associated to the path $\gamma_k(\pi,\lambda)\in\Gamma^{(k),\eta}$ by lemmas 4.1.2 and 4.1.3 satisfy $n(\gamma_k)\leq\|q^{\gamma_k}\|$. Since the sequence $n\varphi_n$ is decreasing monotone also φ_n is and we get $\varphi(\|q^{\gamma_k}\|)\leq\varphi(n(\gamma_k))$, therefore the condition in the statement is sufficient. The proposition is proved. □

4.2. A combinatorial property of Rauzy classes

In this section we prove that any pair $(\beta,\alpha)\in\mathcal{A}^2$ with $\pi^t(\alpha),\pi^b(\beta)>1$ satisfy either property A or property B in definition 4.1.1. For any irreducible Rauzy class $\mathcal{R}$ with alphabet $\mathcal{A}$ let us call $X=X(\mathcal{R})$ and $Y=Y(\mathcal{R})\in\mathcal{A}$ the two letters such that $\pi^t(X)=\pi^b(Y)=1$ for all $\pi\in\mathcal{R}$.

Proposition 4.2.1. *Let $\mathcal{R}$ be any Rauzy class with alphabet $\mathcal{A}$ and $(\beta,\alpha)\in\mathcal{A}^2$ be any ordered pair of letters with $\beta\neq Y$ and $\alpha\neq X$. Then at least one of the following two statements is true:*

a: *There exists an element $\pi\in\mathcal{R}$ such that*

$$\pi^t(\alpha)=\pi^b(\beta)=d \tag{4.15}$$

b: *There exist two (different) elements $\pi,\pi'\in\mathcal{R}$ and two letters $V,V'\in\mathcal{A}$ such that*

$$\begin{aligned}&\{x\in\mathcal{A}\,;\,\pi^t(x)<\pi^t(\alpha)\}\cup\{V\}=\{y\in\mathcal{A}\,;\,\pi^b(y)<\pi^b(\beta)\}\\&\pi^t(V)=\pi^b(\alpha)=d\end{aligned} \tag{4.16}$$

and

$$\begin{aligned}&\{x\in\mathcal{A}\,;\,\pi'^t(x)<\pi'^t(\alpha)\}=\{y\in\mathcal{A}\,;\,\pi'^b(y)<\pi'^b(\beta)\}\cup\{V'\}\\&\pi'^b(V')=\pi'^t(\beta)=d\end{aligned} \tag{4.17}$$

Note: Observe that the case **a** is compatible just with pair of different letters. In case **b**, when $\beta=\alpha$ equation (4.16) implies $\pi^t(\alpha)=d-1$ and $\pi^b(\alpha)=d$ and on the other hand equation (4.17) implies $\pi'^t(\alpha)=d$ and $\pi'^b(\alpha)=d-1$.

Corollary 4.2.2. *For any Rauzy class $\mathcal{R}$ on an alphabet $\mathcal{A}$, any pair $(\beta,\alpha)\in\mathcal{A}^2$ with $\pi^t(\alpha),\pi^b(\beta)>1$ satisfy either property A or property B in definition 4.1.1.*

Proof: (Of proposition 4.2.1.) The proof is by induction on the number of letters d. All Rauzy classes with $d\leq 4$ are easily computable and for these classes the assertion is just a matter of checking a small number of conditions. Therefore we consider a Rauzy class $\mathcal{R}$ on an alphabet $\mathcal{A}$ with $d\geq 5$ letters and suppose that the lemma is true for any Rauzy class $\mathcal{R}'$ on an alphabet $\mathcal{A}'$ with $d'<d$ letters.

Step 1. It is easy to see that there always exists a *standard* $\tilde{\pi}\in\mathcal{R}$, that is an element such that $\tilde{\pi}^t(X)=\tilde{\pi}^b(Y)=1$ and $\tilde{\pi}^t(Y)=\tilde{\pi}^b(X)=d$. Moreover, as shown in [**A,V**], it is possible to find a $\tilde{\pi}$ which is *good* or *degenerate*, where a standard permutation is said good if the permutation that we get deleting the letters X and Y from $\tilde{\pi}$ is still irreducible and is said degenerate if there exists a letter $C\in\mathcal{A}$ different from X and Y that is second or second to last in both the top and bottom lines. Let us consider an element $\tilde{\pi}\in\mathcal{R}$ which is good or degenerate. Let us display also the second letters in the top and bottom lines of $\tilde{\pi}$, that is we write

$$\tilde{\pi}=\begin{pmatrix}X & A & \dots & Y\\ Y & B & \dots & X\end{pmatrix}.$$

Step 2. In the Rauzy diagram of $\mathcal{R}$ we consider the *bottom* cycle $\gamma(\tilde{\pi}, bottom, X)$ of length $d-1$ that starts (and ends) at $\tilde{\pi}$. For any letter $\alpha \in \mathcal{A} \setminus \{X\}$ there exists an unique vertex in $\gamma(\tilde{\pi}, bottom, X)$ with winner X and loser α. We call $\tilde{\pi}(\alpha)$ this vertex. It follows that for any pair (X, α) with $\alpha \neq X$ we find a solution on equation (4.15). With the symmetric argument we show that the *top* cycle $\gamma(\tilde{\pi}, top, Y)$ of length $d-1$ that starts (and ends) at $\tilde{\pi}$ provides us with $d-1$ ordered pairs (β, Y) with $\beta \neq Y$ for which exists a solution of equation (4.15).

Step 3. Now we consider the alphabet $\mathcal{A}_X := \mathcal{A} \setminus \{X\}$ and we apply the operation of *reduction* described in paragraph 2.2.2 (following [**A,G,Y**]). First we split $\mathcal{R}$ in *$\mathcal{A}_X$-colored Rauzy classes*, then from the essential ones we get a *reduced Rauzy class*. Let us consider the *essential* $\mathcal{A}_X$-decorated Rauzy class $\mathcal{R}_X$ that contains the cycle $\gamma(\tilde{\pi}, top, Y)$ and let us call $\mathcal{R}_X^{ess}$ the subset of its essential elements and $\mathcal{R}_X^{red}$ the reduced Rauzy class that we get from it. Let us recall that the operation *red*, restricted to the subset of essential elements in $\mathcal{R}_X$, induces a bijection $red : \mathcal{R}_X^{ess} \to \mathcal{R}_X^{red}$.

Since the letter Y is first in the bottom line and last in the top line of $\tilde{\pi}$ when we delete the letter X from any element in $\mathcal{R}_X$ we get an irreducible permutation, therefore the alphabet of $\mathcal{R}_X^{red}$ is the whole $\mathcal{A}_X$. Moreover any element π^{red} in $\mathcal{R}_X^{red}$ satisfies $\pi^{red\,t}(A) = \pi^{red\,b}(Y) = 1$, therefore by applying the induction hypothesis we get that for any ordered pair of letters $(\beta, \alpha) \in \mathcal{A}^2$ with $\alpha \neq A, X$ and $\beta \neq Y, X$ the proposition is true. Let us fix α, β as above. We know that there exists a solution $\pi^{red} \in \mathcal{R}_X^{red}$ of equation (4.15) or two solutions (π^{red}, V) and (π'^{red}, V') of equations (4.16) and (4.17). We separate the two cases.

Case a: If $\pi^{red} \in \mathcal{R}_X^{red}$ is a solution of equation (4.15), then its (unique) essential preimage $\pi = red^{-1}(\pi^{red}) \in \mathcal{R}_X^{ess}$ is a solution of equation (4.15).

Case b: Here the discussion is more complicated. We first consider the solution $\pi^{red} \in \mathcal{R}_X^{red}$ of equation (4.16), its (unique) essential preimage $\pi \in \mathcal{R}_X^{ess}$ and the letter $V \in \mathcal{A}_X$. We have $\pi^t(X) = 1$, therefore there are two possibilities:

(1) If $\pi^b(X) < \pi^b(\beta)$ then π is a solution of equation (4.16)
(2) If $\pi^b(X) > \pi^b(\beta)$ note that all the pairs (β, α) with $\alpha = \beta$ are automatically excluded and we have

$$\pi = \begin{pmatrix} X & A & & \dots & & \alpha & & \dots & & V \\ Y & & \dots & V & \dots & \beta & \dots & X & \dots & \alpha \end{pmatrix} \quad \texttt{with } \pi^t(\alpha) = \pi^b(\beta)$$

therefore π is not solution of equation (4.16), anyway the following three Rauzy operations

$$\mapsto \begin{pmatrix} X & A & & \dots & & \alpha & & \dots & & V \\ Y & & \dots & V & \dots & \alpha & \dots & \beta & \dots & X \end{pmatrix}$$

$$\mapsto \begin{pmatrix} X & \dots & & V & A & & \dots & & \alpha \\ Y & \dots & V & \dots & \alpha & \dots & \beta & \dots & X \end{pmatrix}$$

$$\mapsto \begin{pmatrix} X & \dots & & V & A & & \dots & & \alpha \\ Y & \dots & V & \dots & \alpha & \dots & X & \dots & \beta \end{pmatrix}$$

give a solution of (4.15). (Note that we may have $V \neq Y$ or $V = Y$ and the argument works in both cases.)

Now we consider the solution $\pi'^{red} \in \mathcal{R}_X^{red}$ of equation (4.17), its (unique) essential preimage $\pi' \in \mathcal{R}_X^{ess}$ and the letter $V' \in \mathcal{A}_X$. Note that we have $Y \neq A, V'$, therefore the general form of π'^{red} is

$$\pi'^{red} = \begin{pmatrix} A & \dots & Y & \dots & \alpha & \dots & \beta \\ Y & & \dots & \beta & \dots & & V' \end{pmatrix} \text{ with } \pi'^t(\alpha) = \pi'^b(\beta) + 1.$$

We still have $\pi'^t(X) = 1$, therefore as before if $\pi'^b(X) < \pi'^b(\beta)$ then π' is a solution of equation (4.17). In the other case, if $\pi'^b(X) > \pi'^b(\beta)$ we have to consider several possibilities. We first separate two cases: $\pi'^b(\beta) < \pi'^b(\alpha) < \pi'^b(X)$ and $\pi'^b(X) < \pi'^b(\alpha) < \pi'^b(V')$.

Case $\pi'^b(X) < \pi'^b(\alpha) < \pi'^b(V')$: In this case all the pairs (β, α) with $\beta = \alpha$ are automatically excluded. We also observe that we cannot have $\pi'^t(V') = \pi'^t(\alpha) - 1$, since in this case π'^{red} would not be irreducible, so let us call $W \neq V'$ the letter that appears just before α in the top line. The general form of π' therefore is

$$\pi' = \begin{pmatrix} X & A & \dots & V' & \dots & W & \alpha & & \dots & & \beta \\ Y & \dots & W & \dots & \beta & \dots & X & \dots & \alpha & \dots & V' \end{pmatrix}.$$

with $\pi'^t(\alpha) = \pi'^b(\beta) + 2$, which is not a solution of (4.17). We apply the following Rauzy steps

$$\pi' \mapsto \begin{pmatrix} X & A & \dots & V' & \alpha & & \dots & & \beta & \dots & W \\ Y & \dots & W & \dots & \beta & \dots & X & \dots & \alpha & \dots & V' \end{pmatrix}$$

$$\mapsto \begin{pmatrix} X & A & \dots & V' & \alpha & & \dots & & \beta & \dots & W \\ Y & \dots & W & \dots & \alpha & \dots & V' & \dots & \beta & \dots & X \end{pmatrix}$$

$$\mapsto \begin{pmatrix} X & & \dots & & \beta & \dots & W & A & \dots & V' & \alpha \\ Y & \dots & W & \dots & \alpha & \dots & V' & \dots & \beta & \dots & X \end{pmatrix}$$

$$\mapsto \begin{pmatrix} X & & \dots & & \beta & \dots & W & A & \dots & V' & \alpha \\ Y & \dots & W & \dots & \alpha & \dots & X & \dots & V' & \dots & \beta \end{pmatrix}$$

and we get a solution of equation (4.15). Note that this sequence of steps does the trick both in the cases $A = V'$ and $A \neq V'$.

Case $\pi'^b(\beta) < \pi'^b(\alpha) < \pi'^b(X)$: We consider two subcases $A = V'$ and $A \neq V'$.

Subcase $A = V'$**:** The general form of π' is

$$\pi' = \begin{pmatrix} X & A & *** & Y & *** & \alpha & & \dots & & \beta \\ Y & *** & & & \beta & \dots & \alpha & \dots & X & \dots & A \end{pmatrix}$$

where the asterisques denote the letters $\{x \in \mathcal{A}\,;\, \pi'^t(x) < \pi'^t(\alpha)\} \setminus \{A\} = \{y \in \mathcal{A}\,;\, \pi'^b(y) < \pi'^b(\beta)\}$ and where $\pi'^t(\alpha) = \pi'^b(\beta) + 2$. We therefore don't have a solution of (4.17). We apply the following Rauzy steps

$$\mapsto \begin{pmatrix} X & A & & \dots & & \beta & *** & Y & *** & \alpha \\ Y & *** & \beta & \dots & \alpha & \dots & X & \dots & A \end{pmatrix}$$

$$\mapsto \begin{pmatrix} X & A & & \dots & & \beta & *** & Y & *** & \alpha \\ Y & *** & \beta & \dots & \alpha & \dots & A & \dots & X \end{pmatrix}$$

$$\mapsto \begin{pmatrix} X & *** & Y & *** & \alpha & A & & \dots & & \beta \\ Y & *** & & & \beta & \dots & \alpha & \dots & A & \dots & X \end{pmatrix}$$

and we get a solution of equation (4.17). Note that in this case we may have $\alpha = \beta$ or $\alpha \neq \beta$ and the sequence of steps works in both cases.

Subcase $A \neq V'$: The general form of π' is

$$\pi' = \begin{pmatrix} X & A & \dots & V' & \dots & \alpha & & \dots & & & \beta \\ Y & \dots & A & \dots & \beta & \dots & \alpha & \dots & X & \dots & V' \end{pmatrix}$$

where both the cases $\alpha = \beta$ and $\alpha \neq \beta$ are possible and $\pi'^t(\alpha) = \pi'^b(\beta)+2$. Again this is not a solution of (4.17). We apply the following Rauzy steps:

$$\pi' \mapsto \begin{pmatrix} X & A & \dots & V' & & \dots & & & \beta & \dots & \alpha \\ Y & \dots & A & \dots & \beta & \dots & \alpha & \dots & X & \dots & V' \end{pmatrix}$$

$$\mapsto \begin{pmatrix} X & A & \dots & V' & & \dots & & & \beta & \dots & \alpha \\ Y & \dots & A & \dots & \beta & \dots & \alpha & \dots & V' & \dots & X \end{pmatrix}$$

$$\mapsto \begin{pmatrix} X & \dots & V' & & \dots & & & \beta & \dots & \alpha & A \\ Y & \dots & A & \dots & \beta & \dots & \alpha & \dots & V' & \dots & X \end{pmatrix}.$$

Calling π'' this last vertex of our Rauzy path, if $\alpha = \beta$ we have

$$\pi'' = \begin{pmatrix} X & \dots & V' & \dots & & \alpha & A \\ Y & \dots & A & \dots & \alpha & V' & X \end{pmatrix}$$

and we get a solution of (4.17) with the two Rauzy steps:

$$\pi'' \mapsto \begin{pmatrix} X & \dots & V' & \dots & & \alpha & A \\ Y & \dots & A & X & \dots & \alpha & V' \end{pmatrix} \mapsto \begin{pmatrix} X & \dots & V' & A & \dots & & \alpha \\ Y & \dots & A & X & \dots & \alpha & V' \end{pmatrix}.$$

In the case $\alpha \neq \beta$ we continue from the general expression of π'' and we get a solution of equation (4.15) with the steps

$$\pi'' \mapsto \begin{pmatrix} X & \dots & V' & & \dots & & & \beta & & \dots & \alpha & A \\ Y & \dots & A & \dots & \alpha & \dots & V' & \dots & X & \dots & \beta \end{pmatrix}$$

$$\mapsto \begin{pmatrix} X & \dots & V' & & \dots & & & \beta & A & \dots & \alpha \\ Y & \dots & A & \dots & \alpha & \dots & V' & \dots & X & \dots & \beta \end{pmatrix}.$$

This completes step 3. The summarized result is that if the proposition is true for any Rauzy class with less that d letters, then it is also true for the Rauzy class $\mathcal{R}$ with d letters that we are considering for the pairs of letters $(\beta, \alpha) \in \mathcal{A}^2$ with $\alpha \neq A, X$ and $\beta \neq Y, X$.

Step 4. Since the letters X and Y (and A and B) play a symmetric role in our proof we can also consider the alphabet $\mathcal{A}_Y := \mathcal{A} \setminus \{Y\}$, the essential $\mathcal{A}_Y$-decorated Rauzy class $\mathcal{R}_Y$ that contains the cycle $\gamma(\tilde{\pi}, bottom, X)$ and the reduced Rauzy class $\mathcal{R}_Y^{red}$ that we get from it. Again the alphabet of $\mathcal{R}_Y^{red}$ is the whole $\mathcal{A}_Y$ and any element π^{red} in $\mathcal{R}_X^{red}$ satisfies $\pi^{red\,b}(B) = \pi^{red\,t}(X) = 1$. Therefore, with a proof symmetric to the one in step 3, that is exchanging the roles of X with Y and of A with B respectively, we get that the proposition is true for any ordered pair of letters $\beta, \alpha)$ with $\beta \neq Y, B$ and $\alpha \neq X, Y$.

Step 5. Combining the results of the preceding steps we get that the proposition is true for any pair of letters as in the statement exept for the only pair $(\beta, \alpha) = (B, A)$. To complete the induction we provide a solution for this pair. We come back to the standard vertex $\tilde{\pi}$. It may be good of degenerate and we consider separately the two cases.

$\tilde{\pi}$ **degenerate.** We have two subcases

(1) $A = B$. In this case

$$\tilde{\pi} = \begin{pmatrix} X & A & \dots & W & \dots & Y \\ Y & A & \dots & & W & X \end{pmatrix}$$

where W denotes the last letter in the bottom line before X. We apply the Rauzy steps

$$\tilde{\pi} \mapsto \begin{pmatrix} X & \dots & W & \dots & Y & A \\ Y & A & \dots & & W & X \end{pmatrix} \mapsto \begin{pmatrix} X & \dots & W & \dots & Y & A \\ Y & A & X & \dots & & W \end{pmatrix}$$
$$\mapsto \begin{pmatrix} X & \dots & W & A & \dots & Y \\ Y & A & X & \dots & & W \end{pmatrix} \mapsto \begin{pmatrix} X & \dots & W & A & \dots & Y \\ Y & \dots & & W & A & X \end{pmatrix}$$
$$\mapsto \begin{pmatrix} X & \dots & Y & \dots & W & A \\ Y & \dots & & W & A & X \end{pmatrix}$$

and we get a solution of equation (4.17). Thanks to the symmetry of $\tilde{\pi}$ we can get also a solution of equation (4.16).

(2) $A \neq B$. Since $d \geq 5$ there exists a letter $C \neq X, Y, A, B$ that is second to last both in top and bottom lines and we have

$$\tilde{\pi} = \begin{pmatrix} X & A & \dots & B & \dots & C & Y \\ Y & B & \dots & A & \dots & C & X \end{pmatrix}.$$

We apply the following Rauzy steps

$$\tilde{\pi} \mapsto \begin{pmatrix} X & A & \dots & B & \dots & C & Y \\ Y & \dots & C & X & B & \dots & A \end{pmatrix} \mapsto \begin{pmatrix} X & A & Y & \dots & B & \dots & C \\ Y & \dots & C & X & B & \dots & A \end{pmatrix}$$
$$\mapsto \begin{pmatrix} X & A & Y & \dots & B & \dots & C \\ Y & \dots & C & B & \dots & A & X \end{pmatrix} \mapsto \begin{pmatrix} X & \dots & B & \dots & C & A & Y \\ Y & \dots & C & B & \dots & A & X \end{pmatrix}$$
$$\mapsto \begin{pmatrix} X & \dots & B & \dots & C & A & Y \\ Y & \dots & A & X & \dots & C & B \end{pmatrix} \mapsto \begin{pmatrix} X & \dots & B & Y & \dots & C & A \\ Y & \dots & A & X & \dots & C & B \end{pmatrix}$$

and we get a solution of equation (4.15).

$\tilde{\pi}$ **good.** ($A \neq B$ is implied.) Let $\hat{\pi}$ be the element obtained from $\tilde{\pi}$ letting Y win one time, that is

$$\hat{\pi} = \begin{pmatrix} X & A & \dots & Y \\ Y & X & B & \dots \end{pmatrix}$$

We consider the alphabet $\mathcal{A}_Y = \mathcal{A} \setminus \{Y\}$ and the $\mathcal{A}_Y$-decorated Rauzy class $\mathcal{R}_Y$ that contains $\hat{\pi}$. We note that $\mathcal{R}_Y$ is an essential decorated Rauzy class and we call $\mathcal{R}_Y^{red}$ its reduction. Since $\tilde{\pi}$ is good then the alphabet of $\mathcal{R}_Y^{red}$ is $\mathcal{A}'' := \mathcal{A} \setminus \{X, Y\}$. As we said in step 1 the rauzy class $\mathcal{R}_Y^{red}$ contains a standard element, thus let us consider any such $\hat{\pi}_{st}$, that is

$$\hat{\pi}_{st} = \begin{pmatrix} A & \dots & B \\ B & \dots & A \end{pmatrix}.$$

An essential pre-image of $\hat{\pi}_{st}$ in the $\mathcal{A}_Y$-decorated Rauzy class $\mathcal{R}_Y$ is an element of the form

$$\tilde{\pi} \mapsto \begin{pmatrix} X & A & \dots & Y & \dots \\ Y & X & B & \dots & A \end{pmatrix}.$$

Letting A win the correct number of times we get

$$\tilde{\pi} \mapsto \begin{pmatrix} X & A & \dots & & Y \\ Y & X & B & \dots & A \end{pmatrix}.$$

which is a solution of equation (4.16). Since the argument is symmetric changing the top line with the bottom one we can get also a solution of equation (4.17). □

Note. The natural claim would have been to say that for the pairs of letters (β, α) with $\alpha \neq \beta$ (and $\beta \neq Y, \alpha \neq X$) we always have a solution of (4.15) and we need to consider equations (4.16) or (4.17) just for the pairs with $\beta = \alpha$. Unfortunately this is not true, for example it can be proved that for all *hyperelliptic* Rauzy classes (with $d \geq 3$) there never exists a solution π for the pair (B, A) considered in step 5.

4.3. Uniform control of the speed of shrinking.

Let us fix any pair of letters (β, α) with $\pi^t(\alpha), \pi^b(\beta) > 1$ for all $\pi \in \mathcal{R}$. Corollary 4.2.2 says that any such pair has property A or B (or both), therefore a pre-reference path η can be associated to it as in definition 4.1.6. Proposition 4.1.12 gives a sufficient criterion to get, for any $(\pi, \lambda) \in \Delta^{(1)}(\mathcal{R})$, infinite reduced triples (β, α, n) that are solutions of equation (1.3). This sufficient criterion is formulated as a shrinking target property for the map $\mathcal{F}_\eta$ defined in equation (4.9). More precisely we look at the iterates $\mathcal{F}_\eta^k(\pi, \lambda)$ and we have a solution for any $k \in \mathbb{N}$ such that the corresponding iterated falls into a shrinking neighborhood localized at the face of the simplex $\Delta^{(1)}_{\pi_{end}}$ opposite respectively to the vertex e_α, in case of property A, or to the vertex e_V, in case of property B. At the k-th iterate the thickness of the shrinking neighborhood is

$$\|\lambda^{(r_k)}\|^{-1} \varphi(\|q^{\gamma_k(\pi,\lambda)}\|).$$

What makes things complicated is that the speed of shrinking of the thickness depends from the starting point (π, λ). In this paragraph we apply well known results on the ergodic theory of the Rauzy-Veech and Zorich algorithms to get a lower bound for this speed of shrinking. Since ergodic properties are going to be used the sufficient criterion that we will get (proposition 4.5.1 in paragraph **??**) will hold no more for any $(\pi, \lambda) \in \Delta^{(1)}(\mathcal{R})$ but just for almost any one (with respect to the Lebesgue measure on $\Delta^{(1)}(\mathcal{R})$). We fix a constant $\theta > 1$ and we introduce the sequence

$$\psi_k := \theta^k \varphi(\theta^k). \tag{4.18}$$

LEMMA 4.3.1. *Let φ_n be a sequence such that $n\varphi(n)$ is decreasing monotone. If $\sum_{n=1}^{\infty} \varphi_n = +\infty$, then for any parameter $\theta > 1$ the sequence ψ_k defined by equation (4.18) has divergent series.*

Proof: The proof is just a classical exercise in calculus. □

For any pair $(\beta, \alpha) \in \mathcal{A}^2$ with $\pi^t(\alpha), \pi^b(\beta) > 1$ for all $\pi \in \mathcal{R}$ let us consider a pre-reference path $\eta : \pi_{start} \to \pi_{end}$ given by lemmas 4.1.2 or 4.1.3 and the associated map $\mathcal{F}_\eta : \Delta^{(1)}(\mathcal{R}) \to \Delta^{(1)}_{\pi_{end}}$ defined by equation (4.9). Recall the notations introduced in paragraph 4.1.2.

PROPOSITION 4.3.2. *There exists a constant $\theta > 1$ (depending on η) such that if ψ_k is the sequence defined in equation (4.18) the following is true. For almost any $(\pi,\lambda) \in \Delta^{(1)}(\mathcal{R})$ there exists an integer N such that for any $k \geq N$ the path $\gamma_k(\pi,\lambda) \in \Gamma^{(k),\eta}$ and Rauzy time $r_k = r_k(\pi,\lambda)$ satisfy*

$$\psi_k \leq \frac{1}{\|\lambda^{(r_k)}\|}\varphi(\|q^{\gamma_k(\pi,\lambda)}\|). \tag{4.19}$$

Proof: For any finite path γ in the Rauzy diagram and any $(\pi,\lambda) \in \Delta^{(1)}_\gamma$ without connections let us denote with $\tilde{\gamma}(\pi,\lambda)$ the shortest segment of the Zorich's path associated to (π,λ) that begins with γ. For any $(\pi,\lambda) \in \Delta^{(1)}_\gamma$ without connections, for any $k \in \mathbb{N}$, the path $\gamma_k \in \Gamma^{(k),\eta}$ such that $(\pi,\lambda) \in \Delta^{(1)}_{\gamma_k}$ and the associated integer $r_k = r_k(\pi,\lambda)$ satisfy:

$$\|q^{\gamma_k(\pi,\lambda)}\| \leq \|q^{\tilde{\gamma}_k(\pi,\lambda)}\| \texttt{ and } \|\lambda^{(r_k)}\| \geq \|{}^tB^{-1}_{\tilde{\gamma}_k(\pi,\lambda)}\lambda\|. \tag{4.20}$$

For any finite path γ in the Rauzy diagram let us call with $T(\gamma)$ its *Zorich's time*, that is the number of Zorich's elementary steps that compose γ. Applying the Oseledet's theorem to the *Zorich's cocycle* we have that there exists a positive constant $\tau > 1$ such that for almost any $(\pi,\lambda) \in \Delta^{(1)}(\mathcal{R})$ and for any $k \in \mathbb{N}$ sufficiently big, the path $\gamma_k \in \Gamma^{(k),\eta}$ such that $(\pi,\lambda) \in \Delta^{(1)}_{\gamma_k}$ satisfy:

$$\|q^{\tilde{\gamma}_k(\pi,\lambda)}\| \leq \tau^{T(\tilde{\gamma}_k(\pi,\lambda))} \texttt{ and } \frac{1}{\|{}^tB^{-1}_{\tilde{\gamma}_k(\pi,\lambda)}\lambda\|} \leq \tau^{T(\tilde{\gamma}_k(\pi,\lambda))}. \tag{4.21}$$

Let us now consider the positive integer function $Z : \Delta^{(1)}(\mathcal{R}) \to \mathbb{N}^*$ defined by

$$Z(\pi,\lambda) := \min\{i \in \mathbb{N}^*; \mathcal{Z}^i(\pi,\lambda) \in \Delta^{(1)}_\eta\}$$

and the first entering map $\tilde{\mathcal{Z}}_\eta : \Delta^{(1)}(\mathcal{R}) \to \Delta^{(1)}_\eta$ of the Zorich map $\mathcal{Z} : \Delta^{(1)}(\mathcal{R}) \to \Delta^{(1)}(\mathcal{R})$ into the simplex $\Delta^{(1)}_\eta$, that is $\tilde{\mathcal{Z}}_\eta(\pi,\lambda) := \mathcal{Z}^{Z(\pi,\lambda)}(\pi,\lambda)$. Finally we consider the *Birkhoff sum* $S_kZ : \Delta^{(1)}(\mathcal{R}) \to \mathbb{N}^*$ of the function Z with respect to the map $\tilde{\mathcal{Z}}_\eta$.

Recall the map $\tilde{R}_\eta : \Delta^{(1)}(\mathcal{R}) \to \Delta^{(1)}_\eta$ defined in paragraph 4.1.2. Since the path η is neat (see definition 4.1.4), corollary 4.1.10 applies to $\mathcal{F}_\eta = \mathbb{P}Q_\eta \circ \tilde{R}_\eta$, then the Rauzy-Veech algorithm, when following any path $\gamma_k \in \Gamma^{(k),\eta}$, pass from $\Delta^{(1)}_\eta$ exactly k times. The map $\mathcal{Z}$ is an acceleration of the Rauzy-Veech algorithm, therefore for any $k \in \mathbb{N}$, any $\gamma_k \in \Gamma^{(k),\eta}$ and any $(\pi,\lambda) \in \Delta^{(1)}_{\gamma_k}$ without connections we have:

$$T(\tilde{\gamma}_k(\pi,\lambda)) \leq S_kZ(\pi,\lambda) + R,$$

where we denoted $R = R(\eta) \in \mathbb{N}$ the number of simple arrows composing the fixed path η.

Since the Zorich's map $\mathcal{Z}$ is ergodic (see theorem 2.1.7 in paragraph 2.1.4) we have that there exists a constant C such that $\lim_{k\to\infty} k^{-1}S_kZ(\pi,\lambda) = C$ for almost any $(\pi,\lambda) \in \Delta^{(1)}(\mathcal{R})$ (C being equal to the inverse of $\mu(\Delta^{(1)}_\eta)$, where μ is the invariant measure for the Zorich's map). Therefore for almost any $(\pi,\lambda) \in \Delta^{(1)}(\mathcal{R})$ there exists an integer N such that for any $k \geq N$ we have

$$S_kZ(\pi,\lambda) \leq Ck$$

(in fact this is true modulo a small change of C, that is taking $C + \epsilon$ instead of C for any choice of $\epsilon > 0$). We deduce that there exist a constant C such that for almost any $(\pi,\lambda)\in\Delta^{(1)}_\gamma$, for any $k\in\mathbb{N}$ sufficiently big, the path $\gamma_k\in\Gamma^{(k),\eta}$ such that $(\pi,\lambda)\in\Delta^{(1)}_{\gamma_k}$ satisfy:

$$T(\tilde{\gamma}_k(\pi,\lambda))\leq Ck. \tag{4.22}$$

Now we put together the results of equations (4.20), (4.21) and (4.22) and defining $\theta:=\tau^C$ we get that for almost any $(\pi,\lambda)\in\Delta^{(1)}(\mathcal{R})$, for any $k\in\mathbb{N}$ sufficiently big, the path $\gamma_k=\gamma_k(\pi,\lambda)\in\Gamma^{(k),\eta}$ such that $(\pi,\lambda)\in\Delta^{(1)}_{\gamma_k}$ and the associated integer $r_k=r_k(\pi,\lambda)$ satisfy:

$$\|q^{\gamma_k(\pi,\lambda)}\|\leq\theta^k \texttt{ and } \frac{1}{\|\lambda^{(r_k)}\|}\leq\theta^k.$$

Finally, since the sequence $n\varphi(n)$ is decreasing monotone, we get that for almost any $(\pi,\lambda)\in\Delta^{(1)}(\mathcal{R})$ there exists an integer N such that for any $k\geq N$ the path $\gamma_k=\gamma_k(\pi,\lambda)\in\Gamma^{(k),\eta}$ such that $(\pi,\lambda)\in\Delta^{(1)}_{\gamma_k}$ satisfy:

$$\theta^k\varphi(\theta^k)\leq\frac{1}{\|\lambda^{(r_k)}\|}\varphi(\|q^{\gamma_k(\pi,\lambda)}\|).$$

The proposition is proved. □

4.4. Local constructions

4.4.1. General case. We recall that if W is any letter in the alphabet $\mathcal{A}$ we denote with $\mathcal{A}_W$ the sub-alphabet $\mathcal{A}\setminus\{W\}$.

Definition 4.4.1. For any $\pi\in\mathcal{R}$, any letter $W\in\mathcal{A}$ and any $\epsilon>0$ we define the set $E(\pi,W,\epsilon)$ of those $\mathcal{A}_W$-colored paths γ starting at π such that $q^\gamma_W>1/\epsilon$ that are minimal with these two properties with respect to the ordering $\prec$ introduced in paragraph 2.1.4. We also define the set $N(\pi,W,\epsilon)$ of those paths ν starting at π that satisfy $q^\nu_W<1/\epsilon$, end with an arrow with winner W and are minimal with these two properties with respect to $\prec$.

It is easy to see from the definition of $E(\pi,W,\epsilon)$ and $N(\pi,W,\epsilon)$ that we have (the union is disjoint by minimality of paths)

$$\Delta^{(1)}_\pi=\Big(\bigsqcup_{\gamma\in E(\pi,W,\epsilon)}\Delta^{(1)}_\gamma\Big)\cup\Big(\bigsqcup_{\eta\in N(\pi,W,\epsilon)}\Delta^{(1)}_\eta\Big)\mod 0. \tag{4.23}$$

Lemma 4.4.2. *For any letter $W\in\mathcal{A}$, any $\pi\in\mathcal{R}$ and any $\epsilon>0$, for any $\gamma\in E(\pi,W,\epsilon)$ we have*

$$(\pi,\lambda)\in\Delta^{(1)}_\gamma\Rightarrow\lambda_W<\epsilon. \tag{4.24}$$

Proof: The vectors $\lambda\in\Delta^{(1)}_\gamma\subset\Delta^{(1)}_\pi$ are linear convex combination of the vectors $v_i:=(1/q^\gamma_i)^tB_\gamma e_i$. Any of these vectors has W coordinate equal to

$$\langle v_i,e_W\rangle=\frac{\langle {}^tB_\gamma e_i,e_W\rangle}{q^\gamma_i}=\frac{\langle e_i,B_\gamma e_W\rangle}{q^\gamma_i}.$$

Since v_W is the most height vertex of the simplex $\Delta^{(1)}_\gamma$ in the W direction then it is the one with biggest W coordinate, whose value is $\frac{\langle B_\gamma e_W,e_W\rangle}{q^\gamma_W}$. Now, by definition of $E(\pi,W,\epsilon)$, the letter W newer wins in the trajectory γ, therefore $B_\gamma e_W=e_W$ and the maximum value of the coordinate W for vectors in $\Delta^{(1)}_\gamma$ is $(q^\gamma_W)^{-1}$, that is smaller that ϵ by definition of $E(\pi,W,\epsilon)$. The lemma is proved. □

Let us consider any pair (β, α) with $\pi^t(\alpha), \pi^b(\beta) > 1$ and a pre-reference path $\eta : \pi_{start} \to \pi_{end}$ given by lemmas 4.1.2 or 4.1.3. If (β, α) has property A then η may be chosen as in lemma 4.1.2 and in this case we know that Rauzy paths ending with η give rise to triples (n, β, α) such that $|T^n u^b_\beta - u^t_\alpha|$ is equal to the length of an interval and these triples are automatically reduced. On the other hand if the pair (β, α) satisfy property B and not property A then the pre-reference path η has to be chosen as in lemma 4.1.3, therefore even after having applied η the condition of having a reduced triple is independent from the past and some more combinatorial work is necessary to get it. This will be the subject of paragraph 4.4.2.

Lemma 4.4.3. *For any pair of letters W and L in $\mathcal{A}$ and any $\pi \in \mathcal{R}$, if we have a path $\gamma : \pi \to \pi'$ satisfying*

$$B_\gamma(e_L - e_W) \in \mathbb{N}^{\mathcal{A}} \tag{4.25}$$

then for any $(\pi, \lambda) \in \Delta^{(1)}_\gamma \subset \Delta^{(1)}_\pi$ we have $\lambda_W \leq \lambda_L$.

Proof: Reasoning in the same way as in lemma 4.4.2, in order to have that $\lambda_W \leq \lambda_L$ for any $\lambda \in \Delta^{(1)}_\gamma$ it is equivalent to have that the condition is true for the vertices v_i of the simplex $\Delta^{(1)}_\gamma$. The condition is $\langle v_i, e_L - e_W\rangle \geq 0$ for any $i = 1..d$, that is

$$\langle {}^t B_\gamma e_i, e_L - e_W\rangle = \langle e_i, {}^t B_\gamma(e_L - e_W)\rangle \geq 0$$

for any $i = 1..d$, that is $B_\gamma(e_L - e_W) \in \mathbb{N}^d$. The lemma is proved. □

4.4.2. Pairs with property B. In this paragraph we fix a pair (β, α) satisfying property B and a pre-reference path $\eta : \pi_{start} \to \pi_{end}$ as in lemma 4.1.3. Since the element $\pi(\beta, \alpha)$ satisfying equation (4.16) is in second to last position in η and since η ends with an arrow having α as winner, then we have $\{x \in \mathcal{A}\,;\, \pi^t_{end}(x) < \pi^t_{end}(\alpha)\} \cup \{V\} = \{y \in \mathcal{A}\,;\, \pi^b_{end}(y) < \pi^b_{end}(\beta)\}$ and $\pi^t_{end}(V) = \pi^b_{end}(\beta)$, that is

$$\pi_{end} = \begin{pmatrix} \dots & L & \dots & \alpha & V & \dots & \\ \dots & V & \dots & L & \beta & \dots & \alpha \end{pmatrix}.$$

where $L \in \mathcal{A}$ is the letter such that $\pi^b_{end}(L) = \pi^b_{end}(\beta) - 1$ and $\pi^t_{end}(L) < \pi^t_{end}(\alpha)$. Let us define the set $\mathcal{A}' := \{x \in \mathcal{A}\,;\, \pi^t_{end}(x) < \pi^t_{end}(\alpha)\}$ and let us call a the number of elements of $\mathcal{A}'$. We will keep these notations fixed in all this paragraph.

Let us consider the essential $(\mathcal{A}\backslash\mathcal{A}')$-decorated Rauzy class $\mathcal{R}_* = \mathcal{R}_*(\pi_{end})$ that contains π_{end} and the associated reduced Rauzy class $\mathcal{R}^{red} = \mathcal{R}^{red}(\pi_{end})$ obtained from $\mathcal{R}_*$ with the operation of reduction described in paragraph 2.2.2. Let us call $\mathcal{A}'' \subset \mathcal{A} \setminus \mathcal{A}'$ the alphabet of $\mathcal{R}^{red}$. Since π_{end} is an essential elements of $\mathcal{R}_*$ then there exists a good letter for π_{end}, furthermore there exists only one good letter and it is evident that it is V.

Lemma 4.4.4. *Let $\hat\gamma : \pi_{end} \to \hat\pi$ be any $\mathcal{A}'$-separated path starting at π_{end} and ending in $\hat\pi$. Then for any letter $x \in \mathcal{A}' \cup \{\alpha\}$ we have:*

$$\hat\pi^t(x) = \pi^t_{end}(x). \tag{4.26}$$

Moreover if $\hat\gamma : \pi_{end} \to \hat\pi$ is maximal $\mathcal{A}'$-separated, then its ending point $\hat\pi$ satisfies $\hat\pi^b(L) = d$ and we have:

$$B_{\hat\gamma} e_V = \sum_{x \in \mathcal{A}\backslash\mathcal{A}'} e_x. \tag{4.27}$$

Proof: If the first statement of the lemma is not true there exists an $\mathcal{A}'$-separated path $\hat{\gamma} : \pi_{end} \to \hat{\pi}$ starting at π_{end} and ending in $\hat{\pi}$ such that equation (4.26) is not true. We can suppose that $\hat{\gamma}$ is minimal with this property, that is $\hat{\pi}$ is the first element in $\hat{\gamma}$ where condition (4.26) does not hold, therefore there exists a letter $x \in \mathcal{A}' \cup \{\alpha\}$ such that

$$\hat{\pi}^t(x) = \pi^t_{end}(x) + 1.$$

Let $\gamma_{last} : \pi \to \hat{\pi}$ be the last arrow in $\hat{\gamma}$ and let W be the letter that wins in γ_{last}. We must have $\pi^t(W) < \pi^t(x)$. Since $\hat{\gamma}$ is $\mathcal{A}'-$separated then $W \in \mathcal{A} \backslash \mathcal{A}'$, therefore π still doesn't satisfy condition (4.26), which is absurd by minimality of $\hat{\gamma}$.

Now let us consider a maximal $\mathcal{A}'$-separated path $\hat{\gamma} : \pi_{end} \to \hat{\pi}$ starting at π_{end} and ending in $\hat{\pi}$. By maximality of $\hat{\gamma}$ there exists a letter $y \in \mathcal{A}'$ such that $\hat{\pi}^t(y) = d$ or $\hat{\pi}^b(y) = d$. By the first part of the lemma the only possibility is $\hat{\pi}^b(y) = d$. Moreover L is the rightmost letter of $\mathcal{A}'$ in the permutation π_{end} and in order to invert its position with respect to any other letter of $y \in \mathcal{A}'$ it has to arrive in last position in the bottom line at least one time, which may happen just at the end of $\hat{\gamma}$ since it is an $\mathcal{A}'$-separated path. Therefore we have $\hat{\pi}^b(L) = d$.

To prove equation (4.27) let us decompose $\hat{\gamma}$ as $\hat{\gamma} = \eta^{(1)}\eta_1 ... \eta^{(m)}\eta_m$, where $m = d - a - 1$ and for any $i = 1..m$ the sub-path $\eta^{(i)}$ is not drifting and η_i is a drifting arrow. Let us write $\hat{\eta}^{(i)} := \eta^{(1)}\eta_1 ... \eta^{(i)}$. For any $i = 1..m$ let us also call α_i and β_i respectively the winner and the loser of the arrow η_i and $\pi_s^{(i)}$ and $\pi_e^{(i)}$ respectively the starting and ending point of η_i. Since V is the only good letter for π we have $\alpha_1 = V$. Then we have

$$B_{\eta^{(1)}} e_V = e_V \texttt{ and } B_{\eta_1} e_V = e_V + e_{\beta_1}$$

and the only good letters for $\pi_e^{(1)}$ are V and β_1. Let us put $\mathcal{I}_0 := \{V\}$. Let us fix $k \leq m$ and suppose by induction that for any $1 \leq i < k$ we have that there exists a subset $\mathcal{I}_i \subset \mathcal{A} \setminus \mathcal{A}'$ with $i + 1$ elements and such that

$$B_{\hat{\eta}^{(i)}}(e_V) = \sum_{x \in \mathcal{I}_{i-1}} e_x \texttt{ and } B_{\hat{\eta}^{(i)}\eta_i}(e_V) = \sum_{x \in \mathcal{I}_i} e_x$$

and the good letters for $\pi_e(i)$ are exactly the $x \in \mathcal{I}_i$. Just observe that the base of the induction hypothesis is satisfied by $\mathcal{I}_0$. Let us consider the path $\hat{\eta}^{(k)} = \hat{\eta}^{(k-1)}\eta_{k-1}\eta^{(k)}$. By the induction hypothesis $B_{\hat{\eta}^{(k-1)}\eta_{k-1}}(e_V) = \sum_{x \in \mathcal{I}_{k-1}} e_x$ and the good letters for $\pi_e(k-1)$ are exactly the $x \in \mathcal{I}_{k-1}$. Since there is no drift for any arrow in $\eta^{(k)}$ then its winner is never in $\mathcal{I}_{k-1}$, therefore we have $B_{\hat{\eta}^{(k)}}(e_V) = \sum_{x \in \mathcal{I}_{k-1}} e_x$. Now let us consider the k-th drifting arrow η_k, its winner α_k and its loser β_k. Since the first part of the lemma says that all the drifting arrows of $\hat{\gamma}$ are top arrows then $\pi_s(k)^b_{\beta_k} = d$, therefore β_k is not an element of $\mathcal{I}_{k-1}$, moreover since η_k is drifting we have $\pi_s(k)^t_{\alpha_k} = d$ and $\pi_s(k)^b_{\alpha_k} < d_b(\pi_s(k))$, therefore $\pi_e(k)^b_{\beta_k} = \pi_s(k)^b_{\alpha_k} + 1$, that is β_k come in good position for $\pi_e(k)$ and putting $\mathcal{I}_k := \mathcal{I}_{k-1} \cup \{\beta_k\}$ the inductive step follows. The lemma is proved. □

Remark 4.4.5. Let us consider any path $\gamma \in E(\pi_{end}, V, \epsilon)$ (definition 4.4.1). Since by definition V never wins in γ and it is the only good letter for π_{end}, then there is no drift for any arrow contained in γ, that is $d(\pi') = d(\pi_{end})$ for any $\pi' \in \mathcal{R}_*$ touched by γ. In other words for any $\gamma \in E(\pi_{end}, V, \epsilon)$ and for any vertex $\pi' \in \mathcal{R}_*$ touched by γ any letter $x \in \mathcal{A}' \cup \{\alpha\}$ has to keep its position in the top line constant and equal to $\pi^t_{end}(x)$ and any letters $x \in \mathcal{A}' \cup \{\beta, V\}$ has to keep its position in the bottom line constant and equal to $\pi^b_{end}(x)$. In particular γ has to be $\mathcal{A}'$-separated.

We fix a path $\gamma \in E(\pi_{end}, V, \epsilon)$ and we introduce the set $\Gamma^{\mathcal{A}'}_\gamma$ of maximal $\mathcal{A}'$-separated paths $\hat{\gamma} : \pi_{end} \to \hat{\pi}$ that begin with γ. Thanks to lemma 4.4.4 for any $\hat{\gamma} \in \Gamma^{\mathcal{A}'}_\gamma$ the ending point $\hat{\pi} \in \mathcal{R}_*$ of $\hat{\gamma}$ satisfies $\hat{\pi}^b(L) = d$, therefore there is a bottom arrow starting at $\hat{\pi}$ that has L as winner. Let us call $\nu(L,a)$ the path starting at $\hat{\pi}$ where L wins exactly $d-a$ times. Let us call $\Gamma(L,a)$ the set of path starting at $\hat{\pi}$ where L wins less than $d-a$ times and then it loses one time.

Definition 4.4.6. Let us consider the fixed pair of letters (β, α) satisfying property B, the letter $V \in \mathcal{A}$ and the pre-reference path $\eta(\beta,\alpha)$ given by lemma 4.1.3 and its last element π_{end}. Let us fix any $\epsilon > 0$. Then we define

$$\mathcal{E}(\pi_{end}, V, \epsilon) := \bigsqcup_{\gamma \in E(\pi_{end}, V, \epsilon)} \bigsqcup_{\hat{\gamma} \in \Gamma^{\mathcal{A}'}_\gamma} \hat{\gamma}\nu(L,a),$$

$$\mathcal{N}(\pi_{end}, V, \epsilon) := N(\pi_{end}, V, \epsilon) \cup \left(\bigsqcup_{\gamma \in E(\pi_{end}, V, \epsilon)} \bigsqcup_{\hat{\gamma} \in \Gamma^{\mathcal{A}'}_\gamma} \bigsqcup_{\zeta \in \Gamma(L,a)} \hat{\gamma}\zeta \right).$$

Lemma 4.4.7. *Let us consider the fixed pair of letters (β, α) satisfying property B, the letter $V \in \mathcal{A}$ and the pre-reference path $\eta : \pi_{start} \to \pi_{end}$ given by lemma 4.1.3. Let us fix any $\epsilon > 0$. For any $\gamma' \in \mathcal{E}(\pi_{end}, V, \epsilon)$ we have*

$$(\pi, \lambda) \in \Delta^{(1)}_{\gamma'} \Rightarrow \lambda_V < \min\{\lambda_L, \epsilon\}. \tag{4.28}$$

Proof: Since any $\gamma' \in \mathcal{E}(\pi_{end}, V, \epsilon)$ begins with a path γ in $E(\pi, V, \epsilon)$ then lemma 4.4.2 tells us that $\lambda_V < \epsilon$ for any $\lambda \in \Delta^{(1)}_{\gamma'}$. We decompose any $\gamma' \in \mathcal{E}(\pi_{end}, V, \epsilon)$ as $\gamma' = \hat{\gamma}\nu(L,a)$, where $\hat{\gamma} \in \Gamma^{\mathcal{A}'}_\gamma$ for some $\gamma \in E(\pi_{end}, V, \epsilon)$. We have

$$B_{\gamma'} e_V = B_{\nu(L,a)}\Big(\sum_{x \in \mathcal{A} \setminus \mathcal{A}'} e_x\Big) = \sum_{x \in \mathcal{A} \setminus \mathcal{A}'} e_x,$$

where the first equality follows from equation (4.27) and the second from the fact that the winner in $\nu(L,a)$ is always L and this letter is not contained in $\mathcal{A} \setminus \mathcal{A}'$. On the other hand we have

$$B_{\gamma'} e_L = B_{\nu(L,a)} e_L = e_L + \Big(\sum_{x \in \mathcal{A} \setminus \mathcal{A}'} e_x\Big).$$

Here the first equality follows since $\hat{\gamma}$ is $\mathcal{A}'$-separated, therefore L newer wins in it, the second follows since equation (4.26) implies that the ending point $\hat{\pi}$ of $\hat{\gamma}$ satisfies $\{x \in \mathcal{A}; \hat{\pi}^t_x > a\} = \mathcal{A} \setminus \mathcal{A}'$ and $\nu(L,a)$ is the concatenation of exactly $d-a$ bottom arrows with winner L. Therefore we have

$$B_{\gamma'}(e_L - e_W) = e_L$$

and the lemma follows from lemma 4.4.3. ☐

4.5. The uniform shrinking target property.

For any $\pi \in \mathcal{R}$, any letter $W \in \mathcal{A}$ and any $\epsilon > 0$ let us introduce the following notation

$$\Delta^{(1)}_{E(\pi,W,\epsilon)} := \bigsqcup_{\gamma \in E(\pi,W,\epsilon)} \Delta^{(1)}_{\gamma}.$$

Moreover if the pair (β, α) has property B and $\eta : \pi_{start} \to \pi_{end}$ and V are respectively a pre-reference path and the letter given by lemma 4.1.3, then we put

$$\Delta^{(1)}_{\mathcal{E}(\pi_{end},V,\epsilon)} := \bigsqcup_{\gamma' \in \mathcal{E}(\pi_{end},V,\epsilon)} \Delta^{(1)}_{\gamma'}.$$

PROPOSITION 4.5.1. *For any pair $(\beta, \alpha) \in \mathcal{A}^2$ with $\pi^t(\alpha), \pi^b(\beta) > 1$ let us consider a pre-reference path $\eta : \pi_{start} \to \pi_{end}$ given by lemmas 4.1.2 or 4.1.3 and the associated map $\mathcal{F}_\eta : \Delta^{(1)}(\mathcal{R}) \to \Delta^{(1)}_{\pi_{end}}$ defined by equation (4.9).*

- *If the pair (β, α) has property A and η is given by lemma 4.1.2, then for almost any $T = (\pi, \lambda) \in \Delta^{(1)}(\mathcal{R})$ in order to have infinitely many triples (β, α, n) reduced for T and solutions of equation (1.3) it is sufficient to have infinitely many solutions $k \in \mathbb{N}$ of*

$$\mathcal{F}^k_\eta(\pi, \lambda) \in \Delta^{(1)}_{E(\pi_{end},\alpha,\psi_k)} \tag{4.29}$$

- *If the pair (β, α) has property B and η is given by lemma 4.1.3, then for almost any $T = (\pi, \lambda) \in \Delta^{(1)}(\mathcal{R})$ in order to have infinitely many triples (β, α, n) reduced for T and solutions of equation (1.3) it is sufficient to have infinitely many solutions $k \in \mathbb{N}$ of*

$$\mathcal{F}^k_\eta(\pi, \lambda) \in \Delta^{(1)}_{\mathcal{E}(\pi_{end},V,\psi_k)} \tag{4.30}$$

Proof: Let $\theta > 1$ be the parameter given by proposition 4.3.2 and let ψ_k be the associated sequence.

Let us suppose that the pair (β, α) has property A and η is given by lemma 4.1.2. Then for any $(\pi, \lambda) \in \Delta^{(1)}(\mathcal{R})$ equation 4.13 in proposition 4.1.12 gives a sufficient condition. Proposition 4.3.2 says that for almost any $(\pi, \lambda) \in \Delta^{(1)}(\mathcal{R})$, to have infinitely many solutions $k \in \mathbb{N}$ of equation 4.13 it is sufficient to have infinitely many solutions of

$$\hat{\lambda}^{(r_k)}_\alpha \leq \psi_k.$$

Equation (4.25) in lemma 4.4.3 implies that if $(\pi_{end}, \lambda) \in \Delta^{(1)}_{E(\pi_{end},\alpha,\psi_k)}$, then $\lambda_\alpha \leq \psi_k$, therefore for almost any $(\pi, \lambda) \in \Delta(\mathcal{R})$, to have infinitely many solutions $k \in \mathbb{N}$ of equation 4.13 it is sufficient to have infinitely many solutions of

$$\hat{\lambda}^{(r_k)} \in \Delta^{(1)}_{E(\pi_{end},\alpha,\psi_k)}.$$

If the pair (β, α) has property B and η is given by lemma 4.1.3 then equation 4.14 in proposition 4.1.12 gives a sufficient condition for any $(\pi, \lambda) \in \Delta^{(1)}(\mathcal{R})$. Applying proposition 4.3.2 we get as before that for almost any $(\pi, \lambda) \in \Delta^{(1)}(\mathcal{R})$, to have infinitely many solutions $k \in \mathbb{N}$ of equation 4.14 it is sufficient to have infinitely many solutions of

$$\hat{\lambda}^{(r_k)}_V \leq \min\{\hat{\lambda}^{(r_k)}_V, \psi_k\}.$$

Equation (4.28) in lemma 4.4.7 implies that if $(\pi_{end}, \lambda) \in \Delta^{(1)}_{\mathcal{E}(\pi_{end},V,\psi_k)}$, then $\lambda_V \leq \min\{\lambda_L, \psi_k\}$, therefore for almost any $(\pi, \lambda) \in \Delta(\mathcal{R})$, to have infinitely many solutions $k \in \mathbb{N}$ of equation 4.14 it is sufficient to have infinitely many solutions of

$$\hat{\lambda}^{(r_k)} \in \Delta^{(1)}_{\mathcal{E}(\pi_{end},V,\psi_k)}.$$

The proposition is proved. □

4.6. The local estimate.

4.6.1. Preliminary facts. In this paragraph we develop the notation that we will use to prove theorem 4.6.1 and we recall some facts from paragraph 2.2.2 in the background.

For any π in any Rauzy class on an alphabet $\mathcal{A}$ with d elements let us call Leb_{d-1} the Lebesgue measure on the simplex $\Delta^{(1)}_\pi$ normalized in order to have $\mathrm{Leb}_{d-1}(\Delta^{(1)}_\pi) = 1/d!$. With this normalization, for any Rauzy path γ starting at π and for the associated vector $q^\gamma = B_\gamma \vec{1}$ we have

$$\mathrm{Leb}_{d-1}(\Delta^{(1)}_\gamma) = \frac{1}{d!} \prod_{\alpha \in \mathcal{A}} \frac{1}{q^\gamma_\alpha}.$$

Given a family Γ_π of Rauzy paths γ starting at π we introduce the notation

$$\mathrm{Leb}_{d-1}(\Gamma_\pi) := \mathrm{Leb}_{d-1}\left(\bigcup_{\gamma \in \Gamma_\pi} \Delta^{(1)}_\gamma \right). \tag{4.31}$$

For any integer $2 \leq k \leq d-1$ and for any $(k-1)$-dimensional sub-simplex $\Delta' \subset \partial \Delta^{(1)}_\pi$ let us call Leb_{k-1} the Lebesgue measure on Δ' normalized in order to have $\mathrm{Leb}_{k-1}(\Delta^{(1)}_\pi) = 1/k!$. Given a proper sub-alphabet $\mathcal{A}'$ of $\mathcal{A}$ with $d' < d$ letters let us consider the $(d'-1)$-simplex $\hat{\Delta}^{(1)}_{\pi,\mathcal{A}'} \subset \partial \Delta^{(1)}_\pi$ whose extremal points are exactly the vectors $e_\alpha \in \mathbb{Z}^d$ with $\alpha \in \mathcal{A}'$. If γ is any $\mathcal{A}'$-colored Rauzy path starting at π, for any $\alpha \in \mathcal{A}'$ we have

$${}^t B_\gamma e_\alpha \in \mathtt{Span}\{e_{\alpha'}; \alpha' \in \mathcal{A}'\}$$

therefore if we call $\hat{\Delta}^{(1)}_{\gamma,\mathcal{A}'} \subset \partial \Delta^{(1)}_\gamma$ the $(d'-1)$-face of $\Delta^{(1)}_\gamma$ spanned by the vectors $(1/q^\gamma_\alpha)^t B_\gamma e_\alpha$ with $\alpha \in \mathcal{A}'$, we have $\hat{\Delta}^{(1)}_{\gamma,\mathcal{A}'} \subset \hat{\Delta}^{(1)}_{\pi,\mathcal{A}'}$. For the sub-simplex $\hat{\Delta}^{(1)}_{\gamma,\mathcal{A}'}$, by the normalization of $\mathrm{Leb}_{d'-1}$ on $\hat{\Delta}^{(1)}_{\pi,\mathcal{A}'}$, we have

$$\mathrm{Leb}_{d'-1}(\hat{\Delta}^{(1)}_{\gamma,\mathcal{A}'}) = \frac{1}{d'!} \prod_{\alpha \in \mathcal{A}'} \frac{1}{q^\gamma_\alpha}.$$

Given a family $\Gamma_{\pi,\mathcal{A}'}$ of $\mathcal{A}'$-colored Rauzy paths γ starting at π we write

$$\mathrm{Leb}_{d'-1}(\Gamma_{\pi,\mathcal{A}'}) := \mathrm{Leb}_{d'-1}\left(\bigcup_{\gamma \in \Gamma_{\pi,\mathcal{A}'}} \hat{\Delta}^{(1)}_{\gamma,\mathcal{A}'} \right).$$

When the sub-alphabet $\mathcal{A}'$ is $\mathcal{A}_W = \mathcal{A} \setminus \{W\}$ for some letter $W \in \mathcal{A}$ we will simply write $\hat{\Delta}^{(1)}_\pi$ to denote the $(d-2)$-simplex $\hat{\Delta}^{(1)}_{\pi,\mathcal{A}_W}$. Similarly for any $\mathcal{A}_W$-colored path γ starting at π we will simply denote with $\hat{\Delta}^{(1)}_\gamma$ the $(d-2)$-simplex $\hat{\Delta}^{(1)}_{\gamma,\mathcal{A}_W}$. As a particular case of the discussion above, $\hat{\Delta}^{(1)}_\gamma$ is a sub-simplex of $\hat{\Delta}^{(1)}_\pi$. By our choice of the normalization of the Lebesgue measure Leb_{d-2} on $\hat{\Delta}_\pi^{(1)}$, for any $\mathcal{A}_W$-colored path γ starting at π we have

$$\mathrm{Leb}_{d-1}(\Delta_\gamma^{(1)}) = \frac{1}{d}\frac{1}{q_W^\gamma}\mathrm{Leb}_{d-2}(\hat{\Delta}_\gamma^{(1)}). \tag{4.32}$$

For any fixed π and $W \in \mathcal{A}$ let us consider the $\mathcal{A}_W$-decorated Rauzy class $\mathcal{R}_W^{col}$ that contains π (see paragraph 2.2.1 in the background). Here we suppose that $\mathcal{R}_W^{col}$ is essential. In this case we can associated to it a reduced Rauzy class $\mathcal{R}_W^{red}$ on some sub-alphabet $\mathcal{A}^{red} \subset \mathcal{A}_W$. We also have a reduction map $red : \mathcal{R}_W^{col} \to \mathcal{R}_W^{red}$, which extends to a map $red : \Pi^{col}(\mathcal{R}_W^{col}) \to \Pi(\mathcal{R}_W^{red})$. The inclusions $\mathcal{A}^{red} \subset \mathcal{A}_W \subset \mathcal{A}$ induce naturally a decomposition

$$\mathbb{R}^{\mathcal{A}} = \mathbb{R}^{\mathcal{A}^{red}} \oplus \mathbb{R}^{\mathcal{A}_W \setminus \mathcal{A}^{red}} \oplus \mathbb{R}^{\mathcal{A} \setminus \mathcal{A}_W}. \tag{4.33}$$

Let us denote $P_{\mathcal{A}^{red}} : \mathbb{R}^{\mathcal{A}} \to \mathbb{R}^{\mathcal{A}^{red}}$ and $P_{\mathcal{A}_W \setminus \mathcal{A}^{red}} : \mathbb{R}^{\mathcal{A}} \to \mathbb{R}^{\mathcal{A}_W \setminus \mathcal{A}^{red}}$ the canonical projections respectively on the first and on the second factors in equation (4.33). For any vector $v \in \mathbb{R}_+^{\mathcal{A}}$ we introduce the simplified notation $\hat{v} := P_{\mathcal{A}^{red}}(v)$. Similarly for paths $\nu \in \Pi^{col}(\mathcal{R}_W^{col})$ we also write $\hat{\nu} := red(\nu) \in \Pi(\mathcal{R}_W^{red})$. The associated matrix $B_{\hat{\nu}}$ acts on the first factor in equation (4.33). Lemma 2.2.1 in paragraph 2.2.2 says that for any $\nu \in \Pi^{col}(\mathcal{R}_W^{col})$ the associated vector $q^\nu = B_\nu \vec{1}$ satisfies

$$\hat{q}^{\hat{\nu}} = P_{\mathcal{A}^{red}}(q^\nu) \texttt{ and } P_{\mathcal{A}_W \setminus \mathcal{A}^{red}}(q^\nu) = \vec{1}. \tag{4.34}$$

Let a be the number of letters of $\mathcal{A}^{red}$. For any $\pi \in \mathcal{R}_W^{col}$ and $\nu \in \Pi^{col}(\mathcal{R}_W^{col})$ let us consider the reduced elements $\hat{\pi} := red(\pi) \in \mathcal{R}_W^{red}$ and $\hat{\nu} = red(\nu) \in \Pi(\mathcal{R}_W^{red})$. Let us also consider the $(a-1)$-simplex $\Delta_{\hat{\pi}}^{(1)}$ corresponding to $\hat{\pi}$ and its $(a-1)$-subsimplex $\Delta_{\hat{\nu}}^{(1)}$ associated to $\hat{\nu}$. Equation (4.34) implies

$$\mathrm{Leb}_{d-2}(\hat{\Delta}_\nu^{(1)}) = \frac{a!}{(d-1)!}\mathrm{Leb}_{a-1}(\Delta_{\hat{\nu}}^{(1)}). \tag{4.35}$$

4.6.2. General case.

THEOREM 4.6.1. *There exists a positive constant C, that depends only from the number d of letters of $\mathcal{A}$, such that for any $\pi \in \mathcal{R}$, any letter $W \in \mathcal{A}$ and any $\epsilon > 0$ we have*

$$\mathrm{Leb}_{d-1}(\Delta_{E(\pi,W,\epsilon)}^{(1)}) \geq C\epsilon. \tag{4.36}$$

Proof: By definition paths in $E(\pi, W, \epsilon)$ are $\mathcal{A}_W$-colored. For any $\gamma \in E(\pi, W, \epsilon)$ we consider the $(d-1)$-simplex $\Delta_\gamma^{(1)}$ and its base $\hat{\Delta}_\gamma^{(1)}$ defined in paragraph 4.6.1. Equation (4.32) in paragraph 4.6.1 implies that it is sufficient to prove that there exist two positive constants $N \in \mathbb{N}$ and C, with $0 < C < 1$, such that for any $\pi \in \mathcal{R}$, any $W \in \mathcal{A}$ and any $\epsilon > 0$ we have

$$\mathrm{Leb}_{d-2}\{\gamma \in E(\pi, W, \epsilon)\, ;\, q_W^\gamma < 2^N/\epsilon\} \geq C, \tag{4.37}$$

then the theorem follows redefining C by $C = C/(d2^N)$.

Let us call $\mathcal{R}_W^{col}$ the decorated Rauzy class that contains π. If $\mathcal{R}_W^{col}$ is not essential then $E(\pi, W, \epsilon)$ contains just one Rauzy path γ. The letter W is the loser of any arrow composing γ, this implies that $q_\alpha^\gamma = 1$ for any $\alpha \in \mathcal{A}_W$. Therefore, if we denote with $\gamma(k)$ the finite Rauzy sub-path of γ composed by its first k arrows, we have $q_W^{\gamma(k)} = k$, hence $1/\epsilon \leq q_W^\gamma \leq 1/\epsilon + 1$ and equation (4.37) follows immediately.

It follows that from now on and during all the proof of the theorem we can suppose that $\mathcal{R}_W^{col}$ is essential. Let us call $\mathcal{R}_W^{red} := red(\mathcal{R}_W^{col})$ the associated reduced Rauzy class and $\mathcal{A}^{red} \subset \mathcal{A}_W$ the corresponding sub-alphabet.

We consider $\mathcal{A}_W$-colored paths $\nu : \pi \to \pi'$ starting at π and such that $q_W^\nu < 1/\epsilon$. Any $\gamma \in E(\pi, W, \epsilon)$ begins with such a ν. For any such ν let us call $E(\pi, W, \epsilon|\nu)$ the set of $\gamma \in E(\pi, W, \epsilon)$ that begin with ν. Let us also call $S(\pi, W, \epsilon|\nu)$ the set of paths $\eta \in \Pi(\mathcal{R}_W^{col})$ starting at the ending point π' of ν and such that the composed path $\gamma = \nu\eta$ is in $E(\pi, W, \epsilon|\nu)$. For ν as before and $\eta \in S(\pi, W, \epsilon|\nu)$ the composed path $\gamma = \nu\eta$ satisfies

$$q^\gamma = B_\eta q^\nu.$$

Definition 4.6.2. An *intermediate path* is an $\mathcal{A}_W$-colored path $\nu : \pi \to \pi'$ starting at π, such that $q_W^\nu < 1/\epsilon$, that satisfies the following property: for any $\mathcal{A}_W$-colored path $\eta : \pi' \to \pi''$ starting from the ending point π' of ν and containing at least one arrow where W loses, we have $(B_\eta q^\nu)_W \geq 1/\epsilon$.

The set of Rauzy paths starting at π is partially ordered by the relation $\prec$ defined by equation (2.8) in paragraph 2.1.4. The partial ordering $\prec$ pass to the set of intermediate paths starting at π. Let us introduce the set $I(\pi, W, \epsilon)$ of the intermediate paths starting at π, minimal with respect to this ordering.

Lemma 4.6.3. *For any $\gamma \in E(\pi, W, \epsilon)$ there exists an unique path $\nu = \nu(\gamma) \in I(\pi, W, \epsilon)$ such that $\gamma \in E(\pi, W, \epsilon|\nu)$. On the other hand for any $\nu \in I(\pi, W, \epsilon)$ the set $E(\pi, W, \epsilon|\nu)$ is not empty.*

Proof: Let us consider any $\gamma \in E(\pi, W, \epsilon)$ and let us decompose it as $\gamma = \gamma'\gamma_{last}$, where γ_{last} is the last arrow of γ. By minimality of paths in $E(\pi, W, \epsilon)$ the arrow γ_{last} has W as loser. The path $\gamma' : \pi \to \pi'$ is of course $\mathcal{A}_W$-colored and satisfies $q_W^{\gamma'} < 1/\epsilon$. Let us call π' the ending point of γ'. Any $\mathcal{A}_W$-colored path $\eta : \pi' \to \pi''$ starts with γ_{last}, since the other arrow starting from π' has W as winner, therefore we can decompose any such η as $\eta = \gamma_{last}\eta'$. It follows that $B_\eta q^{\gamma'} = B_{\eta'} q^\gamma$, that is $(B_\eta q^{\gamma'})_W > q_W^\gamma > 1/\epsilon$ and therefore γ' is intermediate. For any $\gamma \in E(\pi, W, \epsilon)$, among the intermediate beginnings of γ there exist a minimal one, and this one is of course unique by minimality. The second statement is evident. The lemma is proved. □

Remark 4.6.4. Let us decompose any $\gamma \in E(\pi, W, \epsilon)$ as $\gamma = \nu\eta$, where $\nu = \nu(\gamma) \in I(\pi, W, \epsilon)$ is given by lemma 4.6.3 and $\eta \in S(\pi, W, \epsilon|\nu)$. Let us also consider the sub-path $\gamma_{2nd-to-last} \prec \gamma$ that ends with the arrow where the letter W loses for the second to last time in γ. The letter W loses exactly one time in η (at the end of it), therefore $\gamma_{2nd-to-last} \prec \nu$, but in general $\gamma_{2nd-to-last}$ is not intermediate and it does not coincide with ν.

Corollary 4.6.5. *We have a* $\mod 0$ *partition*

$$\hat{\Delta}_\pi^{(1)} = \bigsqcup_{\nu \in I(\pi, W, \epsilon)} \hat{\Delta}_\nu^{(1)}.$$

Proof: The family of simplices $\{\hat{\Delta}_\gamma^{(1)}; \gamma \in E(\pi, W, \epsilon)\}$ form a partition $\mod 0$ of the base $\hat{\Delta}_\pi^{(1)}$. On the other hand any $\gamma \in E(\pi, W, \epsilon)$ starts with the (unique) minimal intermediate path $\nu = \nu(\gamma)$ given by lemma 4.6.3 and for the associated simplices we have $\hat{\Delta}^{(1)}_\gamma \subset \hat{\Delta}^{(1)}_\nu$. This means that the family of all simplices $\hat{\Delta}^{(1)}_\nu$ with $\nu \in I(\pi, W, \epsilon)$ covers (mod 0) the base $\hat{\Delta}^{(1)}_\pi$. By minimality of paths $\nu \in I(\pi, W, \epsilon)$ the simplices $\hat{\Delta}^{(1)}_\nu$ are all disjoint, therefore they form a partition mod 0. The corollary is proved. □

For any integer $k \in \mathbb{N}$ let us consider the set $I(\pi, W, \epsilon|k)$ of paths $\nu \in I(\pi, W, \epsilon)$ with $M(q^\nu) \geq 2^k/\epsilon$. We have $I(\pi, W, \epsilon) = \bigcup_{k=1..\infty} I(\pi, W, \epsilon|k)$ (the union is not disjoint).

LEMMA 4.6.6. *There exist two positive constants C and θ, that depend only from the number d of letters of $\mathcal{A}$, such that for any $\pi \in \mathcal{R}$, any $W \in \mathcal{A}$, any $\epsilon > 0$ and for any $k \in \mathbb{N}^*$ we have*

$$\mathrm{Leb}_{d-2}(I(\pi, W, \epsilon|k)) \leq C k^\theta 2^{-(k-1)}. \tag{4.38}$$

Proof: We decompose any $\nu \in I(\pi, W, \epsilon)$ as $\nu = \nu' \nu_{last}$, where ν_{last} is the last arrow in ν, and we consider the family

$$I'(\pi, W, \epsilon) := \{\nu' \,;\, \nu \in I(\pi, W, \epsilon)\}.$$

We note that the map $I(\pi, W, \epsilon) \to I'(\pi, W, \epsilon); \nu \mapsto \nu'$ is a bijection. This because if for some $\nu' \in I'(\pi, W, \epsilon)$ there exist two paths ν_1 and ν_2 in $I(\pi, W, \epsilon)$ such that $\nu' = \nu'_1 = \nu'_2$, then ν' would be intermediate, which is absurd by minimality of paths in $I(\pi, W, \epsilon)$. For any $k \in \mathbb{N}$ we define the set $I'(\pi, W, \epsilon|k)$ of paths ν' such that the associated ν is in $I(\pi, W, \epsilon|k)$

Let us fix any $k \in \mathbb{N}$ and consider any $\nu' : \pi \to \pi'$ in $I'(\pi, W, \epsilon|k)$. By minimality of paths in $I(\pi, W, \epsilon)$ the sub-path ν' is not intermediate. Therefore there exists an $\mathcal{A}_W$-colored path $\eta' : \pi' \to \pi''$ starting at the ending point π' of ν', containing one arrow where W loses and such that we have $(B_{\eta'} q^{\nu'})_W < 1/\epsilon$. Let us call $X \in \mathcal{A}^{red}$ the letter that wins against W in η'. In terms of the reduced path $\hat{\nu'}$ and the reduced vector $\hat{q}^{\hat{\nu'}}$, since $(B_{\eta'} q^{\nu'})_W < 1/\epsilon$, equation (4.34) implies

$$\hat{q}^{\hat{\nu'}}_X = q^{\nu'}_X < 1/\epsilon.$$

Let us fix $X \in \mathcal{A}^{red}$ and define the set $I(\pi, W, \epsilon|k, X)$ of those paths $\nu \in I(\pi, W, \epsilon|k)$ such that the associated path $\nu' \in I'(\pi, W, \epsilon|k)$ satisfies $q^{\nu'}_X < 1/\epsilon$. We also define the set $I'(\pi, W, \epsilon|k, X)$ of all the paths ν' with the associated ν in $I(\pi, W, \epsilon|k, X)$. We have

$$I(\pi, W, \epsilon|k) = \bigcup_{X \in \mathcal{A}^{red}} I(\pi, W, \epsilon|k, X)$$

(the union is not disjoint). For any $\nu \in I(\pi, W, \epsilon|k, X)$ and the associated ν' we observe that $M(q^\nu) \leq 2M(q^{\nu'})$. It follows that for any $\nu' \in I'(\pi, W, \epsilon|k, X)$ we have $M(q^{\nu'}) \geq 2^{k-1}/\epsilon$. By definition of the set $E(\pi, W, \epsilon)$ we have $q^{\nu'}_W < 1/\epsilon$, therefore equation (4.34) implies that

$$M(q^{\nu'}) = M_{\mathcal{A}^{red}}(q^{\nu'}) = M(\hat{q}^{\hat{\nu'}}),$$

therefore the set $Red(I'(\pi, W, \epsilon|k, X))$ of reduced paths $\hat{\nu'}$ with $\nu' \in I'(\pi, W, \epsilon|X, k)$ is contained in

$$\{\hat{\gamma} \in \Pi(\mathcal{R}^{red}_W) \,;\, \hat{q}^{\hat{\gamma}}_X < 1/\epsilon \texttt{ and } M(\hat{q}^{\hat{\gamma}}) \geq 2^{k-1}/\epsilon\}.$$

Let us call a the cardinality of the reduced alphabet $\mathcal{A}^{red}$. Equation (2.13) in paragraph 2.2.5 implies that there exist two positive constants C and θ that depends only from a such that for any $k \in \mathbb{N}$ we have

$$\mathrm{Leb}_{a-1}\{\hat{\gamma} \in \Pi(\mathcal{R}_W^{red})\,;\, \hat{q}_X^{\hat{\gamma}} < 1/\epsilon \texttt{ and } M(\hat{q}^{\hat{\gamma}}) \geq 2^{k-1}/\epsilon\} \leq Ck^\theta 2^{k-1}$$

that implies

$$\mathrm{Leb}_{a-1}(Red(I'(\pi, W, \epsilon|k, X))) \leq Ck^\theta 2^{k-1}.$$

From the last inequality, using equation (4.35) in theorem 2.2.3 (paragraph 2.2.5) we get

$$\mathrm{Leb}_{d-2}(I'(\pi, W, \epsilon|k, X)) \leq \frac{a!}{(d-1)!} Ck^\theta 2^{k-1}$$

and modulo modifying the constant C it follows trivially that $\mathrm{Leb}_{d-2}(I(\pi, W, \epsilon|k, X)) \leq Ck^\theta 2^{k-1}$. When π and W vary (respectively in $\mathcal{R}$ and W), the cardinality of $\mathcal{A}_W$ is always smaller than d, therefore we can chose a pair of constants C and θ that work for any π and W, these C and θ depend only on d. We sum over all $X \in \mathcal{A}^{red}$ and modulo changing C into aC we get equation (4.38). The lemma is proved. $\square$

For any $\nu \in I(\pi, W, \epsilon)$ and any integer $m \geq 1$ let us consider the set $S(\pi, W, \epsilon|\nu, m)$ of paths $\eta \in S(\pi, W, \epsilon|\nu)$ such that we have $(B_\eta q^\nu)_W \geq 2^m M(q^\nu)$.

Lemma 4.6.7. *There exist two positive constant C and θ, that depend only from the number d of letters of $\mathcal{A}$, such that for any $\nu \in I(\pi, W, \epsilon)$, for the reduced path $\hat{\nu} \in Red(I(\pi, W, \epsilon)) \subset \Pi(\mathcal{R}_W^{red})$ and for any integer $m \geq 1$ we have*

$$P_{\hat{\nu}}(Red(S(\pi, W, \epsilon|\nu, m))) \leq Cm^\theta 2^{-(m-1)}. \tag{4.39}$$

Proof: Let us fix $\nu \in I(\pi, W, \epsilon)$ and a positive integer m. Let us consider any $\eta \in S(\pi, W, \epsilon|\nu, m)$ and let us decompose it as $\eta = \eta'\eta_{last}$, where η_{last} is its last arrow . Since the composed path $\gamma = \nu\eta$ is in $E(\pi, W, \epsilon)$ then the arrow η_{last} has the letter W as loser. Moreover, since ν is minimal intermediate, the sub-path η' is $\{W\}$-separated. Let $Y \in \mathcal{A}^{red}$ be the letter that wins against W in the arrow η_{last}. Since η' is $\{W\}$-separated and obviously $q_W^\nu \leq M(q^\nu)$, then $(B_{\eta'} q^\nu)_Y \geq (2^m - 1)M(q^\nu) \geq 2^{m-1}M(q^\nu)$, that is

$$M(B_{\eta'} q^\nu) \geq 2^{m-1}M(q^\nu).$$

Let us consider the ending point π' of the fixed path ν and for $M \geq m-1$ let $\Gamma(\nu|M)$ be the set of $\{W\}$-separated paths η' in the decorated Rauzy class $\mathcal{R}_W^{col}$ starting at π' such that

$$2^M M(q^\nu) \leq M(B_{\eta'} q^\nu) < 2^{M+1}M(q^\nu).$$

Let us denote $\hat{\Gamma}(\nu|M) := Red(\Gamma(\nu|M))$. Since any path $\eta' \in \Gamma(\nu|M)$ is $\{W\}$-separated, remark 2.2.2 in paragraph 2.2.4 implies that there exists a positive integer s with $s \leq 2(d-1)$ such that the reduced path $\hat{\eta'}$ of η' is a concatenation

$$\hat{\eta'} = \hat{\eta}_1 ... \hat{\eta}_s \tag{4.40}$$

of s paths $\hat{\eta}_i$ not complete with respect to the reduced alphabet $\mathcal{A}^{red}$. We put $\hat{q}^{(0)} := \hat{q}^{\hat{\nu}}$ and $\hat{\eta}^0 := \hat{\nu}$ and for any $i = 1..s$ we define inductively $\hat{\eta}^i := \hat{\eta}^{i-1}\hat{\eta}_i$ and $\hat{q}^{(i)} := B_{\hat{\eta}_i}\hat{q}^{(i-1)}$. We can find s non-negative integers $m_1, .., m_s$ such that for any $i \in \{1, .., s\}$ we have:

$$2^{m_i} M(\hat{q}^{(i-1)}) \leq M(B_{\hat{\eta}_i}\hat{q}^{(i-1)}) < 2^{m_i+1}M(\hat{q}^{(i-1)}). \tag{4.41}$$

It turns out that $m_1, .., m_s$ satisfy the relation:

$$M - s - 1 \le m_1 + .. + m_s \le M. \tag{4.42}$$

Let us fix a positive integer s with $s \le 2(d-1)$ and s non-negative integers $m_1, .., m_s$ satisfying equation (4.42) and let us define $\Gamma(\nu|m_1, .., m_s)$ as the set of $\eta' \in \Gamma(\nu|M)$ such that the corresponding reduced path $\hat{\eta}'$, decomposed as in equation (4.40), satisfies the s conditions in equation (4.41) for the values $m_1, .., m_s$. For any $k \in \{0, .., s-1\}$ we also define the set $\hat{\Gamma}(\nu|m_1, .., m_k)$ of those $\hat{\eta}^k$ that satisfy the first k conditions in equation (4.41) for the first k integers $m_1, .., m_k$. For any $\hat{\eta}^k \in \hat{\Gamma}(\nu|m_1, .., m_k)$ we define the set $\hat{\Gamma}(\nu|m_1, .., m_k|\hat{\eta}^k)$ of those $\hat{\eta}^{k+1} \in \hat{\Gamma}(\nu|m_1, .., m_{k+1})$ that begin with $\hat{\eta}^k$.

Let us fix any s with $s \le 2(d-1)$, any set of s integers $m_1, .., m_s$ satisfying equation (4.42), any $k \in \{0, .., s-1\}$ and any $\hat{\eta}^k \in \hat{\Gamma}(\nu|m_1, .., m_k)$. Equation (2.14) in theorem 2.2.3 (paragraph 2.2.5 in the background) implies that there exist two positive constants C and θ, depending only from the cardinality of $\mathcal{A}^{red}$, such that

$$P_{\hat{\eta}^k}(\hat{\Gamma}(\nu|m_1, .., m_{k+1})) \le C(m_{k+1}+1)^\theta 2^{-m_{k+1}}.$$

Applying this last equation s times we get

$$P_{\hat{\nu}}(\hat{\Gamma}(\nu|m_1, .., m_s)) \le \Pi_{i=1}^s C(m_i+1)^\theta 2^{-m_i} \le C^s M^{s\theta} 2^{-M+s+1}.$$

For any s with $s \le 2(d-1)$ the number of possible vectors $(m_1, .., m_s) \in \mathbb{N}^s$ satisfying equation (4.42) is proportional to M^{s-1}, therefore summing over all the possible $(m_1, .., m_s) \in \mathbb{N}^s$ and all the $s \in \{1, .., 2(d-1)\}$, modulo changing the constants C and θ we get

$$P_{\hat{\nu}}(\hat{\Gamma}(\nu|M)) \le C(M+1)^\theta 2^{-M}.$$

Since $\{\hat{\eta}'; \eta \in S(\pi, W, \epsilon|\nu, m)\} \subset \bigcup_{M \ge m-1} \hat{\Gamma}(\nu|M)$, summing over all $M \ge m-1$ we get

$$P_{\hat{\nu}}\{\hat{\eta}'; \eta \in S(\pi, W, \epsilon|\nu, m)\} \le Cm^\theta 2^{-(m-1)}$$

that trivially implies

$$P_{\hat{\nu}}(Red(S(\pi, W, \epsilon|\nu, m))) \le Cm^\theta 2^{-(m-1)}.$$

Letting π and W vary respectively in $\mathcal{R}$ and $\mathcal{A}$ the cardinality of the alphabet $\mathcal{A}^{red}$ is always less or equal to $d-1$, therefore a pair of constant C and θ can be chosen in order to work for all π and W. These constants will depend only on d. The lemma is proved. □

Here we finish the proof of theorem 4.6.1. For any $k \in \mathbb{N}$ let us define

$$\mathfrak{I}(\pi, W, \epsilon|k) := I(\pi, W, \epsilon) \setminus I(\pi, W, \epsilon|k).$$

For any $\nu \in I(\pi, W, \epsilon)$ and for any integer $m \ge 1$ let us define

$$\mathfrak{S}(\pi, W, \epsilon|\nu, m) := S(\pi, W, \epsilon|\nu) \setminus S(\pi, W, \epsilon|\nu, m).$$

Let us fix any pair of positive integers (k, m). For any $\nu \in \mathfrak{I}(\pi, W, \epsilon|k)$ we have $M(q^\nu) < 2^k/\epsilon$. For any $\nu \in \mathfrak{I}(\pi, W, \epsilon|k)$ and for any $\eta \in \mathfrak{S}(\pi, W, \epsilon|\nu, m)$ we have

$$(B_\eta q^\nu)_W < 2^m M(q^\nu) < 2^{k+m}/\epsilon. \tag{4.43}$$

Let C and θ be the constant that appear in lemmas 4.6.6 and 4.6.7. We take $N \in \mathbb{N}$ such that $CN^\theta 2^{N-1} \ll 1$ and we put $c := 1 - CN^\theta 2^{N-1}$. Equation (4.43) implies that the set $\{\gamma \in E(\pi, W, \epsilon); q_W^\gamma < 2^{2N}/\epsilon\}$ contains

$$\bigsqcup_{\nu \in \mathfrak{I}(\pi,W,\epsilon|N)} \mathfrak{S}(\pi, W, \epsilon|\nu, N).$$

Lemma 2.2.1 in the background implies that for any $\nu \in I(\pi, W, \epsilon)$ we have

$$\mathrm{Leb}_{d-2}(\mathfrak{S}(\pi, W, \epsilon|\nu, N)) = \mathrm{Leb}_{d-2}(\hat{\Delta}_\nu^{(1)})P_{\hat{\nu}}(Red(\mathfrak{S}(\pi, W, \epsilon|\nu, N)))$$

therefore we get

$$\mathrm{Leb}_{d-2}\{\gamma \in E(\pi, W, \epsilon); q_W^\gamma < 2^{2N}/\epsilon\} \geq$$

$$\sum_{\nu \in \mathfrak{I}(\pi,W,\epsilon|N)} \mathrm{Leb}_{d-2}(\hat{\Delta}_\nu^{(1)})P_{\hat{\nu}}(Red(\mathfrak{S}(\pi, W, \epsilon|\nu, N))) \geq c\mathrm{Leb}_{d-2}(\mathfrak{I}(\pi, W, \epsilon|N)) \geq c^2$$

from the results in lemmas 4.6.6 and 4.6.7. Equation (4.37) therefore follows with $C = c^2$. The theorem is proved. □

4.6.3. Pairs with property B..

Proposition 4.6.8. *There exists a positive constant $C > 0$ such that for any pair of letters (β, α) satisfying property B, for any pre-reference path $\eta : \pi_{start} \to \pi_{end}$ given by lemma 4.1.3, for the associated letter V, and for any $\epsilon > 0$ we have*

$$\mathrm{Leb}(\Delta^{(1)}_{\mathcal{E}(\pi_{end},V,\epsilon)}) \geq C\epsilon. \tag{4.44}$$

Proof: Recall the definition of the set $\mathcal{A}' := \{x \in \mathcal{A}; \pi_{end}^t(x) < \pi_{end}^t(\alpha)\}$ introduced in paragraph 4.4.2. With the notation of the same paragraph we call a the number of letters in $\mathcal{A}'$. Let us consider any $\gamma \in E(\pi_{end}, V, \epsilon)$ and the set $\mathcal{E}(\pi_{end}, V, \epsilon|\gamma)$ of those paths $\gamma' \in \mathcal{E}(\pi_{end}, V, \epsilon)$ that begins with γ. We prove that

$$P_\gamma(\mathcal{E}(\pi_{end}, V, \epsilon|\gamma)) \geq \frac{1}{2^{d-a}} \tag{4.45}$$

that implies the proposition with $C' := C2^{-(d-a)}$, where C is the constant appearing in theorem 4.6.1. We recall that for a fixed $\gamma \in E(\pi_{end}, V, \epsilon)$ any $\gamma' \in \mathcal{E}(\pi_{end}, V, \epsilon|\gamma)$ is decomposed as $\gamma' = \hat{\gamma}\nu(L, a)$, where $\hat{\gamma} \in \Gamma_\gamma^{\mathcal{A}'}$ (the set of maximal $\mathcal{A}'-$separated paths beginning with γ, see definition after remark 4.4.5). We have

$$P_\gamma(\mathcal{E}(\pi_{end}, V, \epsilon|\gamma)) = \sum_{\hat{\gamma} \in \Gamma_\gamma^{\mathcal{A}'}} P_\gamma(\Delta^{(1)}_{\hat{\gamma}\nu(L,a)}) =$$

$$\sum_{\hat{\gamma} \in \Gamma_\gamma^{\mathcal{A}'}} P_\gamma(\Delta^{(1)}_{\hat{\gamma}})P_{\hat{\gamma}}(\Delta^{(1)}_{\nu(L,a)}) \geq \inf_{\hat{\gamma} \in \Gamma_\gamma^{\mathcal{A}'}} P_{\hat{\gamma}}(\Delta^{(1)}_{\nu(L,a)})$$

since $\{\Delta^{(1)}_{\hat{\gamma}}; \hat{\gamma} \in \Gamma_\gamma^{\mathcal{A}'}\}$ form a partition $\mod 0$ of $\Delta^{(1)}_\gamma$. We have:

$$\mathrm{Leb}(\Delta^{(1)}_{\gamma'}) = \frac{1}{d!}\left(\prod_{x \in \mathcal{A}} q_x^{\gamma'}\right)^{-1} \texttt{ and } \mathrm{Leb}(\Delta^{(1)}_{\hat{\gamma}}) = \frac{1}{d!}\left(\prod_{x \in \mathcal{A}} q_x^{\hat{\gamma}}\right)^{-1}.$$

The finite path $\nu(L, a)$ is concatenation of $d - a$ bottom arrows. Any of the $d - a$ letters in $\mathcal{A} \setminus \mathcal{A}'$ is the loser of exactly one of these arrows, on the other hand the winner is always the letter L. It follows that for the path $\gamma' = \hat{\gamma}\nu(L, a)$ we have

$$q_x^{\gamma'} = q_x^{\hat{\gamma}} + q_L^{\hat{\gamma}} \texttt{ for } x \in \mathcal{A} \setminus \mathcal{A}' \texttt{ and } q_x^{\gamma'} = q_x^{\hat{\gamma}} \texttt{ for } x \in \mathcal{A}'.$$

Since the trajectories $\hat{\gamma} \in \Gamma_{\gamma}^{\mathcal{A}'}$ are $\{L\}$-separated we have $q_L^{\hat{\gamma}} = 1$, therefore if we decompose any $\gamma' \in \mathcal{E}(\pi, W, \epsilon|\gamma)$ as $\gamma' = \hat{\gamma}\nu(L,a)$ for some $\hat{\gamma} \in \Gamma_{\gamma}^{\mathcal{A}'}$ we have $q_x^{\gamma'} = q_x^{\hat{\gamma}} + 1 \leq 2q_x^{\hat{\gamma}}$ for any $x \in \mathcal{A} \setminus \mathcal{A}'$. Since

$$P_{\hat{\gamma}}(\Delta^{(1)}_{\nu(L,a)}) = \left(\prod_{x\in\mathcal{A}} \frac{q_x^{\hat{\gamma}}}{q_x^{\gamma'}}\right)$$

we get $P_{\hat{\gamma}}(\Delta^{(1)}_{\nu(L,a)}) > 2^{-(d-a)}$ and the result in equation (4.45) follows. The proposition is proved. □

4.7. End of the proof: generalized Borel-Cantelli argument.

We fix any pair $(\beta, \alpha) \in \mathcal{A}^2$ such that $\pi^t(\alpha), \pi^b(\beta) > 1$ for all $\pi \in \mathcal{R}$. We consider a pre-reference path $\eta : \pi_{start} \to \pi_{end}$ as in definition 4.1.6 and the associated map $\mathcal{F}_\eta : \Delta^{(1)}(\mathcal{R}) \to \Delta^{(1)}_{\pi_{end}}$ defined by equation (4.9).

LEMMA 4.7.1. • *If (β, α) has property A, a pre-reference path η can be chosen as in lemma 4.1.2 and in a way such that the following holds. For any $\epsilon > 0$ and for any ν in the sets $E(\pi_{end}, \alpha, \epsilon)$ or $N(\pi_{end}, \alpha, \epsilon)$ (introduced in definition 4.4.1), the simplex $\Delta^{(1)}_\nu$ admits a non-trivial partition into connected components of the domain of the map $\mathcal{F}_\eta$.*

• *If (β, α) has property B, a pre-reference path η can be chosen as in lemma 4.1.3 and in a way such that the following holds. For any $\epsilon > 0$ and for any ν in the sets $\mathcal{E}(\pi_{end}, V, \epsilon)$ or $\mathcal{N}(\pi_{end}, V, \epsilon)$ (introduced in definition 4.4.6), the simplex $\Delta^{(1)}_\nu$ admits a non-trivial partition into connected components of the domain of the map $\mathcal{F}_\eta$.*

Proof: • For any π in the Rauzy class $\mathcal{R}$ and any $\epsilon > 0$ the letter α never wins in paths in $E(\pi, \alpha, \epsilon)$ and wins just one times in paths in $N(\pi, \alpha, \epsilon)$. In view of lemma 4.1.11, to have the required property, it is enough to choose a pre-reference path $\eta : \pi_{start} \to \pi_{end}$ containing at least two arrows with winner α. This is possible as observed in remark 4.1.7 at the end of paragraph 4.1.2. Such η cannot be contained in any path ν in the sets $E(\pi_{end}, \alpha, \epsilon)$ or $N(\pi_{end}, \alpha, \epsilon)$.

• Assume now that the pair (β, α) has the property B. Let us recall the definition of the sub-alphabet $\mathcal{A}'$ of $\mathcal{A}$ given in paragraph 4.4.2 and let us denote with a the number of its elements. In view of lemma 4.1.11, to have the required property, it is enough to choose a pre-reference path $\eta : \pi_{start} \to \pi_{end}$ containing at least $d - a + 2$ arrows with winner V, this is possible as observed in remark 4.1.7 at the end of paragraph 4.1.2. Such an η cannot be a sub-path of any path ν in the sets $\mathcal{E}(\pi_{end}, V, \epsilon)$ and $\mathcal{N}(\pi_{end}, \alpha, \epsilon)$, this because the letter V wins at most $d - a + 1$ times in paths in these sets. The lemma is proved. □

DEFINITION 4.7.2. For any real number M such that $M > 1$ we say that the vector $q \in \mathbb{R}_+^{\mathcal{A}}$ is *M-equilibrated* if $q_i < Mq_j$ for any $i, j \in \mathcal{A}$. We say that the finite Rauzy path η is a *M-equilibrated path* if for any vector $q \in \mathbb{R}_+^{\mathcal{A}}$ the vector $B_\eta q$ is M-equilibrated.

LEMMA 4.7.3. *For any $M > 1$, if η is a M-equilibrated path and γ is any finite Rauzy path ending with η, then we have*

$$\left\|\frac{dP_\gamma}{d\mathrm{Leb}_{d-1}}\right\| \leq M^d \quad \textit{and} \quad \left\|\frac{d\mathrm{Leb}_{d-1}}{dP_\gamma}\right\| \leq M^d. \tag{4.46}$$

Proof: Let γ be any finite path in the Rauzy diagram ending at the element π and consider the probability measure P_γ. For any Rauzy path ν starting at the element π where γ ends we have

$$\frac{P_\gamma(\Delta_\nu^{(1)})}{\text{Leb}_{d-1}(\Delta_\nu^{(1)})} = \frac{\Pi_{\alpha\in\mathcal{A}} q_\alpha^\gamma q_\alpha^\nu}{\Pi_{\alpha\in\mathcal{A}} q_\alpha^{\gamma\nu}}.$$

Since γ ends with η we can write $\gamma = \gamma'\eta$ for some γ' and therefore $q^\gamma = B_\eta q^{\gamma'}$. Since η is an M-equilibrated path it follows that q^γ is an M-equilibrated vector. Moreover, since $q^{\gamma\nu} = B_\nu q^\gamma$, for any $\alpha \in \mathcal{A}$ we have $M^{-1} q_\alpha^\gamma q_\alpha^\nu \leq q_\alpha^{\gamma\nu} \leq M q_\alpha^\gamma q_\alpha^\nu$, and therefore

$$\frac{1}{M^d} \leq \frac{P_\gamma(\Delta_\nu^{(1)})}{\text{Leb}_{d-1}(\Delta_\nu^{(1)})} \leq M^d.$$

When ν varies among all the finite Rauzy paths starting at π the sub-simplices $\Delta_\nu^{(1)}$ form a base of the Borel σ-algebra of $\Delta_\pi^{(1)}$, therefore the estimation in equation (4.46) follows. The lemma is proved. □

LEMMA 4.7.4. *There exists a constant M depending only on the number of intervals d such we can choose a pre-reference path $\eta : \pi_{start} \to \pi_{end}$ as in definition 4.1.6 which satisfies the supplementary condition of lemma 4.7.1 and which is M-equilibrated.*

Proof: A sufficient condition on η for being M-equilibrated is that all the entries of the matrix B_η are positive and that its norm $\|B_\eta\|$ is less that M. Since in any Rauzy class $\mathcal{R}$ any letter wins against any other it is sufficient to consider any finite pre-reference path $\eta : \pi_{start} \to \pi_{end}$ that contains any arrow of the Rauzy diagram $\mathcal{D}$, this will assure that all entries in B_η are positive. Moreover if η is given by lemma 4.1.2 we ask that it contains at least two arrows where the letter α wins, if η is given by lemma 4.1.3 we ask that it contains at least $d-a+2$ arrows where the letter V wins. Pre-reference paths satisfying these requirements exist since lemmas 4.1.2 and 4.1.3 just specify the beginning and the ending part of η. Then it is enough to put $M := \|B_\eta\|$, this value will work for the all pair (β, α). Since the are $(d-1)^2$ such pairs the maximum over the cases is still a finite quantity and will depend just from the number of intervals. □

DEFINITION 4.7.5. Let (β, α) be a pair with $\pi^t(\alpha), \pi^b(\beta) > 1$ for all $\pi \in \mathcal{R}$. A *reference path* for the pair (β, α) is a pre-reference path $\eta : \pi_{start} \to \pi_{end}$ as in definition 4.1.6 that satisfy the supplementary conditions in lemma 4.7.4.

Since reference paths form a subset of pre-reference paths, everything we did for pre-reference paths still work when η is a reference path as in definition 4.7.5. Therefore from now on, for any pair (β, α) such that $\pi^t(\alpha), \pi^b(\beta) > 1$ we consider a reference path $\eta : \pi_{start} \to \pi_{end}$ as in definition 4.7.5 and we associate to it the map $\mathcal{F}_\eta : \Delta^{(1)}(\mathcal{R}) \to \Delta^{(1)}_{\pi_{end}}$ defined in equation 4.1.8. Proposition 4.5.1 still gives a sufficient condition in order to have, for almost every $T \in \Delta^{(1)}(\mathcal{R})$, infinitely many reduced triples (β, α, n) that are solutions of $|T^n u_\beta^b - u_\alpha^t| < \varphi(n)$. For any $k \in \mathbb{N}$ we recall the family of paths $\Gamma^{(k),\eta}$ introduced in paragraph 4.1.2, whose elements are the paths γ_k which give the connected components $\Delta_{\gamma_k}^{(1)}$ of the domain of the k-th iterated of $\mathcal{F}_\eta$. Theorem 4.6.1, proposition 4.6.8 and lemma 4.7.3 imply the following consequence:

COROLLARY 4.7.6. *There exists a constant $C > 0$, depending only on the number of intervals d, such that for any $(\pi, \lambda) \in \Delta^{(1)}(\mathcal{R})$ without connections, any $k \in \mathbb{N}$ and any $\gamma_k \in \Gamma^{(k),\eta}$ such that $(\pi, \lambda) \in \Delta^{(1)}_{\gamma_k}$, we have the following estimates.*

- *If the pair (β, α) has property A and η is given by lemma 4.1.2, then*

$$P_{\gamma_k}(\Delta^{(1)}_{E(\pi_{end},\alpha,\psi_k)}) \geq C\psi_k. \tag{4.47}$$

- *If the pair (β, α) has property B and η is given by lemma 4.1.3, then*

$$P_{\gamma_k}(\Delta^{(1)}_{\mathcal{E}(\pi_{end},V,\psi_k)}) \geq C\psi_k. \tag{4.48}$$

Proof: Let us consider a reference path η as in definition 4.7.5 and the associated map $\mathcal{F}_\eta$. By the definition of $\mathcal{F}_\eta$, for any $k \in \mathbb{N}$, any $\gamma_k \in \Gamma^{(1),\eta}$ ends with η and this one in M-equilibrated. Therefore we can apply lemma 4.7.3 and we get $\|\frac{dP_{\gamma_k}}{d\mathrm{Leb}_{d-1}}\| \leq M^d$ and $\|\frac{d\mathrm{Leb}_{d-1}}{dP_{\gamma_k}}\| \leq M^d$. If η is chosen as in lemma 4.1.2 we apply the estimate in theorem 4.6.1 and we get that

$$P_{\gamma_k}(\Delta^{(1)}_{E(\pi_{end},\alpha,\psi_k)}) \geq M^d\mathrm{Leb}_{d-1}(\Delta^{(1)}_{E(\pi_{end},\alpha,\psi_k)}) \geq CM^d\psi_k.$$

Equation (4.47) follows redefining C as CM^d. If η is chosen as in lemma 4.1.3 we apply the estimate in proposition 4.6.8 and we get that

$$P_{\gamma_k}(\Delta^{(1)}_{\mathcal{E}(\pi_{end},V,\psi_k)}) \geq M^d\mathrm{Leb}_{d-1}(\Delta^{(1)}_{\mathcal{E}(\pi_{end},V,\psi_k)}) \geq C'M^d\psi_k.$$

Equation (4.48) follows redefining C as $C'M^d$. The corollary is proved. □

4.7.1. The Borel-Cantelli argument. Here we finish the proof of proposition 4.0.6. From now on the proof is the same both for pairs with property A and pairs with property B. Therefore to simplify our notation, for any $k \in \mathbb{N}$ we just write $\mathcal{E}_k$ instead of $E(\pi_{end}, \alpha, \psi_k)$ (if the pair (β, α) has property A) or instead of $\mathcal{E}(\pi_{end}, \alpha, \psi_k)$ (if the pair (β, α) has property B). We also write $\mathcal{N}_k$ instead of $N(\pi_{end}, \alpha, \psi_k)$ (if the pair (β, α) has property A) or instead of $\mathcal{N}(\pi_{end}, \alpha, \psi_k)$ (if the pair (β, α) has property B). We also introduce the notation:

$$\Delta^{(1)}(\mathcal{E}_k) := \bigsqcup_{\gamma\in\mathcal{E}_k} \Delta^{(1)}_\gamma \texttt{ and } \Delta^{(1)}(\mathcal{N}_k) := \bigsqcup_{\gamma\in\mathcal{N}_k} \Delta^{(1)}_\gamma.$$

Let us define

$$\mathcal{C}_k := \{(\pi, \lambda) \in \Delta^{(1)}(\mathcal{R}); \mathcal{F}^k_\eta(\pi, \lambda) \in \Delta^{(1)}(\mathcal{N}_k)\}.$$

For any $k \in \mathbb{N}$ and any finite Rauzy path ν recall the set $\Gamma^{(k),\eta}_\nu$ defined in paragraph 4.1.2 of those finite Rauzy path γ_k such that $\Delta^{(1)}_{\gamma_k}$ is a connected component of the domain of $\mathcal{F}^k_\eta|_{\Delta^{(1)}_\nu}$ (the restriction of $\mathcal{F}^k_\eta$ to $\Delta^{(1)}_\nu$).

LEMMA 4.7.7. *Let C be the constant appearing in corollary 4.7.6. For any pair of integers m, n with $m > n$ we have:*

$$\mathrm{Leb}(\bigcap_{k=n}^{m} \mathcal{C}_k) \leq \prod_{k=n}^{m}(1 - C\psi_k). \tag{4.49}$$

Proof: For any $n \in \mathbb{N}$ we have

$$\mathcal{C}_n = \bigsqcup_{\gamma_n\in\Gamma^{(n),\eta}} \bigsqcup_{\nu_n\in\mathcal{N}_n} \Delta^{(1)}_{\gamma_n\nu_n}$$

therefore

$$\mathrm{Leb}(\mathcal{C}_n) = \sum_{\gamma_n \in \Gamma^{(n),\eta}} \sum_{\nu_n \in \mathcal{N}_n} \mathrm{Leb}(\Delta^{(1)}_{\gamma_n \nu_n}) = \sum_{\gamma_n \in \Gamma^{(n),\eta}} \sum_{\nu_n \in \mathcal{N}_n} \mathrm{Leb}(\Delta^{(1)}_{\gamma_n}) P_{\gamma_n}(\Delta^{(1)}_{\nu_n}) =$$

$$\sum_{\gamma_n \in \Gamma^{(n),\eta}} \mathrm{Leb}(\Delta^{(1)}_{\gamma_n}) P_{\gamma_n}(\Delta^{(1)}(\mathcal{N}_n)) \leq (1 - C\psi_n) \sum_{\gamma_n \in \Gamma^{(n),\eta}} \mathrm{Leb}(\Delta^{(1)}_{\gamma_n}) = (1 - C\psi_n)$$

thanks to corollary 4.7.6. According to lemma 4.7.1 the path η is not contained in any $\nu_n \in \mathcal{N}_n$. Corollary 4.1.10 implies that for any $\nu_n \in \mathcal{N}_n$ the simplex $\Delta^{(1)}_{\nu_n}$ admits a non-trivial decomposition into connected components of the domain of the maps $\mathcal{F}_\eta$, that is we have

$$\Delta^{(1)}_{\nu_n} = \bigsqcup_{\gamma_1 \in \Gamma^{\eta}_{\nu_n}} \Delta^{(1)}_{\gamma_1}.$$

For any $\gamma_n \in \Gamma^{(n),\eta}$, any $\nu_n \in \mathcal{N}_n$ and any $\gamma_1 \in \Gamma^{\eta}_{\nu_n}$ we define the path $\gamma_{n+1} = \gamma_n \nu_n \gamma_1$, which corresponds to a connected component $\Delta^{(1)}_{\gamma_{n+1}}$ of the domain of $\mathcal{F}^{(n+1)}_{\eta} | \Delta^1_{\gamma_n \nu_n}$. The decomposition above is equivalent to

$$\Delta^{(1)}_{\gamma_n \nu_n} = \bigsqcup_{\gamma_{n+1} \in \Gamma^{(n+1),\eta}_{\gamma_n \nu_n}} \Delta^{(1)}_{\gamma_{n+1}}.$$

Let us suppose that for $m > n$ we have $\mathrm{Leb}(\bigcap_n^m \mathcal{C}_k) \leq \Pi_{k=n}^m (1 - C\psi_k)$. We also have

$$\bigcap_{k=n}^{m} \mathcal{C}_k = \bigsqcup_{\gamma_n \in \Gamma^{(n),\eta}} \bigsqcup_{\nu_n \in \mathcal{N}_n} \dots \bigsqcup_{\gamma_m \in \Gamma^{(m),\eta}_{\gamma_{m-1}\nu_{m-1}}} \bigsqcup_{\nu_m \in \mathcal{N}_m} \Delta^{(1)}_{\gamma_m \nu_m}.$$

According to lemma 4.7.1 the path η is not contained in any $\nu_m \in \mathcal{N}_m$. Corollary 4.1.10 implies that for any $\nu_m \in \mathcal{N}_m$ the simplex $\Delta^{(1)}_{\nu_m}$ admits a non-trivial decomposition into connected components of the domain of the maps $\mathcal{F}_\eta|_{\Delta^{(1)}_{\nu_m}}$, that is we have

$$\Delta^{(1)}_{\nu_m} = \bigsqcup_{\gamma_1 \in \Gamma^{\eta}_{\nu_m}} \Delta^{(1)}_{\gamma_1}.$$

For any γ_m in the union above, any $\nu_m \in \mathcal{N}_m$ and any $\gamma_1 \in \Gamma^{\eta}_{\nu_m}$ we define the path $\gamma_{m+1} = \gamma_m \nu_m \gamma_1$, which corresponds to a connected component $\Delta^{(1)}_{\gamma_{m+1}}$ of the domain of $\mathcal{F}^{(m+1)}_{\eta} | \Delta^1_{\gamma_m \nu_m}$. The decomposition above is equivalent to

$$\Delta^{(1)}_{\gamma_m \nu_m} = \bigsqcup_{\gamma_{m+1} \in \Gamma^{(m+1),\eta}_{\gamma_m \nu_m}} \Delta^{(1)}_{\gamma_{m+1}}. \tag{4.50}$$

If follows that

$$\bigcap_{k=n}^{m+1} \mathcal{C}_k = \bigsqcup_{\gamma_n \in \Gamma^{(n),\eta}} \bigsqcup_{\nu_n \in \mathcal{N}_n} \dots \bigsqcup_{\gamma_{m+1} \in \Gamma^{(m+1),\eta}_{\gamma_m \nu_m}} \bigsqcup_{\nu_{m+} \in \mathcal{N}_{m+1}} \Delta^{(1)}_{\gamma_{m+1} \nu_{m+1}}.$$

Using the identity $\mathrm{Leb}(\Delta^{(1)}_{\gamma_{m+1}\nu_{m+1}}) = \mathrm{Leb}(\Delta^{(1)}_{\gamma_{m+1}}) P_{\gamma_{m+1}}(\Delta^{(1)}_{\nu_{m+1}})$ and recalling corollary 4.7.6, we have

$$\mathrm{Leb}(\bigcap_{k=n}^{m+1} \mathcal{C}_k) = \sum_{\gamma_n \in \Gamma^{(n),\eta}} \sum_{\nu_n \in \mathcal{N}_n} \dots \sum_{\gamma_{m+1} \in \Gamma^{(m+1),\eta}_{\gamma_m \nu_m}} \mathrm{Leb}(\Delta^{(1)}_{\gamma_{m+1}}) P_{\gamma_{m+1}}(\Delta^{(1)}(\mathcal{N}_{m+1})) \leq$$

$$(1 - C\psi_{m+1}) \sum_{\gamma_n \in \Gamma^{(n),\eta}} \sum_{\nu_n \in \mathcal{N}_n} \cdots \sum_{\gamma_{m+1} \in \Gamma^{(m+1),\eta}_{\gamma_m \nu_m}} \mathrm{Leb}(\Delta^{(1)}_{\gamma_{m+1}}).$$

Equation (4.50) implies that

$$\bigsqcup_{\gamma_n \in \Gamma^{(n),\eta}} \bigsqcup_{\nu_n \in \mathcal{N}_n} \cdots \bigsqcup_{\gamma_{m+1} \in \Gamma^{(m+1),\eta}_{\gamma_m \nu_m}} \Delta^{(1)}_{\gamma_{m+1}} =$$

$$\bigsqcup_{\gamma_n \in \Gamma^{(n),\eta}} \bigsqcup_{\nu_n \in \mathcal{N}_n} \cdots \bigsqcup_{\gamma_m \in \Gamma^{(m),\eta}_{\gamma_{m-1}\nu_{m-1}}} \bigsqcup_{\nu_m \in \mathcal{N}_m} \Delta^{(1)}_{\gamma_m \nu_m} = \bigcap_{k=n}^{m} \mathcal{C}_k,$$

that is

$$\mathrm{Leb}(\bigcap_{k=n}^{m+1} \mathcal{C}_k) \leq (1 - C\psi_{m+1})\mathrm{Leb}(\bigcap_{k=n}^{m} \mathcal{C}_k)$$

and by the induction hypothesis we get that $\mathrm{Leb}(\bigcap_n^{m+1} \mathcal{C}_k) \leq \Pi_{k=n}^{m+1}(1 - C\psi_k)$. The lemma is proved. □

LEMMA 4.7.8. *For any sequence of positive numbers $\{a_n\}_{n\in\mathbb{N}}$, if $\sum_{n\in\mathbb{N}} a_n = \infty$ then $\prod_{n\in\mathbb{N}}(1 - a_n) = 0$.*

Proof: The proof is a classical exercise in calculus. □

For any $N \in \mathbb{N}$ we put $\tilde{\mathcal{C}}_N := \bigcap_{k=N}^{\infty} \mathcal{C}_k$. Thanks to lemmas 4.7.7 and 4.7.8 we have $\mathrm{Leb}(\tilde{\mathcal{C}}_N) = 0$ for any $N \in \mathbb{N}$. Any (π, λ) for which there is just a finite number of solutions of

$$\mathcal{F}^k_\eta(\pi, \lambda) \in \mathcal{E}_k$$

is contained in some $\tilde{\mathcal{C}}_N$ and $\mathrm{Leb}(\bigcup_{N\in\mathbb{N}} \tilde{\mathcal{C}}_N) = 0$, that is, for almost any $(\pi, \lambda) \in \Delta^{(1)}(\mathcal{R})$ there are infinitely many solutions $k \in \mathbb{N}$ of $\mathcal{F}^k_\eta(\pi, \lambda) \in \mathcal{E}_k$. In the two cases of property A or B this means that equations respectively (4.29) or (4.30) have infinite solutions for almost any $(\pi, \lambda) \in \Delta^{(1)}(\mathcal{R})$. Therefore proposition 4.5.1 implies proposition 4.0.6 at the beginning of this chapter. Thanks to 3.3.1 part b) of theorem 1.1.6 follows. This complete the discussion of the divergent case in arbitrary genus.

CHAPTER 5

Divergent case, genus one

In this section M denotes a topological torus and $\Sigma \subset M$ a finite subset with r elements. We consider translation structures X on the pair (M, Σ). Let us fix a singular point $p_0 \in \Sigma$ as base point for the absolute periods $H_1(M, \mathbb{Z})$. If w is an holomorphic 1-form on M the set

$$\Lambda := \{\int_\gamma w; \gamma \in H_1(M, \mathbb{Z})\}$$

defines a *lattice*, that is a co-compact subgroup of $\mathbb{C}$. The holomorphic 1-form w is identified with the constant form dz over $\mathbb{C}/\Lambda$ and is therefore uniquely determined. The moduli space of holomorphic 1-forms on a torus is therefore identified with the space of lattices, that is $\mathrm{GL}(2, \mathbb{R})/\mathrm{SL}(2, \mathbb{Z})$. Let $p_i \neq p_0$ be any other point in Σ different from the base point of the absolute periods. Since w has no zeros on M the only free parameter left to determine a translation structure X is the position of p_i. This position is identified with the coset $\Lambda_i := \Lambda + v_i$ of Λ in $\mathbb{C}$ (where v_i is some element in $\mathbb{C}$) given by

$$\Lambda_i := \{\int_\gamma w; \gamma \texttt{ connects } p_0 \texttt{ to } p_i\}$$

that is the integral of w over the relative periods starting at p_0 and ending at p_i. The datum of a translation structure X on a torus with r marked points is often called *marked torus*. Fixing in p_0 the origin of the associated lattice Λ and calling v_i the position of any other $p_i \neq p_0$ we write

$$X = (\Lambda, v_1, .., v_{r-1}).$$

At any point in Σ the total angle is 2π, therefore the stratum associated to these topological data is $\mathcal{H}(0, .., 0)$, where the coefficient 0 appears r times. Since all the angles at marked points are 2π, for any $p_i \in \Sigma$ there is just an horizontal outgoing separatrix. It follows that the moduli space $\widehat{\mathcal{H}}(0, .., 0)$ defined in paragraph 3.1.1 coincides with $\mathcal{H}(0, .., 0)$. In particular, for any $X \in \mathcal{H}(0, .., 0)$, to specify a configuration of saddle connection it is enough to fix the pair p_i, p_j of points in Σ where the saddle connection starts and ends respectively. Therefore for a marked torus X and for $i, j \in \{0, .., r-1\}$ we redefine the configuration $\mathcal{C}^{(p_j, p_i)}(X)$ as the set of straight segments on X that connect p_j with p_i and that have no other marked point in their interior.

5.1. Some more background.

Theorem 1.4.1 and theorem 1.4.2 concern respectively marked tori and irrational rotations, anyway to prove them we use some arguments which work in general for translation surfaces of any genus and interval exchange transformations.

We collect these arguments in this preliminary paragraph, the reader can skip and come back to it after reading paragraph 5.2.

5.1.1. Action of $\mathrm{SO}(2,\mathbb{R})$ **on** $\Delta^{(1)}(\mathcal{R})$**.** In this paragraph we apply to a translation structure $X(\pi,\lambda,\tau)$ a rotation of its vertical direction and we study what is the effect on the parameter space of length data normalized to one. Let us fix any admissible datum π over the alphabet $\mathcal{A}$ and consider the application

$$\begin{array}{rccc} \rho_\pi: & \Delta_\pi \times \Theta_\pi & \to & \Delta_\pi^{(1)} \\ & (\lambda,\tau) & \mapsto & \|\lambda\|^{-1}\lambda. \end{array} \tag{5.1}$$

For any $\lambda \in \Delta_\pi$ let us consider the linear map $f_{(\pi,\lambda)}$ in $\hom(\mathbb{R}^d, (\vec{1})^\perp)$ defined by

$$f_{(\pi,\lambda)}(\tau) := \langle \tau, \vec{1}\rangle \frac{\lambda}{\|\lambda\|} - \tau. \tag{5.2}$$

Note that $\|\lambda\| = \langle \lambda, \vec{1}\rangle$, but for the vector τ we use the scalar-product notation to stress that $\langle \tau, \vec{1}\rangle$ may also be negative. For any $(\lambda,\tau) \in \Delta_\pi \times \Theta_\pi$ we also consider the function $s_{(\lambda,\tau)} : (-\pi/2, \pi/2) \to \mathbb{R}$ defined by

$$s_{(\lambda,\tau)}(\theta) = \frac{\tan\theta}{\|\lambda\| - \langle \tau, \vec{1}\rangle \tan\theta}. \tag{5.3}$$

Now let us consider any stratum $\widetilde{\mathcal{H}}_i(k_1,..,k_k)$ of the moduli space of translation surfaces marked in p_i. For any $\pi \in \mathfrak{S}_i(k_1,..,k_k)$ it is identified with $\widetilde{\mathcal{H}}(M^\pi, \Sigma^\pi, k)$. We recall the subset $\mho_\pi \subset \Delta_\pi$ defined in paragraph 2.4.3 and the embedding $\mathcal{I}_\pi : \mho_\pi \times \Theta_\pi \to \widetilde{\mathcal{H}}_i(k_1,..,k_r)$ defined in paragraph 2.4.4. $\mathcal{I}_\pi$ has a continuous extension

$$\begin{array}{rccc} I_\pi: & \Delta_\pi \times \Theta_\pi & \to & \widetilde{\mathcal{H}}_i(k_1,..,k_r) \\ & (\lambda,\tau) & \mapsto & X(\pi,\lambda,\tau) \end{array}$$

Now we fix any $(\lambda,\tau) \in \Delta_\pi \times \Theta_\pi$ and we consider consider the associated translation surface $X(\pi,\lambda,\tau) = I_\pi(\lambda,\tau) \in \widetilde{\mathcal{H}}_i(k_1,..,k_r)$. Let $\mathrm{SO}(2,\mathbb{R})X(\pi,\lambda,\tau)$ be the orbit of $X(\pi,\lambda,\tau)$ under the application $\theta \mapsto R_\theta X(\pi,\lambda,\tau)$, where

$$R_\theta := \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

Let $C(\pi,\lambda,\tau)$ be the connected component of $\mathrm{SO}(2,\mathbb{R})X(\pi,\lambda,\tau) \cap I_\pi(\Delta_\pi \times \Theta_\pi)$ that contains $X(\pi,\lambda,\tau)$. Let $I(\pi,\lambda,\tau) \subset (-\pi/2,\pi/2)$ be the maximal interval such that the parametrization

$$\begin{array}{ccc} I(\pi,\lambda,\tau) & \to & C(\pi,\lambda,\tau) \\ \theta & \mapsto & R_\theta X(\pi,\lambda,\tau) \end{array} \tag{5.4}$$

is defined: it is an open interval containing 0. There exists an unique lift

$$\begin{array}{ccc} I(\pi,\lambda,\tau) & \to & \Delta_\pi \times \Theta_\pi \\ \theta & \mapsto & (\lambda_\theta, \tau_\theta) \end{array} \tag{5.5}$$

of the parametrization in equation (5.4) such that for any $\theta \in I(\pi,\lambda,\tau)$ we have $R_\theta X(\pi,\lambda,\tau) = I_\pi(\lambda_\theta,\tau_\theta)$ and for $\theta = 0$ we have $(\lambda_0,\theta_0) = (\lambda,\theta)$. Let ρ_π be the application defined by equation (5.1), let $f_{(\pi,\lambda)}$ be the linear map defined by equation (5.2) and $s_{(\lambda,\tau)}$ be the smooth function defined by equation (5.3). We have the following:

LEMMA 5.1.1. *Let us consider any (λ,τ) in $\Delta_\pi \times \Theta_\pi$, the associated translation surface $X(\pi,\lambda,\tau)$ in $\widetilde{\mathcal{H}}_i(k_1,..,k_r)$ and the interval $I(\pi,\lambda,\tau)$ defined by equation (5.4). The function $s_{(\lambda,\tau)}$ restricted to $I(\pi,\lambda,\tau)$ gives a smooth change of parameter, moreover if $\theta \mapsto (\lambda_\theta,\tau_\theta)$ is the lift of the application $\theta \mapsto R_\theta X(\pi,\lambda,\tau)$ defined in equation (5.5), then the map $\theta \mapsto \rho_\pi(\lambda_\theta,\tau_\theta)$ satisfies*

$$\rho_\pi(\lambda_\theta,\tau_\theta) = \frac{\lambda}{\|\lambda\|} + s_{(\lambda,\tau)}(\theta) f_{(\pi,\lambda)}(\tau), \tag{5.6}$$

that is it is a diffeomorphism which sends $I(\pi,\lambda,\tau)$ onto an open segment $R(\pi,\lambda,\tau)$ in $\Delta_\pi^{(1)}$.

Proof: For the data (π,λ,τ) let us consider the vectors $\xi^t_\alpha, \xi^b_\beta \in \mathbb{C}$ defined in paragraph 2.4.1. We have that $R_\theta X(\pi,\lambda,\tau) \in I_\pi(\Delta_\pi \times \Theta_\pi)$ if and only if

$$\max_{\pi^b(\alpha)>1} \frac{\Re(\xi^b_\alpha)}{\Im(\xi^b_\alpha)} < \tan\theta < \min_{\pi^t(\alpha)>1} \frac{\Re(\xi^t_\alpha)}{\Im(\xi^t_\alpha)}$$

that gives the explicit formula for the endpoints of the interval $I(\pi,\lambda,\tau)$:

$$I(\pi,\lambda,\tau) = \left(\arctan\left(\max_{\pi^b(\alpha)>1} \frac{\Re(\xi^b_\alpha)}{\Im(\xi^b_\alpha)}\right), \arctan\left(\min_{\pi^t(\alpha)>1} \frac{\Re(\xi^t_\alpha)}{\Im(\xi^t_\alpha)}\right)\right).$$

The function $s_{(\lambda,\tau)}$ has a singularity for $\theta = \arctan(\|\lambda\|/\langle\tau,\vec{1}\rangle)$, that is not in $I(\pi,\lambda,\tau)$, therefore is everywhere defined on it. Moreover we have

$$\frac{d}{d\theta} s_{(\lambda,\tau)}(\theta) = \frac{1}{(1+\theta^2)(\|\lambda\| - \langle\tau,\vec{1}\rangle\tan\theta)^2} > 0$$

for all $\theta \in I(\pi,\lambda,\theta)$ and the assertion on $s_{(\lambda,\tau)}$ follows. The explicit formula for the lift $\theta \mapsto (\lambda_\theta,\tau_\theta)$ in equation (5.5) is $\lambda_\theta = \lambda\cos\theta - \tau\sin\theta$ and $\tau_\theta = \lambda\sin\theta + \tau\cos\theta$. After some elementary computations, we get

$$\rho_\pi(\lambda_\theta,\tau_\theta) = \|\lambda_\theta\|^{-1}\lambda_\theta = \frac{\lambda}{\|\lambda\|} + \left(\frac{\tan\theta}{\|\lambda\| - \langle\tau,\vec{1}\rangle\tan\theta}\right)\left(\langle\tau,\vec{1}\rangle\rho(X) - \tau\right).$$

For $\theta \in I(\pi,\lambda,\tau)$ we have $\lambda_\theta \in \mathbb{R}^{\mathcal{A}}_+$, therefore $\langle\lambda_\theta,\vec{1}\rangle = \|\lambda\|\cos\theta - \langle\tau,\vec{1}\rangle\sin\theta > 0$. The lemma is proved. □

5.1.2. Getting reduced triples from non-reduced ones.

LEMMA 5.1.2. *Let T be an i.e.t. without connections. There exist an $\epsilon > 0$ (depending on T) such that the following is true. If (β,α,n) with $\pi^t(\alpha),\pi^b(\beta) > 1$ and $n \in \mathbb{N}$ is a triple non reduced for T and such that*

$$|T^n u^b_\beta - u^t_\alpha| < \epsilon \tag{5.7}$$

then there are two triples (β,x,h) and (y,α,l) reduced for T with $0 \le h,l < n$ that satisfy

$$|T^h u^b_\beta - u^t_x|, |T^l u^b_y - u^t_\alpha| < |T^n u^b_\beta - u^t_\alpha|. \tag{5.8}$$

Proof: First of all we chose

$$\epsilon := 1/2 \min_{x,y\in\mathcal{A}} |u^t_x - u^b_y|$$

(When taking the minimum it is understood that we don't consider the trivial distances $u^t_x - u^t_x = 0$ or $u^b_y - u^b_y = 0$ for all $x,y \in \mathcal{A}$). Since T has no connections we have $\epsilon > 0$. In particular our choice implies that all the intervals I^t_x or I^b_y with $x, y \in \mathcal{A}$ have length at least 2ϵ. Let α, β be letters in $\mathcal{A}$ such that $\pi^t(\alpha), \pi^b(\beta) > 1$ and let (β, α, n) be a triple non reduced for T and such that condition in equation (5.7) holds. Being non reduced there exists some $k \in \{0, .., n\}$ and a letter $z \in \mathcal{A}$ with $u^t_z \in T^{-k}(I(\beta, \alpha, n))$ or $u^b_z \in T^{-k}(I(\beta, \alpha, n))$. We consider the smallest $k \in \{0, .., n\}$ such that the last condition holds. By minimality of k we have that T^{-i} is a translation on $I(\beta, \alpha, n)$ for any $i = 0, .., k$, that is

$$|T^n u^b_\beta - u^t_\alpha| = |T^{n-k} u^b_\beta - T^{-k} u^t_\alpha| < \epsilon.$$

Without any loss in generality we suppose that we have $u^t_z \in T^{-k}(I(\beta, \alpha, n))$. We can also suppose that we have

$$T^{n-k} u^b_\beta < u^t_z < T^{-k} u^t_\alpha. \tag{5.9}$$

Since by our choice of ϵ we have $|u^t_x - u^t_y| > \epsilon$ for any $x \neq y$ the condition above implies $k \geq 1$. Moreover there are no other points u^t_w or u^b_w in $T^{-k}(I(\beta, \alpha, n))$. The first inequality in equation (5.9) implies that $(\beta, z, n-k)$ is a triple with

$$|T^{n-k} u^b_\beta - u^t_z| < |T^n u^b_\beta - u^t_\alpha|.$$

Now we look to the second inequality in equation (5.9). By minimality of k no other singularities of T or T^{-1} are contained in $(u^t_z, T^{-k} u^t_\alpha)$ and in all its iterates $T^i(u^t_z, T^{-k} u^t_\alpha)$ for any $i = 0, .., k$. We apply T one time on the interval. Recalling that for any letter $x \in \mathcal{A}$ we have $\lim_{\epsilon \to 0} T(u^t_x + \epsilon) = u^b_x$, we get

$$u^b_z < T^{-(k-1)} u^t_\alpha.$$

Then we apply T the remaining $k-1$ times and we get that the triple $(z, \alpha, k-1)$ satisfy

$$|T^{k-1} u^b_z - u^t_\alpha| < |T^n u^b_\beta - u^t_\alpha|.$$

Now, if both the two triples are reduced, we are done. If not we call $j(\beta, \alpha, n)$ the number of singularities, both for T and T^{-1}, that are contained in the orbit

$$I(\beta, \alpha, n), T^{-1} I(\beta, \alpha, n), .., T^{-1} I(\beta, \alpha, n).$$

We observe that $j(\beta, z, n-k), j(z, \alpha, k-1) < j(\beta, \alpha, n)$, therefore we can start a descending induction procedure until we get two reduced triples. Observe that in this procedure equation (5.8) keeps true and that the the two letters β and α are always respectively the first letter of the triple (β, x, h) and the second in the triple (y, α, l). The lemma is proved. $\square$

Let $\varphi : \mathbb{N} \to \mathbb{R}_+$ be a sequence such that $n\varphi(n)$ is decreasing monotone.

Corollary 5.1.3. *Let T be an i.e.t. without connections and let $(\beta, \alpha) \in \mathcal{A}^2$ with $\pi^t(\alpha), \pi^b(\beta) > 1$. If there exist infinitely many triples (β, α, n), not reduced for T but solutions of equation (1.3), that is such that*

$$|T^n u^b_\beta - u^t_\alpha| \leq \varphi(n),$$

then there are two letters x, y in $\mathcal{A}$ with $\pi^t(x), \pi^b(y) > 1$ and two sequences of triples (y, α, l_i) and (β, x, h_i) reduced for T and solutions of equation (1.3)

Proof: We first observe that since $n\varphi(n)$ is decreasing monotone, then $\varphi(n) \to 0$ as $n \to \infty$, therefore any sequence of triples (β, α, n_i) that satisfy equation (1.3) eventually satisfy condition (5.7) in lemma 5.1.2.

Let us suppose to have a sequence of triples (β, α, n_i) non reduced for T and satisfying equation (1.3). We construct the two sequences (y, α, l_i) and (β, x, h_i) of reduced triples as follows. Let us first apply lemma 5.1.2 to any (β, α, n_1) with n_1 big enough. We get two reduced triples (y_1, α, l_1) and (β, x_1, h_1). Since $h_1, l_1 < n_1$ and $\varphi(n)$ is decreasing monotone, then equation (5.8) implies that (y_1, α, l_1) and (β, x_1, h_1) are solutions of equation (1.3).

Let us suppose by induction that (y_k, α, l_k) and (β, x_k, h_k) are chosen. Then we take n_{k+1} such that

$$|T^{n_{k+1}} u^b_\beta - u^t_\alpha| < \min\{|T^{h_k} u^b_\beta - u^t_x|, |T^{l_k} u^b_y - u^t_\alpha|\},$$

and we apply lemma 5.1.2 to (β, α, n_{k+1}). We get two reduced triples $(y_{k+1}, \alpha, l_{k+1})$ and $(\beta, x_{k+1}, h_{k+1})$. Since $h_{k+1}, l_{k+1} < n_{k+1}$ and $\varphi(n)$ is decreasing monotone, then equation (5.8) implies that $(y_{k+1}, \alpha, l_{k+1})$ and $(\beta, x_{k+1}, h_{k+1})$ are solutions of equation (1.3).

It may happen that the pairs of letters (x_k, y_k) appearing in these two sequences are not always the same, anyway there exist some pair (x, y) and two subsequences of (y_k, α, h_k) and (β, x_k, l_k) such that this is true. The corollary is proved. □

5.2. Reduction to an arithmetic formulation.

In this paragraph we assume part b) theorem 1.4.2 and we prove theorem 1.4.1. We recall its statement:

If $\int_0^\infty \varphi(t)dt = \infty$ then for any $X \in \mathcal{H}(0, .., 0)$ and for almost any $\theta \in S^1$ the rotated marked torus X_θ has at least $2r-1$ different configurations $\mathcal{C}^{(p_j, p_i)}(X_\theta)$ each one containing infinitely many saddle connection γ which are solutions of equation (1.4), that is:

$$|\tan \text{angle}(\gamma, \partial_y)| \leq \frac{\varphi(|\gamma|)}{|\gamma|}.$$

Remark 5.2.1. For an arbitrary marked torus X we cannot expect that almost any X_θ in its $\mathrm{SO}(2, \mathbb{R})$ orbit is modφ-Liouville, that is we cannot expect that for generic θ any configuration $\mathcal{C}^{(j,i)}(X_\theta)$ contains infinitely many saddle connections satisfying equation (1.4). For example, with $r = 4$ marked points, we can consider the marked torus $X = (\Lambda, v_1, v_2, v_3)$ given by the lattice $\Lambda = \mathbb{Z}^2$ and by the extra marked points $v_1 := (1/2, 0)$, $v_2 := (0, 1/2)$ and $v_3 := (1/2, 1/2)$ (we recall that in this chapter we denote $\Sigma = \{p_0, .., p_{r-1}\}$, where p_0 always coincides with the origin of the lattice Λ). For such X it is easy to check that for any $i = 0, 1, 2, 3$ and for any $\theta \in [0, 2\pi)$ all the configurations $\mathcal{C}^{(i,i)}(X_\theta)$ are empties.

For arbitrary X we also ask if our result is optimal, that is if it is always possible to find, for generic θ, more than $2r-1$ configurations for X_θ containing each one infinitely many solutions of equation (1.4) .

Let us recall that any translation surface X (including marked tori) determines an (unique) abelian differential w_X, that defines an element of $\hom(H_1(M, \Sigma, \mathbb{Z}), \mathbb{C})$. For any $\theta \in [-\pi, \pi)$ let us call w_θ the abelian differential associated to X_θ and also the associated element in $\hom(H_1(M, \Sigma, \mathbb{Z}), \mathbb{R}^2)$.

Lemma 5.2.2. *Let $X \in \mathcal{H}(0, .., 0)$ without any vertical or horizontal connection.*

- *There exists a basis* $\{\tilde{\gamma}_1, \tilde{\gamma}_2\}$ *of the absolute homology* $H_1(M, \mathbb{Z})$ *such that the periods* $w_i := w_X(\tilde{\gamma}_i)$ *of* w_X *satisfy*

$$\Re(w_1), \Re(w_2) > 0 \ \textbf{\textit{and}}\ \Im(w_1) < 0, \Im(w_2) > 0. \tag{5.10}$$

- *There exist an open interval* $I \subset \mathbb{R}$ *containing* 0 *such that for any* $\theta \in I$ *the periods* $w_i(\theta) := w_\theta(\tilde{\gamma}_i)$ *of the rotated marked torus* X_θ *(over the base* $\{\tilde{\gamma}_1, \tilde{\gamma}_2\}$ *of* $H_1(M, \mathbb{Z})$ *constructed above) satisfy equation (5.10).*

Proof: The periods of w_X over the absolute homology $H_1(M, \mathbb{Z})$ form a lattice $\Lambda \subset \mathbb{C}$. The first part of the statement correspond to find a base $\{w_1, w_2\}$ of Λ with the required properties. If Γ is the element in $\mathrm{SL}(2, \mathbb{R})$ defined by

$$\Gamma := \begin{pmatrix} \Re(w_1) & \Re(w_2) \\ \Im(w_1) & \Im(w_2) \end{pmatrix},$$

the changes of base correspond to multiplication on the right by elements of $\mathrm{SL}(2, \mathbb{Z})$. Since X has neither vertical nor horizontal connections, both the quantities

$$a\Re(w_1) + c\Re(w_2) \ \texttt{and}\ b\Im(w_1) + d\Im(w_2)$$

are never zero for any choice of integers $a, b, c, d \in \mathbb{Z}$ with $ad - bc = 1$. Using the generators of $\mathrm{SL}(2, \mathbb{Z})$ it is a trivial exercise to find a base satisfying equation (5.10).

To prove the second part it is sufficient to argue that condition in equation (5.10) is open in $\mathrm{SL}(2, \mathbb{R})$, therefore it keeps true under small rotations. The lemma is proved. □

Let $\mathcal{R}$ be a Rauzy class on an alphabet $\mathcal{A}$ with $d = r + 1$ letters such that there are i.e.t. in $\Delta(\mathcal{R})$ that are rotations. An element $\pi \in \mathcal{R}$ is called *rotational* if $\pi^t(\alpha) - \pi^b(\alpha) = \texttt{const} \mod d$ for any $\alpha \in \mathcal{A}$. Interval exchange transformation with rotational combinatorial data are rotations.

Lemma 5.2.3. *Let* $X \in \mathcal{H}(0, .., 0)$ *without any vertical or horizontal connection. Let* I *be the open interval associated to* X *by lemma 5.2.2. There exist a rotational* π *such that for any* $\theta \in I$ *the rotated marked torus* X_θ *has length-suspension data in* $\Delta_\pi \times \Theta_\pi$*.*

Proof: Let Λ be the lattice in $\mathbb{C}$ associated to X and let $\{w_1, w_2\}$ be the base of Λ given by lemma 5.2.2. The origin of the lattice is identified with the 0 of $\mathbb{C}$. Let S be the open segment on the real line of $\mathbb{C}$ defined by

$$S = (0, \Re(w_1 + w_2)).$$

The interval S projects to an interval in the quotient $\mathbb{C}/\Lambda$, that we still call S. This interval is parallel to the horizontal direction, its left endpoints is p_0 and the right one is on the vertical line passing trough p_0. It follows that the first return map to the section S of the vertical flow on X is a rotation, therefore its combinatorial datum π is rotational. It follows that the marked torus X is representable with length-suspension data (λ, τ) in $\Delta_\pi \times \Theta_\pi$. Now let θ vary in the interval I given by lemma 5.2.2. Let Λ_θ be the lattice in $\mathbb{C}$ associated to X_θ and let $\{w_1(\theta), w_2(\theta)\}$ be the base of Λ_θ given by lemma 5.2.2. Then the embedding in $\mathbb{C}/\Lambda_\theta$ of the horizontal open segment S_θ defined by

$$S_\theta = (0, \Re(w_1(\theta) + w_2(\theta)))$$

gives a section for the vertical flow on X_θ that make the same construction work. Therefore X_θ is suspension of an i.e.t. with the same combinatorial datum π as X, that is a rotation. The lemma is proved. $\square$

For the rotational π determined by lemma 5.2.3 and for $x \in \mathcal{A}$ let us introduce the elements $U_x^{t/b} \in \hom_{\mathbb{Z}}(\mathbb{R}^d, \mathbb{R})$ (with coefficients in $\mathbb{N}$, therefore positive on $\mathbb{R}_+^d$) defined by the formula

$$U_x^{t/b}(\lambda) := \sum_{\pi^{t/b}(y) < \pi^{t/b}(x)} \lambda_y.$$

Their meaning is that if $T = (\pi, \lambda)$ then the singularities for T and T^{-1} are given respectively by $u_x^t = U_x^t(\lambda)$ and $u_y^b = U_y^b(\lambda)$. For the same π let us also call A and M the letters in $\mathcal{A}$ such that respectively $\pi^t(A) = 1$ and $\pi^b(M) = 1$. Since any T in Δ_π is a rotation it is continuous at all u_x^t except for $x = M$. For the same reason T^{-1} is continuous at all u_y^b except for $y = A$. For $T = (\pi, \lambda)$ let us call $\hat{T} = (\pi, \hat{\lambda})$ the corresponding normalized i.e.t. (where $\hat{\lambda} = (\|\lambda\|^{-1}\lambda)$. It easy to see that the rotation number $\alpha = \alpha(T)$ of $T \in \Delta_\pi$ is given by

$$\alpha(T) = U_A^b(\hat{\lambda}). \tag{5.11}$$

In other terms for $\hat{T} : [0,1) \to [0,1)$ we have.

$$\hat{T}x = x + \alpha(T) \mod \mathbb{Z}. \tag{5.12}$$

If π is the rotational element determined by lemma 5.2.3, for any x in $\mathcal{A}$ we consider the relative homology classes $\tilde{\zeta}_x \in H_1(M, \Sigma, \mathbb{Z})$ defined in paragraph 3.2.1. All these elements depend only on π and x.

Let X and I be respectively the flat torus and the open interval in lemma 5.2.2. For any θ in the I we consider the complex numbers $\zeta_x(\theta) := w_\theta(\tilde{\zeta}_x)$. For any $x \in \mathcal{A}$ let us decompose $\zeta_x(\theta)$ as $\zeta_x(\theta) = \lambda_x(\theta) + i\tau_x(\theta)$, where $\lambda_x(\theta)$ and $\tau_x(\theta)$ are real. Lemma 5.2.3 implies that for any $\theta \in I$ the rotated marked torus $R_\theta X$ is representable with the zippered rectangles constructions of paragraph 2.4.1 and moreover it has length-suspension data in $\Delta_\pi \times \Theta_\pi$, that is $\lambda(\theta) \in \Delta_\pi$, $\tau(\theta) \in \Theta_\pi$ and $R_\theta X = X(\pi, \lambda(\theta), \tau(\theta))$ for any $\theta \in I$. In particular the i.e.t. $T_\theta := (\pi, \lambda(\theta))$ is a rotation. As before, the top/bottom singularities of $T_\theta = (\pi, \lambda(\theta))$ are given by $u_x^{t/b}(\theta) = U_x^{t/b}(\lambda(\theta))$ and among these points the only two true discontinuity points are $u_M^t(\theta)$ (for T_θ) and $u_A^b(\theta)$ (for T_θ^{-1}). If for any $\theta \in I$ we put $\hat{\lambda}(\theta) = \|\lambda(\theta)\|^{-1}\lambda(\theta)$ we have a normalized rotation $\hat{T}_\theta = (\pi, \hat{\lambda}(\theta))$. The top singularities for $\hat{T}_\theta$ and $\hat{T}_\theta^{-1}$ are given respectively by $\hat{u}_x^t(\theta) = U_x^t(\hat{\lambda}(\theta))$ and $\hat{u}_y^b(\theta) = U_y^b(\hat{\lambda}(\theta))$. Among these ones the only two true discontinuity point are respectively $\hat{u}_M^t(\theta)$ and $\hat{u}_A^b(\theta)$.

PROPOSITION 5.2.4. *Let us suppose that $\sum_{n=1}^\infty \varphi(n) = \infty$. Then for any pair y, x of letters in $\mathcal{A}$ such that $\pi^t(x), \pi^b(y) > 1$ and almost any θ in the interval I (defined by lemma 5.2.2) there exist infinitely many triples (y, x, n) that are solutions of $|\hat{T}_\theta^n \hat{u}_y^b(\theta) - \hat{u}_x^t(\theta)| \le \varphi(n)$. Anyway these triples may not be all reduced for $\hat{T}_\theta$.*

Proof: (Using part a) of theorem 1.4.2.) Let $(\lambda, \tau) \in \Delta_\pi \times \Theta_\pi$ such that the marked torus X coincides with $X(\pi, \lambda, \tau)$. Lemma 5.1.1 implies the map $\theta \mapsto \hat{T}_\theta$ is the parametrization of a segment in $\Delta_\pi^{(1)}$, in particular the length datum $\hat{\lambda}(\theta)$ of

$\hat{T}_\theta$ is given by

$$\hat{\lambda}(\theta) = \lambda + s_{(\pi,\lambda)}(\theta) f_{(\pi,\lambda)}(\tau)$$

where the linear map $f_{(\pi,\lambda)} : \Theta_\pi \to (\vec{1})$ and the smooth function $s_{(\pi,\lambda)} : I \to \mathbb{R}$ are defined in paragraph 5.1.1. In particular we recall that the function $s_{(\pi,\lambda)}$ is a smooth change of parameter on I. Let y, x be letters in $\mathcal{A}$ such that $\pi^t(x), \pi^b(y) > 1$. By linearity of the functionals U_x^t and U_y^b defined above we have that the singularities $\hat{u}_x^t(\theta)$ and $\hat{u}_y^b(\theta)$ for $\hat{T}_\theta$ are given by

$$\hat{u}_x^t(\theta) = U_x^t(\hat{\lambda}(\theta)) = \hat{u}_x^t(0) + s_{(\pi,\lambda)}(\theta)\hat{v}_x^t$$
$$\hat{u}_y^b(\theta) = U_y^b(\hat{\lambda}(\theta)) = \hat{u}_y^b(0) + s_{(\pi,\lambda)}(\theta)\hat{v}_y^b$$

where $\hat{v}_x^t$ and $\hat{v}_y^b$ are the values respectively of U_x^t and of U_y^b on $f_{(\pi,\lambda)}(\tau)$. The last pair of equations, together with equation (5.12), implies that the quantity $|\hat{T}_\theta^n \hat{u}_y^b(\theta) - \hat{u}_x^t(\theta)|$ equals to

$$\|(\hat{u}_y^b(0) + s_{(\pi,\lambda)}(\theta)\hat{v}_y^b) + n\alpha(\theta) - (\hat{u}_x^t(0) + s_{(\pi,\lambda)}(\theta)\hat{v}_x^t)\|. \tag{5.13}$$

Using equation (5.11) again we see that the rotation number $\alpha(\theta)$ of $\hat{T}_\theta$ is given by

$$\alpha(\theta) = U_A^b(\hat{\lambda}(\theta)) = \hat{u}_A^b(0) + s_{(\pi,\lambda)}(\theta)\hat{v}_A^b$$

with $\hat{v}_A^b = U_A^b(f_{(\pi,\lambda)}\tau)$.

LEMMA 5.2.5. *For any marked torus $X = X(\pi, \lambda, \tau)$ we have $\hat{v}_A^b \neq 0$.*

Proof: We recall from paragraph 5.1.1 that

$$f_{(\pi,\lambda)}(\tau) = \frac{\sum_{x\in\mathcal{A}} \tau_x}{\sum_{y\in\mathcal{A}} \lambda_y}\lambda - \tau.$$

Therefore we have

$$\|\lambda\|\hat{v}_A^b = (\sum_{x\in\mathcal{A}} \tau_x)(\sum_{\pi^b(y)<\pi^b(A)} \lambda_y) - (\sum_{x\in\mathcal{A}} \lambda_x)(\sum_{\pi^b(y)<\pi^b(A)} \tau_y) =$$

$$(\sum_{\pi^b(x)\geq\pi^b(A)} \tau_x)(\sum_{\pi^b(y)<\pi^b(A)} \lambda_y) - (\sum_{\pi^b(x)\geq\pi^b(A)} \lambda_x)(\sum_{\pi^b(y)<\pi^b(A)} \tau_y).$$

Recall the Veech construction of paragraph 2.4.1. For any $x \in \mathcal{A}$ the eight of the interval R_x^t is $h_x = \sum_{\pi^t(y)<\pi^t(x)} \tau_y - \sum_{\pi^b(y)<\pi^b(x)} \tau_y$. Since π is rotational, the condition $\pi^b(x) < \pi^b(A)$ is equivalent to $\pi^t(x) \geq \pi^t(M)$ and $\pi^b(x) \geq \pi^b(A)$ is equivalent to $\pi^t(x) < \pi^t(M)$. Therefore for $\pi^b(x) < \pi^b(A)$ we have

$$h_x = \sum_{\pi^t(y)<\pi^t(M)} \tau_y + \sum_{\pi^t(M)\leq\pi^t(y)<\pi^t(x)} \tau_y - \sum_{\pi^b(y)<\pi^b(x)} \tau_y$$

$$= \sum_{\pi^t(y)<\pi^t(M)} \tau_y = \sum_{\pi^b(x)\geq\pi^b(A)} \tau_y.$$

In the same way we get that for $\pi^b(x) \geq \pi^b(A)$ we have $h_x = -\sum_{\pi^b(y)<\pi^b(A)} \tau_y$. We conclude that $\|\lambda\|\hat{v}_A^b = \text{Area}(X) \neq 0$, ant the lemma is proved. □

Lemma 5.2.5 implies that we can write

$$s_{(\pi,\lambda)}(\theta) = (\alpha(\theta) - \hat{u}_A^b(0))/(\hat{v}_A^b) \tag{5.14}$$

Putting together equations (5.13) and (5.14) we get that for any $y, x \in \mathcal{A}$ such that $\pi^t(x), \pi^b(y) > 1$ there exists real constants $C_{x,y}$ and $D_{x,y}$ (depending on π, λ, τ) such that for any $n \in \mathbb{N}$ we have

$$|\hat{T}_\theta^n \hat{u}_y^b(\theta) - \hat{u}_x^t(\theta)| = \|C_{x,y} + (n + D_{x,y})\alpha(\theta)\|. \tag{5.15}$$

We apply theorem 1.4.2 with the two fixed constants $C_{x,y}$ and $D_{x,y}$ and we get that for almost any $\alpha \in [0, 1)$ there are infinitely many solutions $n \in \mathbb{N}$ of

$$\|C_{x,y} + (n + D_{x,y})\alpha\| \leq \varphi(n).$$

Now we recall that in lemma 5.1.1 we proved that for any data (π, λ, τ) the function $\theta \mapsto s_{(\pi,\lambda)}(\theta)$ is C^∞ and invertible on I. Together with equation (5.14) this implies that the function $I \to [0, 1); \theta \mapsto \alpha(\theta)$ is C^∞ and invertible on I, in particular it preserves sets of zero lebesgue measure. Therefore we get that for any $y, x \in \mathcal{A}$ such that $\pi^t(x), \pi^b(y) > 1$ and for almost any $\theta \in I$ there are infinitely many triples (y, x, n) that are solution of

$$|\hat{T}_\theta^n \hat{u}_y^b(\theta) - \hat{u}_x^t(\theta)| \leq \varphi(n)$$

and proposition is proved. □

5.2.1. Proof of theorem 1.4.1. Let $X \in \mathcal{H}(0, ..0)$. We want to prove a statement that concerns almost any $Y \in \mathrm{SO}(2, \mathbb{R})X$. For any X the set of marked tori Y in $\mathrm{SO}(2, \mathbb{R})X$ that have horizontal or vertical connections form a countable subset of $\mathrm{SO}(2, \mathbb{R})X$. It follows that to prove theorem 1.4.1 it is sufficient to prove that for any $X \in \mathcal{H}(0, .., 0)$ without vertical or horizontal connections there exists an interval $I \subset \mathbb{R}$ containing 0 such that for almost any θ in I the statement holds for the rotated marked torus $R_\theta X$. Given X without vertical or horizontal connections we consider the interval I given by lemma 5.2.2 and let θ vary in I.

We recall that if $\hat{T}_\theta = (\pi, \hat{\lambda}(\theta))$ is the normalized of $T_\theta = (\pi, \lambda(\theta))$, then we have $\lambda(\theta) = \|\lambda(\theta)\|\hat{\lambda}(\theta)$. This implies that

$$|\hat{T}_\theta^n \hat{u}_y^b(\theta) - \hat{u}_x^t(\theta)| = \frac{1}{\|\lambda(\theta)\|}|T_\theta^n u_y^b(\theta) - u_x^t(\theta)|.$$

We decompose I into subsets I_k with $k \in \mathbb{Z}$ such that for any $\theta \in I_k$ we have $2^{k-1} \leq \|\lambda(\theta)\| < 2^k$. Let us suppose that $\sum_{n=1}^\infty \varphi(n) = \infty$. Let us fix any k. Since to multiply φ by a factor 2^k does not affect the divergence (or the convergence) of its series, proposition 5.2 implies that for any y, x such that $\pi^t(x), \pi^b(y) > 1$ and almost any θ in I_k there exists a sequence of triples (y, x, n) that are solutions of

$$|T_\theta^n u_y^b(\theta) - u_x^t(\theta)| \leq \varphi(n).$$

Since the last statement holds for any $k \in \mathbb{Z}$, then it holds for almost any $\theta \in I$. Let us fix y, x such that $\pi^t(x), \pi^b(y) > 1$ and consider a (generic) $\theta \in I$ such that there exist infinitely many triples (y, x, n) as above. There are two cases:

(1) The family of triples (y, x, n) contains a subsequence (y, x, n_i) of triples reduced for T_θ. In this case we keep the subsequence (y, x, n_i).
(2) The family of triples (y, x, n) contains just finitely many reduced triples (possibly 0). In this case we apply corollary 5.1.3 and we get two pairs (y, ν) and (μ, x) in $\mathcal{A}^2$ with $\pi^t(x), \pi^t(\nu), \pi^b(y), \pi^b(\mu) > 1$ that correspond to two infinite families of triples (y, ν, l) and (μ, x, h) reduced for T_θ and solutions of equation (1.3).

Without any loss of generality we can now suppose that $\mathrm{Area}(X) = 1$ and therefore that $\mathrm{Area}(R_\theta X) = 1$ for all $\theta \in I$. (If $\mathrm{Area}(X) \neq 1$, since the multiplication of φ by a constant keep its integral divergent, we can still make the argument work). We can also suppose that $R_\theta X$ has uniquely ergodic vertical flow, since this is true for almost any $\theta \in I$. Then we apply proposition 3.2.3 and the two cases above lead to the following two possible situations: in case (1) the configuration $\mathcal{C}^{(y,x)}(R_\theta X)$ contains infinitely many solutions of equation (1.4), in case (2) the two configurations $\mathcal{C}^{(y,\nu)}(R_\theta X)$ and $\mathcal{C}^{(\mu,x)}(R_\theta X)$ contain infinitely many solutions of equation (1.4).

Now we consider all the pairs $(y, x) \in \mathcal{A}^2$ with $\pi^t(x), \pi^b(y) > 1$, we apply the argument above and we look at all the different configurations $\mathcal{C}^{(w,v)}(R_\theta X)$ that contain infinitely many solutions of equation (1.4) obtained in this way. The worst case is when for all the pairs x, y we always have case (2) and moreover the letters μ and ν that appear applying corollary 5.1.3 are always the same. In this case we have exactly $2r - 1$ different configurations containing each one infinitely many solutions of equation (1.4). In all the other cases the different configurations are always more than $2r - 1$. Theorem 1.4.1 is proved.

5.3. Proof of the arithmetic statement.

This paragraph is devoted to the proof of theorem 1.4.2. We recall the statement. We consider equation (1.9), that is

$$\{(n + x)\alpha - y\} < \varphi(n)$$

where $\varphi : \mathbb{N} \to \mathbb{R}_+$ is a positive sequence such that $n\varphi(n)$ is decreasing monotone and $v = (x, y)$ is any fixed vector in $\mathbb{R}^2$. Then theorem 1.4.2 says:

- **a:** *If $\sum_{n=1}^{\infty} \varphi(n) < +\infty$, then for almost every $\alpha \in \mathbb{R}_+$ there exist finitely many solutions $n \in \mathbb{N}$ of equation (1.9).*
- **b:** *If $\sum_{n=1}^{\infty} \varphi(n) = \infty$ then for almost any α equation (1.9) has infinitely many solutions $n \in \mathbb{N}$.*

Part a) of the theorem correspond to the easy half of the Borel-Cantelli argument and we can prove it directly here. Let us fix any pair of integers m and n in $\mathbb{N}$ and let us consider the set

$$\mathcal{C}_n^m := \{\alpha \in [m, m + 1); \{(n + x)\alpha - y\} < \varphi(n)\}.$$

We also consider the function $f_{n,m} : [m, m + 1) \to [0, 1)$, defined by $f_{n,m}(\alpha) := \{(n+x)\alpha - y\}$, which is piecewise linear and has the same slope on all its branches. The branches of $f_{n,m}$ are defined on sub-intervals of $[m, m + 1)$, if we exclude the leftmost and the rightmost of these intervals all the others have the same length (equal to $1/(n + x)$) and restricted to them $f_{n,m}$ is surjective onto $[0, 1)$. It follows than for any $\epsilon > 0$ and any n big enough, for any subinterval I of $[0, 1)$ we have

$$\mathrm{Leb}(f_{n,m}^{-1}(I)) \leq (1 + \epsilon)\mathrm{Leb}(I).$$

In particular for any m in $\mathbb{N}$ and for any n big enough we have $\mathrm{Leb}(\mathcal{C}_n^m) \leq (1 + \epsilon)\varphi(n)$. Since φ has convergent series it follows that

$$\sum_{n=1}^{\infty} \mathrm{Leb}(\mathcal{C}_n^m) < +\infty$$

and therefore equation (1.9) has just finitely many solutions for almost any α in the interval $[m, m+1)$. Since the same argument works on any other interval $[m, m+1)$ then part a) of theorem 1.4.2 follows.

Part b) of theorem 1.4.2 is more complicated and its proof takes the remaining of this section. In the next two paragraphs we recall some classical facts about the *continued fraction algorithm* (paragraph 5.3.1, see [**K**] for more details) and develop some useful machinery (paragraph 5.3.2, which is a rephrasing of some ideas coming from [**Y2**] at pages 105,106.).

5.3.1. About the classical continued fraction. For two vectors $v, w \in \mathbb{R}^2_+$ we write

$$v \prec w \text{ if } \det(v,w) > 0 \texttt{ and } v \preceq w \text{ if } \det(v,w) \geq 0$$

where (v, w) denotes the matrix with v as first column vector and w as second column vector. Let us define $r_{-2} := (1,0) \in \mathbb{N}^2$ and $r_{-1} := (0,1) \in \mathbb{N}^2$. For any $\alpha \in \mathbb{R}_+$ we define a vector $r_0 = r_0(\alpha) \in \mathbb{N}^2$ by

$$r_0(\alpha) := a_0(\alpha) r_{-1} + r_{-2}$$

where $a_0(\alpha) \in \mathbb{N}$ is such that $a_0 r_{-1} + r_{-2} \preceq (1,\alpha) \prec (a_0+1) r_{-1} + r_{-2}$. Let us write the vector $r_0(\alpha)$ as $r_0(\alpha) = (q_0(\alpha), p_0(\alpha))$, with $q_0(\alpha), p_0(\alpha) \in \mathbb{N}$. Letting α vary in $\mathbb{R}_+$ we introduce the family of integer vectors $Q_0 := \{r_0(\alpha); \alpha \in \mathbb{R}_+\} \subset \mathbb{N}^2$ and the partition $\mathcal{Q}_0 := \{I(r_0); r_0 \in Q_0\}$ whose elements are the intervals $I(r_0)$ with constant value for the function $\alpha \mapsto r_0(\alpha)$.

Now let us suppose that for any $\alpha > 0$ and for all $i < n$ we have defined all the vectors $r_i(\alpha) = (q_i(\alpha), p_i(\alpha))$, with $q_i(\alpha), p_i(\alpha) \in \mathbb{N}$, the associated families of integer vectors $Q_i := \{r_i(\alpha); \alpha \in \mathbb{R}_+\}$ and the partitions $\mathcal{Q}_i := \{I(r_i); r_i \in Q_i\}$ whose elements are the intervals $I(r_i) \subset \mathbb{R}_+$ with specified value for the function $\alpha \mapsto r_i(\alpha)$. Then we define by induction

$$r_n(\alpha) := a_n(\alpha) r_{n-1}(\alpha) + r_{n-2}(\alpha) \tag{5.16}$$

where $a_n(\alpha)$ is the positive integer such that

$$a_n r_{n-1} + r_{n-2} \preceq (1,\alpha) \prec (a_n+1) r_{n-1} + r_{n-2} \text{ if } n \text{ is even}$$

and

$$(a_n+1) r_{n-1} + r_{n-2} \prec (1,\alpha) \preceq a_n r_{n-1} + r_{n-2} \text{ if } n \text{ is odd .}$$

We write $r_n(\alpha) = (q_n(\alpha), p_n(\alpha))$ with $q_n(\alpha), p_n(\alpha) \in \mathbb{N}$ and letting α vary in $\mathbb{R}_+$ we define the family of integer vectors $Q_n := \{r_n(\alpha); \alpha \in \mathbb{R}_+\}$ and the partition $\mathcal{Q}_n := \{I(r_n); r_n \in Q_n\}$ whose elements are the intervals $I(r_n) \subset \mathbb{R}_+$ with specified value for the function $\alpha \mapsto r_n(\alpha)$. For any $n \in \mathbb{N}$ we also consider the σ-algebra generated by the partition $\mathcal{Q}_n$ and we still call it $\mathcal{Q}_n$.

Definition 5.3.1. For any fixed $\alpha \in \mathbb{R}_+$ we say that the vectors $r_n(\alpha) \in \mathbb{N}^2$ defined above are the approximations of α with respect to the continued fraction algorithm.

Let us now consider any interval J contained in $\mathbb{R}_+$ (endpoints included or not) and define

$$i(J) := \min \left\{ k \in \mathbb{N};\, \exists r = (q,p) \in Q_{16k-8} \texttt{ such that } \frac{p}{q} \in \overline{J} \right\}. \tag{5.17}$$

Let us define $J_{i(J)} \subset I$ as the maximal subinterval that is measurable with respect to $\mathcal{Q}_{16i(J)}$. It is easy to show that $J_{i(J)}$ is not empty. There are two possibilities: either $J = J_{i(J)}$ and the endpoints of J are both of the form p/q with $(q,p) \in Q_{16i(J)}$, or $J \setminus J_{i(J)} \neq \emptyset$. In this second case we define a prober subset of J by $\mathcal{J}_{i(J)+1} := J \setminus J_{i(J)}$. If one of the two endpoints of J is of the form p/q with $(q,p) \in Q_{16i(J)}$ then the subset $\mathcal{J}_{i(J)+1}$ consists of just one interval, if not it is a pair of disjoint intervals.

Let us fix any integer $k > i(J)$ and suppose by induction that for any j with $i(J) \leq j < k$ we have defined a decomposition

$$J = J_{i(J)} \cup ... \cup J_j \cup \mathcal{J}_{j+1}, \tag{5.18}$$

of J with the following property: $J_{i(J)} \cup ... \cup J_j$ is the maximal subinterval of J that is measurable with respect to $\mathcal{Q}_j$ and for any l with $i(I) < l \leq j$ the subset J_l is an interval or a pair of disjoint intervals and is measurable with respect to $\mathcal{Q}_l$ but not for $\mathcal{Q}_{l-1}$. Note that this property implies that the rest $\mathcal{J}_{j+1}$ is empty, an interval or a pair of intervals according to the fact that both, just one, or none of the endpoints of J are of the form p/q with $(q,p) \in Q_j$. Then the decomposition of I in equation (5.18) is defined also for k as follows. If J has both its endpoints of the form p/q with $(q,p) \in \mathcal{Q}_{k-1}$, then $\mathcal{J}_k = \emptyset$ and we put also $J_k = \emptyset$ and $\mathcal{J}_{k+1} = \emptyset$. In the other case $\mathcal{J}_k$ is a non-empty subset of J, we define J_k as the maximal subset of $\mathcal{J}_k$ which is measurable with respect to $\mathcal{Q}_k$ and we put $\mathcal{J}_{k+1} := \mathcal{J}_k \setminus J_k$.

LEMMA 5.3.2. *Let $J \subset$ be any interval contained in $\mathbb{R}_+$. For any $k \geq i(J)$ we have*

$$\mathrm{Leb}(\mathcal{J}_{k+1}) \leq (1/2)^{k+1-i(J)} \mathrm{Leb}(J_{i(J)}). \tag{5.19}$$

Moreover if there exists some $k \in \mathbb{N}$ such that $J_k = \emptyset$ then $\mathcal{J}_k = \emptyset$ and J has endpoints in $\mathbb{Q}$.

The proof is consequence of very classical results on the continued fraction algorithm and is left to the reader.

5.3.2. Twisted continued fraction. Let us call $\mathcal{G}$ the semidirect product between $\mathrm{SL}(2,\mathbb{Z})$ and $\mathbb{Z}^2$ defined with the standard action of $\mathrm{SL}(2,\mathbb{Z})$ on $\mathbb{Z}^2$. The elements $(A,b) \in \mathcal{G}$ with $A \in \mathrm{SL}(2,\mathbb{Z})$ and $b \in \mathbb{Z}^2$ act on $\mathrm{GL}(2,\mathbb{R}) \times \mathbb{R}^2$ by

$$(\Lambda, v) \mapsto (\Lambda A, \Lambda b + v).$$

Starting from the base of $\mathbb{Z}^2$ given by $\Lambda := \{r_{-2} = (1,0); r_{-1} = (0,1)\}$, the column vectors of the matrix ΛA, written as $r = (q,p)$, give rise to the approximations p/q for α described in paragraph 5.3.1. For any fixed $v \in \mathbb{R}^2$ the vectors $\{s = \Lambda A b + v\,;\, A \in \mathrm{SL}(2,\mathbb{Z}), b \in \mathbb{Z}^2\}$, written as $s = (q+x, p+y)$, with $p,q \in \mathbb{Z}$ and $q+x, p+y > 0$ and where $v = (x,y)$, give rise to approximations for α of the form $\frac{p+y}{q+x}$. The classical continued fraction gives for any α a sequence of good rational approximations p_n/q_n. Here we want to do the same: for any α we define a sequence of good approximations s_n chosen in the set of points $\frac{p+y}{q+x}$.

Let us fix any $v \in \mathbb{R}^2$. For any $r_0 \in Q_0$ let us put $r'_0 := r_0 + r_{-1}$ and define $\Lambda(r_0)$ as the fundamental domain in $\mathbb{C}$ spanned by the base of $\mathbb{Z}^2$ given by $\{r_0, r'_0\}$. In general for any $k \in \mathbb{N}$ let us introduce the notation $r'_k := r_k + r_{k-1}$, Let us write r'_k as $r'_k = (q'_k, p'_k)$ with $q'_k, p'_k \in \mathbb{N}$ and then define the fundamental domain $\Lambda(r_k)$

spanned by the pair of vectors (r_k, r'_k). For $k \in \mathbb{N}$ and $r_k \in \mathcal{Q}_k$ let also define $v(r_k)$ as the element of minimal length between the vectors $w \in \{v + \mathbb{Z}^2\}$ that satisfy

$$\det(r_k, w), \det(w, r'_k) > 1. \tag{5.20}$$

Observe that with this definition we have

$$\Re(v(r_k)) < 2(q_k + q'_k). \tag{5.21}$$

Let us fix $\alpha > 0$. We observe that for k even the k-th approximant $r_k(\alpha) = (q_k(\alpha), p_k(\alpha))$ satisfies $r_k \preceq (1, \alpha) \prec r'_k$. For any $n \in \mathbb{N}$ we first consider the $16(n-1)$-th approximation $r_{16(n-1)}(\alpha) \in Q_{16(n-1)}$ of α with respect to the continued fraction algorithm. Then we define the n-th *twisted approximation* $s_n(\alpha)$ according to the following two cases:

- If $r_{16(n-1)}(\alpha) \preceq (1, \alpha) \prec v(r_{16(n-1)}(\alpha))$ we define
$$\nu_n(\alpha) := \min\{\nu \in \mathbb{N}; v(r_{16(n-1)}) + \nu r_{16(n-1)} \preceq (1, \alpha)\}.$$
Observe that in this case we always have $\nu_n(\alpha) \geq 1$. Then we define
$$s_n := v(r_{16(n-1)}) + \nu_n r_{16(n-1)}. \tag{5.22}$$
- If $v(r_{16(n-1)}(\alpha)) \preceq (1, \alpha) \prec r'_{16(n-1)}(\alpha)$ we define
$$\nu_n(\alpha) := \max\{\nu \in \mathbb{N}; v(r_{16(n-1)}) + \nu r'_{16(n-1)} \prec (1, \alpha)\}$$
In this case we may also have $\nu_n(\alpha) = 0$. Then we define
$$s_n := v(r_{16(n-1)}) + \nu_n r'_{16(n-1)}. \tag{5.23}$$

For any $n \in \mathbb{N}$ we write $s_n(\alpha) = (k_n(\alpha) + x, j_n(\alpha) + y)$ with $k_n(\alpha), j_n(\alpha) \in \mathbb{Z}$. Observe that we always have $s_n(\alpha) \prec (1, \alpha)$. Letting α vary in $\mathbb{R}_+$ we define the family of vectors

$$P_n := \{s_n(\alpha)\,;\, \alpha \in \mathbb{R}_+\} \subset (\mathbb{Z}^2 + v) \cap \mathbb{R}^2_+$$

and the partition $\mathcal{P}_n := \{I(s_n); s_n \in P_n\}$ whose elements are the intervals $I(s_n) \subset \mathbb{R}_+$ with specified value for the function $\alpha \mapsto s_n(\alpha)$. We also consider the σ-algebra generated by the partition $\mathcal{P}_n$ and we still call it $\mathcal{P}_n$. We observe that for any $n \in \mathbb{N}$ the σ-algebra $\mathcal{P}_n$ is a refinement of $\mathcal{Q}_{16(n-1)}$.

Definition 5.3.3. For any fixed $\alpha \in \mathbb{R}_+$ we say that the vectors $s_n(\alpha) \in (\mathbb{Z}^2 + v) \cap \mathbb{R}^2_+$ defined by equation (5.22) or (5.23) are the *twisted approximations* of α with respect to $v + \mathbb{Z}^2$.

Lemma 5.3.4. *For any $\alpha > 0$ Let us consider the sequences of the approximations r_n (defined by equation (5.16)) and the twisted ones s_n (defined by equations (5.22) or (5.23)). Then we have*

$$k_n + x < q_{16n-8}. \tag{5.24}$$

Proof: We first suppose that we are in the case $r_{16(n-1)}(\alpha) \preceq (1, \alpha) \prec v(r_{16(n-1)}(\alpha))$. In this case we have

$$k_n + x < 2q_{16n-15} + 3q_{16(n-1)}) < q_{16n-13} + q_{16n-14} \leq q_{16n-8}.$$

On the other hand, if we are in the case $v(r_{16(n-1)}(\alpha)) \preceq (1, \alpha) \prec r'_{16(n-1)}(\alpha)$, we have

$$k_n + x < 2q_{16n-14} + 2q_{8n-13} < 2q_{16n-12} \leq q_{16n-8}.$$

The lemma is proved. □

If $s_n(\alpha)$ is defined according to equation (5.22), that is we are in the case $r_{16(n-1)}(\alpha) \preceq (1,\alpha) \prec v(r_{16(n-1)}(\alpha))$, then we define

$$s'_n := s_n - r_{16(n-1)}$$

If $s_n(\alpha)$ is defined according to equation (5.23), that is we are in the case $v(r_{16(n-1)}(\alpha)) \preceq (1,\alpha) \prec r'_{16(n-1)}(\alpha)$, then we define

$$s'_n := s_n + r'_{16(n-1)}.$$

Let us write $s'_n = (k'_n + x, j'_n + y)$, with $k'_n, j'_n \in \mathbb{Z}$. Observe that by definition we have $s_n(\alpha) \prec (1,\alpha) \prec s'_n(\alpha)$. If $s_n(\alpha)$ is defined by equation (5.22) we have

$$\det(s_n, s'_n) = \det(r_{16(n-1)}, v(r_{16(n-1)}))$$

If $s_n(\alpha)$ is defined by equation (5.23) we have

$$\det(s_n, s'_n) = \det(v(r_{16(n-1)}), r'_{16(n-1)})$$

In both cases, recalling that $\det(r_{16(n-1)}, r'_{16(n-1)}) = 1$, equation (5.20) implies

$$1 \leq \det(s_n, s'_n) < 2. \tag{5.25}$$

LEMMA 5.3.5. *For any $n \in \mathbb{N}$ and for any atom $I(s_n)$ of $\mathcal{P}_n$ we have*

$$i(I(s_n)) = n. \tag{5.26}$$

Proof: First of all we observe that any atom $I(s_n)$ of $\mathcal{P}_n$ is contained in some atom $I(r_{16(n-1)})$ of $\mathcal{Q}_{16(n-1)}$, therefore there is no rational $p_{16(n-1)-4}/q_{16(n-1)-8}$ contained in $I(s_n)$, thus $i(J) \geq n$.

Now we prove that $i(I(s_n)) \leq n$. Let us put $k := i(I(s_n))$ and suppose that $k > n$. By definition, there is no rational $r_{16(k-1)-8}$ contained in $I(s_n)$, that means that $I(s_n)$ is a subinterval of some atom of $\mathcal{Q}_{16(k-1)-8}$. Since $k > n$, $I(s_n)$ is a subinterval of some atom $I(r_{16n-8})$ of $\mathcal{Q}_{16n-8}$. We have

$$|I(r_{16n-8})| = \frac{1}{q_{16n-8}q'_{16n-8}}.$$

On the other side we have

$$|I(s_n)| = \frac{|\det(s_n, s'_n)|}{(k_n + x)(k'_n + x)},$$

therefore equation (5.25) implies that

$$q_{16n-8}q'_{16n-8}|\det(s_n, s'_n)| < (k_n + x)(k'_n + x).$$

Equation (5.24) in lemma 5.3.4 says that $k_n + x < q_{16n-8}$, that implies that we have also $k'_n + x < q'_{16n-8}$, therefore the last condition is absurd. The lemma is proved. □

LEMMA 5.3.6. *Let J be any interval contained in some atom $I(s_n)$ of $\mathcal{P}_n$. Then*

$$\text{Leb}(J) \leq (1/2)^{i(J)-n}\text{Leb}(I(s_n)). \tag{5.27}$$

Proof: We put $I := I(s_n)$ and $k := i(J)$. If $k = n$ the lemma is trivially true, thus we can suppose that $k > n$. We use the decomposition of I defined by equation (5.18) until the k-th step:

$$I = I_n \cup ... \cup I_{k-1} \cup \mathcal{I}_k.$$

Since $i(J) = k$ then J is contained in some atom $I(r_{16k-24})$ of $\mathcal{Q}_{16k-24}$, therefore either $J \subset I_l$ for some $l = n, .., k-1$ or $J \subset \mathcal{I}_k$. In the second case the lemma follows immediately from lemma 5.3.2, in the first we just recall that the length of an atom $I(r_n)$ of $\mathcal{Q}_n$ is $1/(q_n q'_n)$ and the uniform estimate $q_n(\alpha) < 2q_{n+2}(\alpha)$ holds for any $n \in \mathbb{N}$ and any $\alpha \in \mathbb{R}$, the lemma is therefore proved. $\square$

5.3.3. A sufficient condition. For any fixed $v = (x, y) \in \mathbb{R}^2$ and for almost every $\alpha \in \mathbb{R}_+$ we look for a condition on α sufficient to have infinite solutions $n \in \mathbb{N}$ of

$$\{(n + x)\alpha - y\} < \varphi(n).$$

For any $\alpha > 0$ we consider the twisted approximations $s_n(\alpha)$ with respect to v. Let us denote $\hat{\alpha} := (1, \alpha) \in \mathbb{R}^2_+$. Since for any $n \in \mathbb{N}$ we have by definition $s_n(\alpha) \prec \hat{\alpha}$, then

$$\det(s_n(\alpha), \hat{\alpha}) = \begin{vmatrix} k_n(\alpha) + x & 1 \\ j_n(\alpha) + y & \alpha \end{vmatrix} = (k_n(\alpha) + x)\alpha - j_n(\alpha) - y > 0.$$

On the other hand

$$|\det(s_n(\alpha), \hat{\alpha},)| \leq \frac{|\hat{\alpha}|}{|s_n(\alpha)|} |\det(s'_n(\alpha), s_n(\alpha))| \leq 2\frac{|\hat{\alpha}|}{|s_n(\alpha)|} \to 0$$

as $n \to \infty$, therefore for n big enough we have $\{(k_n(\alpha) + x)\alpha - y\} = \det(s_n(\alpha), \hat{\alpha})$. For any $n \in \mathbb{N}$ and for any fixed $s_n \in P_n$ and the associated interval $I(s_n) \in \mathcal{P}_n$ we define a bijective linear function $\Upsilon[s_n] : I(s_n) \to [0, 1)$ by the formula

$$\Upsilon[s_n](\alpha) := \frac{k'_n + x}{\det(s_n, s'_n)} \det(s_n, \hat{\alpha}). \tag{5.28}$$

The property of the family of functions $\{\Upsilon[s_n]\}_{s_n \in \mathcal{P}_n}$ that is sufficient to prove theorem 1.4.2 is the following:

PROPOSITION 5.3.7. *If $\psi : \mathbb{N} \to \mathbb{R}_+$ is a positive sequence such that $\sum_{n\in\mathbb{N}} \psi_n = +\infty$ then for almost every $\alpha \in \mathbb{R}_+$ we have infinite solutions $n \in \mathbb{N}$ of*

$$\Upsilon[s_n(\alpha)](\alpha) < \psi_n. \tag{5.29}$$

Proposition 5.3.7 will be proved in the next (and last) paragraph, here we show how it implies theorem 1.4.2.

For any $n \in \mathbb{N}$ and any $s_n \in \mathcal{P}_n$ the function $\Upsilon[s_n](\alpha)$ defined by equation (5.28) satisfies

$$\{(k_n(\alpha) + x)\alpha - y\} = \frac{\det(s_n, s'_n)}{k'_n + x} \Upsilon[s_n](\alpha).$$

Therefore for any $\alpha \in \mathbb{R}_+$ in order to have infinite solutions of equation (1.9) it is sufficient to have infinite solutions $n \in \mathbb{N}$ of

$$\Upsilon[s_n(\alpha)](\alpha) \leq \frac{k'_n(\alpha) + x}{\det(s_n(\alpha), s'_n(\alpha))} \varphi(k_n(\alpha)).$$

It is a well known fact (see [**K**]) that for the standard continued fraction exists a constant $\nu > 0$ such that for almost every $\alpha \in \mathbb{R}_+$ the denominators $q_n = q_n(\alpha)$ satisfy $q_n \leq e^{n\nu}$ for any n big enough. Since for any $\alpha > 0$ and any n we have $k_n(\alpha) + x < q_{8n}(\alpha)$ and $k'_n(\alpha) + x < q_{8n}(\alpha)$ then there exists a positive real number

$\gamma > 0$ such that for any n big enough and for almost every $\alpha \in \mathbb{R}_+$ we have $k_n(\alpha) + x < e^{n\gamma}$ and $k_n(\alpha) + x < e^{n\gamma}$. We put

$$\psi_n := e^{n\gamma}\varphi(e^{n\gamma}).$$

Since by assumption $t \mapsto t\varphi(t)$ is monotone, recalling that for equation (5.25) in paragraph 5.3.2 we always have $1 \leq \det(s_n, s'_n) < 2$, for almost every $\alpha \in \mathbb{R}_+$ and for n big enough we have

$$\psi_n < \frac{k'_n(\alpha) + x}{\det(s_n(\alpha), s'_n(\alpha))}\varphi(k_n(\alpha)).$$

It follows that to prove theorem 1.4.2 it is sufficient to prove for almost any $\alpha \in \mathbb{R}_+$ there exist infinite solutions $n \in \mathbb{N}$ of $\Upsilon[s_n(\alpha)](\alpha) < \psi_n$. It is a simple computation to see that if $\sum_{n\in\mathbb{N}} \varphi(n) = \infty$ that also $\sum_{n\in\mathbb{N}} \psi(n) = \infty$, therefore the sequence ψ_n satisfy the hypothesis in proposition 5.3.7 and theorem 1.4.2 follows.

5.3.4. The sufficient condition has total measure. In this paragraph we prove proposition 5.3.7. Let us define

$$\mathcal{C}_n := \{\alpha \in \mathbb{R}_+ \,;\, \Upsilon[s_n(\alpha)](\alpha) \geq \psi_n\} \tag{5.30}$$

and $\mathcal{C} := \bigcup_{m\in\mathbb{N}} \bigcap_{n\geq M} \mathcal{C}_n$. Proposition 5.3.7 says that $\mathrm{Leb}(\mathcal{C}) = 0$ and in order to show it it is sufficient to prove that for any $M \in \mathbb{N}$ we have $\mathrm{Leb}(\bigcap_{n\geq M} \mathcal{C}_n) = 0$. Thus we fix any $M \in \mathbb{N}$ and we re-define $\mathcal{C} := \bigcap_{n\geq M} \mathcal{C}_n$. To show that $\mathcal{C}$ has zero measure we define a nested family of sets

$$\hat{\mathcal{C}}_n \supset \hat{\mathcal{C}}_{n+1} ...$$

such that $\mathcal{C} \subset \hat{\mathcal{C}}_n$ for any $n \in \mathbb{N}$ and $\mathrm{Leb}(\hat{\mathcal{C}}_n) \to 0$ as $n \to \infty$.

For any atom $I(s_n)$ of $\mathcal{P}_n$ the function $\Upsilon[s_n] : I(s_n) \to [0, 1)$ is bijective and linear, therefore $\Upsilon[s_n]^{-1}[0, \psi_n)$ is a subinterval of $I(s_n)$ and we have

$$|I(s_n) \cap \mathcal{C}_n| = (1 - \psi_n)|I(s_n)|. \tag{5.31}$$

Let us recall the index $i(J) \in \mathbb{N}$ associated by equation (5.17) to any interval J contained in $\mathbb{R}_+$. Lemma 5.3.5 says that $i(I(s_n)) = n$, therefore

$$i(I(s_n) \cap \mathcal{C}_n) \geq n. \tag{5.32}$$

Here we define the sets $\hat{\mathcal{C}}_n$. The first element of the family will be defined for $n = M$. We put $\hat{\mathcal{C}}_M := \mathcal{C}_M$. The connected components of $\hat{\mathcal{C}}_M$ are the subintervals of the atoms $I(s_M)$ of the σ-algebra $\mathcal{P}_M$ defined by the condition $\Upsilon[s_M](\alpha) \geq \psi_M$. We observe that equation (5.32) implies that for any connected component J of $\hat{\mathcal{C}}_M$ we have $i(J) \geq M$. For $n > M$ the definition of $\hat{\mathcal{C}}_n$ is less simple and is given by induction.

Let us fix $n \in \mathbb{N}$ and suppose that for any integer $k < n$ the sets $\hat{\mathcal{C}}_k$ are defined. We also assume by induction that for any $k \in \{1, .., n-1\}$ and for any connected component J of $\hat{\mathcal{C}}_k$ we have $i(J) \geq k$. Now we complete the inductive step. Let us consider a connected component J of $\hat{\mathcal{C}}_{n-1}$. By your inductive hypothesis there are two cases.

- if $i(J) > n-1$ then we put $J \cap \hat{\mathcal{C}}_n := J$, that is the interval pass unchanged to step n of the iterative construction. We observe that in this case we trivially have $i(J) \geq n$.

- if $i(J) = n-1$ we decompose it as $J = J_{n-1} \cup \mathcal{J}_n$ according to equation (5.18). The subinterval J_{n-1} is $\mathcal{Q}_{8(n-1)}$-measurable, therefore $\mathcal{P}_n$-measurable and we put $J_{n-1} \cap \hat{\mathcal{C}}_n := J_{n-1} \cap \mathcal{C}_n$. We observe that equation (5.32) implies that if J' is any connected component in this intersection we have $i(J') \geq n$. The other part remains unchanged, that is we put $\mathcal{J}_n \cap \hat{\mathcal{C}}_n := \mathcal{I}_n$. As we have seen in the discussion of the decomposition in equation (5.17), the rest $\mathcal{I}_n$ is empty or consists of one or two intervals. If J' is a connected component of $\mathcal{I}_n$ it cannot contain any atom of $\mathcal{Q}_{8(n-1)}$, therefore $i(J') \geq n$.

The inductive step is complete, therefore the sets $\hat{\mathcal{C}}_n$ are defined for all $n \in \mathbb{N}$ and they have the property that if J is a connected component of $\hat{\mathcal{C}}_n$, than we have $i(J) \geq n$.

For any $n \in \mathbb{N}$ and any $M \leq k < n$ let us define

$$\Gamma_k^n := \max_{A \subset \{k+1,..n\}} \{(7/8)^{n-k-\sharp(A)} \prod_{i \in A} (1 - \psi_i/2)\}.$$

For any $n > M$ we also set $\Gamma_n^n := 1$.

LEMMA 5.3.8. *For any $n \in \mathbb{N}$, any $0 \leq k < n$ and any connected component J of $\hat{\mathcal{C}}_k$ with $i(J) = k$ we have*

$$\mathrm{Leb}(J \cap \hat{\mathcal{C}}_n) \leq \Gamma_k^n \mathrm{Leb}(J). \tag{5.33}$$

Proof: During all the proof of this lemma, for any interval J we will use the notation $|J| := \mathrm{Leb}(J)$. We first show that if the interval J is a connected component of $\hat{\mathcal{C}}_{n-1}$ with $i(J) = n-1$ then

$$|J \cap \hat{\mathcal{C}}_n| \leq (1 - \psi_n/2)|J|.$$

To see this we consider the decomposition $J = J_{n-1} \cup \mathcal{J}_n$ defined in equation (5.18) and we recall that $J \cap \hat{\mathcal{C}}_n = (J_{n-1} \cap \mathcal{C}_n) \cup \mathcal{J}_n$, this is possible since J_{n-1} is $\mathcal{Q}_{8(n-1)}$-measurable and therefore $\mathcal{P}_n$-measurable. Equation (5.31) implies $|J_{n-1} \cap \mathcal{C}_n| = (1-\psi_n)|J_{n-1}|$ and thus

$$|J \cap \hat{\mathcal{C}}_n| = (1-\psi_n)|J_{n-1}| + |\mathcal{J}_n| = \left((1-\psi_n)\frac{|J_{n-1}|}{|J|} + \frac{|\mathcal{J}_n|}{|J|}\right)|J|$$

$$= \left(1 - (1 - \frac{|\mathcal{J}_n|}{|J|})\psi_n\right)|J| \leq (1 - \psi_n/2)|J|,$$

where the last inequality follows from lemma 5.3.2. For any fixed $n \in \mathbb{N}$ the proof of the lemma is by descending induction on k. The argument above proves the first step of the induction, that corresponds to $k = n-1$.

Now, for any fixed n, we suppose by induction hypothesis that the lemma is proved for any $j \in \{k, .., n-1\}$ and we consider a connected component J of $\hat{\mathcal{C}}_{k-1}$ with $i(J) = k-1$. We decompose J as $J = J_{k-1} \cup \mathcal{J}_k$ according to equation (5.18). We have that J_{k-1} is $\mathcal{Q}_{8(k-1)}$-measurable and therefore $\mathcal{P}_k$-measurable, thus

$$|J \cap \hat{\mathcal{C}}_n| = |(J \cap \hat{\mathcal{C}}_k) \cap \hat{\mathcal{C}}_n| = |(J_{k-1} \cap \mathcal{C}_k) \cap \hat{\mathcal{C}}_n| + |\mathcal{J}_k \cap \hat{\mathcal{C}}_n|$$

Our aim is to show that $|J \cap \hat{\mathcal{C}}_n| \leq \Gamma_{k-1}^n |J|$, in order to do that we have to take account of the two summands in the last equation. We first consider the second one. Let us write $\mathcal{J}_k = J'_1 \cup J'_2$, where J'_1 and j'_2 are the connected components of $\mathcal{J}_k$. We observe that, as a consequence of lemma 5.3.2, for both $\epsilon = 1, 2$, either J'_ϵ is empty, or it musts satisfy $i(J'_\epsilon) = k$. In the second case we can apply the inductive hypothesis to J'_ϵ and we get $|J'_\epsilon \cap \hat{\mathcal{C}}_n| \le \Gamma^n_k |J'_\epsilon|$. In the first case this last inequality keeps trivially true. Summing over $\epsilon = 1, 2$ we have

$$|\mathcal{J}_k \cap \hat{\mathcal{C}}_n| \le \Gamma^n_k |\mathcal{J}_k|. \tag{5.34}$$

Here we pass to the first summand. The connected components of $\hat{\mathcal{C}}_k$ belonging to $J_{k-1} \cap \mathcal{C}_k$ are the intervals J' that arise from the intersection of an atom of $\mathcal{P}_k$ with $\mathcal{C}_k$. For these intervals the index $l = i(J')$ defined by equation (5.17) may take all the values from k to ∞. For any such value l we define the set $\mathcal{P}_{k,l}(J_{k-1})$ of those $s_k \in P_k$ such that, if $I(s_k)$ is the associated atom of $\mathcal{P}_k$, then $J'(s_k) := I(s_k) \cap \mathcal{C}_k$ has index $i(J'(s_k)) = l$. We have

$$J_{k-1} \cap \mathcal{C}_k = \bigcup_{l=k}^{\infty} \bigcup_{s_k \in \mathcal{P}_{k,l}(J_{k-1})} J'(s_k)$$

We separate 3 cases.

- If $l = k$. For any $s_k \in \mathcal{P}_{k,k}(J_{k-1})$ equation (5.31) says that
$$|J'(s_k)| = (1 - \psi_k)|I(s_k)|.$$
Moreover since $J'(s_k)$ is a connected component of $\hat{\mathcal{C}}_k$ with $i(J'(s_k)) = k$ the inductive hypothesis of the lemma implies that
$$|J'(s_k) \cap \hat{C}_n| \le \Gamma^n_k |J'(s_k)|,$$
which together with the first inequality gives us
$$|I(s_k) \cap \hat{C}_n| \le (1 - \psi_k)\Gamma^n_k |I(s_k)|.$$
- $l \in \{k+1, .., n-1\}$. As a consequence of the iterative construction of the sets $\hat{\mathcal{C}}_n$, for any $s_k \in \mathcal{P}_{k,l}(J_{k-1})$ the interval $J'(s_k)$ keeps untouched until the step $l+1$, that is it is a connected component of $\hat{\mathcal{C}}_l$ with $i(J(f_k)) = l$. The inductive hypothesis implies
$$|J'(s_k) \cap \hat{C}_n| \le \Gamma^n_l |J'(s_k)|.$$
On the other hand, since $J'(s_k)$ is contained in some atom $I(s_k)$ of $\mathcal{P}_k$, lemma 5.3.6 implies
$$|J'(s_k)| \le (1/2)^{l-k}|I(s_k)| \le (3/4)^{l-(k-1)}|I(s_k)|,$$
where the last inequality holds since $l > k$. Therefore we have
$$|I(s_k) \cap \hat{C}_n| \le (3/4)^{l-(k-1)}\Gamma^n_l |I(s_k)|$$
- If $l \ge n$. The iterative construction of $\hat{\mathcal{C}}_n$ says that for any $s_k \in \mathcal{D}_{k,l}(J_{k-1})$ the interval $J'(s_k)$ pass untouched the n-th step, that is $J'(s_k)$ is a connected component of $\hat{\mathcal{C}}_n$. Lemma 5.3.6 gives
$$|I(s_k) \cap \hat{\mathcal{C}}_n| = |J'(f_k)| \le (1/2)^{l-k}|I(s_k)| \le (3/4)^{n-(k-1)}|I(s_k)|.$$

Summing the components arising from the tree cases above we get

$$|J_{k-1} \cap \hat{\mathcal{C}}_n| \le \max\left((1-\psi_k)\Gamma^n_k, \max_{k<l\le n}\{(3/4)^{l-(k-1)}\Gamma^n_l\}\right)|J_{k-1}|.$$

We observe that

$$\max\left((1-\psi_k)\Gamma^n_k, \max_{k<l\le n}\{(3/4)^{l-(k-1)}\Gamma^n_l\}\right) \le$$

$$\max\{(1-\psi_k), 3/4\} \max\left(\Gamma_k^n, \max_{k<l\leq n}\{(3/4)^{l-k}\Gamma_l^n\}\right)$$
$$= \max\{(1-\psi_k), (1-1/4)\}\Gamma_k^n = (1-\min\{\psi_k, 1/4\})\,\Gamma_k^n,$$

where the second to last equality follows from the definition of Γ_k^n. Therefore we get

$$|J_{k-1}\cap\hat{\mathcal{C}}_n| \leq (1-\min\{\psi_k, 1/4\})\,\Gamma_k^n |J_{k-1}|. \tag{5.35}$$

Summing (5.34) and (5.35) we get

$$|J\cap\hat{\mathcal{C}}_n| \leq (1-\min\{\psi_k, 1/4\})\,\Gamma_k^n|J_{k-1}| + \Gamma_k^n|\mathcal{J}_k|$$
$$= \left((1-\min\{\psi_k, 1/4\})\left(1-\frac{|\mathcal{J}_k|}{|J|}\right) + \frac{|\mathcal{J}_k|}{|J|}\right)\Gamma_k^n|J|$$
$$= \left(1-\left(1-\frac{|\mathcal{J}_k|}{|J|}\right)\min\{\psi_k, 1/4\}\right)\Gamma_k^n|J|.$$

Since by hypothesis $i(J) = k$, lemma 5.3.2 implies that $|\mathcal{J}_k| \leq |J|/2$, therefore we have

$$1-\left(1-\frac{|\mathcal{J}_k|}{|J|}\right)\min\{\psi_k, 1/4\} \leq 1-(1-1/2)\min\{\psi_k, 1/4\}$$
$$= 1-\min\{\psi_k/2, 1/8\} = \max\{(1-\psi_k/2), 7/8\}.$$

It follows that

$$|J\cap\hat{\mathcal{C}}_n| \leq \max\{(1-\psi_k/2), 7/8\}\Gamma_k^n|J| = \Gamma_{k-1}^n|J|$$

by definition of Γ_{k-1}^n. The lemma is proved. □

Now let us consider any connected component J of $\hat{\mathcal{C}}_M$. The inductive definition of the sets $\hat{\mathcal{C}}_n$ implies that J is a connected component of $\hat{\mathcal{C}}_k$ up to $k = i(J)$. Lemma 5.3.8 implies that for any $n > i(J)$ we have

$$\mathrm{Leb}(J\cap\hat{\mathcal{C}}_n) \leq \Gamma_{i(J)}^n \mathrm{Leb}(J).$$

It is easy to see that $\sum_{n\in\mathbb{N}}\psi(n) = \infty$ implies that

$$\lim_{n\to\infty}\Pi_{k=1}^n(1-\psi(k)) = 0,$$

and this last condition implies that for any fixed $k\in\mathbb{N}$ we have $\lim_{n\to\infty}\Gamma_k^n = 0$. It follows that for any connected component J of $\hat{\mathcal{C}}_M$ we have $\mathrm{Leb}(J\cap\hat{\mathcal{C}}_n)\to 0$ as $n\to\infty$. Therefore $\mu(J\cap\mathcal{C}) = 0$ for any J and we deduce that $\mu(\mathcal{C}) = 0$. Proposition 5.3.7 is proved.

Bibliography

[A,G,Y] A.Avila, S.Gouezel, J.-C.Yoccoz: *"Exponential mixing for the Teichmüller flow"*, Publications mathématiques de l'HIES 104 (2006), 143-211.

[A,V] A.Avila, M.Viana: *"Simplicity of Lyapunov spectra: Proof of the Zorich-Kontsevich conjecture"*, Acta Mathematica 198 (2007),1-56.

[BoLa] C.Boissy, E.Lanneau: *"Dynamics and geometry of the Rauzy-Veech induction for quadratic differentials"*, preprint.

[DaNo] C.Danthony, A.Nogueira: *"Involutions linéaires et feuilletages mesurés"*, C.R. Acad. Sci. Paris Sér I Math., 307, 409-412, (1988).

[E,M] A.Eskin, H.Masur: *"Asintotic formulas on flat surfaces"*, Ergodic Theory and Dynamical Systems, 21, no.2, 443-478, (2002).

[KeMaS] S.Kerckhoff, H.Masur, J.Smillie: *"Ergodicity of billiards flows and quadratic differentials"*, Annals of Math, 124, 293-311, (1986).

[K] Khinchin: *"Continued fractions"*, english translation; P.Noordhoff, Groningen, (1963).

[KZ] M.Kontsevich, A.Zorich: *"Connected components of the moduli spaces of Abelian differentials"*, Invent. Math., 153:3, 631-678, (2003).

[Ma1] H.Masur: *"Interval excange transformation and measured foliations"*, Ann. of Math., 115, 169-200, (1982).

[Ma2] H.Masur: *"Lower bounds for the number of saddle connections and closed trajectories of a quadratic differential"*, Holomorphic functions and Moduli, Vol 1, (Berkeley,CA,1986), 215-228, Math. Sci.Res. Inst. Publ.,10, Springer, New-York - Berlin (1988).

[Ma3] H.Masur: *"Logarithmic law for geodesic in moduli space"*, Contemporary Mathematics, Vol 150, 229-245, (1993).

[M,M,Y] S.Marmi, P.Moussa, J.-C.Yoccoz: *The cohomologycal equation for Roth type interval exchange transformations.* J.American Math. Soc., 18, (2005), 823-872.

[Ser] C.Series: *"The modular surface and continued fractions"*, J. London Math. Soc, (2) 31, 307, 69-80, (1985).

[Ve] W.Veech: *"Gauss measures for transformations on the space of interval exchange maps"*, Annals of Mathematics, 115, 201-242, (1982).

[Y1] J.-C.Yoccoz: *"Echanges d'intervalles"*, Cours Collège de France, Janvier-Mars 2005.

[Y2] J.-C.Yoccoz: *"Petits diviseurs en dimension 1"*, Asterisque, 231; (1994).

[Y3] J.-C.Yoccoz: *"Interval exchange maps and translation surfaces"*, CMI summer school course, Centro di ricerca matematica Ennio de Giorgi, Pisa, June-July 2007 (in preparation).

[Z] A.Zorich: *"Finite Gauss measure on the space of interval exchange transformations. Lyapunov exponents"*, Annales de l'Institu Fourier, 46:2, 325-370 (1996).