Scuola Normale Superiore

Tesi di Perfezionamento in Matematica

Triennio 2015–2018

Functional Calculus on Homogeneous Groups

Candidato
Mattia Calzi

Relatore
Prof. Fulvio Ricci

Contents

Introduction

Let $\mathcal{L}$ be a translation-invariant differential operator on $\mathbb{R}^n$. In many situations, the study of $\mathcal{L}$ may be simplified by means of the Fourier transform; indeed, if we conjugate $\mathcal{L}$ with the Fourier transform, we obtain the operator of multiplication by a polynomial. If we consider a left-invariant differential operator $\mathcal{L}$ on a Lie group G, we may still study $\mathcal{L}$ by means of the Fourier transform; however, if G is not commutative, the Fourier transform is less manageable than in the commutative case, so that a different approach is preferable.

A reasonable alternative is provided by the spectral theorem. On the one hand, this method works well for a large class of left-invariant differential operators, considered individually; on the other hand, if we want to consider the interactions between two or more operators, the spectral theorem requires some commutativity assumptions in order to work. Actually, there exists functional calculi for operators which do not commute (see, for example, [65]), but, if one wants to keep a scalar calculus, some relevant properties have to be lost, like multiplicativity. For this reason, we prefer to work under a suitable commutativity assumption. We observe explicitly that the approach we follow is very sensitive to the chosen family of operators. For example, it may be convenient to study the interactions between two (or more) operators if we embed them into a bigger family whose study is simpler. This is the case for abelian groups, where it seems sensible to study translation-invariant operators as polynomials in the partial derivatives.

Now, assume that $(\mathcal{L}_1, \dots, \mathcal{L}_k)$ is a family of formally self-adjoint left-invariant differential operators on G, each one of which induces an essentially self-adjoint operator on $L^2(G)$ with domain $C^\infty_c(G)$; assume that the self-adjoint operators induced by $\mathcal{L}_1, \dots, \mathcal{L}_k$ commute. Then, there is a unique spectral measure μ on $\mathbb{R}^k$ such that

$$\mathcal{L}_j \varphi = \int_{\mathbb{R}^k} \lambda_j \, \mathrm{d}\mu(\lambda) \varphi$$

for every $\varphi \in C^\infty_c(G)$. If $m\colon \mathbb{R}^k \to \mathbb{C}$ is bounded and μ-measurable, we may then associate with m a distribution $\mathcal{K}(m)$ such that

$$m(\mathcal{L}_1, \dots, \mathcal{L}_k)\varphi = \varphi * \mathcal{K}(m)$$

for every $\varphi \in C^\infty_c(G)$. The mapping $\mathcal{K}$ is the desired substitute for the (inverse) Fourier transform.

One may then investigate the similarities between $\mathcal{K}$ and the (inverse) Fourier transform. For instance, one may consider the following questions:

1. does the 'Riemann-Lebesgue' property hold? In other words, if $m \in L^\infty(\mu)$ and $\mathcal{K}(m) \in L^1(G)$, does m necessarily admit a continuous representative?

2. is there a positive Radon measure β on $\mathbb{R}^k$ such that $\mathcal{K}$ extends to an isometry of $L^2(\beta)$ into $L^2(G)$?

3. if such a 'Plancherel measure' β exists, is it possible to find an 'integral kernel' $\chi \in L^1_{\mathrm{loc}}(\beta \otimes \nu_G)$[1] such that, for every $m \in L^\infty(\beta)$ with compact support,

$$\mathcal{K}(m)(g) = \int_{\mathbb{R}^k} m(\lambda)\chi(\lambda, g) \, \mathrm{d}\beta(\lambda)$$

for almost every $g \in G$?

[1] Here, ν_G denotes a fixed (left or right) Haar measure on G.

4. if G is a group of polynomial growth, so that $\mathcal{S}(G)$ can be defined in a reasonable way, does $\mathcal{K}$ map $\mathcal{S}(\mathbb{R}^k)$ into $\mathcal{S}(G)$?

5. if G is a group of polynomial growth and $\mathcal{K}(m) \in \mathcal{S}(G)$ for some $m \in L^\infty(\mu)$, does m necessarily admit a representative in $\mathcal{S}(\mathbb{R}^k)$?

In the following, we shall be particularly interested in Questions 1 and 5; we shall therefore name the affirmative answers to them as Properties (RL) and (S)

Some of these questions have already been addressed in various situations. For instance, the construction of the Plancherel measure of Question 2 dates back to M. Christ [27, Proposition 3] for the case of a homogeneous sub-Laplacian on a stratified group, and was generalized to weighted subcoercive systems of operators on general Lie groups by A. Martini [58, Theorem 3.2.7]. The 'integral kernel' of Question 3 was then introduced by L. Tolomeo [83, Theorem 2.11] for a sub-Laplacian on a group of polynomial growth.

Further, A. Hulanicki [52] showed that Question 4 has an affirmative answer in the setting of a positive Rockland operator on a graded group; his result was then extended by A. Veneruso [85] to some families of operators on the Heisenberg groups, and then by A. Martini [58, Proposition 4.2.1] to the case of a weighted subcoercive system of operators on a group of polynomial growth.

Finally, Property (S) was analysed by F. Astengo, B. Di Blasio and F. Ricci [4], [5], V. Fischer and F. Ricci [40], V. Fischer, F. Ricci and O. Yakimova [41], and A. Martini, F. Ricci and L. Tolomeo [59] for some families of operators associated with some Gelfand pairs and for a sub-Laplacian on a large class of groups of polynomial growth.

Notice that, even though we stressed the usefulness of the preceding construction on non-commutative groups, a thorough study of $\mathcal{K}$ may provide further insight into the chosen operators $\mathcal{L}_1, \dots, \mathcal{L}_k$ even when G is abelian. For example, if we consider the Laplacian Δ on $\mathbb{R}^n$, Property (S) is basically equivalent to Whitney's theorem on C^∞ even functions [86]. In addition, there are families of operators which show pathological behaviours even in dimension 2 (cf. Remark 6.4). It may also happen that two families of differential operators $(\mathcal{L}_1, \dots, \mathcal{L}_k)$ and $(\mathcal{L}'_1, \dots, \mathcal{L}'_{k'})$ give rise to two equivalent functional calculi, even though neither of the two can be written as a polynomial image of the other. A trivial example is given by the two families (Δ^2) and (Δ^3), where Δ is the Laplacian on $\mathbb{R}^n$; we shall present less intuitive examples in Proposition 7.44. As a consequence, some care has to be taken in the choice of the family.

The best setting where one can study Questions 1, 2, and 3 is probably that of weighted subcoercive systems of operators on general Lie groups, while it seems reasonable to study Questions 4 and 5 on groups of polynomial growth. As we mentioned before, this study has been pursued in the case of one sub-Laplacian and in the case of Gelfand pairs. Even though some of the results which follow generalize to weighted subcoercive systems of operators on (unimodular) Lie groups, we shall generally restrict to homogeneous operators on homogeneous groups for simplicity. In this case, a family consisting of one operator automatically satisfies property (RL), while proeprty (S) still provides some difficulties. Nevertheless, even property (RL) can fail in general.

The thesis is divided into two parts: in the first one, which consists of Chapters 1 to 7, we study Rockland families on homogeneous groups; in the second part, which consists of Chapter 8 and is joint work with T. Bruno, we give sharp asymptotic estimates for the heat kernel (and its derivatives) associated with the standard sub-Laplacian on an H-type group.

In Chapter 1, we introduce some auxiliary spaces which will allow us to define a general notion of convolvability for distributions, following L. Schwartz [76]. As one may expect, this definition does not encompass all the situations in which a reasonable notion of convolvability is known; for instance, it does not seem to be fully compatible with the Fourier transform even on $\mathbb{R}^n$. Nevertheless, it does encompass all the cases we need to use; in addition, in this way we are able to avoid defining convolution on various classes of distributions by means of limiting techniques, which, in principle, do not guarantee that the result is unequivocal.

As in the abelian case, this definition of convolution is widely compatible with left- and right-invariant differential operators and thus leads to simple convolvability criteria for distributions in spaces of Sobolev type.

In addition, by means of a remarkable result of J. Dixmier and P. Malliavin [34], we are able to characterize the topology of many spaces of distributions, such as $\mathcal{D}'$ or $\mathcal{E}'$ (at least when the group is separable), by means of the convolution with the elements of C^∞_c. This kind of characterizations are very useful for the study of the kernel transform, since we basically know kernels as convolutors of C_c^∞ into a suitable space.

In Chapter 2, we collect some technical results. To begin with, we briefly study some spaces of smooth and Schwartz functions on not necessarily regular closed subsets of $\mathbb{R}^n$, following [4]. This study is very useful for the study of property (S). Indeed, a kernel determines the corresponding multiplier only up to a negligible set; in particular, continuous multipliers are uniquely determined only on the joint spectrum $\sigma(\mathcal{L}_1, \dots, \mathcal{L}_k)$. Therefore, it is natural to consider Schwartz multipliers only modulo Schwartz functions which vanish on $\sigma(\mathcal{L}_1, \dots, \mathcal{L}_k)$, while for continuous multipliers one may simply consider $C(\sigma(\mathcal{L}_1, \dots, \mathcal{L}_k))$, thanks to the Tietze extension theorem.

We then pass to a generalization of the dual form of the classical Hadamard's lemma to homogeneous group. In the classical situation, the aforementioned result states that a Schwartz function has vanishing moments up to order k if and only if it is the sum of derivatives of order $k+1$ of Schwartz functions. On homogeneous groups we basically replace the notion of 'order' with that of 'homogeneous degree.' Some more care has to be taken, though, because on not necessarily stratified groups one needs to consider sums of (left- or right-invariant) derivatives of homogeneous degree $> k$ in order to prove an analogous statement.

Finally, we shall consider some results on composite functions. These results are particularly useful to treat image families, that is, families of the form $(\mathcal{L}'_1, \dots, \mathcal{L}'_{k'})$, where $\mathcal{L}'_j = P_j(\mathcal{L}_1, \dots, \mathcal{L}_k)$ for some polynomial mapping P_j on $\mathbb{R}^k$ and for every $j = 1, \dots, k'$. By means of the results we prove, under suitable assumptions on the family $(\mathcal{L}_1, \dots, \mathcal{L}_k)$ one may deduce that properties (RL) and (S) hold for the image family $(\mathcal{L}'_1, \dots, \mathcal{L}'_{k'})$. We shall present several applications of these results in Chapter 7.

In Chapter 3, we introduce Rockland families and the general properties of some associated objects, such as the kernel transform, the Plancherel measure of Question 2, the integral kernel of Question 3, and the multiplier transform. Let us mention here that, even though the integral kernel has excellent properties in the second variable, its joint continuity is equivalent to the fact that the multiplier transform, which is basically the adjoint of $\mathcal{K}$, maps L^1 into $C_0(\sigma(\mathcal{L}_1, \dots, \mathcal{L}_k))$. This property, that is, continuity of the integral kernel, is strictly stronger than property (RL), as we show in Remark 7.51. Nonetheless, establishing this property will be crucial in the proof of Theorem 7.45, which completely solves our problem for families whose elements are some sub-Laplacians and the central derivative on a Heisenberg group.

In the last section of the chapter, we relate property (RL) with the study of the Banach $*$-algebra of L^1 kernels; this connection provides more insight into property (RL). For example, it basically shows how (and why) property (RL) fails: in a sense, multipliers corresponding to L^1 kernels may be discontinuous because $\sigma(\mathcal{L}_1, \dots, \mathcal{L}_k)$ is 'too small' with respect to the Gelfand spectrum of the algebra of L^1 kernels.

In Chapter 4, we consider some basic operations on the underlying group, such as taking quotients or products, as well as on the family, such as taking image families. For what concerns quotients, as a corollary of [58, Proposition 3.2.4], which in turn is a generalization of [57, Proposition 2.1], one may prove an analogue of the Poisson formula for weighted subcoercive systems of operators on amenable Lie groups. Nevertheless, in this more general situation some more care is needed. Roughly speaking, this analogue of the Poisson formula states that, if ones has an L^1 kernel f relative to the family $(\mathcal{L}_1, \dots, \mathcal{L}_k)$ with a *continuous* multiplier m, and the canonical projection π of G onto a quotient of it, then the projection $\pi_*(f)$ of f, given by integration on cosets, is a kernel of $\mathrm{d}\pi(\mathcal{L}_1, \dots, \mathcal{L}_k)$ with the same multiplier m. Observe that, even though under suitable conditions we have $\sigma(\mathrm{d}\pi(\mathcal{L}_1, \dots, \mathcal{L}_k)) \subseteq \sigma(\mathcal{L}_1, \dots, \mathcal{L}_k)$, the corresponding spectral measures may be mutually singular; as a consequence, a result of this kind need not hold if we do not assume that m is continuous. Actually, in Chapter 7 we show examples in which m cannot be taken so as to be continuous and $\pi_*(f)$ is not even a kernel associated with $\mathrm{d}\pi(\mathcal{L}_1, \dots, \mathcal{L}_k)$.

Products are much better behaved than quotients, and we are able to prove that properties (RL) and (S) pass from the factors to the product (and conversely). For what concerns image families, we limit ourselves to writing down the basic facts one needs to be able to apply the technical results of Chapter 2. Considering the variety of applications, we refrain from stating general results in this chapter about properties (RL) and (S), since they would become either inefficient or awkwardly cumbersome otherwise. After these remarks on image families, we briefly compare two notions of equivalence between Rockland families. On the one hand, one may consider an 'algebraic' version of equivalence, according to which two families are equivalent if they are (polynomial) images of one another. This kind of equivalence is quite natural and preserves all the main objects taken into consideration. On the other hand, one may consider a more 'functional' notion of equivalence, according to which two families are equivalent if they are (not necessarily polynomial) *functions* of one another; in other words, if the two families have the same kernels, even though corresponding to different multipliers. This kind of equivalence is much weaker, to the point that it does not preserve properties (RL) and (S), in general. Actually, if two families are 'functionally equivalent' but not 'algebraically equivalent,' then at most one of them can satisfy property (S). This leads to a notion of 'functional completeness' for Rockland families; nevertheless, we shall not pursue its study. We only notice that property (S) implies completeness, while the converse fails.

In Chapter 5, we consider a somewhat different problem. We start from a remarkable correspondence between Calderón-Zygmund kernels and Mihlin multipliers 'of infinite order' on abelian groups (cf. [64, Theorems 2.1.11 and 2.2.1]), and we try to generalize it to our setting. It happens that, in order to get the same result for kernels associated with Rockland families, an additional condition is needed (though we do not know if it is necessary): we need that the space of kernels corresponding to Schwartz multipliers is closed in $\mathcal{S}(G)$. We know that this is the case for abelian groups and when property (S) holds, but we do not know if this property holds in general.

In Chapter 6, we consider the case of abelian homogeneous groups. In this case, it is generally useful to consider Rockland families thereon as (polynomial) images of $-i\partial$; indeed, the kernel transform associated with $-i\partial$ is the inverse Fourier transform, so that the study of Rockland families is basically reduced to the study of proper polynomial mappings with homogeneous components. We are then able to prove a simple characterization of the Rockland operators which satisfy property (S). This result, in turn, provides sufficient (but *not* necessary) conditions in order that a positive Rockland operator on a general homogeneous group should satisfy property (S). We then report, for the sake of completeness, some results by Astengo, Di Blasio and Ricci [5], which show that properties (RL) and (S) hold under suitable invariance conditions. Notice, though, that the simplifications due to the presence of a scalar-valued Fourier transform do *not* prevent the appearance of pathological examples, as we show in Remark 6.4.

In Chapter 7, we restrict ourselves to 2-step stratified groups, and to families whose elements are sub-Laplacians or elements of the centre of the Lie algebra. For this kind of families, we derive several sufficient conditions for the validity of properties (RL) and (S). We also present several examples and applications. We refer the reader to the introductions of Sections 7.3 and 7.4 for a more extensive description of our results.

Finally, Chapter 8 treats a different topic; it is joint work with T. Bruno [24]. Here we consider the heat-kernel p associated with the standard sub-Laplacian on a restricted family of homogeneous groups, namely the groups of Heisenberg type. Considering the importance of heat kernels in various parts of analysis, we propose to find sharp asymptotic estimates p and all of its derivatives.[2]

Acknowledgements

I would like to thank my supervisor, prof. Fulvio Ricci, for his patience, guidance, invaluable help, and for his deep and genuine interest in this work. I would also like to thank him for preventing me from excessively proceeding towards sterile formalism, and for the countless occasions in which he listened to me and gave precious advice.

I would like to thank my collaborator, T. Bruno, and his (and my) former advisor, prof. Giancarlo Mauceri. The former for his patience, energy, and everlasting optimism; the latter for the guidance and advice which provided us.

I would like to thank the other Ph.D. students I met, L. Guerra, G. Mascellani, D. Tewodrose, G. E. Comi, R. Bramati, R. Díaz Martín, G. Stefani, G. Apostolidou and S. Cruz Blázquez, to name but a few, who carefully listened to my questions and let me learn something about their field of interest; and with whom I spent many of my best moments in the last three years.

Last but not least, I would like to thank my family, who encouraged me, listened to my problems, counselled, and supported me since I was born.

[2]For technical reasons, we consider only a specific family of differential operators by means of which all the other ones can be written as finite linear combinations with continuous coefficients.

Chapter 1

Convolution and Representations

1.1 Homogeneous Groups

In this section we recall some definitions about Lie groups and homogeneous groups.

Definition 1.1. A (finite-dimensional real) Lie group is a (finite-dimensional real) analytic manifold G endowed with a group structure such that the product is analytic.

If G is a Lie group, then $G \ni g \mapsto g^{-1} \in G$ is an analytic diffeomorphism (cf. [22, Proposition 3 of Chapter III, § 1, No. 1]).

Definition 1.2. Let G be a Lie group and take $g \in G$. We denote by L_g and R_g the analytic diffeomorphisms $g' \mapsto gg'$ and $g' \mapsto g'g^{-1}$ of G onto itself, respectively. If f is a function defined on G, then we define $L_g f \coloneqq f \circ L_{g^{-1}}$ and $R_g f \coloneqq f \circ R_{g^{-1}}$. If T is a distribution on G, then we write $L_g T$ and $R_g T$ instead of $(L_g)_*(T)$ and $(R_g)_*(T)$. In other words, for every $\varphi \in \mathcal{D}(G)$,

$$\langle L_g T, \varphi \rangle = \langle T, \varphi \circ L_g \rangle = \langle T, L_{g^{-1}} \varphi \rangle;$$

analogously for $R_g T$.

Definition 1.3. A (Radon) measure μ on a Lie group G is relatively invariant if there are two functions $\Delta_L, \Delta_R \colon G \to \mathbb{C}^*$ such that $L_g \mu = \Delta_L(g)^{-1} \mu$ and $R_g \mu \coloneqq \Delta_R(g) \mu$ for every $g \in G$. If $\mu \neq 0$, then Δ_L and Δ_R are uniquely determined homomorphisms of G into $\mathbb{C}^*$, and are called the left and right multipliers of μ, respectively (cf. [20, Chapter VIII, § 1]).

The measure μ is a left or right Haar measure if it is positive, non-zero, relatively invariant, and satisfies $\Delta_R = \chi_G$ or $\Delta_L = \chi_G$, respectively.

Notice that any non-zero left-invariant n-form on a Lie group of dimension n induces a left Haar measure (cf. [22, Proposition 55 of Chapter III, § 3, No. 16]). Since left or right Haar measures are unique up to a multiplicative constant (cf. [20, Theorem 1 of Chapter VIII, § 1, No. 2]), they have an analytic density with respect to Lebesgue measure in every local chart. The same then holds for relatively invariant measures (cf. [20, Corollary to Proposition 10 of Chapter VIII, § 1, No. 8] and [22, Theorem 1 of Chapter III, § 8, No. 1]).

Definition 1.4. A Lie group G is unimodular if it possesses a left and right Haar measure.

Definition 1.5. Let G be a Lie group, and let X be a differential operator on G. Then, X is left-invariant if $(L_g)_*(X) = X$ for every $g \in G$; analogously, X is right-invariant if $(R_g)_* X = X$ for every $g \in G$.

Proposition 1.6. *Let G be a Lie group. Then, the following hold:*

- *to every $v \in T_e(G)$ there corresponds a unique left-invariant vector field X and a unique right-invariant vector field Y on G such that $v = X_e = Y_e$;*

- *there is a unique analytic mapping $\exp_G \colon T_e(G) \to G$ such that $\exp_G(0) = e_G$, $T_0(\exp_G) = I_{T_e(G)}$ and the mapping $\mathbb{R} \ni t \mapsto \exp_G(tv) \in G$ is a group homomorphism for every $v \in T_e(G)$.*

See [22, Theorem 4 of Chapter III, § 6, No. 4] for a proof of the second assertion.

Definition 1.7. The Lie algebra $\mathfrak{g}$ of G is the vector space $T_e(G)$ endowed with the bracket $[v, w] := [X_v, X_w]_e = -[Y_v, Y_w]_e$ for every $v, w \in T_e(G)$, where X_v, X_w are left-invariant vector fields, Y_v, Y_w are right-invariant vector fields, $(X_v)_e = (Y_v)_e = v$ and $(X_w)_e = (Y_w)_e = w$.

The mapping $\exp_G$ of Proposition 1.6 is the exponential map of G.

Definition 1.8. We denote by $\mathfrak{U}_{\mathbb{C}}(\mathfrak{g})$ the complexification of the enveloping algebra of the Lie algebra $\mathfrak{g}$ of G. We shall generally identify the elements of $\mathfrak{g}$ and $\mathfrak{U}_{\mathbb{C}}(\mathfrak{g})$ with left-invariant vector fields and left-invariant differential operators, respectively.

Definition 1.9. A homogeneous group G is a simply connected nilpotent Lie group G whose Lie algebra is endowed with a graduation of type $(\mathbb{R}_+^*, +)$. In other words, $\mathfrak{g} = \bigoplus_{r>0} \mathfrak{g}_r$ and $[\mathfrak{g}_r, \mathfrak{g}_s] \subseteq \mathfrak{g}_{r+s}$ for every $r, s > 0$. We denote by Q the homogeneous dimension of G, that is, $\sum_{r>0} r \dim \mathfrak{g}_r$.

If G is a homogeneous group, then $\exp_G$ is a diffeomorphism (cf. [22, Proposition 13 of Chapter III, § 9, No. 5]). In addition, one may endow $\mathfrak{g}$, and consequently G, with a family of dilations such that

$$s \cdot X = \sum_{r>0} s^r X_r$$

for every $s > 0$ and for every $X \in \mathfrak{g}$, where X_r is the component of X in $\mathfrak{g}_r$ for every $r > 0$. Then, $[s \cdot X, s \cdot Y] = s \cdot [X, Y]$ and $(s \cdot g)(s \cdot h) = s \cdot (gh)$ for every $X, Y \in \mathfrak{g}$, for every $g, h \in G$, and for every $s > 0$.

It turns out that, if ν is the Lebesgue measure on $\mathfrak{g}$, then $(\exp_G)_*(\nu)$ is a left and right Haar measure on G, so that G is unimodular (cf. [43, Proposition 1.2]).

Definition 1.10. Let G be a homogeneous group and take $\delta \in \mathbb{C}$. Then:

- a function f on G is homogeneous of homogeneous degree δ if $f(r\,\cdot\,) = r^\delta f$ for every $r > 0$;
- a distribution T on G is homogeneous of homogeneous degree δ if $(r\,\cdot\,)_*(T) = r^\delta T$ for every $r > 0$;
- a differential operator X on G is homogeneous of degree δ if $(r\,\cdot\,)_* X = r^\delta X$ for every $r > 0$.

Notice that we depart from the notion of homogeneous degree defined in [43] for what concerns distributions, since we prefer to consider distributions as odd n-currents instead of even 0-currents. In this way, any Haar measure on G is homogeneous of homogeneous degree $-Q$, instead of 0. In addition, if a differential operator X is homogeneous of homogeneous degree δ, then the distribution X_e is homogeneous of homogeneous degree δ, and the converse holds if X is either left- or right-invariant. Finally, the elements of $\mathfrak{g}_r$ are homogeneous of degree r for every $r > 0$.

Definition 1.11. Let G be a homogeneous group. A homogeneous norm on G is a symmetric proper homogeneous mapping $|\cdot| \colon G \to \mathbb{R}_+$ with homogeneous degree 1. We shall generally assume that, in addition, $|\cdot|$ is of class C^∞ on $G \setminus \{ e \}$.

Let us now state two results on 'polar decomposition.' We state them a little more generally in order to meet some later needs. The proof is inspired by that of [43, Proposition 1.15].

Proposition 1.12. *Let G be a locally compact group which operates continuously on the left on a locally compact space X; let $|\cdot| \colon X \to G$ be a continuous function on X such that $|g \cdot x| = g|x|$ for every $g \in G$ and for every $x \in X$. Take $f \in C(G; \mathbb{C}^*)$ and $\mu \in \mathcal{M}(X)$, and assume that $(g\,\cdot\,)_*(\mu) = f(g)^{-1}\mu$ for every $g \in G$. Define $S := \{ x \in X : |x| = e \}$. Then, S is a locally compact subspace of X and there is a unique $\widetilde{\mu} \in \mathcal{M}(S)$ such that*

$$\int_X \varphi(x) \,\mathrm{d}\mu(x) = \int_{G \times S} \varphi(g \cdot s) f(g) \,\mathrm{d}(\nu_G \otimes \widetilde{\mu})(g, s)$$

for every $\varphi \in C_c(X)$; here, ν_G denotes a fixed *left Haar measure on G.*

The classical situation occurs when $G = \mathbb{R}_+^*$, X is a homogeneous group (minus the identity), G acts on X by means of the dilations, $|\cdot|$ is a homogeneous norm on X, and $f\colon r \mapsto r^\alpha$ for some $\alpha \in \mathbb{C}$.

Proof. Define $\Phi\colon X \ni x \mapsto (|x|, |x|^{-1} \cdot x) \in G \times S$ and observe that Φ is continuous. Further, the inverse of Φ is clearly the continuous mapping $G \times S \ni (g, s) \mapsto g \cdot s \in X$, so that Φ is a homeomorphism. In addition, since $|\cdot|$ is continuous, S is closed in X; hence, S is locally compact.

Next, notice that the measure $\mu' \coloneqq \left(f^{-1} \circ |\cdot|\right) \cdot \mu$ is invariant under the action of G: indeed, if $\mu = 0$ there is nothing to prove; otherwise, f is clearly a homomorphism. Hence, we may assume that $f = \chi_G$.

Now, take $\varphi \in C_c(S)$ and $\psi \in C_c(G)$, and define

$$F(\psi, \varphi) \coloneqq \int_{G\times S} (\psi \otimes \varphi) \,\mathrm{d}\Phi_*(\mu).$$

Then,

$$F(\psi(g\,\cdot\,), \varphi) = F(\psi, \varphi)$$

for every $g \in G$, so that the mapping $\psi \mapsto F(\psi, \varphi)$ is a left-invariant measure on G. Therefore, [20, Corollary to Proposition 10 of Chapter VIII, § 1, No. 8] implies that there is a unique $\langle \widetilde{\mu}, \varphi\rangle \in \mathbb{C}$ such that

$$F(\psi, \varphi) = \langle \widetilde{\mu}, \varphi\rangle \langle \nu_G, \psi\rangle$$

for every $\psi \in C_c(G)$. Now, take $\psi \in C_c(G)$ so that $\langle \nu_G, \psi\rangle = 1$. Then,

$$|\langle \widetilde{\mu}, \varphi\rangle| = |F(\psi, \varphi)| \leqslant |\Phi_*(\mu)|(\mathrm{Supp}(\psi) \times \mathrm{Supp}(\varphi)) \|\psi\|_\infty \|\varphi\|_\infty,$$

so that $\widetilde{\mu}$ is a Radon measure on S. Finally, since clearly

$$\int_{G\times S} (\psi \otimes \varphi) \,\mathrm{d}\Phi_*(\mu) = F(\psi, \varphi) = \int_{G\times S} (\psi \otimes \varphi) \,\mathrm{d}(\nu_G \otimes \widetilde{\mu})$$

by Fubini's theorem, and since $C_c(G) \otimes C_c(S)$ is dense in $C_c(G \times S)$, the assertion follows. □

Corollary 1.13. *Keep the hypotheses and the notation of Proposition 1.12, and assume that $f = \chi_G$. Take $h \in L^1_{\mathrm{loc}}(\mu)$ so that $h(g\,\cdot\,) = h$ for every $g \in G$. Then, there is a unique $\widetilde{h} \in L^1_{\mathrm{loc}}(\widetilde{\mu})$ such that*

$$\int_X \varphi(x) h(x) \,\mathrm{d}\mu(x) = \int_{G\times S} \varphi(g \cdot s) \widetilde{h}(s) \,\mathrm{d}(\nu_G \otimes \widetilde{\mu})(g, s)$$

for every $\varphi \in C_c(X)$.

In addition, take $p \in [1, \infty]$, a ν_G-integrable subset K_1 of G such that $\nu_G(K_1) > 0$, and a $\widetilde{\mu}$-measurable subset K_2 of S; assume that $\chi_{K_1 \cdot K_2} h \in L^p(\mu)$. Then, $\chi_{K_2} \widetilde{h} \in L^p(\widetilde{\mu})$ and

$$\left\| \chi_{K_2} \widetilde{h} \right\|_{L^p(\widetilde{\mu})} \leqslant \nu_G(K_1)^{-\frac{1}{p}} \|\chi_{K_1\cdot K_2} h\|_{L^p(\mu)}$$

One may also consider the case in which μ and h are relatively G-invariant. The first part of the result is then easily deduced from the present assertion, while the L^p estimates become less transparent.

Proof. Define Φ as in the proof of Proposition 1.12, and set $\beta \coloneqq h \cdot \mu$, so that Proposition 1.12 implies that there is a unique $\widetilde{\beta} \in \mathcal{M}(S)$ such that $\Phi_*(\beta) = \nu_G \otimes \widetilde{\beta}$. Let us prove that $\widetilde{\beta}$ is a measure with base $\widetilde{\mu}$. Indeed, let K be a $\widetilde{\mu}$-negligible compact subset of S, and let H be a compact neighbourhood of e in G. Then, $H \times K$ is a $(\nu_G \otimes \widetilde{\mu})$-negligible compact subset of $G \times S$ by [19, Corollary 3 to Proposition 5 of Chapter V, § 8, No. 3] and the remark following its proof. Then, $\Phi^{-1}(H \times K)$ is a μ-negligible compact subset of X by [19, Proposition 7 of Chapter V, § 6, No. 4] (applied with $\pi = \Phi^{-1}$ and $A = \Phi^{-1}(H \times K)$). Therefore, $\Phi^{-1}(H \times K)$ is β-negligible, so that $H \times K$ is $(\nu_G \otimes \widetilde{\beta})$-negligible again by [19, Proposition 7 of Chapter V, § 6, No. 4]. Since H is not ν_G-negligible, [19, Corollary 1 to Proposition 7 of Chapter V, § 8, No.

3] and the remark following its proof imply that K is $\widetilde{\beta}$-negligible. Therefore, [19, Theorem 2 of Chapter V, § 5, No. 5] implies that there is a unique $\widetilde{h} \in L^1_{\mathrm{loc}}(\widetilde{\mu})$ such that $\widetilde{\beta} = \widetilde{h} \cdot \widetilde{\mu}$.

Finally, take p, K_1, and K_2 as in the statement. Then, for every $\varphi \in C_c(S)$, Fubini's theorem implies that

$$\begin{aligned}\left|\int_S \varphi\chi_{K_2}\widetilde{h}\,\mathrm{d}\widetilde{\mu}\right| &= \frac{1}{\nu_G(K_1)}\left|\int_{G\times S}[\chi_{K_1}\otimes(\varphi\chi_{K_2})](g,s)\widetilde{h}(s)\,\mathrm{d}(\nu_G\otimes\widetilde{\mu})(g,s)\right| \\ &= \frac{1}{\nu_G(K_1)}\left|\int_X [[\chi_{K_1}\otimes(\varphi\chi_{K_2})]\circ\Phi]h\,\mathrm{d}\mu\right| \\ &\leqslant \frac{1}{\nu_G(K_1)}\|h\chi_{K_1\cdot K_2}\|_{L^p(\mu)}\|(\chi_{K_1}\otimes\varphi)\circ\Phi\|_{L^{p'}(\mu)}.\end{aligned}$$

In addition,

$$\|(\chi_{K_1}\otimes\varphi)\circ\Phi\|_{L^{p'}(\mu)} = \|\chi_{K_1}\otimes\varphi\|_{L^{p'}(\nu_G\otimes\widetilde{\mu})} = \nu_G(K_1)^{\frac{1}{p'}}\|\varphi\|_{L^{p'}(\widetilde{\mu})},$$

Then, the arbitrariness of φ implies that

$$\|\chi_{K_2}\widetilde{h}\|_{L^p(\widetilde{\mu})} \leqslant \nu_G(K_1)^{-\frac{1}{p}}\|h\chi_{K_1\cdot K_2}\|_{L^p(\mu)},$$

whence the result. □

Finally, let us present a way to associate to every polynomial mapping on $\mathfrak{g}^*$ a differential operator on G. This procedure is called 'symmetrization,' but we shall present it in a more algebraic way. Since we are interested in formally self-adjoint differential operators, we shall actually consider polynomials on $i\mathfrak{g}^*$, in order that real polynomials be mapped to formally self-adjoint operators, and conversely.

Definition 1.14. Let A be a unital $\mathbb{R}$-algebra and V a finite-dimensional vector subspace of A. Let P be a polynomial mapping on V^*, and let (P_k) be the family of the homogeneous components of P. Let, for each $k \in \mathbb{N}$,

$$\widetilde{P}_k\colon (V^*)^{\otimes k} \to \mathbb{R}$$

be the linear mapping associated with the symmetric k-multilinear mapping which induces P_k. Let $m_k\colon V^{\otimes k} \to A$ be the linear mapping associated with the k-multilinear mapping $(x_j) \mapsto x_1 \cdots x_k$, and let $\iota_k\colon ((V^*)^{\otimes k})^* \to V^{\otimes k}$ be the canonical isomorphism. Consider the linear mapping

$$\sum_{k\in\mathbb{N}} m_k \circ \iota_k \circ {}^t\widetilde{P}_k\colon \mathbb{R}^* \to A.$$

Then, we define P_A as the element of A which canonically corresponds to that mapping, that is, its value at $I_{\mathbb{R}}$.

The following result shows that the abstract definition of P_A looks simpler if computed with the aid of a basis.

Lemma 1.15. *Keep the notation of Definition 1.14. Let $(e_j)_{j\in J}$ be a basis of V and let $(e_j^*)_{j\in J}$ be the associated dual basis. Then,*

$$P_A = \sum_{k\in\mathbb{N}}\sum_{j_1,\dots,j_k\in J}\widetilde{P}_k(e^*_{j_1}\otimes\cdots\otimes e^*_{j_k})\,e_{j_1}\cdots e_{j_k}.$$

Proof. Notice that we may assume that $P = P_k$ for some $k \in \mathbb{N}$. Then,

$$\begin{aligned}\left\langle m_k\circ\iota_k\circ{}^t\widetilde{P}_k, I_{\mathbb{R}}\right\rangle &= \left\langle m_k\circ\iota_k, \widetilde{P}_k\right\rangle \\ &= \left\langle m_k, \sum_{j_1,\dots,j_k\in J}\widetilde{P}_k(e^*_{j_1}\otimes\cdots\otimes e^*_{j_k})\,e_{j_1}\otimes\cdots\otimes e_{j_k}\right\rangle \\ &= \sum_{j_1,\dots,j_k\in J}\widetilde{P}_k(e^*_{j_1}\otimes\cdots\otimes e^*_{j_k})\,e_{j_1}\cdots e_{j_k},\end{aligned}$$

whence the result. □

Proposition 1.16. *Let G be a Lie group with Lie algebra $\mathfrak{g}$. Then, the mapping*

$$\lambda_G\colon \operatorname{Pol}(\mathfrak{g}^*;\mathbb{C}) \ni P \mapsto [P(i^{-1}\cdot)]_{\mathfrak{U}_\mathbb{C}(\mathfrak{g})} \in \mathfrak{U}_\mathbb{C}(\mathfrak{g})$$

is an isomorphism of vector spaces. In addition, for every polynomial mapping P on $\mathfrak{g}^*$ the following hold:

1. *if G is a homogeneous group, then P is homogeneous (with respect to the graduation $(\mathfrak{g}^*_r)_{r>0}$ of $\mathfrak{g}^*$) if and only if $\lambda_G(P)$ is homogeneous; in this case, the homogeneous degrees are the same;*

2. *$\lambda_G(\overline{P}) = \lambda_G(P)^*$ with respect to a right Haar measure. In particular, $\lambda_G(P)$ is formally self-adjoint with respect to a right Haar measure if and only if P is real;*

3. *if G' is a Lie group with Lie algebra $\mathfrak{g}'$, and if $\pi\colon G \to G'$ is a homomorphism of Lie groups, then $\mathrm{d}\pi(\lambda_G(P)) = \lambda_{G'}(P \circ {}^t\mathrm{d}\pi)$.*

Proof. **0.** Let Φ_1 be the mapping $\operatorname{Pol}(\mathfrak{g}^*;\mathbb{C}) \ni P \mapsto [P(i^{-1}\cdot)]_{\mathcal{S}_\mathbb{C}(\mathfrak{g})} \in \mathcal{S}_\mathbb{C}(\mathfrak{g})$, where $\mathcal{S}_\mathbb{C}(\mathfrak{g})$ is the complexification of the symmetric algebra over $\mathfrak{g}$, and observe that Φ_1 is an isomorphism thanks to Lemma 1.15. Then, let Φ_2 be the isomorphism of $\mathcal{S}_\mathbb{C}(\mathfrak{g})$ onto the graded algebra $\mathfrak{G}$ associated with $\mathfrak{U}_\mathbb{C}(\mathfrak{g})$ (cf. [21, Theorem 1 of Chapter 1, § 2, No. 7]). Finally, let Φ_3 be the canonical isomorphism of vector spaces of $\mathfrak{U}_\mathbb{C}(\mathfrak{g})$ onto $\mathfrak{G}$. Then, clearly $\Phi_2 \circ \Phi_1 = \Phi_3 \circ \lambda_G$, so that λ_G is an isomorphism of vector spaces.

1. Suppose that G is a homogeneous group, and let $(X_j)_{j\in J}$ be a basis of $\mathfrak{g}$ with homogeneous elements. Then, also the dual basis (X_j^*) of $\mathfrak{g}^*$ has homogeneous elements, so that the assertion follows easily from Lemma 1.15 and **0** above.

2. Since the elements of $i\mathfrak{g}$ are formally self-adjoint with respect to a right Haar measure, the assertion follows easily from Lemma 1.15 and **0** above.

3. The third assertion follows easily from Lemma 1.15. □

In order to show that the mapping $\operatorname{Pol}(\mathfrak{g}^*;\mathbb{C}) \ni P \mapsto P_{\mathfrak{U}_\mathbb{C}(\mathfrak{g})}$ is the so-called symmetrization, it suffices to observe that this map is linear and that the polynomial mapping $v \mapsto \langle v, X\rangle^k$ has image X^k for every $X \in \mathfrak{g}$, as one sees immediately by means of Lemma 1.15 (cf. [48, Theorem 4.3] and [13, Chapter IV, § 5, No. 11]).

1.2 The Space $\mathcal{B}(M)$

In this section, M denotes a locally compact space. The results of this section are based on [76, 77].

Definition 1.17. We define $\mathcal{B}(M)$ as the set of bounded continuous functions on M with values in $\mathbb{C}$, endowed with the supremum norm.

We shall denote by $\mathcal{B}_c(M)$ the space $\mathcal{B}(M)$ endowed with the finest locally convex topology which induces the topology of $C(M)$, that is, the topology of compact convergence, on the bounded subsets of $\mathcal{B}(M)$.

The topology of $\mathcal{B}_c(M)$ is not always easy to handle. Nonetheless, we have the following result.

Proposition 1.18. *Let (E,τ_1) be a locally convex space, and $\mathfrak{B}$ a translation-invariant set of subsets of E. Let τ_2 be the locally convex topology induced by the semi-norms which are continuous on every element of $\mathfrak{B}$. Then, τ_2 is the finest locally convex topology on E which induces the same topology as τ_1 on each element of $\mathfrak{B}$.*

If, in addition, τ_1 is metrizable, then every sequentially continuous semi-norm on (E,τ_2) is continuous.

Proof. Observe first that τ_2 is finer than every locally convex topology which induces the same topology as τ_1 on every element of $\mathfrak{B}$; in particular, it is finer than τ_1. Conversely, take $B \in \mathfrak{B}$, and let us prove that τ_2 induces the same topology as τ_1 on B; it will suffice to prove that it induces a coarser topology. Take $v \in B$, and let V be a neighbourhood of v in B with respect to τ_2. Then, there is a semi-norm ρ which is continuous with respect to τ_2 and satisfies

$(v + \rho^{-1}([0,1])) \cap B \subseteq V$. Now, there is a neighbourhood U of 0 in $B - v$, endowed with the topology induced by τ_1, such that $\rho(U) \subseteq [0,1]$, so that $U \subseteq \rho^{-1}([0,1])$. Now, this implies that $v + U \subseteq (v + \rho^{-1}([0,1])) \cap B \subseteq V$; since $v + U$ is a neighbourhood of v in B with respect to the topology induced by τ_1, the first assertion follows. The second assertion follows easily. □

For further reference, we state also the following immediate corollary.

Corollary 1.19. *Let F be a locally convex space and H a set of linear mappings from $\mathcal{B}_c(M)$ into F. Then, the following conditions are equivalent:*

1. *H is equicontinuous;*

2. *for every bounded subset B of $\mathcal{B}(M)$, the set $\{ T|_B : T \in H \}$ is equicontinuous.*

If, in addition, M is countable at infinity, then the preceding conditions are equivalent to the following one:

3. *H is sequentially equicontinuous.*

Lemma 1.20. *Let B be a bounded subset of $\mathcal{B}(M)$. Then, there is a bounded subset B' of $\mathcal{B}(M)$ contained in $C_c(M)$ and such that the closure of B' in $\mathcal{B}_c(M)$ contains B.*

Proof. Indeed, let $\mathcal{K}$ be the set of compact subsets of M, and take, for every $K \in \mathcal{K}$, some $\tau_K \in C_c(M)$ such that $\chi_K \leqslant \tau_K \leqslant \chi_M$; endow $\mathcal{K}$ with the section filter induced by $\subseteq$. Define $B' \coloneqq \{ \varphi\tau_K : \varphi \in B, K \in \mathcal{K} \}$. Then, it is clear that B' is bounded in $\mathcal{B}(M)$. In addition, for every $\varphi \in B$ the *bounded* filtered family $(\tau_K \varphi)$ converges to φ in $C(M)$, hence in $\mathcal{B}_c(M)$. □

Proposition 1.21. *The following hold:*

1. *the canonical bilinear mapping $\mathcal{M}^1(M) \times \mathcal{B}_c(M) \ni (\mu, \varphi) \mapsto \int_M \varphi \, \mathrm{d}\mu \in \mathbb{C}$ induces an isomorphism of $\mathcal{M}^1(M)$ onto the strong dual of $\mathcal{B}_c(M)$;*

2. *a subset B of $\mathcal{B}_c(M)$ is bounded (resp. relatively compact) if and only if it is uniformly bounded (resp. and equicontinuous);*

3. *$\mathcal{B}_c(M)$ is Hausdorff and quasi-complete.*

Proof. Take $\mu \in \mathcal{M}^1(M)$. In order to prove that μ induces a continuous linear functional on $\mathcal{B}_c(M)$, by Corollary 1.19 it suffices to observe that μ is continuous on every bounded subset of $\mathcal{B}_c(M)$ with respect to locally uniform convergence. However, this follows from [19, Proposition 21 of Chapter IV, § 5, No. 11].

Conversely, every element of $\mathcal{B}_c(M)'$ induces a unique element of $\mathcal{M}^1(G)$ since the inclusion $C_0(M) \subseteq \mathcal{B}_c(M)$ is continuous and dense by Lemma 1.20. Therefore, $\mathcal{B}_c(M)' = \mathcal{M}^1(M)$ as vector spaces. Consequently, a subset B of $\mathcal{B}_c(M)$ is bounded if and only if $\langle \mu, B \rangle$ is bounded for every $\mu \in \mathcal{M}^1(M)$. Now, this is true if B is uniformly bounded. Conversely, if B is bounded in $\mathcal{B}_c(M)$, then B induces a pointwise bounded subset of $\mathcal{M}^1(M)'$, which is then bounded since $\mathcal{M}^1(M)$ is barrelled. Hence, $\sup_{x \in M, \varphi \in B} |\langle \delta_x, \varphi \rangle| < \infty$ since $\{ \delta_x : x \in M \}$ is bounded in $\mathcal{M}^1(M)$; hence, B is uniformly bounded.

Now, let B be a bounded subset of $\mathcal{B}_c(M)$; then Lemma 1.20 shows that there is a bounded subset B' of $C_0(M)$ such that B is contained in the closure of B' in $\mathcal{B}_c(M)$. Therefore,

$$\sup_{\varphi \in B'} |\langle \mu, \varphi \rangle| \geqslant \sup_{\psi \in B} |\langle \mu, \psi \rangle|$$

for every $\mu \in \mathcal{M}^1(M)$, so that the semi-norm $\mu \mapsto \sup_{\psi \in B} |\langle \mu, \psi \rangle|$ is continuous on $\mathcal{M}^1(M)$. Hence, $\mathcal{M}^1(M)$ is the strong dual of $\mathcal{B}_c(M)$. The other assertions follow from [17, Corollary 3 to Theorem 2 of §§ 1 and 2]. □

Remark 1.22. Assume that M is not compact. Then, $\mathcal{B}_c(M)$ is not semi-barrelled. In particular, it is neither barrelled nor bornological.

Indeed, let (K_n) be a strictly increasing sequence of compact subsets of M whose union N is not compact. Then, the subset $\{ \delta_x : x \in N \}$ of $\mathcal{M}^1(M)$ is clearly bounded but not equicontinuous on $\mathcal{B}_c(M)$ thanks to Corollary 1.19. By the same reference, it is easily seen that the set $\{ \delta_x : x \in K_n \}$ is equicontinuous for every $n \in \mathbb{N}$. Hence, $\mathcal{B}_c(M)$ is not semi-barrelled.

Definition 1.23. We shall denote by $\mathcal{M}^1_c(M)$ the space $\mathcal{M}^1(M)$ endowed with the topology of uniform convergence on the compact subsets of $\mathcal{B}_c(M)$.

Proposition 1.24. *$\mathcal{M}^1_c(M)$ is a Hausdorff space. It is complete if M is paracompact.*

Proof. The first assertion is clear. Then, assume that M is paracompact. Let $\mathcal{M}^1_k(M)$ be the space $\mathcal{M}^1(M)$ endowed with the topology of uniform convergence on the compact subsets of $C_0(M)$, so that $\mathcal{M}^1_k(M)$ is complete by [18, Proposition 12 of Chapter III, § 3, No. 8]. In addition, the identity mapping $\mathcal{M}^1_c(M) \to \mathcal{M}^1_k(M)$ is continuous; let us prove that the topology of $\mathcal{M}^1_c(M)$ is defined by a family of semi-norms which are lower semi-continuous on $\mathcal{M}^1_k(M)$. Indeed, take a relatively compact subset B of $\mathcal{B}_c(M)$. Let $\mathcal{C}$ be a partition of M into σ-compact open subspaces of M (cf. [16, Theorem 5 of Chapter I, § 9, No. 10]). Take, for every $C \in \mathcal{C}$, an increasing sequence $(\tau_{C,j})$ of elements of $C_c(M)$ such that $0 \leqslant \tau_{C,j} \leqslant \chi_C$ for every $j \in \mathbb{N}$, and such that $\mathrm{Supp}(\tau_{C,j})$ is contained in the interior of the set where $\tau_{C,j+1}$ equals 1. Then, it is easily seen that $B'_{C,j} \coloneqq \{ \tau_{C,j}\varphi \colon \varphi \in B \}$ is relatively compact in $C_0(M)$ for every $C \in \mathcal{C}$ and for every $j \in \mathbb{N}$, and that their union B' is relatively compact in $\mathcal{B}_c(M)$. Therefore, the semi-norms of the form $\sup_{\varphi \in B'} |\langle \cdot , \varphi \rangle|$, as B runs through the relatively compact subsets of $\mathcal{B}_c(M)$, define the topology of $\mathcal{M}^1_c(M)$ and are lower semi-continuous on $\mathcal{M}^1_k(M)$. Then, the assertion follows from [16, Corollary to Proposition 7 of Chapter II, § 3, No. 3]. □

We now give some characterizations of the compact subsets of $\mathcal{M}^1_c(M)$. We begin with an elementary result of functional analysis. Recall that a semi-Montel space is a Hausdorff locally convex space whose bounded subsets are relatively compact.

Proposition 1.25. *Let E be a locally convex space, F a semi-Montel space, and H an equicontinuous subset of $\mathcal{L}(E;F)$. Then, H is relatively compact in $\mathcal{L}_c(E;F)$.*

Proof. Notice that also the closure $\overline{H}$ of H in $\mathcal{L}_s(E;F)$ is equicontinuous by [18, Proposition 4 of Chapter III, § 3, No. 4], hence compact by [18, Corollary 1 to Proposition 4 of Chapter III, § 3, No. 4]. Now, [18, Proposition 5 of Chapter III, § 3, No. 4] implies that the topologies induced by $\mathcal{L}_c(E;F)$ and $\mathcal{L}_s(E;F)$ on $\overline{H}$ coincide, so that $\overline{H}$ is a compact subset of $\mathcal{L}_c(E;F)$. □

Proposition 1.26. *Assume that M is paracompact, and let H be a subset of $\mathcal{M}^1_c(M)$. Then, the following conditions are equivalent:*

1. *H is equicontinuous on $\mathcal{B}_c(M)$;*
2. *H is relatively compact;*
3. *H satisfies Prokhorov's condition, that is, H is bounded and*[1]

$$\lim_{K \in \mathcal{K}} \sup_{\mu \in H} |\mu|(M \setminus K) = 0.$$

Proof. **1 $\implies$ 2.** This follows from Proposition 1.25.

2 $\implies$ 3. It is clear that H is bounded; assume by contradiction that there is $\varepsilon_0 > 0$ such that for every compact subset K of M there is $\mu \in H$ such that $|\mu|(M \setminus K) > \varepsilon_0$.

Suppose first that M is σ-compact, and let (V_h) be a covering of M by relatively compact open subsets such that $\overline{V_h} \subseteq V_{h+1}$ for every $h \in \mathbb{N}$. Then, it is clear that we may construct by induction a strictly increasing sequence (h_j) of elements of $\mathbb{N}$, a sequence (K_j) of compact subsets of M and a sequence (μ_j) of elements of H such that $K_j \subseteq V_{h_{j+1}} \setminus \overline{V}_{h_j}$ and $|\mu_j|(K_j) > \varepsilon_0$. Now select, for every $j \in \mathbb{N}$, some $\varphi_j \in \mathcal{D}^0(V_{h_{j+1}} \setminus \overline{V}_{h_j})$ such that $\|\varphi_j\|_\infty \leqslant 1$ and $\langle \mu_j, \varphi_j \rangle = \varepsilon_0$. Then, the set $\{ \varphi_j \colon j \in \mathbb{N} \}$ is compact in $\mathcal{B}_c(M)$. Since H is precompact in $\mathcal{M}^1_c(M)$, we may then find $\mu'_1, \dots, \mu'_h$ in H such that for every $\mu \in H$ there is $\ell \in \{ 1, \dots, h \}$ such that

$$|\langle \mu - \mu'_\ell, \varphi_j \rangle| < \frac{\varepsilon_0}{2}$$

for every $j \in \mathbb{N}$. Taking $\mu = \mu_j$, we see that there are $\ell \in \{ 1, \dots, h \}$ and an infinite subset N of $\mathbb{N}$ such that

$$|\langle \mu'_\ell, \varphi_j \rangle| > \frac{\varepsilon_0}{2}$$

[1] Here, $\mathcal{K}$ denotes the set of compact subsets of M, endowed with the section filter relative to the ordering $\subseteq$.

for every $j \in N$. Since $\|\varphi_j\|_\infty \leqslant 1$ for every $j \in \mathbb{N}$, this implies that $|\mu'_\ell|(V_{h_{j+1}} \setminus \overline{V}_{h_j}) > \frac{\varepsilon_0}{2}$ for every $j \in N$, so that $\mu'_\ell \notin \mathcal{M}^1(M)$: contradiction.

Now, consider the general case. By [16, Theorem 5 of Chapter I, § 9, No. 10], there is a partition of $\mathcal{C}$ of M whose elements are σ-compact open subsets of M. Now, take $\varepsilon > 0$ and assume by contradiction that there is an uncountable subset $\mathcal{C}'$ of $\mathcal{C}$ and a family $(\mu_C)_{C\in\mathcal{C}'}$ of elements of H such that $|\mu_C|(C) > \varepsilon$ for every $C \in \mathcal{C}'$. Then, for every $C \in \mathcal{C}'$ we may find $\varphi_C \in \mathcal{D}(C)$ such that $\|\varphi\|_\infty \leqslant 1$ and $\langle \mu_C, \varphi_C\rangle = \varepsilon$. Clearly, $\{\, \varphi_C : C \in \mathcal{C}' \,\}$ is a compact subset of $\mathcal{B}_c(M)$; reasoning as before, this leads to a contradiction. Therefore, there is a countable union of components of M which contains the support of each element of H, so that the preceding argument applies with minor modifications.

3 $\implies$ 1. Let B be a bounded subset of $\mathcal{B}_c(M)$. Take $\varepsilon > 0$, and define $C_1 := \sup_{\varphi\in B} \|\varphi\|_\infty$ and $C_2 := \sup_{\mu\in H} \|\mu\|$. Let K be a compact subset of M such that

$$|\mu|(M \setminus K) \leqslant \frac{\varepsilon}{3C_1 + 1}.$$

for every $\mu \in H$. Now, assume that $\varphi_1, \varphi_2 \in B$ satisfy

$$|\varphi_1(x) - \varphi_2(x)| \leqslant \frac{\varepsilon}{3C_2 + 1}$$

for every $x \in K$. Then,

$$|\langle \mu, \varphi_1 - \varphi_2\rangle| \leqslant \int_K |\varphi_1 - \varphi_2| \,\mathrm{d}|\mu| + \int_{M\setminus K} |\varphi_1| \,\mathrm{d}|\mu| + \int_{M\setminus K} |\varphi_2| \,\mathrm{d}|\mu| \leqslant \varepsilon,$$

for every $\mu \in H$. Then, Corollary 1.19 completes the proof. □

Corollary 1.27. *Assume that M is paracompact. Then, $\mathcal{B}_c(M)$ is canonically identified with $(\mathcal{M}^1_c(M))'_c$. In addition, $\mathcal{B}_c(M)$ is complete.*

Proof. Since $\mathcal{B}_c(M)$ is quasi-complete by Proposition 1.21, the topology of $\mathcal{M}^1_c(M)$ is coarser than the Mackey topology $\tau(\mathcal{M}^1(M), \mathcal{B}_c(M))$; in addition, it is finer than the weak topology $\sigma(\mathcal{M}^1(M), \mathcal{B}_c(M))$. Therefore, the dual of $\mathcal{M}^1_c(M)$ is canonically identified with $\mathcal{B}_c(M)$ as a vector space by [18, Theorem 1 of Chapter IV, § 1, No. 1]. The first assertion then follows from Proposition 1.26 and [18, Corollary 1 to Proposition 7 of Chapter III, § 3, No. 5].

Now, let us prove that $\mathcal{B}_c(M)$ is complete. Indeed, by means of Proposition 1.26 and the above, we see that the topology of $\mathcal{B}_c(M)$ is induced by a family of semi-norms which are lower semi-continuous with respect to the topology of $\mathcal{E}^0(M)$. Taking into account the fact that $\mathcal{E}^0(M)$ is complete, [16, Proposition 7 of Chapter II, § 3, No. 3] leads to the conclusion. □

Proposition 1.28. *The inclusion $\mathcal{E}'^0_c(M) \subseteq \mathcal{M}^1_c(M)$ is continuous and dense.*

Proof. Since the inclusion $\mathcal{B}_c(M) \subseteq \mathcal{E}^0(M)$ is continuous and dense, it follows that the inclusion $\mathcal{E}'^0_c(M) \subseteq \mathcal{M}^1_c(M)$ is continuous. Then, take $\mu \in \mathcal{M}^1_c(M)$ and define $\mathcal{K}$ and (τ_K) as in the proof of Lemma 1.20. Let us prove that $(\tau_K \mu)$ converges to μ in $\mathcal{M}^1_c(M)$. Indeed, it suffices to observe that this filtered family converges pointwise to μ and that it is equicontinuous on $\mathcal{B}_c(M)$ by Proposition 1.26. □

Observe that, even though our definition of $\mathcal{B}_c(M)$ is inspired by [78, p. 203], the topology we put on $\mathcal{B}_c(\mathbb{R}^n)$ is strictly finer than the topology of the corresponding space $\mathbf{B}^0_c$ of [77], that is, the supremum between the topology induced by that of $\mathcal{E}^0(\mathbb{R}^n)$ and that of uniform convergence on the compact subsets of $\mathcal{M}^1(\mathbb{R}^n)$.

Indeed, [77] and our definition of $\mathcal{B}_c(\mathbb{R}^n)$ imply that the identity $\mathcal{B}_c(\mathbb{R}^n) \to \mathbf{B}^0_c$ is continuous. On the other hand, the topology of $\mathbf{B}^0_c$ is that of uniform convergence on the sets of the form $B_1 + B_2$, where B_1 is relatively compact in $\mathcal{M}^1(\mathbb{R}^n)$ and B_2 is bounded in $\mathcal{E}'^0(\mathbb{R}^n)$, that is, is bounded in $\mathcal{M}^1(\mathbb{R}^n)$ and its elements are supported in a fixed compact subset of $\mathbb{R}^n$. On the other hand, Corollary 1.27 implies that the topology of $\mathcal{B}_c(\mathbb{R}^n)$ is that of uniform convergence on the bounded subsets of $\mathcal{M}^1(\mathbb{R}^n)$ which satisfy Prokhorov's condition.

Now, the set $B := \bigcup_{j\in\mathbb{N}} \{\, 2^{-j}\delta_x : x \in B(0, 2^j) \,\}$ satisfies Prokhorov's condition. However, if $B \subseteq B_1 + B_2$ for some B_1, B_2 as above, then there is $k \in \mathbb{N}$ such that $(\chi_{\mathbb{R}^n} - \chi_{B(0,2^k)}) \cdot B$ is precompact in $\mathcal{M}^1(\mathbb{R}^n)$. Therefore, also the set $\{\, \delta_x \colon x \in B(0, 2^{k+1}) \setminus B(0, 2^k) \,\}$ is precompact in $\mathcal{M}^1(\mathbb{R}^n)$; since this latter set is infinite and discrete, this is absurd Since the bornology generated by the sets of the form $B_1 + B_2$ as above is adapted, the assertion follows from [18, Proposition 2 of Chapter III, § 3, No. 1].

1.3 Spaces of Sobolev Type

In this section, G denotes a Lie group of dimension n, and $\mathbf{X} = (X_1, \dots, X_n)$ and $\mathbf{Y} = (Y_1, \dots, Y_n)$ two bases of left- and right-invariant vector fields on G which agree at the identity e of G. We denote by β a relatively invariant positive measure on G with left and right multipliers Δ_L and Δ_R, respectively.

Definition 1.29. Take $p \in]1, \infty]$ and $k_1, k_2 \in \mathbb{N} \cup \{\, \infty \,\}$, and define $W^{k_1,k_2,p}(\beta)$ as the set of $f \in L^p(\beta) \cdot \beta$ such that $\mathbf{Y}^{\alpha_1}\mathbf{X}^{\alpha_2} f \in L^p(\beta) \cdot \beta$ for every α_1, α_2 of length at most k_1, k_2, respectively. We shall endow $W^{k_1,k_2,p}(\beta)$ with the topology induced by the canonical mapping $f \mapsto (\mathbf{Y}^{\alpha_1}\mathbf{X}^{\alpha_2} f)_{|\alpha_1| \leqslant k_1, |\alpha_2| \leqslant k_2}$ into the product of the $L^p(\beta) \cdot \beta$.

If $p \in [1, \infty[$, then we define analogously the spaces $W^{k_1,k_2,1}(G)$, $W^{k_1,k_2,1}_c(G)$, $W^{k_1,k_2,p}_0(\beta)$, $W^{k_1,k_2,\infty}_0(G)$, $\mathcal{B}^{k_1,k_2}(G)$, and $\mathcal{B}^{k_1,k_2}_c(G)$, starting with $\mathcal{M}^1(G)$, $\mathcal{M}^1_c(G)$, $L^p(\beta)$, $C_0(G)$, $\mathcal{B}(G)$, and $\mathcal{B}_c(G)$, respectively.

Proposition 1.30. *Take $p \in [1, \infty]$ and $k_1, k_2 \in \mathbb{N} \cup \{\, \infty \,\}$. Then, the following hold:*

- *$W^{k_1,k_2,1}_c(G)$ and $\mathcal{B}^{k_1,k_2}_c(G)$ are complete Hausdorff spaces;*
- *$W^{k_1,k_2,p}(\beta)$ and $\mathcal{B}^{k_1,k_2}(G)$ are Fréchet spaces;*
- *$W^{k_1,k_2,p}(\beta)$ is reflexive if $p \in]1, \infty[$;*
- *$\mathcal{D}(G)$ is dense in $\mathcal{B}^{k_1,k_2}_c(G)$, while $\mathcal{D}(G) \cdot \beta$ is dense in $W^{k_1,k_2,1}_c(G)$;*
- *$W^{k_1,k_2,p}_0(\beta) \cdot \beta$ is the closure of $\mathcal{D}(G) \cdot \beta$ in $W^{k_1,k_2,p}(\beta)$.*

Proof. The first three properties follow easily from the fact that the continuous linear mapping

$$W^{k_1,k_2,p}(\beta) \ni f \mapsto (\mathbf{Y}^{\alpha_1}\mathbf{X}^{\alpha_2} f) \in \prod_{|\alpha_1| \leqslant k_1, |\alpha_2| \leqslant k_2} L^p(\beta) \cdot \beta$$

is closed and one-to-one (with suitable modifications for the other spaces). The remaining assertions are proved with standard techniques: one first reduces to distributions with compact support by means of suitable truncations; smooth approximations are then obtained by convolution. □

Definition 1.31. Take $p \in [1, \infty]$ and $k_1, k_2 \in \mathbb{N} \cup \{\, \infty \,\}$. Then, define $W^{-k_1,-k_2,p}(\beta)$ and $\mathcal{B}^{-k_1,-k_2}(G)$ as the strong duals of $W^{k_1,k_2,p'}_0(\beta)$ and $W^{k_1,k_2,1}_c(G)$, respectively.

If $p \in]1, \infty]$, then define $W^{-k_1,-k_2,p}_c(\beta)$, $W^{-k_1,-k_2,1}_c(G)$, and $\mathcal{B}^{-k_1,-k_2}_c(G)$ as the duals of $W^{k_1,k_2,p'}_0(\beta)$, $\mathcal{B}^{k_1,k_2}_c(G)$, and $W^{k_1,k_2,1}_c(G)$, respectively, endowed with the topology of compact convergence.

Proposition 1.32. *Take $k_1, k_2 \in \mathbb{N} \cup \{\, \infty \,\}$ and $p \in [1, \infty]$; let (F, F_0) be one of the pairs*

$$(W^{-k_1,-k_2,p}(\beta), L^p(\beta) \cdot \beta) \quad (W^{-k_1,-k_2,1}_c(G), \mathcal{M}^1_c(G)) \quad (\mathcal{B}^{-k_1,-k_2}(G), \mathcal{B}(G) \cdot \beta),$$

and let B be a subset of F. Then, B is equicontinuous if and only if there are $h_1, h_2 \in \mathbb{N}$ with $h_1 \leqslant k_1$ and $h_2 \leqslant k_2$, and a bounded family $(\mu_{T,\alpha})_{T \in B, |\alpha_1| \leqslant h_1, |\alpha_2| \leqslant h_2}$ of elements of F_0 such that

$$T = \sum_{\substack{|\alpha_1| \leqslant h_1 \\ |\alpha_2| \leqslant h_2}} (\mathbf{Y}^{\alpha_1}\mathbf{X}^{\alpha_2})^\dagger \mu_{T,\alpha}$$

for every $T \in B$.

If, in addition, $(F, F_0) = (W^{-k_1,-k_2,p}(\beta), L^p(\beta) \cdot \beta)$ and B is compact, then the family $(\mu_{T,\alpha})_{T \in B, |\alpha_1| \leqslant h_1, |\alpha_2| \leqslant h_2}$ can be chosen so as to be compact.

Proof. Consider the closed embedding of the proof of Proposition 1.30, and apply [18, Propositions 10 and 15 of Chapter IV, § 1]. ☐

Corollary 1.33. *Take $k_1, k_2 \in \mathbb{N} \cup \{ \infty \}$. Then, $W^{-k_1,-k_2,1}(G)$ is canonically identified with the strong dual of $\mathcal{B}_c^{k_1,k_2}(G)$.*

Proof. It is easily seen that the inclusion $W_0^{k_1,k_2,\infty}(G) \subseteq \mathcal{B}_c^{k_1,k_2}(G)$ is continuous and dense, so that there is a continuous injection of $\mathcal{B}_c^{k_1,k_2}(G)'$ into $W^{-k_1,-k_2,1}(G)$. Now, this injection is onto by Proposition 1.32, so that it only remains to prove that every bounded subset of $\mathcal{B}_c^{k_1,k_2}(G)$ is contained in the closure, in $\mathcal{B}_c^{k_1,k_2}(G)$, of a bounded subset of $W_0^{k_1,k_2,\infty}(G)$. We leave the details to the reader. ☐

Corollary 1.34. *Take $k_1, k_2 \in \mathbb{N} \cup \{ \infty \}$ and $p \in]1, \infty[$. Then, $\mathcal{D}(G) \cdot \beta$ is dense in each one of the spaces $W^{-k_1,-k_2,p}(\beta)$, $W_c^{-k_1,-k_2,1}(G)$, and $W_c^{-k_1,-k_2,\infty}(\beta)$. In addition, the inclusion $\mathcal{E}'^{k_1+k_2}(G) \subseteq W^{-k_1,-k_2,1}(G)$ is continuous and dense.*

Definition 1.35. Take $k_1, k_2 \in \mathbb{N} \cup \{ \infty \}$ and $p \in [1, \infty]$. Then, we shall define $W^{-k_1,k_2,p}(\beta)$ as the set of $T \in W^{-k_1,0,p}(\beta)$ such that $\mathbf{X}^\alpha T \in W^{-k_1,0,p}(\beta)$ for every α with length at most k_2. We shall endow $W^{-k_1,k_2,p}(\beta)$ with the topology induced by the mapping $T \mapsto (\mathbf{X}^\alpha T) \in \prod_{|\alpha| \leqslant k_2} W^{-k_1,0,p}(\beta)$. We shall denote by $W_c^{-k_1,k_2,p}(\beta)$ the space $W^{-k_1,k_2,p}(\beta)$ endowed with the topology induced by the mapping $T \mapsto (\mathbf{X}^\alpha T) \in \prod_{|\alpha| \leqslant k_2} W_c^{-k_1,0,p}(\beta)$.

The spaces $W^{k_1,-k_2,p}(\beta)$ and $W_c^{k_1,-k_2,p}(\beta)$ are defined analogously.

Observe that this is consistent with the preceding notation when $k_1 = 0$.

Definition 1.36. Take $k_1, k_2 \in \mathbb{N} \cup \{ \infty \}$. Then, we shall define $\mathcal{B}_c^{-k_1,k_2}(G)$ as the set of $T \in \mathcal{B}_c^{-k_1,0}(G)$ such that $\mathbf{X}^\alpha T \in \mathcal{B}_c^{-k_1,0}(G)$ for every α with length at most k_2. We shall endow $\mathcal{B}_c^{-k_1,k_2}(G)$ with the topology induced by the mapping $T \mapsto (\mathbf{X}^\alpha T) \in \prod_{|\alpha| \leqslant k_2} \mathcal{B}_c^{-k_1,0}(G)$.

The space $\mathcal{B}_c^{k_1,-k_2}(G)$ is defined analogously.

Observe that this definition is consistent with the preceding notation when $k_1 = 0$.

Definition 1.37. Take $k_1, k_2 \in \mathbb{Z} \cup \{ \pm\infty \}$. Then, we shall say that a subset K of $W_c^{k_1,k_2,1}(G)$ is strictly compact if $\mathbf{Y}^\alpha \mathbf{X}^\beta K$ is equicontinuous on $\mathcal{B}_c^{(k_1)_-,(k_2)_-,1}(G)$ for every α, β with length at most $(k_1)_+, (k_2)_+$, respectively.

Analogous definition switching the roles of $\mathcal{B}$ and W^1.

Proposition 1.38. *Take $\mu \in \mathcal{M}_+(G)$. If $\mu \in W^{-\infty,-\infty,1}(G)$, then $\mu \in \mathcal{M}^1(G)$.*

Notice that the same assertion does *not* hold if μ is not positive. For example, one may take $\varphi \in \mathcal{E}(G) \cap \mathcal{L}^1(\beta)$ such that $X_1^\dagger \varphi \notin \mathcal{L}^1(\beta)$. Then, $X_1^\dagger(\varphi \cdot \beta) \in W^{0,-1,1}(\beta)$, but $X_1^\dagger(\varphi \cdot \beta) = (X_1^\dagger \varphi) \cdot \beta \notin \mathcal{M}^1(\beta)$.

Proof. Indeed, $\langle \mu, \tau_K \rangle$ is bounded as K runs through the set of compact subsets of G, where τ_K denotes an element of $\mathcal{D}(G)$ such that $\chi_K \leqslant \tau_K \leqslant \chi_G$. Since μ is positive, we have $\mu^*(G) < \infty$, so that $\mu \in \mathcal{M}^1(G)$. ☐

We conclude this section with some Sobolev embeddings.

Definition 1.39. Let $(M, \mathbf{g})$ be a Riemannian manifold, and take $k \in \mathbb{N}$ and $p \in [1, \infty[$. Let ∇ be the Levi-Civita connection on M and let ω be the canonical volume form. Define, for every $\varphi \in C^\infty(M)$,

$$\|\varphi\|_{W^{k,p}(M,\mathbf{g})} := \left(\sum_{j=0}^{k} \int_M |\nabla^j \varphi|^p \omega \right)^{\frac{1}{p}}.$$

Then, $W^{k,p}(M, \mathbf{g})$ is defined as the completion of the set of $\varphi \in C^\infty(M)$ such that $\|\varphi\|_{W^{k,p}(M,\mathbf{g})}$ is finite with respect to the norm $\| \cdot \|_{W^{k,p}(M,\mathbf{g})}$.

Lemma 1.40. *Let $\mathbf{g}$ be a left-invariant Riemannian metric on G, and take $k \in \mathbb{N}$ and $p \in [1, \infty[$; assume that β is left-invariant. Then, the following hold:*

1. *the spaces $W^{k,p}(G, \mathbf{g})$ and $W_0^{0,k,p}(\beta)$ are canonically isomorphic as locally convex spaces;*

2. *if* $h \in \mathbb{N}$, $q \in [1, \infty[$, *and* $\frac{1}{p} \geqslant \frac{1}{q} \geqslant \frac{1}{p} - \frac{h}{n} > 0$, *then* $W_0^{0,k+h,p}(\beta)$ *embeds canonically into* $W_0^{0,k,q}(\beta)$;

3. *if* $h \in \mathbb{N}$ *and* $\frac{h}{n} > \frac{1}{p}$, *then* $W_0^{0,k+h,p}(\beta)$ *embeds canonically into* $\mathcal{B}^{0,k}(G)$.

Proof. The proof of the first assertion is left to the reader. As for what concerns the other assertions, by [6, Theorem 2.21], it suffices to observe that $(G, \mathbf{g})$ is a complete Riemannian manifold, and that its curvature and exponential map are left invariant, so that its curvature is bounded and its injectivity radius is strictly positive. □

Proposition 1.41. *Assume that* β *is both left- and right-invariant, and take* $k_1, k_2 \in \mathbb{N} \cup \{\infty\}$, $h_1, h_2 \in \mathbb{N}$ *and* $p, q \in [1, \infty[$. *Then, the following hold:*

1. *if* $\frac{1}{p} \geqslant \frac{1}{q} \geqslant \frac{1}{p} - \frac{h_1+h_2}{n} > 0$, *then* $W_0^{k_1+h_1,k_2+h_2,p}(\beta)$ *embeds canonically into* $W_0^{k_1,k_2,q}(\beta)$;

2. *if* $\frac{h_1+h_2}{n} > \frac{1}{p}$ *and either* $\frac{h_1}{n} \neq \frac{1}{p}$ *or* $\frac{h_2}{n} \neq \frac{1}{p}$, *then* $W_0^{k_1+h_1,k_2+h_2,p}(\beta)$ *embeds canonically into* $\mathcal{B}^{k_1,k_2}(G)$.

Proof. **1.** Take $f \in W_0^{k_1+h_1,k_2+h_2,p}(\beta)$. Take $q' \in [1, \infty]$ so that $\frac{1}{q'} = \frac{1}{p} - \frac{h_1}{n}$. Then, for every α_1 with length at most k_1 we have $\mathbf{Y}^{\alpha_1} f \in W_0^{h_1,k_2+h_2,p}(\beta)$, so that $\mathbf{Y}^{\alpha_1} f \in W_0^{0,k_2+h_2,q'}(\beta)$ by Lemma 1.40. In the same way we see that for every α_2 with length at most k_2 we have $\mathbf{Y}^{\alpha_1}\mathbf{X}^{\alpha_2} f \in L^q(\beta)$, whence our assertion.

2. Assume, for the sake of definiteness, that $\frac{h_1}{n} \neq \frac{1}{p}$. If $\frac{h_1}{n} < \frac{1}{p}$, then choose q' as in **1**. Then, Lemma 1.40 implies that $\mathbf{Y}^{\alpha_1} f \in W^{0,h_2,q'}(\beta)$ for every α_1 of length at most k_1, so that another application of Lemma 1.40 shows that $\mathbf{Y}^{\alpha_1}\mathbf{X}^{\alpha_2} f \in \mathcal{B}(G)$ for every α_2 of length at most k_2, whence the result. If, on the contrary, $\frac{h_1}{n} > \frac{1}{p}$, then Lemma 1.40 leads directly to the conclusion. □

Proposition 1.42. *Take* $k_1, k_2 \in \mathbb{N} \cup \{\infty\}$. *Then,* $W^{k_1+1,k_2,1}(G)$ *and* $W^{k_1,k_2+1,1}(G)$ *embed continuously into* $W_0^{k_1,k_2,1}(\beta)$.

Proof. We may reduce to proving that $W^{0,1,1}(G)$ embeds into $L^1(\beta)$. Now, by means of suitable truncations, we may reduce to prove the assertion for measures with compact support; then, by means of a finite partition of the unity we may reduce to the case in which that support is contained in the domain of a local chart. Then, the assertion follows from [2, Exercise 3.2]. □

Corollary 1.43. *Assume that* β *is both left- and right-invariant, and take* $k_1, k_2 \in \mathbb{N} \cup \{\infty\}$ *such that* $k_1 + k_2 = \infty$ *and* $p \in [1, \infty[$. *Then, the following hold:*

1. $W^{k_1,k_2,p}(\beta) = W_0^{k_1,k_2,p}(\beta) \cdot \beta$ *and the inclusion* $W_0^{k_1,k_2,p}(\beta) \subseteq \mathcal{B}^{k_1,k_2}(G)$ *is continuous;*

2. $W^{k_1,k_2,\infty}(\beta) = \mathcal{B}^{k_1,k_2}(G) \cdot \beta$.

Proof. The first assertion is a consequence of Propositions 1.41 and 1.42. Next, take $f \in W^{k_1,k_2,\infty}(\beta)$. Then, for every $\varphi \in \mathcal{D}(G)$ we have $\varphi \cdot f \in W^{k_1,k_2,1}(G)$, so that $\varphi \cdot f \in \mathcal{D}(G) \cdot \beta$. The assertion follows from the arbitrariness of φ. □

1.4 Convolution

In this section, G will denote a Lie group of dimension n, and β a relatively invariant measure on G with left and right multipliers Δ_L, Δ_R, respectively. The definition of convolution we adopt is a straightforward generalization of [76]. See also [32] and the references therein for an account of other equivalent definitions of convolution on $\mathbb{R}^n$.

Definition 1.44. Take $T_1, T_2 \in \mathcal{D}'(G)$. We say that T_1 and T_2 are convolvable if

$$[\varphi \circ (\,\cdot\,)](T_1 \otimes T_2) \in W^{-\infty,-\infty,1}(G \times G)$$

for every $\varphi \in \mathcal{D}(G)$. In this case, we define

$$\langle T_1 * T_2, \varphi \rangle := \langle [\varphi \circ (\,\cdot\,)](T_1 \otimes T_2), \chi_{G \times G} \rangle$$

for every $\varphi \in \mathcal{D}(G)$.

Observe that Proposition 1.38 implies that, if two measures μ_1, μ_2 are convolvable *as measures*, in the sense that the mapping $\cdot : G \times G \to G$ is $(\mu_1 \otimes \mu_2)$-proper, that is, $(|\mu_1| \otimes |\mu_2|)$-proper, then μ_1 and μ_2 are convolvable in the sense of Definition 1.44. Nevertheless, the converse fails, in general, if μ_1 or μ_2 is not positive (cf. Remark 1.56).

Lemma 1.45. *Let T_1 and T_2 be two convolvable distributions on G. Then, the mapping $\varphi \mapsto \langle T_1 * T_2, \varphi\rangle$ defines a distribution which is supported in the closure of* $\operatorname{Supp}(T_1)\operatorname{Supp}(T_2)$*.*

Proof. Indeed, let (τ_K) be a filtered family which is bounded in $\mathcal{B}_c^{\infty,\infty}(G\times G)$, has elements in $\mathcal{D}(G\times G)$, and converges to $\chi_{G\times G}$ in $\mathcal{B}_c^{\infty,\infty}(G\times G)$.[2] Then, for every $\varphi \in \mathcal{D}(G)$, the family

$$\langle T_1 \otimes T_2, [\varphi \circ (\,\cdot\,)]\tau_K\rangle$$

is bounded and converges to $\langle T_1 * T_2, \varphi\rangle$. Therefore, the mappings $\varphi \mapsto \langle T_1 \otimes T_2, [\varphi \circ (\,\cdot\,)]\tau_K\rangle$ are equicontinuous on $\mathcal{D}(G)$, so that their pointwise limit $T_1 * T_2$ is continuous on $\mathcal{D}(G)$. The assertion concerning the support follows from the fact that, if the support of φ does not intersect the closure of $\operatorname{Supp}(T_1)\operatorname{Supp}(T_2)$, then $\varphi \circ (\,\cdot\,)$ vanishes on a neighbourhood of the support $\operatorname{Supp}(T_1)\times\operatorname{Supp}(T_2)$ of $T_1 \otimes T_2$. □

The following simple result will be very useful in the sequel.

Proposition 1.46. *Take $T_1, T_2 \in \mathcal{D}'(G)$. Then, T_1 and T_2 are convolvable if and only if $\check{T}_2$ and $\check{T}_1$ are convolvable. In this case,*

$$(T_1 * T_2)\check{} = \check{T}_2 * \check{T}_1.$$

Proof. Suppose that T_1 and T_2 are convolvable, and take $\varphi \in \mathcal{D}(G)$. Then, $[\check{\varphi} \circ (\,\cdot\,)](T_1 \otimes T_2) \in W^{-\infty,-\infty,1}(G\times G)$, so that its image under the reflection $(g_1, g_2) \mapsto (g_2^{-1}, g_1^{-1})$, that is, $[\varphi \circ (\,\cdot\,)](\check{T}_2 \otimes \check{T}_1)$, belongs to $W^{-\infty,-\infty,1}(G)$, and has the same value at $\chi_{G\times G}$.[3] Therefore, $\check{T}_2$ and $\check{T}_1$ are convolvable and $(T_1 * T_2)\check{} = \check{T}_2 * \check{T}_1$. The converse implication follows if one applies the preceding result to $\check{T}_2$ and $\check{T}_1$. □

Proposition 1.47. *Take $T_1, T_2 \in \mathcal{D}'(G)$ and a continuous (hence analytic) representation χ of G in $\mathbb{C}^*$. Then, T_1 and T_2 are convolvable if and only if χT_1 and χT_2 are convolvable. In this case,*

$$\chi(T_1 * T_2) = (\chi T_1) * (\chi T_2).$$

Proof. Take $\varphi \in \mathcal{D}(G)$. Then,

$$[\varphi \circ (\,\cdot\,)][(\chi T_1) \otimes (\chi T_2)] = [(\chi\varphi) \circ (\,\cdot\,)](T_1 \otimes T_2).$$

Since the mapping $\varphi \mapsto \chi\varphi$ induces and automorphism of $\mathcal{D}(G)$, it is clear that T_1 and T_2 are convolvable if and only if χT_1 and χT_2 are convolvable. In addition, in this case $\chi(T_1 * T_2) = (\chi T_1) * (\chi T_2)$. □

Definition 1.48. If T_1 and T_2 are two convolvable distributions and $T_1 * T_2 = f \cdot \beta$ for some $f \in L^1_{\mathrm{loc}}(\beta)$, then we shall also denote f by $T_1 *^\beta T_2$, or even $T_1 * T_2$ if no ambiguity is to be feared. If, in addition, $T_1 = f_1 \cdot \beta$ for some $f_1 \in L^1_{\mathrm{loc}}(\beta)$, then we shall also write $f_1 *^\beta T_2$, or even $f_1 * T_2$, instead of $(f_1 \cdot \beta) * T_2$. We shall use similar conventions also if $T_2 = f_2 \cdot \beta$ for some $f_2 \in L^1_{\mathrm{loc}}(\beta)$.

Definition 1.49. We shall say that two distributions T_1 and T_2 on G are transversally convolvable if $\cdot :\ \operatorname{Supp}(T_1)\times\operatorname{Supp}(T_2) \to G$ is a proper mapping.

Lemma 1.50. *Let A, B be two closed subsets of G. Then, the following facts are equivalent:*

1. *$\cdot : A \times B \to G$ is proper;*

2. *$A \cap KB^{-1}$ is compact for every compact subset K of G;*

3. *$B \cap A^{-1}K$ is compact for every compact subset K of G.*

[2] For example, if V is a compact neighbourhood of e in $G \times G$ and $\psi \in \mathcal{D}(G\times G)$ equals 1 on V, then one may define $\tau_K := \psi *^\beta \chi_{VKV} *^\beta \psi$ for every compact subset K of $G \times G$.

[3] It suffices to observe that the reflection $(g_1, g_2) \mapsto (g_2^{-1}, g_1^{-1})$ induces an automorphism of $\mathcal{B}_c^{\infty,\infty}(G\times G)$.

Proof. **1** $\implies$ **2** $\wedge$ **3.** Indeed, if K is a compact subset of G, then $\mathrm{pr}_1([(\,\cdot\,)^{-1}(K)]\cap(A\times B)) = A\cap KB^{-1}$ and $\mathrm{pr}_2([(\,\cdot\,)^{-1}(K)]\cap(A\times B)) = B\cap A^{-1}K$.

2 $\wedge$ **3** $\implies$ **1.** Indeed, by **1** we see that $[(\,\cdot\,)^{-1}(K)]\cap(A\times B)\subseteq[A\cap KB^{-1}]\times[B\cap A^{-1}K]$. Since $[(\,\cdot\,)^{-1}(K)]\cap(A\times B)$ is closed, it is compact.

2 $\iff$ **3.** Indeed, let K be a compact subset of G, and assume that **2** holds. Then, $A\cap KB^{-1}$ is compact, so that $A^{-1}\cap BK^{-1}$ is compact. Therefore, $(A^{-1}\cap BK^{-1})K$ is compact. Since $A^{-1}K\cap B$ is closed and contained in $(A^{-1}\cap BK^{-1})K$, it is compact. Hence **3** holds. In the same way one proves the converse implication. ☐

Proposition 1.51. *Take two transversally convolvable distributions T_1 and T_2 on G. Then, T_1 and T_2 are convolvable. In addition, if V is a compact neighbourhood of e in G, $\varphi_1,\varphi_2\in\mathcal{E}(G)$,*

$$\chi_{V\,\mathrm{Supp}(T_1)}\leqslant\varphi_1\leqslant\chi_{V^2\,\mathrm{Supp}(T_1)},\qquad \chi_{\mathrm{Supp}(T_2)V}\leqslant\varphi_2\leqslant\chi_{\mathrm{Supp}(T_2)V^2},$$

and $\varphi\in\mathcal{D}(G)$, then the mappings

$$g_1\mapsto\varphi_1(g_1)\langle T_2,\varphi(g_1\,\cdot\,)\rangle \qquad\text{and}\qquad g_2\mapsto\varphi_2(g_2)\langle T_1,\varphi(\,\cdot\,g_2)\rangle$$

belong to $\mathcal{D}(G)$, and

$$\langle T_1*T_2,\varphi\rangle=\langle T_1,g_1\mapsto\varphi_1(g_1)\langle T_2,\varphi(g_1\,\cdot\,)\rangle\rangle=\langle T_2,g_2\mapsto\varphi_2(g_2)\langle T_1,\varphi(\,\cdot\,g_2)\rangle\rangle.$$

Proof. Observe that, for every $\varphi\in\mathcal{D}(G)$, the intersection of the support of $\varphi\circ(\,\cdot\,)$ with the support of $T_1\otimes T_2$ is compact, so that $[\varphi\circ(\,\cdot\,)](T_1\otimes T_2)$ has compact support. Hence, T_1 and T_2 are convolvable. Now, observe that Lemma 1.50 implies that $\cdot : V^2\,\mathrm{Supp}(T_1)\times\mathrm{Supp}(T_2)V^2\to G$ is still proper, so that Lemma 1.50 again implies that the mappings

$$g_1\mapsto\varphi_1(g_1)\langle T_2,\varphi(g_1\,\cdot\,)\rangle \qquad\text{and}\qquad g_2\mapsto\varphi_2(g_2)\langle T_1,\varphi(\,\cdot\,g_2)\rangle,$$

which are clearly of class C^∞, have compact support. Finally, take a filtered family $(\tau_K)_{K\in\mathcal{K}}$ which is bounded and converges to $\chi_{G\times G}$ in $\mathcal{B}^{\infty,\infty}_c(G\times G)$ and whose elements belong to $\mathcal{D}(G\times G)$. Then, for every $\varphi\in\mathcal{D}(G)$,

$$\begin{aligned}\langle T_1*T_2,\varphi\rangle&=\langle[\varphi\circ(\,\cdot\,)](T_1\otimes T_2),\chi_{G\times G}\rangle\\&=\lim_{K,\mathcal{K}}\langle[\varphi\circ(\,\cdot\,)](T_1\otimes T_2),\tau_K\rangle\\&=\lim_{K,\mathcal{K}}\langle T_1,g_1\mapsto\langle T_2,\varphi(g_1\,\cdot\,)\tau_K(g_1,\,\cdot\,)\rangle\rangle;\end{aligned}$$

since the mappings $g_1\mapsto\langle T_2,\varphi(g_1\,\cdot\,)\tau_K(g_1,\,\cdot\,)\rangle$ converge locally uniformly with every left-invariant derivative to $g_1\mapsto\langle T_2,\varphi(g_1\,\cdot\,)\rangle$, the first equality follows easily from the preceding remarks. The other equality is proved similarly. ☐

Corollary 1.52. *Keep the hypotheses and the notation of Proposition 1.51. Assume further that $T_1=\psi_1\cdot\beta$ for some $\psi_1\in\mathcal{E}^{r_1+r_2}(G)\cdot\beta$, and that $T_2\in\mathcal{D}'^{r_2}(G)$. Then, $T_1*T_2\in\mathcal{E}^{r_1}(G)\cdot\beta$ and*

$$(\psi_1*^\beta T_2)(g_1)=\big\langle T_2,\varphi_2\,\psi_1(g_1\,\cdot\,^{-1})\Delta_R^{-1}\big\rangle.$$

Analogously, if $T_2=\psi_2\cdot\beta$ for some $\psi_2\in\mathcal{E}^{r_1+r_2}(G)\cdot\beta$, and $T_1\in\mathcal{D}'^{r_1}(G)$, then $T_1*T_2\in\mathcal{E}^{r_2}(G)\cdot\beta$ and

$$(T_1*^\beta\psi_2)(g_2)=\big\langle T_1,\varphi_1\,\psi_2(\,\cdot\,^{-1}g_2)\Delta_L^{-1}\big\rangle.$$

The proof is standard and is omitted

Corollary 1.53. *Take three non-empty closed subsets F_1,F_2,F_3 of G such that the mapping*

$$F_1\times F_2\times F_3\ni(g_1,g_2,g_3)\mapsto g_1g_2g_3\in G$$

is proper. Take $T_1,T_2,T_3\in\mathcal{D}'(G)$, and assume that T_j is supported in F_j for every $j=1,2,3$. Then, the following hold:

- *T_1 and T_2 are transversally convolvable;*
- *T_2 and T_3 are transversally convolvable;*

- $T_1 * T_2$ *and* T_3 *are transversally convolvable;*
- T_1 *and* $T_2 * T_3$ *are transversally convolvable;*
- $(T_1 * T_2) * T_3 = T_1 * (T_2 * T_3)$.

The proof is standard and is omitted.

Corollary 1.54. *Take* $r_1, r_2 \in \mathbb{N} \cup \{\infty\}$ *and three non-empty closed subsets* F_1, F_2, F_3 *of* G *such that the mapping*

$$F_1 \times F_2 \times F_3 \ni (g_1, g_2, g_3) \mapsto g_1 g_2 g_3 \in G$$

is proper. Take $T_1 \in \mathcal{D}'^{r_1}(G)$, $T_2 \in \mathcal{D}'^{r_2}(G)$, *and* $\varphi_3 \in \mathcal{E}^{r_1+r_2}(G)$, *and assume that they are supported in* F_1, F_2, *and* $\check{F}_3$, *respectively. Then, the following hold:*

- T_1 *and* T_2 *are transversally convolvable;*
- φ_3 *and* $\Delta_R \check{T}_2$ *are transversally convolvable;*
- $\Delta_L \check{T}_1$ *and* φ_3 *are transversally convolvable;*
- *the following equalities hold:*[4]

$$\langle T_1 * T_2, \varphi_3 \rangle = \big\langle T_1, \varphi_3 *^\beta (\Delta_R \check{T}_2) \big\rangle = \big\langle T_2, (\Delta_L \check{T}_1) *^\beta \varphi_3 \big\rangle.$$

Proof. By Corollaries 1.52 and 1.53 and Propositions 1.46 and 1.47, we see that the ordered pairs (T_1, T_2), $(T_1 * T_2, \Delta_L \check{\varphi}_3)$, $(T_2, \Delta_L^{-1} \check{\varphi}_3)$ and $(T_1, T_2 * (\Delta_L^{-1} \varphi_3))$ are transversally convolvable, and that

$$\begin{aligned} \langle T_1 * T_2, \varphi_3 \rangle &= [(T_1 * T_2) *^\beta (\Delta_L^{-1} \check{\varphi}_3)](e) \\ &= [T_1 *^\beta (T_2 *^\beta (\Delta_L^{-1} \check{\varphi}_3))](e) \\ &= \big\langle T_1, [T_2 *^\beta (\Delta_L^{-1} \check{\varphi}_3)]\check{}\, \Delta_L^{-1} \big\rangle \\ &= \big\langle T_1, [(\Delta_R^{-1} \varphi_3) *^\beta \check{T}_2] \Delta_R \big\rangle \\ &= \big\langle T_1, \varphi_3 *^\beta (\Delta_R \check{T}_2) \big\rangle. \end{aligned}$$

In the same way one may prove the other equality. □

The following result is the main achievement of Definition 1.44. It will enable us to simplify several proofs in the following.

Proposition 1.55. *Take* $T_1, T_2 \in \mathcal{D}'(G)$ *such that* T_1 *and* T_2 *are convolvable, a left-invariant differential operator* X *and a right-invariant differential operator* Y *of* G*. Then, the following hold:*

1. YT_1 *and* XT_2 *are convolvable, and*

$$YX(T_1 * T_2) = (YT_1) * (XT_2);$$

2. *suppose that* X *and* Y *are vector fields which coincide at* e*. Then,* XT_1 *and* T_2 *are convolvable if and only if* T_1 *and* YT_2 *are convolvable. If this is the case, then*

$$(XT_1) * T_2 = T_1 * (YT_2).$$

Notice that the both assertions may fail if we do not assume that T_1 and T_2 are convolvable, even if we assume that the stated formulae make sense. Indeed, assume that $G = \mathbb{R}$, that β is Lebesgue measure, and that $X = Y$ is the first derivative. Then:

- if $T_1 = I_\mathbb{R} \cdot \beta$ and $T_2 := \beta$, then T_1 and $XT_2 = 0$ are convolvable, but $XT_1 = T_2$ and T_2 are *not* convolvable;

[4] In order to make the stated formulae more concise, we write $\langle T_1 * T_2, \varphi_3 \rangle$ instead of $\langle T_1 * T_2, \psi_{1,2}\, \varphi_3 \rangle$, where $\psi_{1,2}$ is some element of $\mathcal{E}(G)$ which equals 1 on a neighbourhood of the support of $T_1 * T_2$, subject to the condition that the intersection $\mathrm{Supp}(\psi_{1,2}) \cap \mathrm{Supp}(\varphi_3)$ is compact. Analogous considerations for the other cases.

- if $T_1 = \operatorname{sgn} \cdot \beta$ and $T_2 = \beta$, then T_1 and $XT_2 = 0$ are convolvable, $XT_1 = 2\delta_0$ and T_2 are convolvable, but
$$(XT_1) * T_2 = 2\beta \neq 0 = T_1 * (XT_2).$$

Proof. **1.** Take $\varphi \in \mathcal{D}(G)$, and assume first that Y is the identity. Proceeding by induction, we may assume that X is a vector field. Now,

$$\langle X(T_1 * T_2), \varphi \rangle = \langle T_1 * T_2, X^\dagger \varphi \rangle = \langle [(X^\dagger \varphi) \circ (\,\cdot\,)](T_1 \otimes T_2), \chi_{G\times G} \rangle.$$

Define X_2' on $G \times G$ so that $(X_2'\psi)(g_1, g_2) := X(\psi(g_1, \,\cdot\,))(g_2)$ for every $\psi \in \mathcal{E}(G \times G)$ and for every $g_1, g_2 \in G$, so that X_2' is a left-invariant vector field on $G \times G$. Then,

$$[(X^\dagger \varphi) \circ (\,\cdot\,)](T_1 \otimes T_2) = [\varphi \circ (\,\cdot\,)][T_1 \otimes (XT_2)] + X_2'^\dagger [[\varphi \circ (\,\cdot\,)](T_1 \otimes T_2)],$$

so that $[\varphi \circ (\,\cdot\,)][T_1 \otimes (XT_2)] \in W^{-\infty,-\infty,1}(G \times G)$. Hence, T_1 and XT_2 are convolvable by the arbitrariness of φ, and

$$\langle [(X^\dagger \varphi) \circ (\,\cdot\,)](T_1 \otimes T_2), \chi_{G\times G} \rangle = \langle [\varphi \circ (\,\cdot\,)][T_1 \otimes (XT_2)], \chi_{G\times G} \rangle,$$

that is,
$$\langle X(T_1 * T_2), \varphi \rangle = \langle T_1 * (XT_2), \varphi \rangle,$$
whence the result in this case. Now, by means of Proposition 1.46 we get the assertion also in the case in which X is the identity while Y is arbitrary. The general assertion then follows.

2. Suppose that XT_1 and T_2 are convolvable, and take $\varphi \in \mathcal{D}(G)$. Observe that, with notation analogous to that of **1**,

$$\begin{aligned} X_1'[\varphi \circ (\,\cdot\,)](g_1, g_2) &= X[R_{g_2}\varphi](g_1) = X_e[L_{g_1^{-1}} R_{g_2} \varphi] \\ &= Y[L_{g_1^{-1}} \varphi](g_2) = Y_2'[\varphi \circ (\,\cdot\,)](g_1, g_2) \end{aligned}$$

for every $g_1, g_2 \in G$. Then,

$$[\varphi \circ (\,\cdot\,)][(XT_1) \otimes T_2] = [\varphi \circ (\,\cdot\,)][T_1 \otimes (YT_2)] + (X_1' - Y_2')[[\varphi \circ (\,\cdot\,)](T_1 \otimes T_2)].$$

Since T_1 and T_2 are convolvable, $[\varphi \circ (\,\cdot\,)][T_1 \otimes (YT_2)] \in W^{-\infty,-\infty,1}(G \times G)$, so that the arbitrariness of φ implies that T_1 and YT_2 are convolvable. In addition,

$$\langle [\varphi \circ (\,\cdot\,)][(XT_1) \otimes T_2], \chi_{G\times G} \rangle = \langle [\varphi \circ (\,\cdot\,)][T_1 \otimes (YT_2)], \chi_{G\times G} \rangle.$$

that is,
$$\langle (XT_1) * T_2, \varphi \rangle = \langle T_1 * (YT_2), \varphi \rangle,$$
whence the second assertion. The converse implication may be proved by means of Proposition 1.46. □

Remark 1.56. Take $G = \mathbb{R}$, and define $\varphi := e^{-|\,\cdot\,|}$ and $\psi := e^{-|\,\cdot\,|} \sin(e^{2|\,\cdot\,|})$; let β be Lebesgue measure. Then, $T_1 := \varphi \cdot \beta$ and $T_2 := \psi \cdot \beta$ are convolvable, so that T_1' and T_2 are convolvable, T_1 and T_2' are convolvable, and

$$(T_1 * T_2)' = T_1' * T_2 = T_1 * T_2'.$$

Now, observe that T_1, T_2, and T_1' are bounded measures, so that $T_1' * T_2$ is a bounded measure; in addition, T_2' is a measure. Nevertheless, T_1 and T_2' are *not* convolvable as measures, that is, $+\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is not $(T_1 \otimes |T_2'|)$-proper. Indeed, $T_2' - 2\cos(e^{2|\,\cdot\,|}) e^{|\,\cdot\,|} \operatorname{sgn} \cdot \beta$ is a bounded measure, so that it will suffice to show that the positive measures T_1 and $\left|\cos(e^{2|\,\cdot\,|})\right| e^{|\,\cdot\,|} \cdot \beta$ are not convolvable. However, Tonelli's theorem implies that

$$\begin{aligned} \int^*_{+^{-1}([-R,R])} e^{-|x|} \left|\cos\left(e^{2|y|}\right)\right| e^{|y|} \,\mathrm{d}(x,y) &\geqslant \int^*_{[R,+\infty[} \left|\cos\left(e^{2y}\right)\right| e^{y} (e^{R-y} - e^{-R-y}) \,\mathrm{d}y \\ &= \sinh(R) \int^*_{[e^{2R},+\infty[} \frac{|\cos(z)|}{z} \,\mathrm{d}z = +\infty, \end{aligned}$$

for every $R > 0$, whence our claim.

We shall now state without proof some hypocontinuity results for convolution on various spaces. See [78] for some proofs in the abelian case.

Corollary 1.57. *Take $r_1, r_2 \in \mathbb{N} \cup \{\infty\}$; then, the bilinear mappings*

$$*^\beta \colon \mathcal{E}^{r_1}(G) \times \mathcal{D}^{r_2}(G) \to \mathcal{E}^{r_1+r_2}(G), \qquad *^\beta \colon \mathcal{D}^{r_1}(G) \times \mathcal{E}^{r_2}(G) \to \mathcal{E}^{r_1+r_2}(G)$$
$$*^\beta \colon \mathcal{D}^{r_1}(G) \times \mathcal{D}^{r_2}(G) \to \mathcal{D}^{r_1+r_2}(G),$$

are hypocontinuous relative to the bounded subsets of each factor.

Corollary 1.58. *Take $r_1, r_2 \in \mathbb{N} \cup \{\infty\}$; then, the bilinear mappings*

$$*^\beta \colon \mathcal{D}'^{r_1}(G) \times \mathcal{D}^{r_1+r_2}(G) \to \mathcal{E}^{r_2}(G), \qquad *^\beta \colon \mathcal{D}^{r_1+r_2}(G) \times \mathcal{D}'^{r_2}(G) \to \mathcal{E}^{r_1}(G)$$
$$*^\beta \colon \mathcal{E}'^{r_1}(G) \times \mathcal{E}^{r_1+r_2}(G) \to \mathcal{E}^{r_2}(G), \qquad *^\beta \colon \mathcal{E}^{r_1+r_2}(G) \times \mathcal{E}'^{r_2}(G) \to \mathcal{E}^{r_1}(G)$$
$$*^\beta \colon \mathcal{E}'^{r_1}(G) \times \mathcal{D}^{r_1+r_2}(G) \to \mathcal{D}^{r_2}(G), \qquad *^\beta \colon \mathcal{D}^{r_1+r_2}(G) \times \mathcal{E}'^{r_2}(G) \to \mathcal{D}^{r_1}(G),$$

where the spaces $\mathcal{E}'$ and $\mathcal{D}'$ are endowed with the topology of bounded (resp. compact) convergence, are hypocontinuous relative to the bounded subsets of each space $\mathcal{E}'$, $\mathcal{D}'$ and to the bounded (resp. compact) subsets of each space $\mathcal{E}$, $\mathcal{D}$.

Corollary 1.59. *Take $r_1, r_2 \in \mathbb{N} \cup \{\infty\}$; then, the bilinear mappings*

$$* \colon \mathcal{E}'^{r_1}(G) \times \mathcal{D}'^{r_2}(G) \to \mathcal{D}'^{r_1+r_2}(G), \qquad * \colon \mathcal{D}'^{r_1}(G) \times \mathcal{E}'^{r_2}(G) \to \mathcal{D}'^{r_1+r_2}(G)$$
$$* \colon \mathcal{E}'^{r_1}(G) \times \mathcal{E}'^{r_2}(G) \to \mathcal{E}'^{r_1+r_2}(G),$$

where all spaces are endowed with the topology of bounded (resp. compact, pointwise) convergence are hypocontinuous relative to the equicontinuous (resp. equicontinuous, finite) subsets of each factor.

The following theorem is a generalization of an remarkable result by J. Dixmier and P. Malliavin [34, Theorem 3.1]. We begin with a definition.

Definition 1.60. Let V, W be two vector spaces over $\mathbb{R}$ or $\mathbb{C}$, and let A be a subset of V. We shall say that a mapping $T \colon A \to W$ is linear if $T\left(\sum_{j\in J} \lambda_j v_j\right) = \sum_{j\in J} \lambda_j T(v_j)$ for every finite set J, for every $(\lambda_j) \in \mathbb{F}^J$, and for every $(v_j) \in A^J$ such that $\sum_{j \in J} \lambda_j v_j \in A$.

In other words, T is linear if it is the restriction to A of a linear mapping defined on V.

Theorem 1.61. *Let G be a Lie group. Let V be a neighbourhood of e in G and F a Fréchet space which is continuously embedded in $\mathcal{E}(G)$. Assume that $F *^\beta X \subseteq F$ for every $X \in \mathfrak{U}(\mathfrak{g})$. Let B be a bounded subset of F. Then, there are a finite family $(\psi_\gamma)_{\gamma \in \{0,1\}^n}$ of elements of $\mathcal{D}(V)$, and a uniformly continuous linear mapping T on B with values in a bounded subset of $F^{\{0,1\}^n}$ such that*

$$\varphi = \sum_{\gamma \in \{0,1\}^n} T_\gamma(\varphi) *^\beta \psi_\gamma,$$

and such that $\operatorname{Supp}(T_\gamma(\varphi)) \subseteq \operatorname{Supp}(\varphi)$ *for every $\gamma \in \{0,1\}^n$ and for every $\varphi \in B$.*

Proof. Take $\varepsilon > 0$ so that the mapping

$$\Phi \colon]-\varepsilon, \varepsilon[^n \ni (t_1, \dots, t_n) \mapsto \exp_G(t_n X_n) \cdots \exp_G(t_1 X_1)$$

is a diffeomorphism of $]-\varepsilon, \varepsilon[^n$ onto an open subset of V. Let $(\rho_j)_{j \in \mathbb{N}}$ be an incresing sequence of continuous semi-norms on F which define the topology of F, and let $X_1, \dots, X_n$ be a basis of the Lie algebra of G. Set, for every $m, j \in \mathbb{N}$,[5]

$$M_{m,j} \coloneqq \sup_{\varphi \in B} \left\lVert \varphi *^\beta X_1^{2m} \right\rVert_{\rho_j} < +\infty.$$

[5]Observe that the mapping $F \ni \varphi \mapsto \varphi * X \in F$ is continuous for every $X \in \mathfrak{U}(\mathfrak{g})$ thanks to the closed graph theorem, so that $B * X_1^{2m}$ is bounded in F.

Then, [34, Lemma 2.5 and Remark 2.6] imply that there are a sequence (c_m) in $\mathbb{R}_+$ and $g_0^{(1)}, g_1^{(1)} \in \mathcal{D}(]-\varepsilon,\varepsilon[)$ such that $\sum_{m\in\mathbb{N}} c_m M_{m,j} < +\infty$ for every $j \in \mathbb{N}$, and

$$\sum_{m=0}^{\infty} (-1)^m c_m \partial^{2m} g_0^{(1)} = \delta_0 + g_1^{(1)}$$

in $\mathcal{E}'(\mathbb{R})$. Let $\mu_0^{(1)}, \mu_1^{(1)}$ be the images of the measures $g_0^{(1)} \cdot \nu_\mathbb{R}$ and $g_1^{(1)} \cdot \nu_\mathbb{R}$ under the mapping $\mathbb{R} \ni t \mapsto \exp_G(tX_1) \in G$. Then, it is clear that

$$\sum_{m=0}^{\infty} \left[(-1)^m c_m X_1^{2m} * \mu_0^{(1)}\right] = \delta_e + \mu_1^{(1)}$$

in $\mathcal{E}'(G)$. In particular,

$$\varphi = \sum_{m=0}^{\infty} \left[(-1)^m c_m \left(\varphi *^\beta X_1^{2m}\right) *^\beta \mu_0^{(1)}\right] - \varphi *^\beta \mu_1^{(1)}$$

in $\mathcal{E}(G)$ for every $\varphi \in B$. Now, thanks to our choice of the sequence (c_m), the Weierstrass criterion and the completeness of F imply that the sum $\sum_{m\in\mathbb{N}} (-1)^m c_m \, \varphi *^\beta X_1^{2m}$ converges in F to some $T_0^{(1)}(\varphi)$, uniformly as φ runs through B. Therefore, $T_0^{(1)}(B)$ is bounded in F and the mapping $B \ni \varphi \mapsto T_0^{(1)}(\varphi) \in F$ is uniformly continuous.

Define $T_1^{(1)}(\varphi) \coloneqq -\varphi$, so that $T_j^{(1)}$ is a uniformly continuous linear mapping from B into a bounded subset of F for $j = 0, 1$. Now, fix $\varphi \in B$; since F embeds continuously into $\mathcal{E}(G)$, $\sum_{m=0}^{\infty} \left[(-1)^m c_m (\varphi *^\beta X_1^{2m}) *^\beta \mu_0^{(1)}\right]$ converges to $T_0^{(1)}(\varphi) * \mu_0^{(1)}$ in $\mathcal{E}(G)$, so that

$$\varphi = T_0^{(1)}(\varphi) * \mu_0^{(1)} + T_1^{(1)}(\varphi) * \mu_1^{(1)}.$$

In addition, $T_j^{(1)}(\varphi)$ is clearly supported in $\operatorname{Supp}(\varphi)$ for $j = 0, 1$, and $B_1 \coloneqq T_0^{(1)}(B) \cup T_1^{(1)}(B)$ is bounded in F.

Define inductively, for $k = 2, \dots, n$, bounded subsets B_k of F, measures $\mu_0^{(k)}$, $\mu_1^{(k)}$ on G and uniformly continuous linear mappings $T_0^{(k)}, T_1^{(k)}$ from B_{k-1} into F such that:

- $B_k = T_0^{(k)}(B_{k-1}) \cup T_1^{(k)}(B_{k-1})$ is bounded in F;
- there are $g_0^{(k)}, g_1^{(k)} \in \mathcal{D}(]-\varepsilon,\varepsilon[)$ such that $\mu_0^{(k)}$ and $\mu_1^{(k)}$ are the image of the measures $g_0^{(k)} \cdot \nu_\mathbb{R}$ and $g_1^{(k)} \cdot \nu_\mathbb{R}$ under the mapping $t \mapsto \exp_G(tX_k)$;
- $\operatorname{Supp}\left(T_j^{(k)}(\varphi)\right) \subseteq \operatorname{Supp}(\varphi)$ for $j = 0, 1$ and for every $\varphi \in B_{k-1}$;
- $\varphi = T_0^{(k)}(\varphi) *^\beta \mu_0^{(k)} + T_1^{(k)}(\varphi) *^\beta \mu_1^{(k)}$ for every $\varphi \in B_{k-1}$.

Take $\gamma \in \{0,1\}^n$ and set $T_\gamma \coloneqq T_{\gamma_n}^{(n)} \circ \cdots \circ T_{\gamma_1}^{(1)}$ and $\mu_\gamma \coloneqq \mu_{\gamma_n}^{(n)} * \cdots * \mu_{\gamma_1}^{(1)}$. Then, T is a uniformly continuous linear mapping from B into a bounded subset of $F^{\{0,1\}^n}$ such that $\operatorname{Supp}(T_\gamma(\varphi)) \subseteq \operatorname{Supp}(\varphi)$ for every $\gamma \in \{0,1\}^n$ and for every $\varphi \in B$; in addition,

$$\varphi = \sum_{\gamma \in \{0,1\}^n} T_\gamma(\varphi) *^\beta \mu_\gamma$$

for every $\varphi \in B$. It remains to prove that the μ_γ have the required properties. However, μ_γ is clearly the image measure $\Phi\left(\left(g_{\gamma_n}^{(n)} \otimes \cdots \otimes g_{\gamma_1}^{(1)}\right) \cdot \nu_{\mathbb{R}^n}\right)$, so that μ_γ has density $\left(g_{\gamma_n}^{(n)} \otimes \cdots \otimes g_{\gamma_1}^{(1)}\right) \circ \Phi^{-1} \in \mathcal{D}(V)$ with respect to $\Phi(\nu_{\mathbb{R}^n})$. Finally, $\Phi(\nu_{\mathbb{R}^n})$ has a density of class C^ω with respect to β, whence the result. ☐

Theorem 1.62. *Let V be an open neighbourhood of e in G. Endow $\mathcal{D}'(G)$ with the topology of bounded (resp. pointwise) convergence. Then, the mapping*

$$\Phi \colon \mathcal{D}'(G) \ni T \mapsto [\varphi \mapsto T * \varphi] \in \mathcal{L}_s(\mathcal{D}(V); \mathcal{D}'(G))$$

is an isomorphism onto its image.

Proof. Notice first that Corollary 1.58 implies that Φ is a continuous linear mapping. Conversely, let $\mathfrak{F}$ be a filter on $\mathcal{D}'(G)$, and assume that $\Phi(\mathfrak{F})$ converges to 0. Take a bounded (resp. finite) subset B of $\mathcal{D}(G)$, and apply Theorem 1.61. Then, we find a finite family $(\psi_j)_{j=1}^{2^n}$ of elements of $\mathcal{D}(V)$ and a continuous linear mapping L from $\Delta_L \check{B}$ into $\mathcal{D}(G)^{2^n}$ such that

$$\varphi = \sum_{j=1}^{2^n} \psi_j * L_j(\varphi)$$

for every $\varphi \in \Delta_L \check{B}$. Since $\mathfrak{F} * \psi_j$ converges to 0 in $\mathcal{D}'(G)$, Corollary 1.58 (resp. Corollary 1.52) implies that

$$\mathfrak{F} *^\beta \varphi = \sum_{j=1}^{2^n} (\mathfrak{F} * \psi_j) *^\beta L_j(\varphi)$$

converges to 0 in $\mathcal{E}(G)$ (resp. pointwise), uniformly as φ runs through $\Delta_L \check{B}$. Hence, $\mathfrak{F} = (\mathfrak{F} * (\Delta_L \check{\cdot}))(e)$ converges uniformly to 0 on B. By the arbitrariness of B, we infer that $\mathfrak{F}$ converges to 0 in $\mathcal{D}'(G)$. Therefore, Φ is one-to-one and its inverse is continuous on $\Phi(\mathcal{D}'(G))$. □

Corollary 1.63. *Let $\Phi\colon \mathcal{D}(G) \to \mathcal{D}'_w(G)$ be a left-invariant continuous linear operator. Then, there is a unique $T \in \mathcal{D}'(G)$ such that $\Phi(\varphi) = \varphi * T$ for every $\varphi \in \mathcal{D}(G)$.*

Proof. Let (φ_j) be a sequence of elements of $\mathcal{D}(G)$ which converges to δ_e in $\mathcal{E}'^0(G)$. Then, for every $\varphi \in \mathcal{D}(G)$,

$$\lim_{j\to\infty} \varphi * \Phi(\varphi_j) = \lim_{j\to\infty} \Phi(\varphi *^\beta \varphi_j) = \Phi(\varphi)$$

in $\mathcal{D}'_w(G)$, hence in $\mathcal{D}'(G)$ since $\mathcal{D}(G)$ is a Montel space. Therefore, Theorem 1.62 implies that $(\Phi(\varphi_j))$ is a Cauchy sequence in $\mathcal{D}'(G)$, so that it converges to some T. It is then clear that $\Phi(\varphi) = \varphi * T$ for every $\varphi \in \mathcal{D}(G)$. □

Corollary 1.64. *Let B be a subset of $\mathcal{D}'(G)$. Then, $B * \varphi$ is bounded in $\mathcal{D}'(G)$ for every $\varphi \in \mathcal{D}(V)$ if and only if B is bounded in $\mathcal{D}'(G)$.*

Theorem 1.65. *Assume that G is countable at infinity, and let V be an open neighbourhood of e in G. Endow $\mathcal{E}'(G)$ with the topology of bounded (resp. pointwise) convergence. Then, the mapping*

$$\Phi\colon \mathcal{E}'(G) \ni T \mapsto [\varphi \mapsto T * \varphi] \in \mathcal{L}_s(\mathcal{D}(V); \mathcal{E}'(G))$$

is an isomorphism onto its image.

The proof is similar to that of Theorem 1.62 and is omitted.

Corollary 1.66. *Let V be an open neighbourhood of e in G. Let B be a subset of $\mathcal{D}'(G)$, and assume that $B * \varphi$ is bounded in $\mathcal{E}'(G)$ for every $\varphi \in \mathcal{D}(V)$. Then, B is contained and bounded in $\mathcal{E}'(G)$.*

Proof. **1.** Assume first that G in countable at infinity. Take $T \in B$, define $\Phi_T\colon \mathcal{D}(V) \ni \varphi \mapsto T * \varphi \in \mathcal{E}'(G)$, and observe that Φ_T is continuous by [18, Theorem 1 and Corollary to Theorem 2 of Chapter III, § 6]. Let W be a neighbourhood of e in G such that $W^2 \subseteq V$, and let (φ_j) be a sequence of elements of $\mathcal{D}(W)$ which converges to δ_e in $\mathcal{E}'(G)$. Then, $\Phi_T(\varphi_j) * \varphi = \Phi_T(\varphi_j *^\beta \varphi)$ converges to $\Phi_T(\varphi)$ in $\mathcal{E}'(G)$ for every $\varphi \in \mathcal{D}(W)$. Hence, Theorem 1.65 implies that $(\Phi_T(\varphi_j))$ is a Cauchy sequence in $\mathcal{E}'(G)$; since it converges to T in $\mathcal{D}'(G)$, it follows that $T \in \mathcal{E}'(G)$. Then, Theorem 1.65 implies that B is bounded in $\mathcal{E}'(G)$.

2. Consider the general case. Let F be the closure of the union of the supports of the elements of B. We want to prove that F is compact in G. Indeed, by **1** we see that $F \cap G'$ is compact for every separable open subgroup G' of G. Let G_e be the component of G which contains e, and let A be any separable open subset of G. Then, $G'_A := \bigcup_{m\in\mathbb{N}} (G_e \cup A \cup A^{-1})^m$ is a separable open subgroup of G which contains A. Therefore, $F \cap G'_A$ is compact. Now, if F is not compact, there must be a sequence (C_j) of distinct components of G such that $F \cap C_j \neq \emptyset$. However, $C := \bigcup_{j\in\mathbb{N}} C_j$ is a separable open subset of G, so that $F \cap C \subseteq F \cap G'_C$ is compact, which is absurd. Therefore, F is compact.

Let V be a relatively compact neighbourhood of F. If we apply **1** to G'_V, we then see that $B|_{G'_V}$ is equicontinuous with respect to the topology of $\mathcal{E}(G'_V)$. Therefore, B is contained in $\mathcal{E}'(G)$ and bounded therein. □

Corollary 1.67. *Let V be an open neighbourhood of e in G, $\mathfrak{F}$ a filter with countable base on $\mathcal{E}'(G)$ and $T \in \mathcal{E}'(G)$. Assume that $\mathfrak{F} * \varphi$ converges to $T * \varphi$ in $\mathcal{E}'_w(G)$ for every $\varphi \in \mathcal{D}(V)$. Then, $\mathfrak{F}$ converges to T in $\mathcal{E}'(G)$.*

Proof. We may reduce to the case in which $\mathfrak{F}$ is the elementary filter associated with a sequence (T_j). Since the $\{ T_j * \varphi \colon j \in \mathbb{N} \}$ is bounded in $\mathcal{E}'(G)$ for every $\varphi \in \mathcal{D}(V)$, Corollary 1.66 implies that $\{ T_j \colon j \in \mathbb{N} \}$ is bounded in $\mathcal{E}'(G)$. Therefore, there is a separable subgroup G' of G such that each T_j is supported in G'. In addition, since $\mathcal{E}(G)$ is a Montel space, $T_j * \varphi$ converges to $T * \varphi$ in $\mathcal{E}'(G)$ for every $\varphi \in \mathcal{D}(V)$. Hence, Theorem 1.65 implies that $T_j|_{G'}$ converges to $T|_{G'}$ in $\mathcal{E}'(G')$. The assertion follows. □

Corollary 1.68. *Let $\Phi \colon \mathcal{D}(G) \to \mathcal{E}'_w(G)$ be a left-invariant continuous linear operator. Then, there is a unique $T \in \mathcal{E}'(G)$ such that $\Phi(\varphi) = \varphi * T$ for every $\varphi \in \mathcal{D}(G)$.*

The proof is similar to that of Corollary 1.63 and is omitted.

1.5 Convolution on Spaces of Sobolev Type

In this section, G will denote a Lie group endowed with a non-zero relatively invariant positive measure β with left and right multipliers Δ_L and Δ_R, respectively. We shall first give some general criteria for convolvability.

Proposition 1.69 (Young's Inequality). *Take $k_1, k_2, k_3 \in \mathbb{Z} \cup \{ \pm\infty \}$, and $p_1, p_2, p_3 \in [1, \infty]$ such that $\frac{1}{p_1'} + \frac{1}{p_2'} = \frac{1}{p_3'}$. In addition, take $T_1 \in W^{k_1,k_2,p_1}(\beta)$ and $T_2 \in W^{-k_2,k_3,p_2}(\beta)$. Then, the following hold:*

1. *$\Delta_L^{1/p_2'} T_1$ and $\Delta_R^{1/p_1'} T_2$ are convolvable;*
2. *$\big(\Delta_L^{1/p_2'} T_1\big) * \big(\Delta_R^{1/p_1'} T_2\big) \in W^{k_1,k_3,p_3}(\beta)$ and, if $k_1 = k_2 = k_3 = 0$,*
$$\Big\| \big(\Delta_L^{1/p_2'} T_1\big) * \big(\Delta_R^{1/p_1'} T_2\big) \Big\|_{p_3} \leqslant \|T_1\|_{p_1} \|T_2\|_{p_2};$$
3. *if $k_1 = k_2 = k_3 = 0$ and $T_2 = f_2 \cdot \beta$ for some $f_2 \in L^{p_2}(\beta)$, then*
$$\big[\big(\Delta_L^{1/p_2'} T_1\big) *^\beta \big(\Delta_R^{1/p_1'} T_2\big)\big](g_2) = \int_G \big(\Delta_R^{1/p_1'} f_2\big)(g_1^{-1} g_2) \Delta_L^{1/p_2}(g_1^{-1}) \,\mathrm{d}T_1(g_1)$$
for β-almost every $g_2 \in G$;
4. *if $k_1 = k_2 = k_3 = 0$ and $T_1 = f_1 \cdot \beta$ for some $f_1 \in L^{p_1}(\beta)$, then*
$$\big[\big(\Delta_L^{1/p_2'} T_1\big) *^\beta \big(\Delta_R^{1/p_1'} T_2\big)\big](g_1) = \int_G \big(\Delta_L^{1/p_2'} f_1\big)(g_1 g_2^{-1}) \Delta_R^{1/p_1}(g_2^{-1}) \,\mathrm{d}T_2(g_2)$$
for β-almost every $g_1 \in G$.

For the proof, use the classical Young's inequality to prove convolvability when $k_1 = k_2 = k_3 = 0$, and then make use of Proposition 1.55 to establish the general case. The details are left to the reader.

Proposition 1.70. *The bilinear mappings*
$$*^\beta \colon (\Delta_L \mathcal{M}^1(G)) \times \mathcal{B}(G) \to \mathcal{B}(G) \qquad\qquad *^\beta \colon \mathcal{B}(G) \times (\Delta_R \mathcal{M}^1(G)) \to \mathcal{B}(G)$$
are well-defined and continuous.

Proof. Take $\mu_1 \in \mathcal{M}^1(G)$ and $\varphi_2 \in \mathcal{B}(G)$. Then, Proposition 1.69 shows that
$$((\Delta_L \mu_1) *^\beta \varphi_2)(g_1) = \int_G \varphi_2(g_2^{-1} g_1) \,\mathrm{d}\mu_1(g_2)$$
for β-almost every $g_1 \in G$, so that $(\Delta_L \mu_1) *^\beta \varphi_2$ has a continuous representative by dominated convergence. The assertion follows. □

Observe that $\check{}$ induces an isomorphism between $W^{k_1,k_2,p}(\beta)$ and $(\Delta_L \Delta_R)^{-1/p'} W^{k_2,k_1,p}(\beta)$ for every $k_1, k_2 \in \mathbb{Z} \cup \{ \pm\infty \}$ and for every $p \in [1,\infty]$.

Corollary 1.71. *Take $p_1, p_2, p_3 \in [1,\infty]$ such that $\frac{1}{p_1'} + \frac{1}{p_2'} = \frac{1}{p_3'}$. In addition, take (T_1, T_2, φ_3) in $\big(\Delta_L^{1/p_2'} W^{0,0,p_1}(\beta)\big) \times \big(\Delta_R^{1/p_1'} W^{0,0,p_2}(\beta)\big) \times W_0^{0,0,p_3'}(\beta)$ if $p_3 \neq 1$, in $\mathcal{M}^1(G) \times \mathcal{M}^1(G) \times \mathcal{B}(G)$ otherwise. Then,*

$$\langle T_1 * T_2, \varphi_3 \rangle = \big\langle T_1, \varphi_3 *^\beta (\Delta_R \check{T}_2) \big\rangle = \big\langle T_2, (\Delta_L \check{T}_1) *^\beta \varphi_3 \big\rangle.$$

Proof. If $p_1 \neq 1$, then Proposition 1.69 shows that $\varphi_3 *^\beta (\Delta_R \check{T}_2) \in \Delta_L^{-1/p_2'} W_0^{0,0,p_1'}(\beta)$, while $\varphi_3 *^\beta (\Delta_R \check{T}_2) \in \Delta_L^{-1/p_2'} \mathcal{B}(G)$ if $p_1 = 1$ thanks to Proposition 1.70 and [20, Proposition 14 of Chapter VIII, § 4, No. 5]. Therefore, the trilinear mapping

$$(T_1, T_2, \varphi_3) \mapsto \langle T_1 * T_2, \varphi_3 \rangle - \big\langle T_1, \varphi_3 *^\beta (\Delta_R \check{T}_2) \big\rangle$$

is well-defined and continuous on the stated domain. In addition, Corollary 1.54 implies that it vanishes if at least two among T_1, T_2 and φ_3 have compact support. Hence, the assertion follows by approximation. The second equality is proved similarly. □

Corollary 1.72. *Take $p_1, p_2, p_3 \in [1,\infty]$ such that $\frac{1}{p_1'} + \frac{1}{p_2'} = \frac{1}{p_3'}$ (resp. and such that $p_3 \neq 1$), and $k_1, k_2, k_3 \in \mathbb{Z} \cup \{ \pm\infty \}$. Let $\mathfrak{S}_1$ and $\mathfrak{S}_2$ be the set of bounded (resp. compact) subsets of $\Delta_L^{1/p_2'} W^{k_1,k_2,p_1}(\beta)$ and $\Delta_R^{1/p_1'} W^{-k_2,k_3,p_2}(\beta)$, respectively. Then, the bilinear mapping*

$$*\colon \big(\Delta_L^{1/p_2'} W^{k_1,k_2,p_1}(\beta)\big) \times \big(\Delta_R^{1/p_1'} W^{-k_2,k_3,p_2}(\beta)\big) \to W^{k_1,k_3,p_3}(\beta)$$
$$(\textit{resp. } *\colon \big(\Delta_L^{1/p_2'} W_c^{k_1,k_2,p_1}(\beta)\big) \times \big(\Delta_R^{1/p_1'} W_c^{-k_2,k_3,p_2}(\beta)\big) \to W_c^{k_1,k_3,p_3}(\beta))$$

is well-defined and $(\mathfrak{S}_1, \mathfrak{S}_2)$-hypocontinuous.

Proof. **1.** Observe that by means of Proposition 1.55 we may reduce to the case in which $k_1, k_3 \leqslant 0$. In addition, by means of Proposition 1.46 we may further assume that $k_2 \geqslant 0$. Let $\mathfrak{S}_3$ be the set of bounded (resp. compact) subsets of $W_0^{-k_1,-k_3,p_3'}(\beta)$. Take $B_j \in \mathfrak{S}_j$ for $j = 1,2,3$. Then, Proposition 1.32 implies that for every γ with length at most k_2 there is a finite subset $P_{B_1,\gamma}$ of $\{ \alpha \in \mathbb{N}^n \colon |\alpha| \leqslant |k_1| \}$ and a bounded (resp. compact) family $(\mu_{T_1,\alpha,\gamma})_{T_1 \in B_1, \alpha \in P_{B_1,\gamma}}$ of elements of $\Delta_L^{1/p_3} \Delta_R^{-1/p_1'} W^{0,0,p_1}(\beta)$ such that

$$\mathbf{Y}^\gamma (\Delta_L \check{T}_1) = \sum_{\alpha \in P_{B_1,\gamma}} \mathbf{X}^\alpha \mu_{T_1,\alpha,\gamma}$$

for every $T_1 \in B_1$. By the same reference, there is a finite set P_{B_2} of $(\alpha,\gamma) \in \mathbb{N}^n \times \mathbb{N}^n$ such that $|\alpha| \leqslant k_2$ and $|\gamma| \leqslant |k_3|$, and a bounded (resp. compact) family $(\mu_{T_2,\alpha,\gamma})_{T_2 \in B_2, (\alpha,\gamma) \in P_{B_2}}$ of elements of $\Delta_R^{1/p_1'} W^{0,0,p_2}(\beta)$ such that

$$T_2 = \sum_{(\alpha,\gamma) \in P_{B_2}} \mathbf{Y}^\alpha (\mathbf{X}^\gamma)^\dagger \mu_{T_2,\alpha,\gamma}.$$

2. Now, take $T_1 \in \Delta_L^{1/p_2'} W^{k_1,k_2,p_1}(\beta)$, $T_2 \in B_2$ and $\varphi_3 \in B_3$. Then, Proposition 1.55 implies that

$$\begin{aligned}
\langle T_1 * T_2, \varphi_3 \rangle &= \sum_{(\alpha,\gamma) \in P_{B_2}} \big\langle [X_n^{\alpha_n} \cdots X_1^{\alpha_1} T_1] * [(\mathbf{X}^\gamma)^\dagger \mu_{T_2,\alpha,\gamma}], \varphi_3 \big\rangle \\
&= \sum_{(\alpha,\gamma) \in P_{B_2}} \langle [X_n^{\alpha_n} \cdots X_1^{\alpha_1} T_1] * \mu_{T_2,\alpha,\gamma}, \mathbf{X}^\gamma \varphi_3 \rangle \\
&= \sum_{(\alpha,\gamma) \in P_{B_2}} \langle X_n^{\alpha_n} \cdots X_1^{\alpha_1} T_1, (\mathbf{X}^\gamma \varphi_3) * (\Delta_R \check{\mu}_{T_2,\alpha,\gamma}) \rangle,
\end{aligned}$$

where the third equality follows by means of Corollary 1.71. In particular, the set of $(\mathbf{X}^\gamma \varphi_3) * (\Delta_R \check{\mu}_{T_2,\alpha,\gamma})$, as T_2 runs through B_2 and φ_3 runs through B_3, is bounded (resp. compact) in the space $\Delta_L^{-1/p_2'} W_0^{-k_1,0,p_1'}(\beta)$ if $p_1 \neq 1$, in $\Delta_L^{-1/p_2'} \mathcal{B}_c^{-k_1,0}(G)$ otherwise. Therefore, the mappings $\Delta_L^{1/p_2'} W^{k_1,k_2,p_1}(\beta) \ni T_1 \mapsto T_1 * T_2 \in W^{k_1,k_3,p_3}(\beta)$ (resp. $\Delta_L^{1/p_2'} W_c^{k_1,k_2,p_1}(\beta) \ni T_1 \mapsto T_1 * T_2 \in W_c^{k_1,k_3,p_3}(\beta)$), as T_2 runs through B_2, are equicontinuous.

3. Next, take $T_1 \in B_1$, $T_2 \in \Delta_R^{1/p_1'} W^{-k_2,k_3,p_2}(\beta)$ and $\varphi_3 \in B_3$. Then, Proposition 1.55 implies that, for every α with length at most $|k_1|$ and for every γ with length at most k_2,

$$\begin{aligned}\mathbf{X}^\alpha \mathbf{Y}^\gamma((\Delta_L \check{T}_1) * \varphi_3) &= [\mathbf{Y}^\gamma(\Delta_L \check{T}_1)] * (\mathbf{X}^\alpha \varphi_3)\\ &= \sum_{\delta \in P_{B_1,\gamma}} (\mathbf{X}^\delta \mu_{T_1,\delta,\gamma}) * (\mathbf{X}^\alpha \varphi_3)\\ &= \sum_{\delta \in P_{B_1,\gamma}} \mu_{T_1,\delta,\gamma} * (Y_n^{\delta_n} \cdots Y_1^{\delta_1} \mathbf{X}^\alpha \varphi_3).\end{aligned}$$

In particular, Propositions 1.69 and 1.70 imply that the set of $\check{T}_1 * \varphi_3$, as T_1 runs through B_1 and φ_3 runs through B_3, is bounded (resp. compact) in $\Delta_R^{-1/p_1'} W_0^{k_2,-k_3,p_2'}(\beta)$ if $p_2 \neq 1$, in $\Delta_R^{-1/p_1'} \mathcal{B}_c^{k_2,-k_3}(\beta)$ otherwise. Then, by means of Corollary 1.71 we see that

$$\langle T_1 * T_2, \varphi_3 \rangle = \langle T_2, (\Delta_L \check{T}_1) * \varphi_3 \rangle.$$

Then, the maps $\Delta_R^{1/p_1'} W^{-k_2,k_3,p_2}(\beta) \ni T_2 \mapsto T_1 * T_2 \in W^{k_1,k_3,p_3}(\beta)$ (resp. $\Delta_R^{1/p_1'} W_c^{-k_2,k_3,p_2}(\beta) \ni T_2 \mapsto T_1 * T_2 \in W_c^{k_1,k_3,p_3}(\beta)$), as T_1 runs through B_1, are equicontinuous. ☐

Proposition 1.73. *Take $k_1, k_2, k_3 \in \mathbb{Z} \cup \{\pm\infty\}$. Then, the following bilinear mappings*

$$\begin{aligned}&*^\beta \colon (\Delta_L W_c^{k_1,k_2,1}(G)) \times \mathcal{B}_c^{-k_2,k_3}(G) \to \mathcal{B}_c^{k_1,k_3}(G)\\ &*^\beta \colon \mathcal{B}_c^{k_1,k_2}(G) \times (\Delta_R W_c^{-k_2,k_3,1}(G)) \to \mathcal{B}_c^{k_1,k_3}(G)\\ &* \colon W_c^{k_1,k_2,1}(G) \times W_c^{-k_2,k_3,1}(G) \to W_c^{k_1,k_3,1}(G)\end{aligned}$$

are well-defined and hypocontinuous for the sets of strictly compact subsets of each factor.

Proof. **1.a.** Assume first that $k_1 = k_2 = k_3 = 0$. Let B_1 be a compact subset of $\mathcal{M}_c^1(G)$ and let B_2 be a bounded subset of $\mathcal{B}_c(G)$. Then, Proposition 1.26 shows that B_1 satisfies Prokhorov's condition. Set $C_1 \coloneqq \sup_{\mu_1 \in B_1} \|\mu_1\|_1$ and $C_2 \coloneqq \sup_{\varphi_2 \in B_2} \|\varphi_2\|_\infty$, and take $\varepsilon > 0$ and a compact subset K_3 of G. Then, there is compact subset K_1 of G such that

$$|\mu_1|(G \setminus K_1) < \frac{\varepsilon}{3C_2 + 1}$$

for every $\mu_1 \in B_1$. Take $\varphi_1, \varphi_2 \in B_2$ so that

$$|\varphi_1(g) - \varphi_2(g)| \leqslant \frac{\varepsilon}{3C_1 + 1}$$

for every $g \in K_3^{-1} K_1$. Then,

$$\begin{aligned}\big|((\Delta_L \mu_1) *^\beta \varphi_1)(g_1) - ((\Delta_L \mu_1) *^\beta \varphi_2)(g_1)\big| &\leqslant \int_{K_1} |\varphi_1(g_2^{-1} g_1) - \varphi_2(g_2^{-1} g_1)| \,\mathrm{d}|\mu_1|(g_2)\\ &+ \int_{G \setminus K_1} |\varphi_1(g_2^{-1} g_1)| \,\mathrm{d}|\mu_1|(g_2) + \int_{G \setminus K_1} |\varphi_2(g_2^{-1} g_1)| \,\mathrm{d}|\mu_1|(g_2) \leqslant \varepsilon.\end{aligned}$$

Therefore, the mappings $\mathcal{B}_c(G) \ni \varphi \mapsto (\Delta_L \mu_1) *^\beta \varphi \in \mathcal{B}_c(G)$, as μ_1 runs through B_1, are equicontinuous.

1.b. Let B_2 be a compact subset of $\mathcal{B}_c(G)$, and let B_3 be a compact subset of $\mathcal{M}_c^1(G)$. Take $\varepsilon > 0$, and define $C_3 \coloneqq \sup_{\mu_3 \in B_3} \|\mu_3\|_1$, $C_2 \coloneqq \sup_{\varphi_2 \in B_2} \|\varphi_2\|_\infty$, and take a compact subset K_3 of G such that

$$|\mu_3|(G \setminus K_3) < \frac{\varepsilon}{3C_2 + 1}$$

for every $\mu_3 \in B_3$. In addition, fix $g_1 \in G$ and let V be a compact neighbourhood of g_1 in G such that

$$|\varphi_2(g_1'^{-1} g_2) - \varphi_2(g_1^{-1} g_2)| < \frac{\varepsilon}{3C_3 + 1}$$

for every $\varphi \in B_2$, for every $g_1' \in V$, and for every $g_2 \in K_3$. Arguing as in **1.a**, we then see that

$$|((\Delta_L \mu_3) * \check{\varphi}_2)(g_1') - ((\Delta_L \mu_3) * \check{\varphi}_2)(g_1)| < \varepsilon$$

for every $g_1' \in V$, for every $\varphi_2 \in B_2$, and for every $\mu_3 \in B_3$. Therefore, the set of $\mu_3 *^\beta \check{\varphi}_2$, as μ_3 runs through B_3 and φ_2 runs through B_2, is relatively compact in $\mathcal{B}_c(G)$. Now, Corollary 1.71 implies that

$$\langle (\Delta_L \mu_1) *^\beta \varphi_2, \mu_3 \rangle = \langle \mu_1, (\Delta_L \mu_3) *^\beta \check{\varphi}_2 \rangle$$

for every $\mu_1 \in \mathcal{M}^1_c(G)$, for every $\varphi_2 \in B_2$ and for every $\mu_3 \in B_3$. Since $\mathcal{B}_c(G)$ is isomorphic to $(\mathcal{M}^1_c(G))'_c$ by Corollary 1.27, it follows that the mappings $\mathcal{M}^1_c(G) \ni \mu_1 \mapsto (\Delta_L \mu_1) *^\beta \varphi_2 \in \mathcal{B}_c(G)$, as φ_2 runs through B_2, are equicontinuous.

1.c. The assertions concerning the bilinear mapping $*\colon \mathcal{M}^1_c(G) \times \mathcal{M}^1_c(G) \to \mathcal{M}^1_c(G)$ follow by transposition from **1.a**–**b**.

2. The proof of the general case proceeds as that of Corollary 1.72. □

Corollary 1.74. *Take $k_1, k_2 \in \mathbb{Z} \cup \{ \pm\infty \}$ and $p \in [1, \infty]$. Then, $\mathcal{D}(G) \cdot \beta$ is dense in $W^{k_1,k_2,p}_c(\beta)$.*

Proof. By means of suitable truncations, we may approximate every element of $W^{k_1,k_2,p}_c(\beta)$ by distributions with compact support; since $\mathcal{E}'^0_c(G)$ embeds continuously into $W^{-k_2,k_2,1}_c(G)$, by convolution with a suitable approximate identity the assertion follows by Corollary 1.72 and Proposition 1.73. □

Proposition 1.75. *Take $k_1, k_2, k_3, k_4 \in \mathbb{Z} \cup \{ \pm\infty \}$. In addition, take $p_1, p_2, p_3 \in [1, \infty]$ such that $\frac{1}{p_1'} + \frac{1}{p_2'} + \frac{1}{p_3'} \leqslant 1$. Take $T_1 \in \Delta_L^{1/p_2' + 1/p_3'} W^{k_1,k_2,p_1}(\beta)$, $T_2 \in \Delta_L^{1/p_3'} \Delta_R^{1/p_1'} W^{-k_2,-k_3,p_2}(\beta)$ and $T_3 \in \Delta_R^{1/p_1' + 1/p_2'} W^{k_3,k_4,p_3}(\beta)$. Then*

$$T_1 * (T_2 * T_3) = (T_1 * T_2) * T_3.$$

Proof. Indeed, Corollary 1.72 implies that the trilinear mapping

$$(T_1, T_2, T_3) \mapsto T_1 * (T_2 * T_3) - (T_1 * T_2) * T_3$$

is separately continuous from

$$\big(\Delta_L^{1/p_2' + 1/p_3'} W^{k_1,k_2,p_1}(\beta)\big) \times \big(\Delta_L^{1/p_3'} \Delta_R^{1/p_1'} W^{-k_2,-k_3,p_2}(\beta)\big) \times \big(\Delta_R^{1/p_1' + 1/p_2'} W^{k_3,k_4,p_3}(\beta)\big)$$

into $W^{k_1,k_4,p_4}(\beta)$, where $\frac{1}{p_4'} = \frac{1}{p_1'} + \frac{1}{p_2'} + \frac{1}{p_3'}$. In addition, it vanishes if at least two among T_1, T_2, T_3 are compactly supported by Corollary 1.53. Since one at most among p_1, p_2, p_3 can be ∞, the assertion follows. □

Theorem 1.76. *Take $p \in [1, \infty]$ and a neighbourhood V of e in G. Endow each space W with the topology of bounded (resp. compact, pointwise – on the corresponding space $\mathcal{B}_c$ if $p = 1$) convergence. Then, the following hold:*

1. *the mapping*

$$\Phi\colon W^{0,-\infty,p}(\beta) \ni T \mapsto [\varphi \mapsto T * \varphi] \in \mathcal{L}_s(\mathcal{D}(V); W^{0,-\infty,p}(\beta))$$

is an isomorphism onto its image;

2. *the mapping*

$$\Phi\colon W^{-\infty,0,p}(\beta) \ni T \mapsto [\varphi \mapsto \varphi * T] \in \mathcal{L}_s(\mathcal{D}(V); W^{-\infty,0,p}(\beta))$$

is an isomorphism onto its image;

3. *the mapping*[6]

$$\Phi\colon W^{-\infty,-\infty,p}(\beta) \ni T \mapsto [(\varphi_1, \varphi_2) \mapsto \varphi_1 * T * \varphi_2] \in \mathcal{L}_s(\mathcal{D}(V), \mathcal{D}(V); W^{-\infty,-\infty,p}(\beta))$$

is an isomorphism onto its image.

[6] By $\mathcal{L}_s(\mathcal{D}(V), \mathcal{D}(V); W^{-\infty,-\infty,p}(\beta))$ we mean the space of separately continuous bilinear mappings, endowed with the topology of pointwise convergence. Thus $\mathcal{L}_s(\mathcal{D}(V), \mathcal{D}(V); W^{-\infty,-\infty,p}(\beta))$ is canonically isomorphic to $\mathcal{L}_s(\mathcal{D}(V); \mathcal{L}_s(\mathcal{D}(V); W^{-\infty,-\infty,p}(\beta)))$.

The proof is similar to that of Theorem 1.62 and is omitted.

Corollary 1.77. *Take $p \in [1,\infty]$. Then, the following hold:*

1. *let $\Phi\colon \mathcal{D}(G) \to W^{0,-\infty,p}(\beta)$ be a right-invariant continuous linear operator. Then, there is a unique $T \in W^{0,-\infty,p}(\beta)$ such that $\Phi(\varphi) = T * \varphi$ for every $\varphi \in \mathcal{D}(G)$;*

2. *let $\Phi\colon \mathcal{D}(G) \to W^{-\infty,0,p}(\beta)$ be a left-invariant continuous linear operator. Then, there is a unique $T \in W^{-\infty,0,p}(\beta)$ such that $\Phi(\varphi) = \varphi * T$ for every $\varphi \in \mathcal{D}(G)$.*

The proof is similar to that of Corollary 1.63 and is omitted.

Corollary 1.78. *Take $p \in [1,\infty]$, and let V be a neighbourhood of e in G. Endow each space W with the topology of bounded (resp. compact, pointwise – on the corresponding space $\mathcal{B}_c$ if $p = 1$) convergence. Let B be a subset of $\mathcal{D}'(G)$. Then, the following hold:*

1. *if $B * \varphi$ is contained and bounded (resp. relatively compact) in $W^{0,-\infty,p}(\beta)$ for every $\varphi \in \mathcal{D}(V)$, then B is contained and bounded (resp. relatively compact) in $W^{0,-\infty,p}(\beta)$;*

2. *if $\varphi * B$ is contained and bounded (resp. relatively compact) in $W^{-\infty,0,p}(\beta)$ for every $\varphi \in \mathcal{D}(V)$, then B is contained and bounded (resp. relatively compact) in $W^{-\infty,0,p}(\beta)$;*

3. *if B is contained in $W^{-\infty,-\infty,p}(\beta)$ and $\varphi_1 * B * \varphi_2$ is bounded (resp. relatively compact) in $W^{-\infty,-\infty,p}(\beta)$ for every $\varphi_1, \varphi_2 \in \mathcal{D}(V)$, then B is bounded (resp. relatively compact) in $W^{-\infty,-\infty,p}(\beta)$.*

Corollary 1.79. *Let B be a subset of $\mathcal{D}'(G)$ and let V be a neighbourhood of e in G. Then, the following conditions are equivalent:*

1. *B is contained and bounded in $W^{0,-\infty,\infty}(\beta)$ (resp. $W^{-\infty,0,\infty}(\beta)$);*

2. *$B *^\beta \varphi$ (resp. $\varphi *^\beta B$) is bounded in $\mathcal{B}(G)$ for every $\varphi \in \mathcal{D}(V)$;*

3. *the set of $L_g(\Delta_L^{-1} T)$ (resp. $R_g(\Delta_R^{-1} T)$), as g runs through G and T runs through B, is bounded in $\mathcal{D}'(G)$.*

Proof. **1** $\iff$ **2.** This follows from Corollaries 1.72 and 1.78.

2 $\iff$ **3.** Indeed, if $T \in B$ and $\varphi \in \mathcal{D}(V)$, then

$$(T *^\beta \varphi)(g) = \langle T, \Delta_L^{-1} L_g \check{\varphi} \rangle$$

for every $g \in G$. □

Corollary 1.80. *Assume that G is a homogeneous group, and take $\varphi \in \mathcal{S}(G)$ with integral 1. Define $\varphi_j \coloneqq 2^{jQ}\varphi(2^j\,\cdot\,)$ for every $j \in \mathbb{N}$, where Q is the homogeneous dimension of G. Then, $(\varphi_j \cdot \beta)$ converges to δ_e in $W^{\infty,-\infty,1}(G)$ and in $W^{-\infty,\infty,1}(G)$.*

Proof. We prove only the first assertion. Observe that, for every right-invariant differential operator Y on G and for every $\psi \in \mathcal{D}(G)$, the sequence $((Y\varphi_j) *^\beta \psi)$ converges to $Y\psi$ in $\mathcal{S}(G)$, hence in $W_0^{0,-\infty,1}(G)$. Then, Theorem 1.76 implies that $Y(\varphi_j \cdot \beta)$ converges to $Y\delta_e$ in $W^{0,-\infty,1}(G)$, whence the result by the arbitrariness of Y. □

Lemma 1.81. *Let G be a homogeneous group. Take $\varphi \in W^{0,1,1}(G)$, a bounded family $(\mu_t)_{t>0}$ of elements of $\mathcal{M}^1(G)$ and $\psi \in L^\infty(G)$. Define, for every $t > 0$, $\varphi_t \coloneqq (t\,\cdot)_*\varphi$, and assume that $\mu_t *^\beta \psi *^\beta \varphi_t$ (resp. $\check{\varphi}_t *^\beta \psi *^\beta \mu_t$) converges pointwise almost everywhere to some $\widetilde{\psi}$ as $t \to +\infty$. Then, $\widetilde{\psi}$ is constant almost everywhere on G.*

Notice that, if $\mu_t = \delta_e$ for every $t > 0$, then the assertion becomes: 'if $\psi * \varphi_t$ converges pointwise almost everywhere to some $\widetilde{\psi}$, then $\widetilde{\psi}$ is constant almost everywhere on G.'

Proof. Indeed, for every $j = 1, \dots, n$ we have, by Proposition 1.55,

$$X_j(\mu_t * \psi * \varphi_t) = \mu_t * \psi * (X_j\varphi_t) = t^{-\mathrm{d}_j} \mu_t * \psi * [(X_j\varphi)_t],$$

where $(X_j\varphi)_t \coloneqq (t\,\cdot)_*(X_j\varphi)$ and d_j is the homogeneous degree of X_j. Since $\mu_t * \psi * [(X_j\varphi)_t]$ is bounded in $L^\infty(G) \cdot \beta$, and since $\mu_t * \psi * \varphi_t$ converges to $\widetilde{\psi} \cdot \beta$ in $\mathcal{D}'_w(G)$ by the dominated convergence theorem, it is easily seen that $X_j\widetilde{\psi} = 0$. By the arbitrariness of j, we then infer that $\widetilde{\psi}$ is constant almost everywhere. The second assertion follows reasoning on the opposite group of G. □

1.6 Representations

We start with some results on general representations and then we pass to group homomorphisms.

Definition 1.82. Let π be a continuous unitary representation of G into a hilbertian space H, and take $k \in \mathbb{N} \cup \{ \infty \}$. Then, $C^k(\pi)$ denotes the set of $v \in H$ such that the mapping $G \ni g \mapsto \pi(g) \cdot v \in H$ is of class C^k, endowed with the topology induced by that of $\mathcal{E}^k(G; H)$. We shall define $C^{-k}(\pi)$ as the strong dual of $C^k(\pi)$.

Observe that $C^k(\pi)$ is a hilbertian space if k is finite, a Montel space if k is infinite; in particular, $C^k(\pi)$ is reflexive for every $k \in \mathbb{Z} \cup \{ \pm\infty \}$. Therefore, the scalar product on H induces a $\overline{\cdot}$-semiliner continuous embedding of H into $C^{-k}(\pi)$ which turns $C^\infty(\pi)$ into a dense subspace of $C^{-k}(\pi)$, for every $k \in \mathbb{N} \cup \{ \infty \}$. For every $k \in \mathbb{Z} \cup \{ \pm\infty \}$, we shall then extend $\langle \cdot | \cdot \rangle$ to a hypocontinuous sesquilinear form (relative to the bounded subsets of each factor) on $C^k(\pi) \times C^{-k}(\pi)$ which is compatible with the duality on both factors. Then, π induces continuous representations of G in $C^k(\pi)$ for every $k \in \mathbb{Z} \cup \{ \pm\infty \}$.

Proposition 1.83. *Let π be a continuous unitary representation of G into a hilbertian space H. Take $k_1, k_2 \in \mathbb{Z} \cup \{ \pm\infty \}$ and $T \in W^{-\infty,-\infty,1}(G)$, and define*

$$\langle \pi(T) \cdot v_1 | v_2 \rangle \coloneqq \langle T, g \mapsto \langle \pi(g) \cdot v_1 | v_2 \rangle \rangle$$

for every $v_1 \in C^\infty(\pi)$ and for every $v_2 \in C^\infty(\pi)$. Then, π induces three continuous linear mappings

$$W^{k_1,k_2,1}(G) \to \mathcal{L}(C^{-k_2}(\pi); C^{k_1}(\pi))$$
$$W_c^{k_1,k_2,1}(G) \to \mathcal{L}_c(C^{-k_2}(\pi); C^{k_1}(\pi))$$
$$W_c^{k_1,k_2,1}(G) \to \mathcal{L}(C^{-k_2}(\pi); C_c^{k_1}(\pi)),$$

where $C_c^{k_1}(\pi)$ is the dual of $C^{-k_1}(\pi)$ endowed with the topology of compact convergence.

Proof. Observe first that, if $h_1, h_2 \in \mathbb{N} \cup \{ \infty \}$, $v_1 \in C^{h_1}(\pi)$, $v_2 \in C^{h_2}(\pi)$, $|\alpha_1| \leqslant h_1$ and $|\alpha_2| \leqslant h_2$, then

$$\mathbf{Y}^{\alpha_2} \mathbf{X}^{\alpha_1} \langle \pi(\,\cdot\,) \cdot v_1 | v_2 \rangle = \langle \pi(\,\cdot\,) \cdot \mathrm{d}\pi(\mathbf{X}^{\alpha_1}) \cdot v_1 | \mathrm{d}\pi(\mathbf{Y}^{\alpha_2})^* \cdot v_2 \rangle,$$

so that $\langle \pi(\,\cdot\,) \cdot v_1 | v_2 \rangle \in \mathcal{B}_c^{h_2,h_1}(G)$. The assertion then follows easily when $k_1, k_2 \leqslant 0$, and then when $k_1, k_2 \geqslant 0$ by transposition. We leave to the reader the details of the case $k_1 k_2 < 0$. □

Definition 1.84. Let π be a continuous unitary representation of G into a hilbertian space H, and take $T \in W^{-\infty,-\infty,1}(G)$. Then, we shall define $\pi(T)$ as in Proposition 1.83.

Proposition 1.85. *Let π be a continuous unitary representation of G into a hilbertian space H. Take $k_1, k_2, k_3 \in \mathbb{Z} \cup \{ \pm\infty \}$, $T_1 \in W^{k_1,k_2,1}(G)$, and $T_2 \in W^{-k_2,k_3,1}(G)$. Then,*

$$\pi(T_1 * T_2) = \pi(T_1) \cdot \pi(T_2).$$

Proof. Observe first that the assertion is clear if $T_1, T_2 \in \mathcal{D}(G) \cdot \beta$. Now, the left-hand side of the asserted formula is a separately continuous function of (T_1, T_2) from $W_c^{k_1,k_2,1}(G) \times W_c^{-k_2,k_3,1}(G)$ into $\mathcal{L}_c(C^{-k_3}(\pi); C^{k_1}(\pi))$ by Propositions 1.73 and 1.83. The same holds for the right-hand side by Proposition 1.83. Since $\mathcal{D}(G) \cdot \beta$ is dense in $W_c^{k_1,k_2,1}(G)$ and in $W_c^{-k_2,k_3,1}(G)$ by Corollary 1.74, the assertion follows. □

Proposition 1.86. *Let G be a postliminal unimodular separable Lie group, and let $\nu_{\widehat{G}}$ be the Plancherel measure on $\widehat{G}$ corresponding to a Haar measure β on G (cf. [33, 18.8.2]).*[7] *Let H be the canonical field of hilbertian spaces on $\widehat{G}$ (cf. [33, 8.6.1]). Choose a measurable field π of*

[7] When we speak of measurable fields of hilbertian spaces, vectors, etc., we shall always refer to the $\nu_{\widehat{G}}$-completion of the standard Borel structure of $\widehat{G}$.

continuous unitary representations on $\widehat{G}$ such that $\pi_\zeta \in \zeta$ and π_ζ is a representation on H_ζ for every $\zeta \in \widehat{G}$ (cf. [33, 8.6.2]). Take $k \in \mathbb{Z}_+^*$. Then, the isomorphism[8]

$$\mathcal{F}\colon L^2(G) \to \int_{\widehat{G}}^{\oplus} \mathcal{L}^2(H_\zeta)\,\mathrm{d}\nu_{\widehat{G}}(\zeta)$$

induces a (unique) continuous linear mapping[9]

$$W^{-k,0,2}(G) \to \int_{\widehat{G}}^{\oplus} \mathcal{L}^2(C^k(\pi_\zeta); H_\zeta)\,\mathrm{d}\nu_{\widehat{G}}(\zeta).$$

Proof. **1.** Since G is separable, we may find a countable dense subset D of $\mathcal{D}(G)$. Then, let $(v_j)_{j\in\mathbb{N}}$ be a sequence of measurable vector fields with values in the H_ζ such that $(v_{j,\zeta})$ is total in H_ζ for every $\zeta \in \widehat{G}$. Since $\pi_\zeta(D)\cdot H_\zeta$ is dense in $C^k(\pi_\zeta)$, the countable family $((\pi_\zeta(\varphi)\cdot v_{j,\zeta})_\zeta)_{j\in\mathbb{N},\varphi\in D}$ is total in $C^k(\pi_\zeta)$. Therefore, we may define the measurable vector fields with values in the $C^k(\pi_\zeta)$ as the $v \in \prod_{\zeta\in\widehat{G}} C^k(\pi_\zeta)$ such that the mapping

$$\zeta \mapsto \langle v_\zeta | \pi_\zeta(\varphi)\cdot v_{j,\zeta}\rangle_{C^k(\pi_\zeta)}$$

is measurable for every $j \in \mathbb{N}$ and for every $\varphi \in D$.

2. Take $f \in L^2(G)$ and choose a finite family $(f_\gamma)_{|\gamma|\leqslant k}$ of elements of $L^2(G)$ such that $f = \sum_{|\gamma|\leqslant k} \mathbf{Y}^\gamma f_\gamma$. Let us prove that

$$\mathcal{F}f(\pi_\zeta) = \sum_{|\gamma|\leqslant k} (\mathcal{F}f_\gamma)(\pi_\zeta)\cdot \mathrm{d}\pi_\zeta(\mathbf{X}^\gamma)$$

on $C^\infty(\pi_\zeta)$ for $\nu_{\widehat{G}}$-almost every $\zeta \in \widehat{G}$. Indeed, take a sequence (φ_k) of elements of $\mathcal{D}(G)\cdot\beta$ which converges to δ_e in $\mathcal{E}'^0_c(G)$. Then,

$$\mathcal{F}(\varphi_k *^\beta f)(\pi_\zeta) = \sum_{|\gamma|\leqslant k} (-1)^{|\gamma|} (\mathcal{F}f_\gamma)(\pi_\zeta)\cdot \pi_\zeta(\mathbf{Y}^\gamma \check{\varphi}_k)$$

for $\nu_{\widehat{G}}$-almost every $\zeta \in \widehat{G}$. Now, up to a subsequence $\mathcal{F}(f *^\beta \varphi_k)(\pi_\zeta)$ converges to $\mathcal{F}f(\pi_\zeta)$ in $\mathcal{L}^2(H_\zeta)$ for $\nu_{\widehat{G}}$-almost every $\zeta \in \widehat{G}$, while Proposition 1.83 implies that $(-1)^{|\gamma|}\pi_\zeta(\mathbf{Y}^\gamma\check{\varphi}_k)$ converges to $\mathrm{d}\pi_\zeta(\mathbf{X}^\gamma)$ in $\mathcal{L}_c(C^k(\pi_\zeta); H_\zeta)$. The assertion follows.

Now, it is clear that the mapping $\zeta \mapsto \mathrm{d}\pi_\zeta(\mathbf{X}^\alpha)$ is a measurable field of linear mappings from the $C^k(\pi_\zeta)$ into the H_ζ, and that $\|\mathrm{d}\pi_\zeta(\mathbf{X}^\alpha)\|_{\mathcal{L}(C^k(\pi_\zeta);H_\zeta)} \leqslant 1$ for every $\zeta \in \widehat{G}$. Therefore, $\mathcal{F}f$ is a class of measurable fields of linear mappings from the $C^k(\pi_\zeta)$ into the H_ζ, and

$$\|(\mathcal{F}f)(\zeta)\|_{\mathcal{L}^2(C^k(\pi_\zeta);H_\zeta)} \leqslant \sum_{|\gamma|\leqslant k} \|(\mathcal{F}f_\gamma)(\zeta)\|_{\mathcal{L}^2(H_\zeta;H_\zeta)}$$

for $\nu_{\widehat{G}}$-almost every $\zeta \in \widehat{G}$. Therefore,

$$\|\mathcal{F}f\|_{\int_{\widehat{G}}^{\oplus} \mathcal{L}^2(C^k(\pi_\zeta);H_\zeta)\,\mathrm{d}\nu_{\widehat{G}}(\zeta)} \leqslant \sum_{|\gamma|\leqslant k} \|f_\gamma\|_{L^2(G)}.$$

Now, observe that the mapping

$$\prod_{|\gamma|\leqslant k} L^2(G) \ni (f_\gamma) \mapsto \sum_{|\gamma|\leqslant k} \mathbf{Y}^\gamma f_\gamma \in W^{-k,0,2}(G)$$

is onto by Proposition 1.32; therefore, the mapping

$$f \mapsto \inf_{f=\sum_{|\gamma|\leqslant k}\mathbf{Y}^\gamma f_\gamma} \sqrt{\sum_{|\gamma|\leqslant k} \|f_\gamma\|^2_{L^2(G)}}$$

[8] Define $\mathcal{F}f(\pi_\zeta) = \pi_\zeta(\check{f})$ for $f \in L^1(G)\cap L^2(G)$ and for every $\zeta \in \widehat{G}$.

[9] Once a basis $(X_1,\dots,X_n)$ of $\mathfrak{g}$ has been fixed, endow $C^k(\pi_\zeta)$, for every $\zeta \in \widehat{G}$, with the hilbertian norm $v \mapsto \sqrt{\sum_{k=1}^n \|\mathrm{d}\pi(X_k)\cdot v\|^2}$. The topological vector space $\int_{\widehat{G}}^{\oplus} \mathcal{L}^2(C^k(\pi_\zeta), H_\zeta)\,\mathrm{d}\nu_{\widehat{G}}(\zeta)$ does not depend on the choice of $(X_1,\dots,X_k)$. See the proof for the definition of measurable vector fields.

is a norm which defines the topology of $W^{-k,0,2}(G)$. Hence, the mapping

$$\mathcal{F}\colon L^2(G) \mapsto \int_{\widehat{G}}^{\oplus} \mathcal{L}^2(C^k(\pi_\zeta); H_\zeta)\,\mathrm{d}\nu_{\widehat{G}}(\zeta).$$

is continuous with respect to the topology of $W^{-k,0,2}(G)$. Since $L^2(G)$ is dense in $W^{-k,0,2}(G)$ by Corollary 1.34, the assertion follows. □

Definition 1.87. Keep the hypotheses and the notation of Proposition 1.86, and take an element T of $W^{-\infty,0,2}(G)$, so that $T \in W^{-k,0,2}(G)$ for some $k \in \mathbb{N}$. Then, we shall define $\mathcal{F}T$ as in Proposition 1.86.

In addition, for every $\zeta \in \widehat{G}$ we shall write ζ instead of π_ζ by a slight abuse of notation.

Corollary 1.88. *Keep the hypotheses and the notation of Proposition 1.86, and take $\gamma \in \mathbb{N}^n$ and $T \in W^{-\infty,0,2}(G)$. Then,*

$$\mathcal{F}(\mathbf{Y}^\gamma T)(\pi) = \mathcal{F}T(\pi)\mathrm{d}\pi(\mathbf{X}^\gamma)$$

on $C^\infty(\pi)$ for $\nu_{\widehat{G}}$-almost every $\pi \in \widehat{G}$.

Proof. Indeed, equality holds if $T \in \mathcal{D}(G)\cdot\beta$. The assertion then follows by density thanks to Corollary 1.34. □

Corollary 1.89. *Keep the hypotheses and the notation of Proposition 1.86, and take $T \in W^{-\infty,0,1}(G) \subseteq W^{-\infty,0,2}(G)$. Then, $\pi(\check{T}) = \mathcal{F}T(\pi)$ for $\nu_{\widehat{G}}$-almost every $\pi \in \widehat{G}$. In particular, if $X \in \mathfrak{U}_{\mathbb{C}}(\mathfrak{g})$, then $\mathcal{F}X(\pi) = \mathrm{d}\pi(X)$ for $\nu_{\widehat{G}}$-almost every $\pi \in \widehat{G}$.*

Proof. Take a relatively compact open neighbourhood V of e in G and a sequence (φ_j) of elements of $\mathcal{D}(V)\cdot\beta$ which converges to δ_e in $\mathcal{E}_c'^0(G)$. Then, $\check{T} *^\beta \varphi_j \in W^{0,\infty,1}(G) \subseteq L^1(G) \cap L^2(G)$, so that $\pi(\check{T} * \varphi_j) = \mathcal{F}(\check{\varphi}_j *^\beta T)(\pi)$ for $\nu_{\widehat{G}}$-almost every $\pi \in \widehat{G}$. Now, $\pi(\check{T} * \varphi_j)$ converges to $\pi(\check{T})$ in $\mathcal{L}(C^\infty(\pi); H_\pi)$ for every $\pi \in \widehat{G}$, thanks to Proposition 1.83. Next, choose $k \in \mathbb{N}$ so that $T \in W^{-k,0,2}(G)$; then, $\check{\varphi}_j * T$ converges to T in $W^{-k,0,2}(G)$ by Corollary 1.72. Indeed, the endomorphisms $\check{\varphi}_j * \cdot$ of $W^{-k,0,2}(G)$ are equicontinuous since the sequence $(\check{\varphi}_j)$ is bounded in $\mathcal{E}'^0(G)$, hence in $W^{-k,k,1}(G)$; in addition, they converge to the identity on the dense subspace $\mathcal{D}(G)\cdot\beta$, so that the assertion follows. The proof is complete. □

Proposition 1.90. *Let G, G' be two Lie groups, and let $\pi\colon G \to G'$ be a continuous (hence analytic) homomorphism of G into G'. Take $k_1, k_2 \in \mathbb{Z} \cup \{\pm\infty\}$. Then, the following hold:*

1. *the formula, for $\varphi \in \mathcal{D}(G')$ and $T \in W^{-\infty,-\infty,1}(G)$,*

$$\langle \pi_*(T), \varphi\rangle \coloneqq \langle T, \varphi \circ \pi\rangle$$

defines an element $\pi_*(T)$ of $W^{-\infty,-\infty,1}(G')$;

2. *the mappings*

$$W^{k_1,k_2,1}(G) \ni T \mapsto \pi_*(T) \in W^{k_1,k_2,1}(G')$$
$$W_c^{k_1,k_2,1}(G) \ni T \mapsto \pi_*(T) \in W_c^{k_1,k_2,1}(G')$$

are continuous;

3. *if $T \in W^{-\infty,-\infty,1}(G)$ and X, Y are a left- and right-invariant differential operators on G, respectively, then*

$$\mathrm{d}\pi(Y)^\dagger \mathrm{d}\pi(X)^\dagger \pi_*(T) = \pi_*(Y^\dagger X^\dagger T).$$

Proof. Suppose first that $k_1, k_2 \leqslant 0$. Then, the mapping

$$\mathcal{B}_c^{-k_1,-k_2}(G') \ni \varphi \mapsto \varphi \circ \pi \in \mathcal{B}_c^{-k_1,-k_2}(G)$$

is well-defined and continuous, since for every α_1, α_2 of length at most $-k_1, -k_2$ respectively, and for every $\varphi \in \mathcal{B}_c^{-k_1,-k_2}(G')$,

$$\mathbf{Y}^{\alpha_1}\mathbf{X}^{\alpha_2}(\varphi \circ \pi) = [\mathrm{d}\pi(\mathbf{Y})^{\alpha_1}\mathrm{d}\pi(\mathbf{X})^{\alpha_2}\varphi] \circ \pi.$$

Therefore, all the assertions follow easily in this case.

The remaining cases follow easily from **3** and preceding one. □

Definition 1.91. Let G, G' be two Lie groups, and let $\pi\colon G \to G'$ be a continuous homomorphism of G into G'. Take $T \in W^{-\infty,-\infty,1}(G)$. Then, we shall define $\pi_*(T)$ as in Proposition 1.90.

Proposition 1.92. *Let G, G' be two Lie groups, and let $\pi\colon G \to G'$ be a continuous homomorphism of G* onto *G'. Let $\beta_G, \beta_{G'}$ be two left Haar measures on G, G' respectively. Then, there is a unique left Haar measure β_H on $H := \ker \pi$ such that, with a slight abuse of notation,*

$$\int_G \varphi \,\mathrm{d}\beta_G = \int_{G'} \int_H \varphi(gh) \,\mathrm{d}\beta_H(h) \,\mathrm{d}\beta_{G'}(\pi(g))$$

for every $\varphi \in \mathcal{D}^0(G)$. In addition, if $f \in L^1(\beta_G)$ and we define

$$\pi_*(f)(\pi(g)) := \int_H f(gh) \,\mathrm{d}\beta_H(h)$$

for β'_G-almost every $\pi(g) \in G'$, then π_* is well-defined and maps $L^1(G)$ onto *$L^1(G')$ with norm 1. Finally,*

$$\pi_*(f \cdot \beta_G) = \pi_*(f) \cdot \beta_{G'}.$$

Proof. The first assertion is a consequence of [20, Proposition 10 of Chapter VII, § 2, No. 7]. The second assertion follows from [20, Propositions 4, 5 and Corollary to Proposition 9 of Chapter VII, § 2] ☐

Definition 1.93. Let G, G' be two Lie groups, and let $\pi\colon G \to G'$ be a continuous homomorphism of G *onto* G'. Let $\beta_G, \beta_{G'}$ be two left Haar measures on G, G' respectively. Then, we shall define β_H and π_* as in Proposition 1.92.

Corollary 1.94. *Keep the hypotheses and the notation of Proposition 1.92. Assume further that G and G' are unimodular, and take $f \in L^1(\beta_G)$. Then,*

$$\pi_*(f)(\pi(g)) = \int_H \varphi(gh) \,\mathrm{d}\beta_H(h) = \int_H \varphi(hg) \,\mathrm{d}\beta_H(h)$$

for $\beta_{G'}$-almost every $\pi(g)$.

Proof. Apply Proposition 1.92 to the opposites of G and G'. Then, we find a unique right Haar measure β'_H on H such that (with an abuse of notation)

$$\int_G \varphi \,\mathrm{d}\beta_G = \int_{G'} \int_H \varphi(hg) \,\mathrm{d}\beta'_H(h) \,\mathrm{d}\beta_{G'}(\pi(g))$$

for every $\varphi \in \mathcal{D}^0(G)$. However, Proposition 1.92 implies that

$$\int_G \varphi \,\mathrm{d}\beta_G = \int_G \check{\varphi} \,\mathrm{d}\beta_G = \int_{G'} \pi_*(\check{\varphi}) \,\mathrm{d}\beta_{G'} = \int_{G'} \pi_*(\check{\varphi})\check{} \,\mathrm{d}\beta_{G'},$$

and that

$$\pi_*(\check{\varphi})\check{}(\pi(g)) = \int_H \varphi(h^{-1}g) \,\mathrm{d}\beta_H(h) = \int_H \varphi(hg) \,\mathrm{d}\beta_H(h)$$

since H is unimodular by [20, Proposition 10 of Chapter VII, § 2, No. 7]. Therefore, $\beta_H = \beta'_H$. ☐

Corollary 1.95. *Keep the hypotheses and the notation of Proposition 1.92. Then, the following hold:*

1. *π_* maps $W_0^{k_1,k_2,1}(\beta_G)$ continuously into $W_0^{k_1,k_2,1}(\beta_{G'})$ for every $k_1, k_2 \in \mathbb{N} \cup \{ \infty \}$;*
2. *$\mathrm{d}\pi(Y)\mathrm{d}\pi(X)^\dagger \pi_*(\varphi) = \pi_*(Y X^\dagger \varphi)$ for every right- and left-invariant differential operators Y, X on G of order at most $k_1, k_2 \in \mathbb{N}$, respectively, and for every $\varphi \in W_0^{k_1,k_2,1}(\beta_G)$.*

Proof. The assertions follow from Propositions 1.90 and 1.92. ☐

Corollary 1.96. *Keep the hypotheses and the notation of Proposition 1.92, and let $\mathcal{L}$ be a hypoelliptic left- or right-invariant differential operator on G. Then, $\mathrm{d}\pi(\mathcal{L})$ is hypoelliptic.*

Proof. **1.** Let us first show that there is a positive function $\tau \in \mathcal{E}(G)$ such that $\mathrm{Supp}(\tau)\cap\pi^{-1}(K)$ is compact for every compact subset K of G', and such that $\int_H \tau(gh)\,\mathrm{d}\beta_H(h) = 1$ for every $g \in G$. Indeed, take, for every $g' \in G'$, some positive $\tau_{g'} \in \mathcal{D}(G)$ such that $\tau_{g'}(g) > 0$ for some $g \in \pi^{-1}(g')$. Let $U_{g'}$ be the set where $\tau_{g'}$ does not vanish, and observe that $U_{g'}$ is an open set, so that $\pi(U_{g'})$ is an open neighbourhood of g' in G'. Since G' is paracompact, there is a partition of unity $(\lambda_{g'})_{g'\in G'}$ with elements in $\mathcal{D}(G')$ and subordinate to the covering $(U_{g'})$. Then, the mapping $\tau' := \sum_{g'}(\lambda_{g'}\circ\pi)\tau_{g'}$ is a positive element of $\mathcal{E}(G)$, its support has a compact intersection with $\pi^{-1}(K)$ for every compact subset K of G'; in addition, τ does not vanish identically on any fibre of π. Then, we may define $\pi_*(\tau')$ with the same formula used in Proposition 1.92, and clearly $\pi_*(\tau')$ is a positive nowhere-vanishing function on G'. In addition, for every $\varphi \in \mathcal{D}(G')$ we have $\pi_*(\tau')\varphi = \pi_*(\tau'(\varphi\circ\pi)) \in \mathcal{D}(G')$ since $\tau'(\varphi\circ\pi) \in \mathcal{D}(G)$ (see **2** below), so that $\pi_*(\tau) \in \mathcal{E}(G')$. Hence, it suffices to define $\tau := \frac{\tau'}{\pi_*(\tau')\circ\pi}$.

2. We shall assume that $\mathcal{L}$ is right-invariant; the other case follows considering the opposite group of G. Observe that Corollary 1.95 together with the classical (local) Sobolev embeddings shows that π_* maps $\mathcal{D}(G)$ continuously into $\mathcal{D}(G')$; one may prove that this mapping is actually onto, but we shall not need this fact. Then, we may define by transposition a continuous linear mapping $\pi^*\colon \mathcal{D}'(G') \to \mathcal{D}'(G)$. Then, take $T \in \mathcal{D}'(G')$ and assume that $\mathrm{d}\pi(\mathcal{L})T = \psi\cdot\beta_{G'} + S$, where ψ is the extension by 0 of an element of $\mathcal{E}(U)$ for some open subset U of G', while $S \in \mathcal{D}'(G')$ and $\mathrm{Supp}(S)\cap U = 0$; let us prove that T equals an element of $\mathcal{E}(G')\cdot\beta_{G'}$ on U. Since the assertion is local, we may assume that T is compactly supported and that U is relatively compact. Now, Proposition 1.92 and Corollary 1.95 show, by transposition, that

$$\mathcal{L}\pi^*(T) = \pi^*(\mathrm{d}\pi(\mathcal{L})T) = \pi^*(\psi\cdot\beta_{G'}) + \pi^*(S) = (\psi\circ\pi)\cdot\beta_G + \pi^*(S).$$

Since clearly $\mathrm{Supp}(\pi^*(S))\cap\pi^{-1}(U) = \emptyset$ and since $\mathcal{L}$ is hypoelliptic, this implies that $\pi^*(T) = \zeta\cdot\beta_G + S'$, where ζ is the extension by 0 of an element of $\mathcal{E}(\pi^{-1}(U))$, while $S' \in \mathcal{D}'(G)$ and $\mathrm{Supp}(S)\cap\pi^{-1}(U) = \emptyset$.

Now, take τ as in **1**. Then, $\tau\cdot\pi^*(T) \in \mathcal{E}'(G)$ since we assumed that T is compactly supported, so that

$$\langle \pi_*(\tau\cdot\pi^*(T)), \varphi\rangle = \langle \pi^*(T), \tau(\varphi\circ\pi)\rangle = \langle T, \pi_*(\tau(\varphi\circ\pi))\rangle = \langle T, \varphi\rangle$$

for every $\varphi \in \mathcal{D}(G')$, thanks the properties of τ. Therefore, T equals $\pi_*(\tau\zeta)\cdot\beta_{G'}$ on U, whence the assertion since $\pi_*(\tau\zeta)$ is the extention by 0 of an element of $\mathcal{E}(U)$. □

Proposition 1.97. *Keep the hypotheses and the notation of Proposition 1.92. Assume further that G and G' are homogeneous groups, and that π is homogeneous. Then, the following hold:*

1. *π_* induces a strict morphism of $\mathcal{S}(G)$* onto *$\mathcal{S}(G')$ which has a continuous linear section;*
2. *π_* induces a strict morphism of $\mathcal{D}(G)$* onto *$\mathcal{D}(G')$ which has a continuous linear section.*

Proof. Let us prove that the mapping $\pi_*\colon \mathcal{S}(G)\to\mathcal{S}(G')$ has a continuous linear section. Let $\mathfrak{g}$ (resp. $\mathfrak{h}, \mathfrak{g}'$) be the Lie algebra of G (resp. H, G'). Apply Lemma A.22 to the homogeneous ideal $\mathfrak{h}$ of $\mathfrak{g}$, thus finding a strong Malcev basis (in the sense of [28, second note after Theorem 1.1.13]) $(X_j)_{j=1}^n$ of $\mathfrak{g}$ consisting of homogeneous elements such that $\mathfrak{h} = \bigoplus_{j=1}^m X_j\mathbb{R}$, where $m := \dim\mathfrak{h}$. Let $\mathfrak{h}' := \bigoplus_{j=m+1}^n X_j\mathbb{R}$. Then, $\mathrm{d}\pi$ induces an isomorphism of homogeneous vector spaces between $\mathfrak{h}'$ and $\mathfrak{g}'$. Furthermore, the mapping $\Phi\colon \mathfrak{h}\oplus\mathfrak{h}' \ni (Y,X)\mapsto \exp_G(Y)\exp_G(X)\in G$ is a polynomial diffeomorphism by [28, Proposition 1.2.7]. Therefore, Φ induces an isomorphism between $\mathcal{S}(\mathfrak{h}\oplus\mathfrak{h}')$ and $\mathcal{S}(G)$. Now, take $\psi \in \mathcal{S}(G')$. Then, $\psi\circ\exp_{G'} \in \mathcal{S}(\mathfrak{g}')$, so that $\widetilde{\psi} := \psi\circ\exp_{G'}\circ\mathrm{d}\pi \in \mathcal{S}(\mathfrak{h}')$. Next, take $\varphi \in \mathcal{D}(H)$ so that $\int_H \varphi(y)\,\mathrm{d}\beta'_H(y) = 1$, and observe that β'_H is the push-forward of the Lebesgue measure on $\mathfrak{h}$ along $\exp_H$, which is the restriction to $\mathfrak{h}$ of $\exp_G$. Define $\tau_\psi := (\varphi\otimes\widetilde{\psi})\circ\Phi^{-1}$. By the preceding remarks, $\tau_\psi \in \mathcal{S}(G)$; in addition, $\tau_\psi \in \mathcal{D}(G)$ if $\psi \in \mathcal{D}(G')$. Furthermore, for every $X \in \mathfrak{h}'$,

$$\begin{aligned}\pi_*(\tau_\psi)(\pi(\exp_G(X))) &= \int_H \tau_\psi(y\exp_G(X))\,\mathrm{d}\beta'_H(y)\\ &= \int_{\mathfrak{h}} \varphi(Y)\widetilde{\psi}(X)\,\mathrm{d}Y\\ &= \psi(\pi(\exp_G(X)))\end{aligned}$$

since $\pi(\exp_G(X)) = \exp_{G'}(\mathrm{d}\pi(X))$. Since the set of $\pi(\exp_G(X))$ as X runs through $\mathfrak{h}'$ is G', this proves that $\pi_*(\tau_\psi) = \psi$. Now, it is easily seen that the mapping $\psi \mapsto \tau_\psi$ is linear and continuous from $\mathcal{S}(G')$ into $\mathcal{S}(G)$ and also from $\mathcal{D}(G')$ into $\mathcal{D}(G)$. This completes the proof. ☐

Proposition 1.98. *Let G, G' be two Lie groups, and let π be a continuous homomorphism of G onto G'. Let $\beta_{G'}$ be a relatively invariant measure on G' with right multiplier Δ'_R, and define $\varpi(g) \cdot f := \Delta'^{1/2}_R(\pi(g)) f(\,\cdot\,\pi(g))$ for every $f \in L^2(\beta_{G'})$ and for every $g \in G$. Then, the following hold:*

1. *ϖ is a continuous unitary representation of G in $L^2(\beta_{G'})$;*
2. *$C^k(\varpi) = W^{0,k,2}_0(\beta_{G'})$ for every $k \in \mathbb{N} \cup \{\infty\}$;*
3. *if $T \in W^{-k_2,-k_1,1}(G)$, and $f_1 \in C^{k_1}(\varpi)$, then*

$$\varpi(T) \cdot f_1 = (f_1 \cdot \beta_{G'}) * \big(\Delta'^{1/2}_R \pi(\check{T})\big).$$

Proof. **1.** It is clear that $\varpi(g)$ is a unitary operator on $L^2(\beta_{G'})$ for every $g \in G$. Since continuity is clear on $\mathcal{D}(G')$, and since $\mathcal{D}(G')$ is dense in $L^2(\beta_{G'})$, the assertion follows.

2. Take a (real) left-invariant vector field X on G, and $f \in \operatorname{dom}(\mathrm{d}\varpi(X))$. Then, for every $\varphi \in \mathcal{D}(G')$,

$$\begin{aligned}
\langle \mathrm{d}\varpi(X) \cdot f | \varphi \rangle &= X\langle \varpi(\,\cdot\,) f | \varphi\rangle(e)\\
&= \lim_{t\to 0} \frac{\Big\langle \Delta'^{1/2}_R\big(e^{t\mathrm{d}\pi(X)}\big) f\big(\,\cdot\, e^{t\mathrm{d}\pi(X)}\big) - f \Big| \varphi \Big\rangle}{t}\\
&= \lim_{t\to 0} \frac{\Big\langle f \Big| \Delta'_R\big(e^{-\frac{t}{2}\mathrm{d}\pi(X)}\big)\varphi\big(\,\cdot\, e^{-t\mathrm{d}\pi(X)}\big) - \varphi \Big\rangle}{t}\\
&= \big\langle f \big| \mathrm{d}\pi(X)^\dagger \varphi + \tfrac{1}{2}(\mathrm{d}\pi(X)\Delta'_R)(e')\varphi \big\rangle,
\end{aligned}$$

so that $\mathrm{d}\pi(X) f = \mathrm{d}\varpi(X)\cdot f - \frac{1}{2}(\mathrm{d}\pi(X)\Delta'_R)(e') f \in L^2(\beta_{G'})$. Conversely, assume that $\mathrm{d}\pi(X) f \in L^2(\beta_{G'})$; then

$$\begin{aligned}
\lim_{t\to 0} \frac{\varpi\big(e^{t\mathrm{d}\pi(X)}\big)\cdot f - f}{t} &= \lim_{t\to 0} \fint_0^t \Delta'^{1/2}_R\big(e^{s\mathrm{d}\pi(X)}\big)\Big[\tfrac{1}{2}(\mathrm{d}\pi(X)\Delta'_R)(e') f\big(\,\cdot\, e^{s\mathrm{d}\pi(X)}\big)\\
&\quad + \mathrm{d}\pi(X) f\big(\,\cdot\, e^{s\mathrm{d}\pi(X)}\big)\Big]\,\mathrm{d}s = \mathrm{d}\pi(X) f + \tfrac{1}{2}(\mathrm{d}\pi(X)\Delta'_R)(e') f
\end{aligned}$$

in $L^2(\beta_{G'})$, so that $f \in \operatorname{dom}(\mathrm{d}\varpi(X))$. It then follows that $C^k(\varpi) = W^{0,k,2}_0(\beta_{G'})$ for every $k \in \mathbb{N} \cup \{\infty\}$.

3. For every $f_2 \in C^\infty(\varpi)$,

$$\begin{aligned}
\varpi(T) \cdot (f_1, f_2) &= \Big\langle T, g \mapsto \Big\langle \Delta'^{1/2}_R(\pi(g)) f_1(\,\cdot\,\pi(g)) \cdot \beta_{G'}, \overline{f_2}\Big\rangle\Big\rangle\\
&= \Big\langle \pi(T), \Delta'^{-1/2}_R \big[\check{\overline{f_2}} *^{\beta_{G'}} (\Delta'_R f_1)\big]\Big\rangle\\
&= \Big\langle \big(\Delta'^{-1/2}_R \pi(T)\big) * (f_1 \cdot \beta_{G'})\check{}, \check{\overline{f_2}}\Big\rangle\\
&= \Big\langle (f_1 \cdot \beta_{G'}) * \big(\Delta'^{1/2}_R \pi(\check{T})\big), \overline{f_2}\Big\rangle,
\end{aligned}$$

whence the result. ☐

Definition 1.99. Let G, G' be two Lie groups, endowed with two right Haar measures β_G and $\beta_{G'}$, and let π be a continuous homomorphism of G *onto* G'. Then, we shall define the right quasi-regular representation of G in G' as the representation ϖ of Proposition 1.98.

Chapter 2

Some Useful Tools

2.1 Smooth Functions on Closed Sets

In this section, all manifolds are supposed to be over $\mathbb{R}$, of class C^∞, and locally compact, that is, Hausdorff and locally of finite dimension.

Definition 2.1. Let M be a manifold, k an element of $\mathbb{N} \cup \{\infty\}$, x an element of M, E a Hausdorff locally convex space, and $f_1, f_2 \colon M \to E$ two functions. We say that f_1 and f_2 have contact of order greater than k at x if $f_1(x) = f_2(x)$ and there is a local chart (U, φ) of M at x such that $[(f_1 - f_2) \circ \varphi^{-1}](y) = o(|y - \varphi(x)|^h)$ for $y \to \varphi(x)$ and for every $h \in \mathbb{N}$, $h \leqslant k$.[1]

Consider the equivalence relation on $\mathcal{E}^k(M; E)$ defined as follows: 'f_1 and f_2 have contact of order greater than k at x.' We denote by $J^k_x(M; E)$ the quotient of $\mathcal{E}^k(M; E)$ with respect to this equivalence relation. Its elements are called jets of order k at x. We denote by j^k_x the canonical projection and by $T^{(k)}_x(M)$ the dual of $J^k_x(M)$.

Lemma 2.2. *Let M be a manifold, k an element of $\mathbb{N} \cup \{\infty\}$, x an element of M and E a Fréchet space. Then, the following hold:*

1. *$T^{(\infty)}_x(M)$ is the inductive limit of the $T^{(h)}_x(M)$, as h runs through $\mathbb{N}$;*

2. *$J^k_x(M; E)$ is canonically isomorphic to $J^k_x(M) \widehat{\otimes} E$;*

3. *if $\mathfrak{B}$ is a basis of $T^{(k)}_x(M)$, then the mapping*

$$J^k_x(M; E) \ni \varphi \mapsto (X\varphi)_X \in E^{\mathfrak{B}}$$

is an isomorphism.

In particular, the bilinear mapping

$$T^{(k)}_x(M) \times J^k_x(M; E) \ni (X, \varphi) \mapsto X\varphi \in E$$

is well-defined and separating in $J^k_x(M; E)$ (and also in $T^{(k)}_x(M)$, provided that $E \neq 0$).

Proof. Observe first that we may assume that M is an open subset of $\mathbb{R}^n$ for some $n \in \mathbb{Z}^*_+$, and that $E \neq 0$. Then, $\mathfrak{B}_0 := (\partial^\alpha_x)_{|\alpha| \leqslant k}$ is a basis of $T^{(k)}_x(M)$. Next, observe that $\mathcal{E}(M)$ is nuclear thanks to [84, Corollary to Theorem 51.5 and Proposition 50.1]. In addition, [84, Theorem 44.1] shows that $\mathcal{E}(M; E)$ is canonically isomorphic to $\mathcal{E}(M) \widehat{\otimes}_\varepsilon E$, hence to $\mathcal{E}(M) \widehat{\otimes} E$ by [84, Theorem 50.1]. Then, define

$$\Phi \colon \mathcal{E}(M) \ni \varphi \mapsto (X\varphi)_X \in \mathbb{R}^{\mathfrak{B}_0},$$

so that Φ is a continuous linear mapping and $\ker \Phi = \ker j^k_x$. Next, observe that $\Phi \widehat{\otimes} I_E$ canonically corresponds to the mapping

$$\Phi_E \colon \mathcal{E}(M; E) \ni \varphi \mapsto (X\varphi)_X \in E^{\mathfrak{B}_0},$$

[1] This property then holds for every local chart (U, φ) at x.

whose kernel is the kernel of j^k_x on $\mathcal{E}(M;E)$. Then, [45, Proposition 3 of Chapter I, § 1, No. 2] implies that $\ker \Phi_E$ is the closed subspace of $\mathcal{E}(M;E)$ generated by the elements of the form $\varphi \otimes v$ with $\varphi \in \ker \Phi$ and $v \in E$. In addition, [45, Proposition 3 of Chapter I, § 1, No. 2] again implies that this set is also the kernel of the canonical mapping $\mathcal{E}(M)\widehat{\otimes} E \to J^k_x(M)\widehat{\otimes} E$, which is *onto* by [84, Proposition 43.9]. Hence, **2** follows.

Now, **3** is clear if $E = \mathbb{R}$ and $k \in \mathbb{N}$, since both spaces have the same finite dimension. In addition, if $E = \mathbb{R}$ and $k = \infty$, then the canonical mapping of **3** is onto by [50, Theorem 1.2.6]; since both spaces are Fréchet spaces, **3** follows from the open mapping theorem in this case. Observe that also **1** follows by means of [18, Proposition 15 of Chapter IV, § 1, No. 5].

Finally, the general case of **3** follows from the case $E = \mathbb{R}$, **2**, and [45, Proposition 6 of Chapter I, § 1, No. 3]. ☐

Definition 2.3. Let M be a manifold, F a closed subset of M, k an element of $\mathbb{N} \cup \{ \infty \}$, and E a Hausdorff locally convex space. Then, we denote by $\mathcal{E}_{M,k}(F;E)$ the quotient

$$\mathcal{E}(M;E) \big/ \{ f \in \mathcal{E}(M;E) \colon j^k(f) = 0 \text{ on } F \}.$$

We shall omit to denote k is F is the closure of an open set.

Lemma 2.4. *Let M be a manifold, $k \in \mathbb{N} \cup \{ \infty \}$, $x \in M$ and E a Hausdorff locally convex space. Then, $J^k_x(M;E)$ is canonically isomorphic to $\mathcal{E}_{M,k}(\{ x \};E)$.*

Proof. Indeed, this is clear if $k = \infty$. Then, assume that $k \in \mathbb{N}$, and take $\varphi \in \mathcal{E}^k(M;E)$. Working in local coordinates, it is easily seen that there is a function $\psi \in \mathcal{E}(M;E)$ such that $j^k_x(\varphi) = j^k_x(\psi)$. The assertion follows. ☐

Proposition 2.5. *Let M be a paracompact manifold, F a closed subset of M, k an element of $\mathbb{N} \cup \{ \infty \}$ and E a Fréchet space. Then, $\mathcal{E}_{M,k}(F)\widehat{\otimes} E$ is canonically isomorphic to $\mathcal{E}_{M,k}(F;E)$.*

Proof. Since the projective tensor product is compatible with products (cf. [84, Exercise 43.8]), we may assume that M is countable at infinity, so that $\mathcal{E}(M)$ is nuclear thanks to [84, Corollary to Theorem 51.5 and Proposition 50.1]. In addition, we may assume that $E \neq 0$. Now, the proof of [84, Theorem 44.1] can be generalized to prove that $\mathcal{E}(M;E)$ is canonically isomorphic to $\mathcal{E}(M)\widehat{\otimes}_\varepsilon E$, hence to $\mathcal{E}(M)\widehat{\otimes} E$ by [84, Theorem 50.1]. Then, define

$$\Phi \colon \mathcal{E}(M) \ni \varphi \mapsto j^k(\varphi) \in \prod_{x \in F} J^k_x(M)$$

so that Φ is a continuous linear mapping with kernel $\{ f \in \mathcal{E}(M) \colon j^k(f) = 0 \text{ on } F \}$. Next, observe that $\Phi \widehat{\otimes} I_E$ canonically corresponds to the mapping

$$\mathcal{E}(M;E) \ni \varphi \mapsto j^k(\varphi) \in \prod_{x \in F} J^k_x(M;E)$$

thanks to Lemma 2.2 and [45, Proposition 6 of Chapter I, § 1, No. 3]. Therefore, [45, Proposition 3 of Chapter I, § 1, No. 2] implies that $\{ f \in \mathcal{E}(M;E) \colon j^k(f) = 0 \text{ on } F \}$ is the closed subspace of $\mathcal{E}(M;E)$ generated by the elements of the form $\varphi \otimes v$ with $\varphi \in \ker \Phi$ and $v \in E$. Now, [45, Proposition 3 of Chapter I, § 1, No. 2] again implies that this set is also the kernel of the canonical mapping $\mathcal{E}(M)\widehat{\otimes} E \to \mathcal{E}_{M,k}(F)\widehat{\otimes} E$, which is *onto* by [84, Proposition 43.9]. The assertion follows. ☐

Definition 2.6. Let M be a manifold of class C^∞, F a closed subset of M, k_1, k_2 two elements of $\mathbb{N} \cup \{ \infty \}$, E a Hausdorff locally convex space and x an element of F. Then, define $J^{k_1}_{M,k_2,x}(F;E)$ as the quotient of $J^{k_1}_x(M;E)$ by $\{ j^{k_1}_x(\varphi) \colon \varphi \in \mathcal{E}(M;E),\ j^{k_2}(\varphi) = 0 \text{ on } F \}$.

Definition 2.7. Let G be a homogeneous group, F a closed subset of G, k an element of $\mathbb{N} \cup \{ \infty \}$, and E a Fréchet space. Then, we denote by $\mathcal{S}_{G,k}(F;E)$ the quotient

$$\mathcal{S}(G;E) \big/ \{ f \in \mathcal{S}(G;E) \colon j^k(f) = 0 \text{ on } F \}.$$

We shall omit to denote k if F is the closure of an open set.

Proposition 2.8. *Let G be a homogeneous group, F a closed subset of G, k an element of $\mathbb{N} \cup \{ \infty \}$, and E a Fréchet space. Then, the mapping $(\varphi, v) \mapsto \varphi \otimes v$ induces a canonical isomorphism*

$$\mathcal{S}_{G,k}(F) \widehat{\otimes} E \to \mathcal{S}_{G,k}(F; E).$$

Proof. The proof proceeds along the lines of that of Proposition 2.5. □

Corollary 2.9. *Let G_1, G_2 be two homogeneous groups, and let F_1, F_2 be two closed subsets of G_1, G_2, respectively. Let E be a Fréchet space. Then,*

$$\mathcal{S}_{G_1 \times G_2, k}(F_1 \times F_2; E) \cong \mathcal{S}_{G_1,k}(F_1) \widehat{\otimes} \mathcal{S}_{G_2,k}(F_2) \widehat{\otimes} E$$

canonically for every $k \in \{ 0, \infty \}$.

Proof. By Proposition 2.8, we may assume that $E = \mathbb{R}$. Then, observe that the composition of the canonical mappings

$$\begin{aligned} \mathcal{S}(G_1 \times G_2) \cong \mathcal{S}(G_1; \mathcal{S}(G_2)) \to \mathcal{S}(G_1; \mathcal{S}_{G_2,k}(F_2)) \\ \to \mathcal{S}_{G_1,k}(F_1; \mathcal{S}_{G_2,k}(F_2)) \cong \mathcal{S}_{G_1,k}(F_1) \widehat{\otimes} \mathcal{S}_{G_2,k}(F_2) \end{aligned}$$

is onto. In addition, its kernel is the set of $\varphi \in \mathcal{S}(G_1 \times G_2)$ such that $j^k(\varphi)$ vanishes on $F_1 \times F_2$, whence the result. □

Definition 2.10. Let G be a homogeneous group, V an open subset of G and E a Fréchet space. Then, we shall denote by $\widetilde{\mathcal{S}}_G(V; E)$ the set of $\varphi \in \mathcal{E}(V)$ such that for every $\alpha \in \mathbb{N}^n$ and for every $k \in \mathbb{N}$ the function

$$(1 + |\cdot|)^k \mathbf{X}^\alpha \varphi$$

is bounded on V. We shall endow $\widetilde{\mathcal{S}}_G(V; E)$ with the semi-norms

$$\varphi \mapsto \sup_{x \in V} (1 + |x|)^k \|(\mathbf{X}^\alpha \varphi)(x)\|_\rho$$

as k runs through $\mathbb{N}$, α runs through $\mathbb{N}^n$, and ρ runs through the set of continuous semi-norms on E.

Proposition 2.11. *Let G be a homogeneous group, V an open subset of G and E a Fréchet space. Then, $\widetilde{\mathcal{S}}_G(V; E)$ is a Fréchet space and j^0 maps $\mathcal{S}_G(\overline{V}; E)$ continuously into $\widetilde{\mathcal{S}}_G(V; E)$.*

Proposition 2.12. *Let G be a homogeneous group and V a dilation-invariant open subset of G which is minimally smooth in the sense of [79, Chapter VI, § 3.3]. Then, the canonical mapping $\mathcal{S}(G) \to \widetilde{\mathcal{S}}_G(V)$ admits a continuous linear section. In particular, $\mathcal{S}_G(V)$ and $\widetilde{\mathcal{S}}_G(\overline{V})$ are canonically isomorphic.*

Proof. Notice that we may assume that G is abelian. In addition, observe that [79, Theorem 5 of Chapter VI, § 3.1] shows that there is a linear extension operator $\Phi \colon L^\infty(V) \to L^\infty(G)$ which maps $W^{k,\infty}(V)$ continuously into $W^{k,\infty}(G)$ for every $k \in \mathbb{N} \cup \{ \infty \}$.

Next, take $\tau_0, \tau_1 \in \mathcal{D}_+(G)$ so that $0 \notin \mathrm{Supp}(\tau_1)$ and so that the function

$$\tau_2 \coloneqq \tau_0^2 + \sum_{j \in \mathbb{N}} \tau_1(2^{-j} \cdot)^2$$

never vanishes. Then, $\sqrt{\tau_2} \in \mathcal{E}(G)$, so that, up to replacing τ_0 and τ_1 with $\frac{\tau_0}{\sqrt{\tau_2}}$ and $\frac{\tau_1}{\sqrt{\tau_2}}$, respectively, we may assume that $\tau_2 = \chi_G$. Now, take $\varphi \in \widetilde{\mathcal{S}}_G(V)$ and define

$$\Psi(\varphi) \coloneqq \tau_0 \Phi(\tau_0 \varphi) + \sum_{j \in \mathbb{N}} \tau_1(2^{-j} \cdot) \Phi(\tau_1 \varphi(2^j \cdot))(2^{-j} \cdot).$$

Then, our choice of τ_0 and τ_1 shows that $\Psi(\varphi)$ is well defined, and that $\Psi(\varphi) = \varphi$ on V.

Let us prove that Ψ maps $\widetilde{\mathcal{S}}_G(V)$ continuously into $\mathcal{S}(G)$. Indeed, take $k, N \in \mathbb{N}$. Then, it is clear that there is a constant $C_{k,N} > 0$ such that

$$\max_{|\alpha| \leqslant k} \left\| \partial^\alpha (\tau_1 \varphi(2^j \cdot)) \right\|_\infty \leqslant C_{k,N} 2^{-jN}$$

for every $j \in \mathbb{N}$. Hence, there is a constant $C'_{k,N} > 0$ such that

$$\max_{|\alpha| \leqslant k} |\partial^\alpha \Psi(\varphi)| \leqslant C'_{k,N} \chi_{\mathrm{Supp}(\tau_0)} + C'_{k,N} \sum_{j \in \mathbb{N}} 2^{-jN} \chi_{2^j \cdot \mathrm{Supp}(\tau_1)}.$$

Therefore, there is a constant $C''_{k,N} > 0$ such that

$$\max_{|\alpha| \leqslant k} |\partial^\alpha \Psi(\varphi)| \leqslant \frac{C''_{k,N}}{(1 + |\cdot|)^N}.$$

This proves that $\Psi(\varphi) \in \mathcal{S}(G)$. Now, either analysing more carefully the preceding estimates, or applying the closed graph theorem, we see that Ψ is actually continuous. □

We conclude this section with a result on the composition with homogeneous polynomial mappings.

Proposition 2.13. *Take two homogeneous groups E_1, E_2, a dilation-invariant closed subset C of E_1, an open subset V of $\mathbb{R}^n$, $r \in \mathbb{N} \cup \{ \infty \}$, and a mapping $P \colon V \to \mathrm{Pol}(E_1; E_2)$ of class C^r such that $P(v)(t \cdot) = t \cdot P(v)$ and $P(v)(x_1) \neq 0$ for every $v \in V$, for every $t > 0$, and for every non-zero $x_1 \in C$. Then, the mapping*

$$V \ni v \mapsto [\varphi \mapsto \varphi \circ P(v)] \in \mathcal{L}(\mathcal{S}(E_2); \mathcal{S}_{E_1,k}(C))$$

is well-defined and of class C^r for every $k \in \mathbb{N} \cup \{ \infty \}$.

Proof. **1.** Observe that we may assume that both E_1 and E_2 are abelian. Let S_1 be the unit sphere of E_1 associated with some homogeneous norm, and fix a compact subset K of V. Then, the mapping $v \mapsto P(v) \in C(S_1 \cap C; E_2)$ is continuous since $\mathrm{Pol}(E_1; E_2)$ embeds continuously into $\mathcal{E}(E_1; E_2)$. In addition, the mapping $V \times E_1 \ni (v, x_1) \mapsto P(v)(x_1) \in E_2$ is continuous, so that

$$\min_{v \in K} \min_{x_1 \in S_1 \cap C} |P(v)(x_1)| > 0.$$

By compactness, we may find a neighbourhood U_K of $S_1 \cap C$ in S_1 such that

$$\inf_{v \in K} \inf_{x_1 \in U_K} |P(v)(x_1)| > 0.$$

Then, take $\tau_K \in \mathcal{B}^\infty(E_1)$ such that τ_K equals 1 on a neighbourhood of C, and $\mathrm{Supp}(\tau_K) \subseteq B(0,1) \cup (\mathbb{R}_+^* \cdot U_K)$.

Next, let $\mathcal{O}_M(E_1; E_2)$ be the set of $\varphi \in \mathcal{E}(E_1)$ such that $\varphi \mathcal{S}(E_1) \subseteq \mathcal{S}(E_1; E_2)$, endowed with the topology induced by the mapping $\varphi \mapsto [\psi \mapsto \varphi \psi] \in \mathcal{L}(\mathcal{S}(E_1); \mathcal{S}(E_1; E_2))$. Then, $\mathrm{Pol}(E_1; E_2)$ embeds continuously into $\mathcal{O}_M(E_1; E_2)$, so that the mapping $P \colon V \to \mathcal{O}_M(E_1; E_2)$ is of class C^r.

2. Keep the notation of **1** above. Let us prove that the mapping

$$\Phi_K \colon K \ni v \mapsto [\varphi \mapsto (\varphi \circ P(v)) \tau_K] \in \mathcal{L}(\mathcal{S}(E_2); \mathcal{S}(E_1))$$

is well-defined and that $\Phi_K(K)$ is bounded in $\mathcal{L}_s(\mathcal{S}(E_2); \mathcal{S}(E_1))$. Take $\varphi \in \mathcal{S}(E_2)$ and $h_1, h_2 \in \mathbb{N}$. Then, Faà di Bruno's formula implies that[2]

$$(1 + |x_1|)^{h_1} (\Phi_K(v)(\varphi))^{(h_2)}(x_1) = (1 + |x_1|)^{h_1} \sum_{h_2 = h_3 + h_4} \frac{h_2!}{h_3! h_4!} \tau_K^{(h_3)}(x_1) \cdot$$
$$\cdot \sum_{\sum_{p=1}^{h_4} p \ell_p = h_4} \frac{h_4!}{\ell!} \varphi^{(|\ell|)}(P(v)(x_1)) \prod_{p=1}^{h_4} \left(\frac{P(v)^{(p)}(x_1)}{p!} \right)^{\ell_p}.$$

for every $v \in K$ and for every $x_1 \in E_1$. Now, observe that $P(K)$ is bounded in $\mathcal{O}_M(E_1; E_2)$ by **1**, and that $\tau_K \in \mathcal{B}^\infty(E_1)$. Therefore, there are $h_5 \in \mathbb{N}$ and a constant $C > 0$ such that

$$|(1 + |x_1|)^{h_1} (\Phi(v)(\varphi))^{(h_2)}(x_1)| \leqslant C (1 + |x_1|)^{h_1 + h_5} \chi_{\mathrm{Supp}(\tau_K)}(x_1) \sum_{h_3 = 0}^{h_2} |\varphi^{(h_3)}(P(v)(x_1))|$$

[2] Cf. Corollary A.24 for the notation.

for every $v \in K$ and for every $x_1 \in E_1$. Next, observe that there is a constant $C' > 0$ such that

$$|\varphi^{(h_3)}(x_2)| \leqslant \frac{C'}{(1+|x_2|)^{h_1+h_5}}$$

for every $x_2 \in E_2$ and for every $h_3 = 0, \dots, h_2$. In addition, by **1** and the homogeneity of P, we see that there is a constant $C'' > 0$ such that

$$(1 + |P(v)(x_1)|) \geqslant C''(1 + |x_1|)$$

for every $v \in K$ and for every $x_1 \in \mathrm{Supp}(\tau_K)$. Therefore,

$$|(1+|x_1|)^{h_1}(\Phi_K(v)(\varphi))^{(h_2)}(x_1)| \leqslant \frac{CC'}{C''^{h_1+h_5}}(h_2+1)$$

for every $x_1 \in E_1$. By the arbitrariness of $h_1, h_2 \in \mathbb{N}$, the assertion follows.

3. Now, let us prove that Φ_K is continuous. Since $\Phi_K(K)$ is bounded in $\mathcal{L}_s(\mathcal{S}(E_2); \mathcal{S}(E_1))$ and since $\mathcal{S}(E_2)$ is barrelled, we see that $\Phi_K(K)$ is equicontinuous. Since, in addition, $\mathcal{S}(E_2)$ is a Montel space, by means of [18, Proposition 5 of Chapter III, § 3, No. 4] we see that it will suffice to show that the mapping $K \ni v \mapsto \Phi_K(v) \cdot \varphi \in \mathcal{S}(E_1)$ is continuous for every $\varphi \in \mathcal{S}(E_2)$. Now, also $\mathcal{S}(E_1)$ is a Montel space; in addition, the set of Dirac measures is total in $\mathcal{S}'(E_1)$. By the same reference, it will then suffice to show that the mapping $K \ni v \mapsto \langle \Phi_K(v)(\varphi), \delta_{x_1} \rangle$ is continuous for every $\varphi \in \mathcal{S}(E_1)$ and for every $x_1 \in E_1$. However, this is clear, whence the result.

By the arbitrariness of K, and since V is locally compact, we then see that the mapping

$$\Phi \colon V \ni v \mapsto [\varphi \mapsto \varphi \circ P(v)] \in \mathcal{L}(\mathcal{S}(E_2); \mathcal{S}_{E_1,k}(C))$$

is well-defined and continuous. Hence, the assertion follows if $r = 0$.

4. Now, let K' be the interior of K and assume that $r \geqslant 1$. Let us prove that Φ_K is differentiable on K', with derivative[3]

$$\Phi_K'(v)(\varphi) = \Phi_K(\varphi')P'(v)$$

for every $v \in K'$ and for every $\varphi \in \mathcal{S}(E_2)$. Observe first that the mapping

$$\Phi_{K,1} \colon K' \ni v \mapsto [\varphi \mapsto \Phi_K(\varphi')P'(v)] \in \mathcal{L}(\mathbb{R}^n; \mathcal{L}(\mathcal{S}(E_2); \mathcal{S}(E_1)))$$

is continuous thanks to **3** above and Lemma A.25. Fix $v_0 \in K'$ and a compact convex neighbourhood U of v_0 in K'. Since $\mathcal{L}(\mathcal{S}(E_2); \mathcal{S}(E_1))$ is a complete space, [19, Proposition 8 of Chapter VI, § 1, No. 2] implies that the integral

$$\int_{[0,1]} \Phi_{K,1}(v_0 + tv) \cdot v \,\mathrm{d}t$$

exists in $\mathcal{L}(\mathcal{S}(E_2); \mathcal{S}(E_1))$ for every $v \in U - v_0$. In addition, for every $\varphi \in \mathcal{S}(E_2)$ and for every $x_1 \in E_1$,

$$\int_{[0,1]} \Phi_{K,1}(v_0 + tv)(\varphi)(x_1) \cdot v \,\mathrm{d}t = \Phi_K(v_0 + v)(\varphi)(x_1) - \Phi_K(v_0)(\varphi)(x_1),$$

so that Φ_K is differentiable at v_0 and its derivative is $\Phi_{K,1}$. The assertion follows.

5. Now, assume that $r \in \mathbb{Z}_+^*$. Then, **4** implies that

$$\Phi_K'(v) = \Phi_K(\,\cdot\,')P'(v)$$

for every $v \in K'$. Arguing by induction, we may assume that Φ_K is of class C^{r-1}, so that by means of Lemma A.25 we see that Φ_K' is of class C^r on K'. By the arbitrariness of K, we see that Φ is of class C^r on V. The assertion for $r = \infty$ follows. □

[3]By an abuse of notation, we write $\Phi_K(\varphi')P'(v)$ instead of $\sum_{j=1}^{n_2} \Phi_K(\partial_j \varphi)P_j'(v)$, where we identified E_2 with $\mathbb{R}^{n_2}$ for some $n_2 > 0$.

2.2 Hadamard's Lemma

In this section, G denotes a homogeneous group of dimension n and homogeneous dimension Q. We denote by $|\cdot|$ a homogeneous norm on G, and by $\mathbf{X} = (X_j)_{j=1,\dots,n}$ a basis of homogeneous left-invariant vector fields on G; for every γ we denote by d_γ the homogeneous degree of $\mathbf{X}^\gamma = \prod_{j=1}^n X_j^{\gamma_j}$. We shall assume that $\mathrm{d}_1 \leqslant \dots \leqslant \mathrm{d}_n$. We denote by $\mathbf{Y} = (Y_j)_{j=1}^n$ the basis of right-invariant vector fields on G which corresponds to $\mathbf{X}$.

We shall denote by $\mathbf{d}\cdot\mathbb{N}^n$ the set of the sums $\sum_{j=1}^n d_j k_j$, as $k = (k_1,\dots,k_n) \in \mathbb{N}^n$. In other words, $\mathbf{d}\cdot\mathbb{N}^n$ is the set of homogeneous degrees of the left-invariant differential operators on G.

We identify the Lie algebra $\mathfrak{g}$ of G with $\mathbb{R}^n$ by means of the basis (X_j), and then we identify G with $\mathbb{R}^n$ by means of the exponential map. Therefore, $x = (x_j)$ means $x = \exp\Big(\sum_{j\in J} x_j X_j\Big)$.

We shall prove here a technical lemma about the correspondences between $\mathbf{X}^\beta$, $\mathbf{Y}^\beta$ and ∂^β. The main idea is that the 'principal terms' of these operators coincide; nevertheless, we shall need some precise description of the terms of higher degree which appear in the decompositions, for example, of $\mathbf{X}^\beta$ in terms of either the $\mathbf{Y}^{\beta'}$ or the $\partial^{\beta'}$. This result is based on [43, Chapter 1, C].

Lemma 2.14. *Take $\beta \in \mathbb{N}^n$. Then, there are unique families with finite support $(P_{k,\beta,\beta'})$, $k = 1,\dots,6$, of functions on G such that*

$$\mathbf{X}^\beta = \sum_{\beta'} P_{1,\beta,\beta'}\partial^{\beta'} \qquad \mathbf{Y}^\beta = \sum_{\beta'} P_{2,\beta,\beta'}\partial^{\beta'}$$
$$\partial^\beta = \sum_{\beta'} P_{3,\beta,\beta'}\mathbf{X}^{\beta'} \qquad \partial^\beta = \sum_{\beta'} P_{4,\beta,\beta'}\mathbf{Y}^{\beta'}$$
$$\mathbf{X}^\beta = \sum_{\beta'} P_{5,\beta,\beta'}\mathbf{Y}^{\beta'} \qquad \mathbf{Y}^\beta = \sum_{\beta'} P_{6,\beta,\beta'}\mathbf{X}^{\beta'}.$$

Furthermore, for every β,β' and for every $k = 1,\dots,6$, $P_{k,\beta,\beta'}$ is a homogeneous polynomial of homogeneous degree $\mathrm{d}_{\beta'} - \mathrm{d}_\beta$, $\partial_\ell P_{k,\beta,\beta'} = 0$ if $\mathrm{d}_\ell \geqslant \max_{\beta'_h \neq 0} \mathrm{d}_h$ and $k = 1,2$, and one of the following conditions holds:

(i) $P_{k,\beta,\beta'} = 0$;

(ii) $\beta = \beta'$;

(iii) $|\beta| > |\beta'|$;

(iv) $|\beta| = |\beta'|$ *and* $\mathrm{d}_{\beta'} > \mathrm{d}_\beta$.

Finally, $P_{k,\beta,\beta} = 1$.

By the proof, it follows that $\partial_\ell P_{k,\beta,\beta'} = 0$ if $\mathrm{d}_\ell = \mathrm{d}_n$, for every $k = 1,\dots,6$.

Proof. **1.** The existence and uniqueness of the families $(P_{k,\beta,\beta'})$ follows from standard arguments. The fact that the $P_{k,\beta,\beta'}$ are homogeneous of degree $\mathrm{d}_{\beta'} - \mathrm{d}_\beta$ follows easily from the uniqueness of the decomposition and from the homogeneity of $\mathbf{X}^\beta, \mathbf{Y}^\beta$ and ∂^β. Let us prove the other assertions by induction on the length of β. If $\beta = 0$, then there is nothing to prove.

2. Let us treat here the case in which $|\beta| = 1$. Notice first that the group law of G is represented by the Hausdorff series in the chosen system of coordinates, so that there is a finite family of polynomials $(P_j)_{j=1,\dots,n}$ such that

$$(x_j)\cdot(y_j) = (x_j + y_j + P_j((x_h)_{\mathrm{d}_h<\mathrm{d}_j};(y_h)_{\mathrm{d}_h<\mathrm{d}_j}))$$

for every $(x_j),(y_j) \in G$. This implies that

$$X_j(x) = \partial_j + \sum_{\mathrm{d}_{j'}>\mathrm{d}_j} \partial_{2,j}P_{j'}((x_h)_{\mathrm{d}_h<\mathrm{d}_{j'}};0)\partial_{j'}$$
$$Y_j(y) = \partial_j + \sum_{\mathrm{d}_{j'}>\mathrm{d}_j} \partial_{1,j}P_{j'}(0;(y_h)_{\mathrm{d}_h<\mathrm{d}_{j'}})\partial_{j'},$$

so that, for every $x \in G$,

$$P_{1,e_j,e_{j'}}(x) = \partial_{2,j} P_{j'}((x_h)_{\mathrm{d}_h < \mathrm{d}_{j'}}; 0)$$
$$P_{2,e_j,e_{j'}}(x) = \partial_{1,j} P_{j'}(0; (x_h)_{\mathrm{d}_h < \mathrm{d}_{j'}})$$

if $\mathrm{d}_{j'} > \mathrm{d}_j$, $P_{1,e_j,e_j} = P_{2,e_j,e_j} = 1$, and $P_{1,e_j,e_{j'}} = P_{2,e_j,e_{j'}} = 0$ in the remaining cases. Hence, the assertion holds for $k = 1, 2$ if β has length 1.

3. Now, consider the general case; then, there are j and γ such that $\mathbf{X}^\beta = X_j \mathbf{X}^\gamma$. By the inductive hypothesis,

$$\mathbf{X}^\gamma = \sum_{\gamma'} P_{1,\gamma,\gamma'} \partial^{\gamma'},$$

where $P_{1,\gamma,\gamma'}$ is a homogeneous polynomial of degree $\mathrm{d}_{\gamma'} - \mathrm{d}_\gamma$ which satisfies one of the conditions (i) to (iv), and $\partial_\ell P_{1,\gamma,\gamma'} = 0$ if $\mathrm{d}_\ell \geqslant \max_{\gamma'_h \neq 0} \mathrm{d}_h$. Therefore,

$$P_{1,\beta,\beta'} = \sum_{\gamma', j' \colon \beta' = \gamma' + e_{j'}} P_{1,e_j,e_{j'}} P_{1,\gamma,\gamma'} + \sum_{j'} P_{1,e_j,e_{j'}} \partial_{j'} P_{1,\gamma,\beta'}.$$

It is then clear that $P_{1,\beta,\beta'}$ is a polynomial, and that $\partial_\ell P_{1,\beta,\beta'} = 0$ if $\mathrm{d}_\ell \geqslant \max_{\beta'_h \neq 0} \mathrm{d}_h$. Now, assume that $P_{1,\beta,\beta'} \neq 0$. If $P_{1,e_j,e_{j'}} \partial_{j'} P_{1,\gamma,\beta'} \neq 0$ for some j', then $|\beta| > |\gamma| \geqslant |\beta'|$ by the inductive hypothesis, so that (iii) holds. If there is no such j', then there are γ' and j' such that $\beta' = \gamma' + e_{j'}$ and $P_{1,e_j,e_{j'}} P_{1,\gamma,\gamma'} \neq 0$. Then, $\mathrm{d}_{j'} \geqslant \mathrm{d}_j$, $\mathrm{d}_{\gamma'} \geqslant \mathrm{d}_\gamma$ and $|\gamma'| \leqslant |\gamma|$ by the inductive hypothesis. Next, assume that $|\beta| = |\beta'|$. If $\mathrm{d}_j < \mathrm{d}_{j'}$, then $\mathrm{d}_{\beta'} = \mathrm{d}_{\gamma'} + \mathrm{d}_{j'} > \mathrm{d}_\gamma + \mathrm{d}_j = \mathrm{d}_\beta$ and we are done. Otherwise, the inductive hypothesis implies that $j = j'$, so that $\gamma = \gamma'$ and $\beta = \beta'$; furthermore, $P_{1,\beta,\beta} = P_{1,e_j,e_j} P_{1,\gamma,\gamma} = 1$. This completes the proof for $k = 1$. The case $k = 2$ follows applying the case $k = 1$ to the opposite group of G.

4. Now, consider the case $k = 3$. Then, we can solve the equations

$$\mathbf{X}^\beta = \sum_{\beta'} P_{1,\beta,\beta'} \partial^{\beta'}$$

for $\partial^{\beta'}$, thus getting the family $(P_{3,\beta,\beta'})$. More precisely,

$$P_{3,\beta,\beta'} = \sum_{\ell \in \mathbb{N}} (-1)^\ell \sum_{\mathrm{Card}(\beta^{(1)}, \dots, \beta^{(\ell)}) = \ell} P_{1,\beta,\beta^{(1)}} P_{1,\beta^{(1)},\beta^{(2)}} \cdots P_{1,\beta^{(\ell)},\beta'},$$

where the term corresponding to $\ell = 0$ must be interpreted as $P_{1,\beta,\beta'}$. Hence, $P_{3,\beta,\beta'}$ is a polynomial which satisfies one of the conditions (i) to (iv). This completes the proof for $k = 3$. The case $k = 4$ follows applying the case $k = 3$ to the opposite group of G.

5. Next, consider the case $k = 5$. Then,

$$\begin{aligned}\mathbf{X}^\beta &= \mathbf{Y}^\beta + \sum_{\beta'} (P_{1,\beta,\beta'} - P_{2,\beta,\beta'}) \partial^{\beta'} \\ &= \mathbf{Y}^\beta + \sum_{\beta'} \sum_{\beta''} (P_{1,\beta,\beta'} - P_{2,\beta,\beta'}) P_{4,\beta',\beta''} \mathbf{Y}^{\beta''},\end{aligned}$$

so that

$$P_{5,\beta,\beta''} = \delta_{\beta,\beta''} + \sum_{\beta'} (P_{1,\beta,\beta'} - P_{2,\beta,\beta'}) P_{4,\beta',\beta''}.$$

It then follows that $P_{5,\beta,\beta''}$ is a polynomial and satisfies one of the conditions (i) to (iv). This completes the proof for $k = 5$. The case $k = 6$ follows applying the case $k = 5$ to the opposite group of G. □

As a corollary of the preceding analysis, we state the following result. It essentially shows that we can construct Taylor polynomials of any sufficiently differentiable function *with respect to either the left- or the right-invariant derivatives.* This happens to be an important tool, since it will allow us to deal with the (possibly) non-commutative structure of G and to take greater care of the homogeneous structure of G. For example, one may need to approximate sufficiently smooth functions f with polynomials P_f in such a way that all the left-invariant derivatives of f and P_f, up to a fixed *homogeneous* degree, at a given point are equal.

Corollary 2.15. *Take $x \in G$ and $\beta \in \mathbb{N}^n$. Then, there are two unique polynomials $P_{x,\beta}$ and $Q_{x,\beta}$ such that*

$$(\mathbf{X}^{\beta'} P_{x,\beta})(x) = \delta_{\beta,\beta'} \qquad \text{and} \qquad (\mathbf{Y}^{\beta'} Q_{x,\beta})(x) = \delta_{\beta,\beta'}$$

for every β'. Further, $P_{x,\beta}$ and $Q_{x,\beta}$ have homogeneous degree d_β.

This result is essentially [43, Proposition 1.30], so that the proof is omitted.

We now consider an analogue of the 'dual form' of Hadamard's lemma on homogeneous groups. For the sake of completeness, we state and prove also the classical versions.

Lemma 2.16 (Hadamard). *Let F be a Fréchet space, and take $k \in \mathbb{N}$, and $f \in \mathcal{S}(\mathbb{R}^n; F)$. Then, the following conditions are equivalent:*

1. *$f^{(j)}(0) = 0$ for every $j = 0, \dots, k$;*
2. *there is a finite family $(f_\beta)_{|\beta|=k+1}$ of elements of $\mathcal{S}(\mathbb{R}^n; F)$ such that $f = \sum_{|\beta|=k+1} (\,\cdot\,)^\beta f_\beta$.*

Proof. **1** $\implies$ **2.** Let φ be a C^∞ function such that $\chi_{B(0,1)} \leqslant \varphi \leqslant \chi_{B(0,2)}$. By Taylor's formula,

$$\begin{aligned}(f\varphi)(x) &= \int_{[0,1]} (f\varphi)^{(k+1)}(tx) \cdot x^{k+1} \frac{(1-t)^k}{k!} \, \mathrm{d}t \\ &= \sum_{|\beta|=k+1} x^\beta f_{0,\beta}(x),\end{aligned}$$

where

$$f_{0,\beta}(x) = \varphi\Big(\frac{x}{2}\Big) \frac{k+1}{\beta!} \int_{[0,1]} \partial^\beta(\varphi f)(tx)(1-t)^k \, \mathrm{d}t.$$

Notice that $f_{0,\beta} \in \mathcal{D}_{\overline{B}(0,4)}(\mathbb{R}^n; F)$ for every β of length $k+1$. Next, define

$$f_{\infty,\beta}(x) \coloneqq \frac{(k+1)!}{\beta!} \frac{x^\beta}{\|x\|^{2(k+1)}} (1-\varphi(x)) f(x).$$

Then,

$$\sum_{|\beta|=k+1} x^\beta f_{\infty,\beta}(x) = (1-\varphi(x)) f(x),$$

and each $f_{\infty,\beta}$ belongs to $\mathcal{S}(\mathbb{R}^n; F)$. It suffices to define $f_\beta \coloneqq f_{0,\beta} + f_{\infty,\beta}$.

2 $\implies$ **1.** Obvious. □

The following corollary, which corresponds to Lemma 2.16 via the Fourier transform, is the result which we wish to generalize. It is worthwhile to consider a vector-valued version.

Corollary 2.17. *Let F be a Fréchet space, and take $\varphi \in \mathcal{S}(\mathbb{R}^n; F)$ and $k \in \mathbb{N}$. Then, the following conditions are equivalent:*

1. *$\int_{\mathbb{R}^n} \varphi P \, \mathrm{d}\mathcal{H}^n = 0$ for every polynomial P of degree at most k;*
2. *there is a finite family $(\varphi_\beta)_{|\beta|=k+1}$ of elements of $\mathcal{S}(\mathbb{R}^n; F)$ such that $\varphi = \sum_{|\beta|=k+1} \partial^\beta \varphi_\beta$.*

Proof. The assertion follows from Lemma 2.16 by means of the Fourier transform if $F = \mathbb{C}$, hence if $F = \mathbb{R}$ by taking the real parts.

Now, define $\mathcal{S}_k(\mathbb{R}^n; F)$ as the set of $\varphi \in \mathcal{S}_k(\mathbb{R}^n; F)$ such that $\langle \varphi | P \rangle = 0$ for every polynomial P of degree at most k, and consider the continuous linear mapping

$$\tau_{k,F} \colon \prod_{|\beta|=k+1} \mathcal{S}(\mathbb{R}^n; F) \ni (\varphi_\beta) \mapsto \sum_{|\beta|=k+1} \partial^\beta \varphi_\beta \in \mathcal{S}_k(\mathbb{R}^n; F).$$

The assertion then means that $\tau_{k,F}$ is onto. Next, observe that

$$\prod_{|\beta|=k+1} \mathcal{S}(\mathbb{R}^n; F) \cong \Bigg(\prod_{|\beta|=k+1} \mathcal{S}(\mathbb{R}^n) \Bigg) \widehat{\otimes} F$$

canonically, thanks to Proposition 2.8 and [84, Exercise 43.8]. Let us prove that the canonical isomorphism $\mathcal{S}(\mathbb{R}^n;F)\cong\mathcal{S}(\mathbb{R}^n)\widehat{\otimes}F$ induces an isomorphism $\mathcal{S}_k(\mathbb{R}^n;F)\cong\mathcal{S}_k(\mathbb{R}^n)\widehat{\otimes}F$. Indeed, let B_k be a basis of the space of polynomials of degree at most k on $\mathbb{R}^n$. Then, $\mathcal{S}_k(\mathbb{R}^n;F)$ is the kernel of the linear mapping

$$\Phi_{k,F}\colon \mathcal{S}(\mathbb{R}^n;F)\ni\varphi\mapsto(\langle\varphi|P\rangle)\in F^{B_k},$$

which corresponds to $\Phi_{k,\mathbb{R}}\widehat{\otimes}I_F$ under the canonical identifications. Taking into account the fact that $\mathcal{S}(\mathbb{R}^n)$ is nuclear, the assertion follows by means of [45, Proposition 3 of Chapter I, § 1, No. 2] and [84, Proposition 43.7].

Therefore, by means of [84, Proposition 43.9] we see that $\tau_{k,F}$ is onto, whence the result. □

Corollary 2.18. *Let F be a Fréchet space, $\varphi\in\mathcal{S}(G;F)$ and $k\in\mathbf{d}\cdot\mathbb{N}^n$. Then, the following conditions are equivalent:*

1. *$\langle\varphi|P\rangle=0$ for every polynomial P of homogeneous degree $<k$;*
2. *there is a family with finite support $(\varphi_{k,\beta})_{\mathrm{d}_\beta\geqslant k}$ of elements of $\mathcal{S}(G;F)$ such that $\varphi=\sum_{\mathrm{d}_\beta\geqslant k}\mathbf{X}^\beta\varphi_{k,\beta}$;*
3. *there is a family with finite support $(\widetilde{\varphi}_{k,\beta})_{\mathrm{d}_\beta\geqslant k}$ of elements of $\mathcal{S}(G;F)$ such that $\varphi=\sum_{\mathrm{d}_\beta\geqslant k}\mathbf{Y}^\beta\widetilde{\varphi}_{k,\beta}$.*

If, in addition, $\{\,X_j\colon \mathrm{d}_j=\mathrm{d}_1\,\}$ generates the Lie algebra of G, then the preceding conditions are equivalent to the following ones:

2′. *there is a finite family $(\psi_{k,\beta})_{\mathrm{d}_\beta=k}$ of elements of $\mathcal{S}(G;F)$ such that $\varphi=\sum_{\mathrm{d}_\beta=k}\mathbf{X}^\beta\psi_{k,\beta}$;*

3′. *there is a finite family $(\widetilde{\psi}_{k,\beta})_{\mathrm{d}_\beta=k}$ of elements of $\mathcal{S}(G;F)$ such that $\varphi=\sum_{\mathrm{d}_\beta=k}\mathbf{Y}^\beta\widetilde{\psi}_{k,\beta}$.*

Proof. **1 $\implies$ 2.** We proceed by induction on $k\in\mathbf{d}\cdot\mathbb{N}^n$. If $k=0$, we may take $\varphi_{0,0}\coloneqq\varphi$ and $\varphi_{0,\beta}\coloneqq 0$ for $\beta\neq 0$. Next, assume that $k>0$ and let k' be the greatest element of $\mathbf{d}\cdot\mathbb{N}^n$ which is $<k$; assume, by induction, that the assertion holds for k'. Then, there is a family $(\varphi_{k',\beta})_{\mathrm{d}_\beta\geqslant k'}$ of elements of $\mathcal{S}(G;F)$ such that $\varphi=\sum_{\mathrm{d}_\beta\geqslant k'}\mathbf{X}^\beta\varphi_{k',\beta}$. Take β such that $\mathrm{d}_\beta=k'$; let us prove that $\varphi_{k',\beta}$ has integral 0. Notice that Lemma 2.14 implies that there is a unique family with finite support $(P_{\gamma,\gamma'})$ of homogeneous polynomials such that $\mathbf{X}^\gamma=\sum_{\gamma'}P_{\gamma,\gamma'}\partial^{\gamma'}$; in addition, $P_{\gamma,\gamma}=1$, $P_{\gamma,\gamma'}$ has homogeneous degree $\mathrm{d}_{\gamma'}-\mathrm{d}_\gamma$ and one of the following conditions holds:

(i) $P_{\gamma,\gamma'}=0$;

(ii) $\gamma=\gamma'$;

(iii) $|\gamma|>|\gamma'|$;

(iv) $|\gamma|=|\gamma'|$ and $\mathrm{d}_{\gamma'}>\mathrm{d}_\gamma$.

Write Q_β to denote the monomial $x\mapsto x^\beta$. Then, for every γ such that $\mathrm{d}_\gamma\geqslant k'$

$$\langle\mathbf{X}^\gamma\varphi_{k',\gamma}|Q_\beta\rangle=\sum_{\gamma'}(-1)^{|\gamma|}\Big\langle\varphi_{k',\gamma}\Big|P_{\gamma,\gamma'}\partial^{\gamma'}Q_\beta\Big\rangle.$$

If $P_{\gamma,\gamma'}\partial^{\gamma'}Q_\beta\neq 0$, then $\mathrm{d}_\beta\geqslant\mathrm{d}_{\gamma'}\geqslant\mathrm{d}_\gamma\geqslant k'=\mathrm{d}_\beta$. In particular, both $P_{\gamma,\gamma'}$ and $\partial^{\gamma'}Q_\beta$ are non-zero constants, so that $\gamma'=\beta$. Hence,

$$\langle\mathbf{X}^\gamma\varphi_{k',\gamma}|Q_\beta\rangle=(-1)^{|\gamma|}\beta!\langle\varphi_{k',\gamma}|P_{\gamma,\beta}\rangle.$$

Further, either $\beta=\gamma$ (so that $P_{\gamma,\beta}=P_{\beta,\beta}=1$) or $|\beta|<|\gamma|$. Therefore,

$$\begin{aligned}0=\langle\varphi|Q_\beta\rangle&=\sum_{\mathrm{d}_\gamma=k'}(-1)^{|\gamma|}\beta!\langle\varphi_{k',\gamma}|P_{\gamma,\beta}\rangle\\&=(-1)^{|\beta|}\beta!\int_G\varphi_{k',\beta}(x)\,\mathrm{d}x+\sum_{\substack{\mathrm{d}_\gamma=k'\\|\gamma|>|\beta|}}(-1)^{|\gamma|}P_{\gamma,\beta}(0)\beta!\int_G\varphi_{k',\gamma}(x)\,\mathrm{d}x.\end{aligned}$$

Arguing by descending induction on the length of β (still subject to the condition $\mathrm{d}_\beta = k'$), we then see that $\varphi_{k',\beta}$ has integral 0. Therefore, Corollary 2.17 implies that, for every β such that $\mathrm{d}_\beta = k'$, there is a family $(\varphi_{k',\beta,j})$ of elements of $\mathcal{S}(G;F)$ such that

$$\varphi_{k',\beta} = \sum_{j=1}^{n} \partial_j \varphi_{k',\beta,j}.$$

Now, by Lemma 2.14, for every j there is a family with finite support $(Q_{j,j'})$ of polynomials such that $Q_{j,j'}$ is homogeneous of homogeneous degree $\mathrm{d}_{j'} - \mathrm{d}_j$, and

$$\partial_j = \sum_{j'} Q_{j,j'} X_{j'}.$$

By transposition, this yields

$$\partial_j = \sum_{j'} X_{j'}(Q_{j,j'} \cdot\,).$$

Hence,

$$\varphi_{k',\beta} = \sum_{j,j'} X_{j'}(Q_{j,j'}\varphi_{k',\beta,j}),$$

so that

$$\varphi = \sum_{\mathrm{d}_\beta \geqslant k} \mathbf{X}^\beta \varphi_{k',\beta} + \sum_{\mathrm{d}_\beta = k'} \sum_{j,j'} \mathbf{X}^\beta X_j(Q_{j,j'}\varphi_{k',\beta,j}).$$

The assertion follows easily.

2 $\implies$ 1. Indeed, for every polynomial P of homogeneous degree $< k$,

$$\langle \varphi | P \rangle = \sum_{\mathrm{d}_\beta \geqslant k} (-1)^{|\beta|} \langle \varphi_{k,\beta} | \mathbf{X}^\beta P \rangle = 0$$

since $\mathbf{X}^\beta P$ is a polynomial of homogeneous degree $< k - \mathrm{d}_\beta \leqslant 0$, so that it is 0.

1 $\iff$ 3. This follows from the equivalence **1** $\iff$ **2** applied to the opposite group of G.

Assume from now on that $\{ X_j : \mathrm{d}_j = \mathrm{d}_1 \}$ generate the Lie algebra of G, and define $m \coloneqq \max\{ j : \mathrm{d}_j = \mathrm{d}_1 \}$.

2 $\implies$ 2′. Take $(\varphi_{k,\beta})$ as in **2**. Take β such that $\mathrm{d}_\beta > k$, and expand each X_j, with $j = m+1, \dots, n$, as an iterated commutator of $X_1, \dots, X_m$. If $h \coloneqq \frac{\mathrm{d}_\beta}{\mathrm{d}_1} \in \mathbb{Z}_+^*$, then

$$\mathbf{X}^\beta = \sum_\gamma c_{\beta,\gamma} \mathbf{X}_\gamma$$

for some family of real numbers $(c_{\beta,\gamma})$ with finite support; here, for every $\gamma = (\gamma_1, \dots, \gamma_h) \in \{ 1, \dots, m \}^h$, $\mathbf{X}_\gamma \coloneqq X_{\gamma_1} \dots X_{\gamma_h}$. Split each γ in (γ', γ''), where $\sum_j \mathrm{d}_{\gamma'_j} = k$.[4] Then,

$$\mathbf{X}^\beta \varphi_{k,\beta} = \sum_\gamma c_{\beta,\gamma} \mathbf{X}_{\gamma'}(\mathbf{X}_{\gamma''}\varphi_{k,\beta}).$$

Since each $\mathbf{X}_{\gamma'}$, for $c_{\beta,\gamma} \neq 0$, can be expanded as a linear combination of the $\mathbf{X}^\beta$ with $\mathrm{d}_\beta = k$, the result follows easily.

2′ $\implies$ 2. This is obvious.

3 $\iff$ 3′. Apply the equivalence **2** $\iff$ **2′** to the opposite group of G. □

Below we present an easy converse concerning the validity of **2′** and **3′** of Corollary 2.18. In particular, it shows that the condition imposed in order that conditions **2′** and **3′** hold is optimal.

[4] In other words, γ' has the first $\frac{k}{\mathrm{d}_1} \in \mathbb{N}$ components of γ.

Remark 2.19. Assume that, for every $\varphi \in \mathcal{S}(G)$ such that $\int_G \varphi(x)\,\mathrm{d}x = 0$, there is a finite family $(\varphi_j)_{\mathrm{d}_j=\mathrm{d}_1}$ of elements of $\mathcal{S}(G)$ such that $\varphi = \sum_{\mathrm{d}_j=\mathrm{d}_1} X_j\varphi_j$. Then, $\{ X_j : \mathrm{d}_j = \mathrm{d}_1 \}$ generates $\mathfrak{g}$ as a Lie algebra.

Indeed, let $\mathfrak{h}$ be the ideal of $\mathfrak{g}$ generated by $\{ X_j : \mathrm{d}_j = \mathrm{d}_1 \}$. Define $H \coloneqq \exp(\mathfrak{h})$, so that H is a normal homogeneous subgroup of G. Let G' be the quotient of G by H, and let $\pi\colon G \to G'$ be the canonical projection. Then, Proposition 1.97 implies that there is a strict morphism π_* of $\mathcal{S}(G)$ *onto* $\mathcal{S}(G')$ such that $\pi_*(X\psi) = \mathrm{d}\pi(X)\pi_*(\psi)$ for every $\psi \in \mathcal{S}(G)$ and for every left-invariant differential operator X on G. Now, take φ and (φ_j) as above. Then, $\pi_*(\varphi) = \sum_{\mathrm{d}_j=\mathrm{d}_1} \mathrm{d}\pi(X_j)\pi_*(\varphi_j) = 0$, so that the set $\left\{ \varphi \in \mathcal{S}(G) : \int_G \varphi(x)\,\mathrm{d}x = 0 \right\}$ is contained in the kernel of π_*. Since π_* is *onto*, this proves that $\mathcal{S}(G')$ has dimension at most 1; hence, $G' = \{ e \}$. This means that $\mathfrak{h} = \mathfrak{g}$. Now, $\mathfrak{h}$ is (contained in, hence equal to) the vector space generated by $\{ X_j : \mathrm{d}_j = \mathrm{d}_1 \} \cup [\mathfrak{g}, \mathfrak{g}]$, so that $\{ X_j : \mathrm{d}_j = \mathrm{d}_1 \}$ generates $\mathfrak{g}$ as a Lie algebra (cf. [21, Exercise 4.b of Chapter I, § 4]).

2.3 Composite Functions: Continuous Functions

In this section we develop some tools to deal with the following problem: given three Hausdorff spaces X, Y, Z, a measure μ on X, a μ-proper mapping $\pi\colon X \to Y$,[5] and a function $m\colon Y \to Z$ such that $m \circ \pi$ equals μ-almost everywhere a continuous function, does m equal $\pi_*(\mu)$-almost everywhere a continuous function?

To this end, we introduce the following definition.

Definition 2.20. Let X be a Hausdorff space, Y a set, μ a positive Radon measure on X, and π a mapping of X into Y.

We say that two points x, x' of $\mathrm{Supp}(\mu)$ are (μ,π)-connected if $\pi(x) = \pi(x')$ and there are $x = x_1, \dots, x_k = x' \in \pi^{-1}(\pi(x)) \cap \mathrm{Supp}(\mu)$ such that

$$\mu^\bullet\big(\pi^{-1}(\pi(U_j) \cap \pi(U_{j+1}))\big) > 0$$

for every $j = 1, \dots, k$, for every neighbourhood U_j of x_j in $\mathrm{Supp}(\mu)$ and for every neighbourhood U_{j+1} of x_{j+1} in $\mathrm{Supp}(\mu)$.

We say that μ is π-connected if every two elements of $\pi^{-1}(y) \cap \mathrm{Supp}(\mu)$ are (μ,π)-connected for every $y \in Y$.

Observe that (μ,π)-connectedness actually depends only on the equivalence class of μ and the equivalence relation induced by π on X. In addition, notice that if Y is a topological space and π is open at some point of each fibre (in the support of μ), then μ is clearly π-connected.

We emphasize that, in the definition of (μ,π)-connectedness, the points $x_1, \dots, x_k$ are fixed *before* considering their neighbourhoods. In other words, if for every neighbourhood U of x in $\mathrm{Supp}(\mu)$ and for every neighbourhood U' of x' in $\mathrm{Supp}(\mu)$ we found $x = x_1, \dots, x_k = x'$ and neighbourhoods U_j of x_j in $\mathrm{Supp}(\mu)$ such that $U = U_1$, $U' = U_k$ and

$$\mu^\bullet\big(\pi^{-1}(\pi(U_j) \cap \pi(U_{j+1}))\big) > 0$$

for every $j = 1, \dots, k$, then we would *not* be able to conclude that x and x' are (μ,π)-connected (cf. Remark 2.25 below).

Now we can prove our main result. Notice that, even though its hypotheses are quite restrictive, it still gives rise to important consequences.

Proposition 2.21. *Let X, Y, Z be three Hausdorff spaces, $\pi\colon X \to Y$ a mapping, and μ a π-connected positive Radon measure on X. Assume that π is μ-proper and that there is a disintegration $(\lambda_y)_{y \in Y}$ of μ relative to π[6] such that* $\mathrm{Supp}(\lambda_y) \supseteq \mathrm{Supp}(\mu) \cap \pi^{-1}(y)$ *for $\pi_*(\mu)$-almost every $y \in Y$.*

Take a continuous mapping $m_0\colon X \to Z$ and assume that there is mapping $m_1\colon Y \to Z$ such that $m_0(x) = (m_1 \circ \pi)(x)$ for locally μ-almost every $x \in X$. Then, there is a $\pi_*(\mu)$-measurable mapping $m_2\colon Y \to Z$ such that $m_0 = m_2 \circ \pi$ pointwise *on* $\mathrm{Supp}(\mu)$.

[5] Recall that this means that $\pi_*(\mu)$ is a Radon measure. In other words, for every $y \in Y$ there is an open neighbourhood V of y in Y such that $\pi^{-1}(V)$ is μ-integrable.

[6] By this we mean that λ_y is a probability measure carried by $\pi^{-1}(y)$ for locally $\pi_*(\mu)$-almost every $y \in Y$, and that for every μ-measurable function $f\colon X \to [0, +\infty]$ the function $Y \ni y \mapsto \int_X^\bullet f\,\mathrm{d}\lambda_y$ is $\pi_*(\mu)$-measurable and $\int_X^\bullet f\,\mathrm{d}\mu = \int_Y^\bullet \int_X^\bullet f\,\mathrm{d}\lambda_y\,\mathrm{d}\pi_*(\mu)(y)$.

If π is also proper, then m_2 is actually continuous on the image of π.

Proof. Observe first that there is a locally $\pi_*(\mu)$-negligible subset N of Y such that $m_1 \circ \pi = m_0$ locally λ_y-almost everywhere for every $y \in Y \setminus N$. In addition, we may assume that $\operatorname{Supp}(\mu) = X$ and that, if $y \in Y \setminus N$, then the support of λ_y contains $\pi^{-1}(y)$. Since m_0 is continuous and since $m_1 \circ \pi$ is constant on the support of λ_y, it follows that m_0 is constant on $\pi^{-1}(y)$ for every $y \in Y \setminus N$.

Now, take $y \in \pi(X) \cap N$ and $x_1, x_2 \in \pi^{-1}(y)$. Let $\mathfrak{U}(x_1)$ and $\mathfrak{U}(x_2)$ be the filters of neighbourhoods of x_1 and x_2, respectively. Assume first that $\pi(U_1) \cap \pi(U_2)$ is not locally $\pi_*(\mu)$-negligible for every $U_1 \in \mathfrak{U}(x_1)$ and for every $U_2 \in \mathfrak{U}(x_2)$, and take $U_1 \in \mathfrak{U}(x_1)$ and $U_2 \in \mathfrak{U}(x_2)$. Then, there is $y_{U_1,U_2} \in \pi(U_1) \cap \pi(U_2) \setminus N$, and then $x_{h,U_1,U_2} \in U_h \cap \pi^{-1}(y_{U_1,U_2})$ for $h = 1, 2$. Now, $m_0(x_{1,U_1,U_2}) = m_0(x_{2,U_1,U_2})$ for every $U_1 \in \mathfrak{U}(x_1)$ and for every $U_2 \in \mathfrak{U}(x_2)$; in addition, $x_{h,U_1,U_2} \to x_h$ in X along the product filter of $\mathfrak{U}(x_1)$ and $\mathfrak{U}(x_2)$. Since m_0 is continuous, passing to the limit we see that $m_0(x_1) = m_0(x_2)$. Since μ is π-connected, this implies that m_0 is constant on $P^{-1}(y)$ for *every* $y \in \pi(X)$. The assertion follows. ☐

Here are some useful examples of connected measures.

Proposition 2.22. *Let E_1, E_2 be two finite-dimensional affine spaces, $L\colon E_1 \to E_2$ an affine mapping, C a closed convex subset of E_1 and μ a positive Radon measure on E_1 with support C. Take a μ-measurable subset X of E_1 which carries μ. Then, μ_X is $L|_X$-connected.*[7]

Proof. Notice that we may assume that the following hold:

- E_1 and E_2 are vector spaces and L is linear;
- C has non-empty interior.

Now, take $x \in C$. Since C has non-empty interior, we may find a bounded open convex subset U of C and an open convex neighbourhood V of 0 in $\ker L$ such that $U + V \subseteq C$. Take $r \in]0, 1]$ and $x, y \in C$ such that $y - x \in V$; take $R > 0$ so that $U \subseteq B(x, R)$. Then, for every $u \in U$ we have $y + r(u - x) \in B(y, Rr) \cap [y, y - x + u] \subseteq B(y, Rr) \cap C$; analogously, $x + r(U - x) \subseteq B(x, Rr) \cap C$. Since $L(x) = L(y)$, we infer that

$$L^{-1}(L(B(x, Rr) \cap C) \cap L(B(y, Rr) \cap C)) \supseteq x + r(U - x).$$

Now, $x + r(U - x)$ is a non-empty open subset of $C = \operatorname{Supp}(\mu)$, so that $\mu(x + r(U - x)) > 0$. The arbitrariness of r then implies that x and y are (μ, L)-connected.

In the same way we see that if $x, y \in \operatorname{Supp}(\mu_X) = C \cap X$ and $y - x \in V$, then x, y are $(\mu_X, L|_X)$-connected. The assertion then follows from the fact that X is a countable union of precompact sets. ☐

Before we state Corollary 2.24, we recall the definition of a convex polyhedron.

Definition 2.23. Let E be a finite-dimensional affine space and P a subset of E. Then, P is a convex polyhedron if it is the intersection of a finite number of closed half-spaces. Equivalently, P is a convex polyhedron if it is the convex envelope of a finite family of points and closed half-lines of E.

Corollary 2.24. *Let E_1, E_2 be two finite-dimensional affine spaces, C a convex polyhedron of E_1, $L\colon E_1 \to E_2$ an affine mapping which is proper on C, and μ a positive Radon measure on E_1 with support ∂C. Then, μ is L-connected.*

Proof. **1.** Consider first the case in which C is compact and has non-empty interior, $E_1 = \mathbb{R}^n$, $E_2 = \mathbb{R}^{n-1}$ and $L(x_1, \dots, x_n) = (x_1, \dots, x_{n-1})$ for every $(x_1, \dots, x_n) \in E_n$. Define $C' := L(C)$, so that C' is a compact convex polyhedron of E_2. Now, the functions

$$f_- \colon C' \ni x' \mapsto \min\{\, y \in \mathbb{R} \colon (x', y) \in C \,\}$$

and

$$f_+ \colon C' \ni x' \mapsto \max\{\, y \in \mathbb{R} \colon (x', y) \in C \,\}$$

[7]Here, we denote by μ_X the Radon measure induced by μ on X.

are well-defined; in addition, f_- is convex while f_+ is concave. Therefore, f_- and f_+ are continuous on $\overset{\circ}{C'}$ by [18, Corollary to Proposition 21 of Chapter II, § 2, No. 10]. Now, observe that $f_- \leqslant f_+$; if $f_-(x') = f_+(x')$ for some $x' \in \overset{\circ}{C'}$, then $f_- = f_+$ on C' by convexity, and this contradicts the assumption that C has non-empty interior. Therefore, $\left\{ (x', y) \colon x' \in \overset{\circ}{C'}, f_-(x') < y < f_+(x') \right\}$ is the interior of C, so that $\partial C \cap (\overset{\circ}{C'} \times \mathbb{R})$ is the union of the graphs Γ_- and Γ_+ of the restrictions of f_- and f_+ to $\overset{\circ}{C'}$. Since L induces homeomorphisms of Γ_- and Γ_+ onto $\overset{\circ}{C'}$, it follows that $(x', f_-(x'))$ and $(x', f_+(x'))$ are (μ, L)-connected for every $x' \in \overset{\circ}{C'}$.

Now, take $x' \in \partial C'$, and observe that $\{ (x', y) \colon y \in [f_-(x'), f_+(x')] \} \subseteq \partial C$. Take $y \in [f_-(x'), f_+(x')]$ and an $(n-1)$-dimensional facet F of C which contains (x', y). Observe that the support of $\chi_F \cdot \mu$ is F. Indeed, clearly $\operatorname{Supp}(\chi_F \cdot \mu) \subseteq F$. Conversely, take x in the relative interior of F. Then, every sufficiently small open neighbourhood of x intersects ∂C only on F, so that it is clear that $x \in \operatorname{Supp}(\chi_F \cdot \mu)$. Since F is the closure of its relative interior, the assertion follows. Then, Proposition 2.22 implies that (x', y) and (x', y') are (μ, L)-connected for every $y' \in \mathbb{R}$ such that $(x', y') \in F$. Since ∂C is the (finite) union of its $(n-1)$-dimensional facets, it follows that (x', y) and (x', y') are (μ, L)-connected for every $y' \in [f_-(x'), f_+(x')]$. The assertion follows in this case.

2. Now, consider the general case. Observe first that we may assume that C has non-empty interior. Take $y \in L(\partial C)$ and a closed cube Q in E_2 which contains y in its interior. Then, $C \cap L^{-1}(Q)$ is a compact polyhedron; in addition,

$$\partial C \cap L^{-1}(\overset{\circ}{Q}) = \partial[C \cap L^{-1}(Q)] \cap L^{-1}(\overset{\circ}{Q}).$$

Hence, in order to prove that any two points of $L^{-1}(y) \cap \partial C$ are (μ, L)-connected, we may assume that C is compact. Now, take $x_1, x_2 \in \partial C$ such that $x_1 \neq x_2$ and $L(x_1) = L(x_2)$. Let L' be an affine mapping defined on E_1 such that $L'(x_1) = L'(x_2)$ and such that the fibres of L' have dimension 1. Then, we may apply **1** above and deduce that x_1, x_2 are (μ, L')-connected. It is then easily seen that x_1, x_2 are also (μ, L)-connected, whence the result. □

Remark 2.25. Notice that Corollary 2.24 is false for general convex sets. Indeed, choose $E_1 = \mathbb{R}^3$, $E_2 = \mathbb{R}^2$, $L = \operatorname{pr}_{1,2}$ and

$$C_1 \coloneqq \{ (x, y, z) \in E_1 \colon 2yz \geqslant x^2, z \in [0, 1], y \geqslant 0 \}.$$

Define C as the union of C_1 and $\pi(C_1)$, where π is the reflection along the plane $\operatorname{pr}_3^{-1}(1)$. Then, ∂C is the union of

$$C_1' \coloneqq \{ (x, y, z) \in E_1 \colon 2yz = x^2, z \in [0, 1], y \geqslant 0 \}$$

and $\pi(C_1')$. Choose any continuous function $m_1 \colon C_1' \to \mathbb{C}$, and define $m \colon \partial C \to \mathbb{C}$ so that it equals m_1 on C_1' and $m_1 \circ \pi$ on $\pi(C_1')$. Then, m is clearly continuous. In addition, it is clear that C_1' intersects the fibres of L at most at one point except for $L^{-1}(0, 0)$. Since m can be chosen so that it is *not* constant on $\{ (0, 0) \} \times [0, 2]$, Proposition 2.21 shows that $\chi_{\partial C} \cdot \mathcal{H}^2$ cannot be L-connected.

We now consider some results about the disintegration of measures, in order to meet the hypotheses of Proposition 2.21.

Proposition 2.26. *Let, for $j = 1, 2$, E_j be an $\mathcal{H}^{k_j}$-measurable and countably $\mathcal{H}^{k_j}$-rectifiable subset of $\mathbb{R}^{n_j}$. Assume that $k_2 \leqslant k_1$, and let P be a locally Lipschitz mapping from E_1 into E_2. Take a positive $f \in L^1_{\mathrm{loc}}(\chi_{E_1} \cdot \mathcal{H}^{k_1})$, and assume that P is $f \cdot \mathcal{H}^{k_1}$-proper and that $f(x) \operatorname{ap} J_{k_2} P(x) \neq 0$ for $\mathcal{H}^{k_1}$-almost every $x \in E_1$.*[8]

Then, the following hold:

[8]Define $\operatorname{ap} J$ as in [39, 3.2.16]. Recall that $\operatorname{ap} J_{k_2} P(x) = \left\| \bigwedge^{k_2} T_x P \right\|$ if E_1 and P are of class C^1 in a neighbourhood of x.

1. *the mapping*
$$g\colon \mathbb{R}^{n_2} \ni y \mapsto \int_{P^{-1}(y)} \frac{f}{\operatorname{ap} J_{k_2} P} \, \mathrm{d}\mathcal{H}^{k_1-k_2}$$
is well-defined $\mathcal{H}^{k_2}$-almost everywhere and measurable; in addition,
$$P_*(f \cdot \mathcal{H}^{k_1}) = g \cdot \mathcal{H}^{k_2};$$

2. *the measure*
$$\beta_y \coloneqq \frac{1}{g(y)} \frac{f}{\operatorname{ap} J_{k_2} P} \chi_{P^{-1}(y)} \cdot \mathcal{H}^{k_1-k_2}$$
is well-defined and Radon for $P_*(f \cdot \mathcal{H}^{k_1})$-almost every $y \in \mathbb{R}^{n_2}$; in addition, (β_y) is a disintegration of $f \cdot \mathcal{H}^{k_1}$ relative to P;

3. *β_y is equivalent to $\chi_{P^{-1}(y)} \cdot \mathcal{H}^{k_1-k_2}$ for $P_*(f \cdot \mathcal{H}^{k_1})$-almost every $y \in E_2$.*

Proof. Since E_1 is $\mathcal{H}^{k_j}$-measurable and countably $\mathcal{H}^{k_j}$-rectifiable, we may find an increasing sequence $(E_{j,p})_{p\in\mathbb{N}}$ of $\mathcal{H}^{k_j}$-measurable and $\mathcal{H}^{k_j}$-rectifiable subsets of $\mathbb{R}^{n_j}$ such that $E_j \subseteq \bigcup_{p\in\mathbb{N}} E_{j,p}$ $(j = 1, 2)$. We may further assume that $E_{2,p}$ is Borel measurable for every $p \in \mathbb{N}$, and that $E_1 = \bigcup_{p\in\mathbb{N}} E_{j,p}$. Then, $E_{1,p} \cap P^{-1}(E_{2,q})$ is an $\mathcal{H}^{k_1}$-measurable and $\mathcal{H}^{k_1}$-rectifiable subset of $\mathbb{R}^{n_1}$ for every $p, q \in \mathbb{N}$. Take an $\mathcal{H}^{k_1}$-measurable function $h\colon \mathbb{R}^{n_1} \to [0, +\infty]$. Then, [39, Theorem 3.2.22] implies that for $\mathcal{H}^{k_2}$-almost every $y \in E_{2,q}$ the set $P^{-1}(y) \cap E_{1,p}$ is $\mathcal{H}^{k_1-k_2}$-measurable and $\mathcal{H}^{k_1-k_2}$-rectifiable, and that
$$\int_{E_{1,p}\cap P^{-1}(E_{2,q})} h \operatorname{ap} J_{k_2} P \, \mathrm{d}\mathcal{H}^{k_1} = \int_{E_{2,q}} \int_{P^{-1}(y)\cap E_{1,p}} h \, \mathrm{d}\mathcal{H}^{k_1-k_2} \, \mathrm{d}\mathcal{H}^{k_2}(y).$$
Passing to the limit as $p, q \to \infty$, we see that for $\mathcal{H}^{k_2}$-almost every $y \in E_2$, the set $P^{-1}(y)$ is $\mathcal{H}^{k_1-k_2}$-measurable and countably $\mathcal{H}^{k_1-k_2}$-rectifiable, that the mapping $y \mapsto \int_{P^{-1}(y)} h \, \mathrm{d}\mathcal{H}^{k_1-k_2}$ is $\mathcal{H}^{k_2}$-almost everywhere defined and measurable, and that
$$\int_{E_1} h \operatorname{ap} J_{k_2} P \, \mathrm{d}\mathcal{H}^{k_1} = \int_{\mathbb{R}^{n_2}} \int_{P^{-1}(y)} h \, \mathrm{d}\mathcal{H}^{k_1-k_2} \, \mathrm{d}\mathcal{H}^{k_2}(y).$$
Therefore, if Z is the set of zeroes of $f \operatorname{ap} J_{k_2} P$, then $\mathcal{H}^{k_1-k_2}(P^{-1}(y) \cap Z) = 0$ for $\mathcal{H}^{k_2}$-almost every $y \in \mathbb{R}^{n_2}$. Hence, g is well-defined $\mathcal{H}^{k_2}$-almost everywhere and measurable. In addition, for $\mathcal{H}^{k_2}$-almost every $y \in \mathbb{R}^{n_2}$ we have $g(y) = 0$ if and only if $\mathcal{H}^{k_1-k_2}(P^{-1}(y)) = 0$.

Now, take a positive $\varphi \in \mathcal{D}^0(\mathbb{R}^{n_2})$. Then, the preceding remarks imply that
$$\int_{E_1} (\varphi \circ P) f \, \mathrm{d}\mathcal{H}^{k_1} = \int_{\mathbb{R}^{n_2}} \varphi \, g \, \mathrm{d}\mathcal{H}^{k_2}.$$
By the arbitrariness of φ, it follows that $P_*(f \cdot \mathcal{H}^{k_1}) = g \cdot \mathcal{H}^{k_2}$; in particular, $g \in L^1_{\mathrm{loc}}(\mathcal{H}^{k_2})$.

Finally, take $\varphi \in \mathcal{D}^0(\mathbb{R}^{n_1})$. Then,
$$\int_{E_1} \varphi f \, \mathrm{d}\mathcal{H}^{k_1} = \int_{\mathbb{R}^{n_2}} g(y) \int_{E_1} \varphi \, \mathrm{d}\beta_y \, \mathrm{d}\mathcal{H}^{k_2}(y),$$
so that (β_y) is a disintegration of $f \cdot \mathcal{H}^{k_1}$ relative to P (cf. [19, Proposition 2 of Chapter V, § 3, No. 1] and [19, Theorem 1 of Chapter VI, § 3, No. 1]). □

Corollary 2.27. *Let M be an analytic manifold of dimension n and countable at infinite, endowed with a positive Radon measure μ which is equivalent to Lebesgue measure on every local chart. In addition, take $k, h \in \mathbb{N}$ and a μ-proper analytic mapping $P\colon M \to \mathbb{R}^k$ with generic rank h. Then, the following hold:*

1. *$P(M)$ is $\mathcal{H}^h$-measurable and countably $\mathcal{H}^h$-rectifiable;*

2. *$P_*(\mu)$ is equivalent to $\chi_{P(M)}\mathcal{H}^h$;*

3. $\operatorname{Supp}(P_*(\mu)) = \overline{P(M)}$*;*

4. *if* $(\beta_y)_{y\in\mathbb{R}^k}$ *is a disintegration of* μ *relative to* P, *then* $\mathrm{Supp}(\beta_y) = P^{-1}(y)$ *for* $\mathcal{H}^h$*-almost every* $y \in P(M)$.

It is worthwhile for our analysis to consider the case in which M is possibly disconnected.

Proof. Observe first that M may be embedded as a closed submanifold of class C^∞ of $\mathbb{R}^{2n+1}$ by Whitney embedding theorem (cf. [31, Theorem 5 of Chapter 1]). We may therefore assume that $\mu = f \cdot \mathcal{H}^n$ for some $f \in L^1_{\mathrm{loc}}(\chi_M \cdot \mathcal{H}^n)$. Now, [75] implies that the set where P has rank $< h$, which is $\mathcal{H}^n$-negligible by analyticity, has $\mathcal{H}^h$-negligible image under P. Since the image under P of the set where P has rank h is a countable union of analytic submanifolds of $\mathbb{R}^k$ of dimension h, we see that $P(M)$ is $\mathcal{H}^h$-measurable and countably $\mathcal{H}^h$-rectifiable. Therefore, Proposition 2.26 applies. Now, the preceding arguments show that $P^{-1}(y)$ is an analytic submanifold of dimension $n-h$ of M for $\mathcal{H}^h$-almost every $y \in P(M)$. As a consequence, $\mathrm{Supp}(\beta_y) = \mathrm{Supp}\big(\chi_{P^{-1}(y)} \cdot \mathcal{H}^{n-h}\big) = P^{-1}(y)$ for $\mathcal{H}^h$-almost every $y \in P(M)$; for the same reason, we also see that $P_*(\mu)$ is equivalent to $\chi_{P(M)} \cdot \mathcal{H}^h$. Finally, $\mathrm{Supp}(P_*(\mu)) = \overline{P(M)}$ since P is continuous and $\mathrm{Supp}(\mu) = M$. □

2.4 Composite Functions: Schwartz Functions

In this section we shall extend some results by E. Bierstone, P. Milman and G. W. Schwarz to the case of Schwartz functions by means of the techniques developed by F. Astengo, B. Di Blasio and F. Ricci. We shall take advantage of the remarkable works of E. Bierstone, P. Milman and G. W. Schwarz about the composition of smooth functions on analytic manifolds, and we shall refer to [9, 10, 11] for any unexplained definition, in particular for the notion of (Nash) subanalytic sets. As a matter of fact, in the applications we shall only need to know that any convex subanalytic set is automatically Nash subanalytic, since it is contained in an affine space of the same dimension; and that semianalytic sets are Nash subanalytic (cf. [9, Proposition 2.3]).

Our starting point is the following result (cf. [9, Theorem 0.2] and [11, Theorem 0.2.1]).

Theorem 2.28. *Let* C *be a closed subanalytic subset of* $\mathbb{R}^n$ *and let* $P\colon \mathbb{R}^n \to \mathbb{R}^m$ *be an analytic mapping. Assume that* P *is proper on* C *and that* $P(C)$ *is Nash subanalytic. Then, the canonical mapping*

$$\Phi\colon \mathcal{E}(\mathbb{R}^m) \ni \varphi \mapsto \varphi \circ P \in \mathcal{E}_{\mathbb{R}^n}(C)$$

has a closed range, and admits a continuous linear section defined on $\Phi(\mathcal{E}(\mathbb{R}^n))$.

In addition, $\psi \in \mathcal{E}_{\mathbb{R}^n}(C)$ *belongs to the image of* Φ *if and only if for every* $y \in \mathbb{R}^m$ *there is* $\varphi_y \in \mathcal{E}(\mathbb{R}^m)$ *such that* $j^\infty_x(\varphi_y \circ P) = j^\infty_x(\psi)$ *in* $J^\infty_{\mathbb{R}^n,0}(C)$ *for every* $x \in C$ *such that* $P(x) = y$.

In order to simplify the notation, we shall simply say that ψ is a formal composite of P if the second condition of the statement holds.

The following result extends Theorem 2.28 to the case of Schwartz functions. We omit the proof, since it basically consists in repeating that of [5, Theorem 6.1] with minor modifications.

Theorem 2.29. *Let* E_1, E_2 *be two homogeneous groups, and* $P\colon E_1 \to E_2$ *a polynomial mapping such that* $P(r \cdot x) = r \cdot P(x)$ *for every* $r > 0$ *and for every* $x \in E_1$. *Let* C *be a dilation-invariant subanalytic closed subset of* E_1, *and assume that* P *is proper on* C *and that* $P(C)$ *is Nash subanalytic. Then, the canonical mapping*

$$\Phi\colon \mathcal{S}(E_2) \ni \varphi \mapsto \varphi \circ P \in \mathcal{S}_{E_1}(C)$$

has a closed range, and admits a continuous linear section defined on $\Phi(\mathcal{S}(E_2))$. *In addition,* $\psi \in \mathcal{S}_{E_1}(C)$ *belongs to the image of* Φ *if and only if it is a formal composite of* P.

As a matter of fact, in our applications $E_1 = E_{\mathcal{L}_A}$, $E_2 = E_{P(\mathcal{L}_A)}$, and $C = \sigma(\mathcal{L}_A)$. Then, Theorem 2.29 gives sufficient conditions in order that some $f \in \mathcal{S}_{P(\mathcal{L}_A)}$ which has a Schwartz multiplier in $E_{\mathcal{L}_A}$ should have a Schwartz multiplier in $E_{P(\mathcal{L}_A)}$.

Notice, however, that sometimes it is convenient to take C so as to be a subset of $\sigma(\mathcal{L}_A)$ such that $P(C) = \sigma(P(\mathcal{L}_A))$, since $\sigma(\mathcal{L}_A)$ need *not* be subanalytic.

In the following result we give a simple application of Theorem 2.29.

Corollary 2.30. *Let V and W be two finite-dimensional affine spaces, C a subanalytic closed convex cone in V, and L an affine mapping of V into W which is proper on C. Take $m_1 \in \mathcal{S}(V)$, and assume that there is $m_2 \colon W \to \mathbb{C}$ such that $m_1 = m_2 \circ L$ on C. Then, there is $m_3 \in \mathcal{S}(W)$ such that $m_1 = m_3 \circ L$ on C.*

Proof. **1.** Observe first that we may assume that V, W are vector spaces, and that C has vertex 0 and generates V. Indeed, the first three assumptions and the linearity of L are satisfied if we choose suitable 'origins' in both V and W. Then, up to replacing V with the affine space generated by C (which is a vector space since $0 \in C$), also the fourth assumption is satisfied.

Observe, by the way, that $L(C)$ is subanalytic (cf. [10, Theorem 0.1 and Proposition 3.13]), hence Nash subanalytic.

2. Fix $x \in C$. Since C generates V, we may find a free family $(v_j)_{j\in J}$ in C which generates an algebraic complement V' of $\ker L$ in V. In addition, since either $x = 0$ or $x \notin \ker L$, we may assume that $x \in V'$. Let $L' \colon W \to V$ be the composite of the inverse of the restriction of L to V' with the natural immersion of V' in V. Then, L' is a linear section of L.

Define $m' \coloneqq m_1 \circ L'$, so that $m' \in \mathcal{E}(W)$. Next, define $C' \coloneqq V' \cap C$, so that C' is a closed convex cone with non-empty interior in V', since it contains the non-empty open set $\sum_{j\in J} \mathbb{R}_+^* v_j$. Take $z \in C'$ and any $y \in C \cap [x + \ker L]$. Then, $x + z = (L' \circ L)(x + z) = (L' \circ L)(y + z)$, so that $m_1 = m' \circ L$ on $y + C'$. Since m_1 is constant on the intersections of C with the translates of $\ker L$, it is also clear that the same holds on $C \cap (y + C' + \ker L)$. Now, denote by $\mathring{C}'$ the interior of C' in V'. Then, $y + \mathring{C}' + \ker L$ is an open convex set and y is adherent to $C \cap (y + \mathring{C}' + \ker L)$; therefore, the Taylor polynomials of every fixed order of m_1 and $m' \circ L$ about y must coincide on $C \cap (y + \mathring{C}' + \ker L)$, hence on V since $C \cap (y + \mathring{C}' + \ker L)$ has non-empty interior. Since this holds for every $y \in C \cap [x + \ker L]$, Theorem 2.29 implies that there is $m_3 \in \mathcal{S}(W)$ such that $m_1 = m_3 \circ L$ on C. ☐

Chapter 3

Rockland Families

In this chapter, G denotes a homogeneous group; ν_G, or simply ν, denotes a *fixed* Haar measure on G. In addition, we fix a non-empty, commutative, finite family $\mathcal{L}_A = (\mathcal{L}_\alpha)_{\alpha \in A}$ of formally self-adjoint, homogeneous, left-invariant differential operators without constant terms on G.

3.1 Admissible Families

In order that the notion of spectral multipliers relative to the family $\mathcal{L}_A$ make sense, we need to impose some further conditions, which we summarize in the following definition.

Definition 3.1. We say that the family $\mathcal{L}_A$ is admissible if $\mathcal{L}_\alpha$ is essentially self-adjoint on $\mathcal{D}(G)$ for every $\alpha \in A$ and the self-adjoint extensions of the $\mathcal{L}_\alpha$ commute as self-adjoint operators on $L^2(G)$.

If the family $\mathcal{L}_A$ is admissible, then we shall denote by $\mu_{\mathcal{L}_A}$ the spectral measure associated with the family of the self-adjoint extensions of the $\mathcal{L}_\alpha$.

Notice that we do *not* assume that each $\mathcal{L}_\alpha$ is non-zero: indeed, this requirement would create some fuss in some subsequent results, and is totally irrelevant. On the contrary, we shall usually work under an assumption which ensures that *some* $\mathcal{L}_\alpha$ is not zero.

In addition, observe that the condition which we truly need is that the operators $(\mathcal{L}_\alpha, \mathcal{D}(G))$ have commuting left-invariant homogeneous self-adjoint extensions, but we shall not need this generalization.

Definition 3.2. We denote by $E_{\mathcal{L}_A}$ the space $\mathbb{R}^A$ endowed with the dilations

$$r \cdot \lambda := (r^{\delta_\alpha} \lambda_\alpha)_{\alpha \in A}$$

for every $r > 0$ and for every $\lambda = (\lambda_\alpha) \in E_{\mathcal{L}_A}$, where δ_α is the homogeneous degree of $\mathcal{L}_\alpha$ if $\mathcal{L}_\alpha \neq 0$, and is 1 otherwise. Finally, $|\cdot|$ will denote a homogeneous norm on $E_{\mathcal{L}_A}$.

Now it makes sense to consider functions of the family $\mathcal{L}_A$. However, we are not interested in the operators $m(\mathcal{L}_A)$ as much as in their (convolution) kernels; therefore, we shall restrict ourselves to a (still wide) class of multipliers in order that the operators $m(\mathcal{L}_A)$ be defined at least on $\mathcal{D}(G)$. The following simple result proves that the operators $m(\mathcal{L}_A)$ which are defined at least on $\mathcal{D}(G)$ have a (right convolution) kernel.

Lemma 3.3. *Assume that $\mathcal{L}_A$ is admissible and let $m\colon E_{\mathcal{L}_A} \to \mathbb{C}$ be a $\mu_{\mathcal{L}_A}$-measurable function such that $\mathcal{D}(G) \subseteq \mathrm{Dom}(m(\mathcal{L}_A))$. Then, there is a unique $K \in \mathcal{D}'(G)$ such that $m(\mathcal{L}_A) \cdot \varphi = \varphi * K$ for every $\varphi \in \mathcal{D}(G)$.*

In addition, $K \in W^{-\infty,0,2}(G)$ and for every $\varphi \in W^{0,\infty,1}(G)$ we have $\varphi \in \mathrm{Dom}(m(\mathcal{L}_A))$ and $m(\mathcal{L}_A) \cdot \varphi = \varphi * K$.

Proof. Observe first that $m(\mathcal{L}_A)$ is a closed operator on $L^2(G)$; since $\mathcal{D}(G)$ embeds continuously into $L^2(G)$, the operator $m(\mathcal{L}_A)\colon \mathcal{D}(G) \to L^2(G)$ has a closed graph. Then, the closed graph theorem (cf. [18, Proposition 10 of Chapter II, § 4, No. 6]) implies that the operator $m(\mathcal{L}_A)\colon \mathcal{D}(G) \to L^2(G)$ is continuous. Since $m(\mathcal{L}_A)$ is left-invariant, Corollary 1.77 implies that it has a unique (right) convolution kernel $K \in W^{-\infty,0,2}(G)$. Now, Corollary 1.72 implies that the mapping $W^{0,\infty,1}(G) \ni \varphi \mapsto \varphi * K \in L^2(G)$ is continuous. Since $m(\mathcal{L}_A)$ is a closed operator and since $\mathcal{D}(G)$ is dense in $W^{0,\infty,1}(G)$, the last assertion follows. $\square$

The following definition provides some short-hand notation which will be very useful as the study advances. Indeed, it will allow us to avoid long and cumbersome sentences such as 'assume that m is a $\mu_{\mathcal{L}_A}$-measurable function defined on $E_{\mathcal{L}_A}$ with values in $\mathbb{C}$ such that $\mathcal{D}(G)$ is contained in the domain of $m(\mathcal{L}_A)$ and such that the kernel of the operator $m(\mathcal{L}_A)$ belongs to F,' and similar ones.

Definition 3.4. Let F be a locally convex space which is continuously embedded into $\mathcal{D}'_w(G)$. We shall denote by $\mathcal{M}(\mu_{\mathcal{L}_A}; F)$ the set of $\mu_{\mathcal{L}_A}$-measurable mappings $m\colon E_{\mathcal{L}_A} \to \mathbb{C}$ such that $\mathcal{D}(G) \subseteq \mathrm{Dom}(m(\mathcal{L}_A))$ and such that the convolution kernel K of $m(\mathcal{L}_A)$ (cf. Lemma 3.3) belongs to F. We shall denote by $\mathcal{K}_{\mathcal{L}_A}$ the mapping which to each $m \in \mathcal{M}(\mu_{\mathcal{L}_A}; \mathcal{D}'_w(G))$ associates the convolution kernel $\mathcal{K}_{\mathcal{L}_A}(m)$ of $m(\mathcal{L}_A)$. We shall endow $\mathcal{M}(\mu_{\mathcal{L}_A}; F)$ with the topology induced by F through $\mathcal{K}_{\mathcal{L}_A}$.

Let N be the set of $\mu_{\mathcal{L}_A}$-negligible functions. Since $(m + m')(\mathcal{L}_A) = m(\mathcal{L}_A)$ for every $m' \in N$, we shall also define $(m + N)(\mathcal{L}_A) := m(\mathcal{L}_A)$ and $\mathcal{K}_{\mathcal{L}_A}(m + N) := \mathcal{K}_{\mathcal{L}_A}(m)$.

Definition 3.5. Let F be a locally convex space which is continuously embedded into $\mathcal{D}'_w(G)$. We shall denote by $F_{\mathcal{L}_A}$ the space $\mathcal{K}_{\mathcal{L}_A}(\mathcal{M}(\mu_{\mathcal{L}_A}; F))$. We shall denote by $F_{\mathcal{L}_A,0}$ the space $\mathcal{K}_{\mathcal{L}_A}(\mathcal{M}(\mu_{\mathcal{L}_A}; F) \cap C(E_{\mathcal{L}_A}))$.

Definition 3.6 (Riemann-Lebesgue). We say that $\mathcal{L}_A$ satisfies property (RL) if $L^1_{\mathcal{L}_A}(G) = L^1_{\mathcal{L}_A,0}(G)$.

Now, we shall study in detail the elementary properties of the 'kernel transform' $\mathcal{K}_{\mathcal{L}_A}$. Here, 'in detail' means that we shall strive to achieve (almost) the best possible formulation of most results with respect to the weakness of the hypotheses. This will unfortunately make some statements concerning convolution less readable, since this purpose will force us to consider some quite abstract spaces which satisfy suitable conditions.

We first translate the homogeneity of the operators $\mathcal{L}_\alpha$ in terms of $\mu_{\mathcal{L}_A}$ and $\mathcal{K}_{\mathcal{L}_A}$. Then, we interpret in terms of $\mathcal{K}_{\mathcal{L}_A}$ the fact that $\overline{m}(\mathcal{L}_A)$ is the adjoint of $m(\mathcal{L}_A)$. In the results which follow, we translate the fact that $(m_1 m_2)(\mathcal{L}_A)$ is the closure of the composite operator $m_1(\mathcal{L}_A) m_2(\mathcal{L}_A)$ into results about $\mathcal{K}_{\mathcal{L}_A}$ in two different situations: when the expression $m_1(\mathcal{L}_A) \cdot \mathcal{K}_{\mathcal{L}_A}(m_2)$ makes sense; when the expression $\mathcal{K}_{\mathcal{L}_A}(m_2) * \mathcal{K}_{\mathcal{L}_A}(m_1)$ makes sense. Finally, we shall also deal with the following question: when is $m(\mathcal{L}_A) \cdot \psi$ actually equal to $\psi * \mathcal{K}_{\mathcal{L}_A}(m)$?

Proposition 3.7. *Assume that $\mathcal{L}_A$ is admissible. Then,*

$$\mathcal{K}_{\mathcal{L}_A}(m(r\,\cdot\,)) = (r\,\cdot\,)_*(\mathcal{K}_{\mathcal{L}_A}(m))$$

for every $m \in \mathcal{M}(\mu_{\mathcal{L}_A}; \mathcal{D}'_w(G))$ and for every $r > 0$.[1]

Proof. Notice first that, for every $\alpha \in A$ and for every $r > 0$,

$$\begin{aligned} r^{\delta_\alpha} \int_E \lambda_\alpha \,\mathrm{d}\mu_{\mathcal{L}_A}(\lambda) &= r^{\delta_\alpha} \mathcal{L}_\alpha \\ &= (r^{-1}\,\cdot\,) \circ \mathcal{L}_\alpha \circ (r\,\cdot\,) \\ &= (r^{-1}\,\cdot\,) \circ \int_E \lambda_\alpha \,\mathrm{d}\mu_{\mathcal{L}_A}(\lambda) \circ (r\,\cdot\,). \end{aligned}$$

Now, the mappings $m \mapsto \langle \mu_{\mathcal{L}_A}, m(r\,\cdot\,)\rangle$ and $m \mapsto (r^{-1}\,\cdot\,) \circ \langle \mu_{\mathcal{L}_A}, m\rangle \circ (r\,\cdot\,)$, defined on $\mathcal{D}^0(E_{\mathcal{L}_A})$, are spectral measures, so that the arbitrariness of α implies that they are equal. Next, take $m \in \mathcal{M}(\mu_{\mathcal{L}_A}; \mathcal{D}'_w(G))$. Then, for every $\psi \in \mathcal{D}(G)$,

$$\begin{aligned} \psi * \mathcal{K}_{\mathcal{L}_A}(m(r\,\cdot\,)) &= \int_E m(r \cdot \lambda) \,\mathrm{d}\mu_{\mathcal{L}_A}(\lambda) \cdot \psi \\ &= [(\psi(r\,\cdot\,)) * \mathcal{K}_{\mathcal{L}_A}(m)](r^{-1}\,\cdot\,) \\ &= \psi * [(r\,\cdot\,)_*(\mathcal{K}_{\mathcal{L}_A}(m))], \end{aligned}$$

[1] Notice that, if $\mathcal{K}_{\mathcal{L}_A}(m) \in L^1_{\mathrm{loc}}(G) \cdot \nu_G$ and we identify $\mathcal{K}_{\mathcal{L}_A}(m)$ with its density, then this means that

$$\mathcal{K}_{\mathcal{L}_A}(m(r\,\cdot\,)) = r^{-Q} \mathcal{K}_{\mathcal{L}_A}(m)(r^{-1}\,\cdot\,)$$

for every $r > 0$.

whence $\mathcal{K}_{\mathcal{L}_A}(m(r\,\cdot\,)) = (r\,\cdot\,)_*(\mathcal{K}_{\mathcal{L}_A}(m))$ by the arbitrariness of ψ. □

Proposition 3.8. *Assume that $\mathcal{L}_A$ is admissible and take $m \in \mathcal{M}(\mu_{\mathcal{L}_A}; \mathcal{D}'_w(G))$. Then,*

$$\mathcal{K}_{\mathcal{L}_A}(m)^* = \mathcal{K}_{\mathcal{L}_A}(\overline{m}).$$

Proof. For every $\varphi, \psi \in \mathcal{D}(G)$,

$$\begin{aligned}\langle \varphi | \psi * \mathcal{K}_{\mathcal{L}_A}(m)^* \rangle &= \langle \varphi * \mathcal{K}_{\mathcal{L}_A}(m) | \psi \rangle \\ &= \langle m(\mathcal{L}_A) \cdot \varphi | \psi \rangle \\ &= \langle \varphi | \overline{m}(\mathcal{L}_A) \cdot \psi \rangle \\ &= \langle \varphi | \psi * \mathcal{K}_{\mathcal{L}_A}(\overline{m}) \rangle,\end{aligned}$$

whence $\mathcal{K}_{\mathcal{L}_A}(m)^* = \mathcal{K}_{\mathcal{L}_A}(\overline{m})$ by the arbitrariness of φ and ψ. □

Proposition 3.9. *Assume that $\mathcal{L}_A$ is admissible, and take $m_1 \in \mathcal{M}(\mu_{\mathcal{L}_A}; \mathcal{D}'_w(G))$ and $m_2 \in \mathcal{M}(\mu_{\mathcal{L}_A}; \mathrm{Dom}(m_1(\mathcal{L}_A)))$. Then, $m_1 m_2 \in \mathcal{M}(\mu_{\mathcal{L}_A}; \mathcal{D}'_w(G))$ and*

$$\mathcal{K}_{\mathcal{L}_A}(m_1 m_2) = m_1(\mathcal{L}_A) \cdot \mathcal{K}_{\mathcal{L}_A}(m_2).$$

Proof. Assume first that $m_1 \in \mathcal{L}^\infty(\mu_{\mathcal{L}_A})$, $m_2 \in \mathcal{M}(\mu_{\mathcal{L}_A}; L^2(G))$, and $\varphi \in \mathcal{D}(G)$. Then,

$$\begin{aligned}(m_1 m_2)(\mathcal{L}_A) \cdot \varphi &= m_1(\mathcal{L}_A) \cdot m_2(\mathcal{L}_A) \cdot \varphi \\ &= m_1(\mathcal{L}_A) \cdot \int_G \varphi(g) L_g \mathcal{K}_{\mathcal{L}_A}(m_2)\,\mathrm{d}g \\ &= \int_G \varphi(g) L_g m_1(\mathcal{L}_A) \cdot \mathcal{K}_{\mathcal{L}_A}(m_2)\,\mathrm{d}g \\ &= \varphi * [m_1(\mathcal{L}_A) \cdot \mathcal{K}_{\mathcal{L}_A}(m_2)],\end{aligned}$$

where the first equality follows from the fact that $\varphi \in \mathrm{Dom}(m_2(\mathcal{L}_A))$ and $m_2(\mathcal{L}_A) \cdot \varphi \in L^2(G) = \mathrm{Dom}(m_1(\mathcal{L}_A))$, while the third equality follows from the fact that $m_1(\mathcal{L}_A)$ is left-invariant and continuous on $L^2(G)$. The arbitrariness of φ then implies that $\mathcal{K}_{\mathcal{L}_A}(m_1 m_2) = m_1(\mathcal{L}_A) \cdot \mathcal{K}_{\mathcal{L}_A}(m_2)$.

Now, consider the general case. For every $k \in \mathbb{N}$, let E_k be the set of $\lambda \in E_{\mathcal{L}_A}$ such that $|m_1(\lambda)| \leqslant k$ and $|m_2(\lambda)| \leqslant k$. Then,

$$\lim_{k \to \infty} \mathcal{K}_{\mathcal{L}_A}(m_1 m_2 \chi_{E_k}) = \lim_{k \to \infty} (m_1 \chi_{E_k})(\mathcal{L}_A) \cdot \mathcal{K}_{\mathcal{L}_A}(m_2) = m_1(\mathcal{L}_A) \cdot \mathcal{K}_{\mathcal{L}_A}(m_2)$$

in $L^2(G)$, by spectral theory and the above. Now, take $\varphi \in \mathcal{D}(G)$; then $\chi_{E_k}(\mathcal{L}_A) \cdot \varphi \in \mathrm{Dom}((m_1 m_2)(\mathcal{L}_A))$ and converges to φ in $L^2(G)$. In addition,

$$(m_1 m_2)(\mathcal{L}_A) \cdot \chi_{E_k}(\mathcal{L}_A) \cdot \varphi = (m_1 m_2 \chi_{E_k})(\mathcal{L}_A) \cdot \varphi = \varphi * \mathcal{K}_{\mathcal{L}_A}(m_1 m_2 \chi_{E_k})$$

converges to $\varphi * (m_1(\mathcal{L}_A) \cdot \mathcal{K}_{\mathcal{L}_A}(m_2))$ in $L^2(G)$. Since $(m_1 m_2)(\mathcal{L}_A)$ is a closed operator, $\varphi \in \mathrm{Dom}((m_1 m_2)(\mathcal{L}_A))$ and $(m_1 m_2)(\mathcal{L}_A) \cdot \varphi = \varphi * (m_1(\mathcal{L}_A) \cdot \mathcal{K}_{\mathcal{L}_A}(m_2))$. The assertion follows by the arbitrariness of φ. □

Proposition 3.10. *Let F_1, F_2 be two vector subspaces of $\mathcal{D}'(G)$, and endow F_1, F_2 with two Hausdorff locally convex topologies such that the inclusions*

$$F_1 \subseteq L^2(G) \qquad \text{and} \qquad F_2 \subseteq \mathcal{D}'_w(G)$$

are well-defined and continuous; in addition, assume that $\mathcal{D}(G) \cap F_1$ is dense in F_1. Take $m \in \mathcal{M}(\mu_{\mathcal{L}_A}; F_2)$ and assume that one of the following conditions hold:

1. *the bilinear mapping $*\colon F_1 \times F_2 \to L^2(G)$ is well-defined and separately continuous;*

2. *F_1 is $\mathrm{Dom}(m(\mathcal{L}_A))$ with a coarser topology, and the bilinear mapping $*\colon F_1 \times F_2 \to \mathcal{D}'_w(G)$ is well-defined and separately continuous.*

Then, $F_1 \subseteq \mathrm{Dom}(m(\mathcal{L}_A))$ and, for every $\psi \in F_1$,

$$m(\mathcal{L}_A) \cdot \psi = \psi * \mathcal{K}_{\mathcal{L}_A}(m).$$

Proof. **1.** Take $\psi \in F_1$. Let (ψ_k) be a sequence of elements of $\mathcal{D}(G) \cap F_1$ which converges to ψ in F_1; then, (ψ_k) converges to ψ in $L^2(G)$ and $m(\mathcal{L}_A) \cdot \psi_k = \psi_k * \mathcal{K}_{\mathcal{L}_A}(m)$ converges to $\psi * \mathcal{K}_{\mathcal{L}_A}(m)$ in $L^2(G)$. Since $m(\mathcal{L}_A)$ is a closed operator, this proves that $\psi \in \mathrm{Dom}(m(\mathcal{L}_A))$ and that $m(\mathcal{L}_A) \cdot \psi = \psi * \mathcal{K}_{\mathcal{L}_A}(m)$.

2. Take $\psi \in F_1$. For every $k \in \mathbb{N}$, let E_k be the set of $\lambda \in E_{\mathcal{L}_A}$ such that $|m(\lambda)| \leqslant k$. Notice that $\chi_{E_k}(\mathcal{L}_A) \cdot \psi$ converges to ψ in $\mathrm{Dom}(m(\mathcal{L}_A))$ by spectral theory, hence in F_1. Take, for every $k \in \mathbb{N}$, a sequence $(\psi_{k,j})_j$ of elements of $\mathcal{D}(G)$ which converges to $\chi_{E_k}(\mathcal{L}_A) \cdot \psi$ in F_1. Then,

$$\begin{aligned} m(\mathcal{L}_A) \cdot \psi &= \lim_{k \to \infty} (m\chi_{E_k})(\mathcal{L}_A) \cdot \psi \\ &= \lim_{k \to \infty} m(\mathcal{L}_A) \cdot \chi_{E_k}(\mathcal{L}_A) \cdot \psi \\ &= \lim_{k \to \infty} \lim_{j \to \infty} m(\mathcal{L}_A) \cdot \psi_{k,j} \\ &= \lim_{k \to \infty} \lim_{j \to \infty} \psi_{k,j} * \mathcal{K}_{\mathcal{L}_A}(m) \\ &= \lim_{k \to \infty} (\chi_{E_k}(\mathcal{L}_A) \cdot \psi) * \mathcal{K}_{\mathcal{L}_A}(m) \\ &= \psi * \mathcal{K}_{\mathcal{L}_A}(m), \end{aligned}$$

where the limits in the first four equalities are in $L^2(G)$, while the other ones are in $\mathcal{D}'_w(G)$. The assertion follows by the arbitrariness of ψ. □

Proposition 3.11. *Let F_1, F_2, F_3 be three vector subspaces of $\mathcal{D}'(G)$, and endow F_1, F_2, F_3 with three Hausdorff locally convex topologies such that the inclusions*

$$F_1, F_2 \subseteq \mathcal{D}'_w(G) \qquad \text{and} \qquad F_3 \subseteq L^2(G)$$

are well-defined and continuous. Assume that $\mathcal{E}'(G) \cap F_1$ is dense in F_1, and that the following bilinear mappings

$$\begin{aligned} &*\colon F_1 \times F_2 \to \mathcal{D}'_w(G) \\ &*\colon \mathcal{D}(G) \times F_1 \to F_3 \\ &*\colon F_3 \times F_2 \to L^2(G) \end{aligned}$$

are well-defined and separately continuous. Now, take $m_1 \in \mathcal{M}(\mu_{\mathcal{L}_A}; F_1)$ and $m_2 \in \mathcal{M}(\mu_{\mathcal{L}_A}; F_2)$. Then, $m_1 m_2 \in \mathcal{M}(\mu_{\mathcal{L}_A}; \mathcal{D}'_w(G))$ and

$$\mathcal{K}_{\mathcal{L}_A}(m_1 m_2) = \mathcal{K}_{\mathcal{L}_A}(m_1) * \mathcal{K}_{\mathcal{L}_A}(m_2).$$

Proof. Take $\varphi \in \mathcal{D}(G)$. By **1** of Proposition 3.10, we see that $m_1(\mathcal{L}_A) \cdot \varphi = \varphi * \mathcal{K}_{\mathcal{L}_A}(m_1) \in \mathrm{Dom}(m_2(\mathcal{L}_A))$, and that

$$(m_1 m_2)(\mathcal{L}_A) \cdot \varphi = m_2(\mathcal{L}_A) \cdot m_1(\mathcal{L}_A) \cdot \varphi = (\varphi * \mathcal{K}_{\mathcal{L}_A}(m_1)) * \mathcal{K}_{\mathcal{L}_A}(m_2).$$

Now, the trilinear mapping

$$\mathcal{D}(G) \times F_1 \times F_2 \ni (T_1, T_2, T_3) \mapsto (T_1 * T_2) * T_3 - T_1 * (T_2 * T_3) \in \mathcal{D}'_w(G)$$

is well-defined and separately continuous. In addition, it vanishes on $\mathcal{D}(G) \times (\mathcal{E}'(G) \cap F_1) \times F_2$, so that it vanishes identically. Thus,

$$(m_1 m_2)(\mathcal{L}_A) \cdot \varphi = \varphi * (\mathcal{K}_{\mathcal{L}_A}(m_1) * \mathcal{K}_{\mathcal{L}_A}(m_2)),$$

so that $m_1 m_2 \in \mathcal{M}(\mu_{\mathcal{L}_A}; \mathcal{D}'_w(G))$ and $\mathcal{K}_{\mathcal{L}_A}(m_1 m_2) = \mathcal{K}_{\mathcal{L}_A}(m_1) * \mathcal{K}_{\mathcal{L}_A}(m_2)$ by the arbitrariness of φ. □

We state the following easy corollaries to show the usefulness and flexibility of Proposition 3.11.

Corollary 3.12. *Take $m_1, m_3 \in \mathcal{M}(\mu_{\mathcal{L}_A}; W^{-\infty,\infty,1}(G))$ and $m_2 \in \mathcal{M}(\mu_{\mathcal{L}_A}; \mathcal{D}'_w(G))$.*

Then, $m_1 m_2 \overline{m_3} \in \mathcal{M}(\mu_{\mathcal{L}_A}; \mathcal{D}'_w(G))$ and

$$\mathcal{K}_{\mathcal{L}_A}(m_1 m_2 \overline{m_3}) = \mathcal{K}_{\mathcal{L}_A}(m_1) * \mathcal{K}_{\mathcal{L}_A}(m_2) * \mathcal{K}_{\mathcal{L}_A}(m_3)^*.$$

Proof. Observe first that $\mathcal{K}_{\mathcal{L}_A}(m_2) \in W^{-\infty,0,2}(G)$ by Lemma 3.3. Therefore, if $m_3 = \chi_{E_{\mathcal{L}_A}}$ the assertion follows from Proposition 3.11, applied with $F_1 = W^{-\infty,\infty,1}(G)$, $F_2 = W^{-\infty,0,2}(G)$, and $F_3 = W^{0,\infty,1}(G)$. The general case follows from the preceding one by means of Proposition 3.8. □

As a particular case, we have the following result.

Corollary 3.13. *Take $m \in \mathcal{M}(\mu_{\mathcal{L}_A}; \mathcal{D}'_w(G))$ and $P \in \mathbb{C}[A]$. Then, $mP \in \mathcal{M}(\mu_{\mathcal{L}_A}; \mathcal{D}'_w(G))$ and*[2]

$$\mathcal{K}_{\mathcal{L}_A}(mP) = P(\mathcal{L}_A)\mathcal{K}_{\mathcal{L}_A}(m) = P\big(\mathcal{L}_A^R\big)\mathcal{K}_{\mathcal{L}_A}(m).$$

Corollary 3.14. *Assume that $W^{\infty,\infty,1}_{\mathcal{L}_A}(G) = W^{\infty,\infty,1}_{\mathcal{L}_A,0}(G)$ and that there is $f \in \mathcal{S}_{\mathcal{L}_A}(G)$ such that $\int_G f \,\mathrm{d}\nu_G = 1$. Then, $W^{-\infty,-\infty,1}_{\mathcal{L}_A}(G) = W^{-\infty,-\infty,1}_{\mathcal{L}_A,0}(G)$.*

In particular, if there is $f \in \mathcal{S}_{\mathcal{L}_A}(G)$ such that $\int_G f \,\mathrm{d}\nu_G = 1$, then property (RL) can be tested on kernels in $W^{\infty,\infty,1}_{\mathcal{L}_A}(G)$, and then holds for kernels in $W^{-\infty,-\infty,1}_{\mathcal{L}_A}(G)$.

Proof. Take a bounded continuous function τ such that $f = \mathcal{K}_{\mathcal{L}_A}(\tau)$; define $\tau_j := \tau(2^{-j}\,\cdot)$ for every $j \in \mathbb{N}$, and take $m \in \mathcal{M}(\mu_{\mathcal{L}_A}; W^{-\infty,-\infty,1}(G))$. Then, Corollary 3.12 implies that

$$\mathcal{K}_{\mathcal{L}_A}\Big(m|\tau_j|^2\Big) = \mathcal{K}_{\mathcal{L}_A}(\tau_j) * \mathcal{K}_{\mathcal{L}_A}(m) * \mathcal{K}_{\mathcal{L}_A}(\tau_j)^*.$$

Therefore, $\mathcal{K}_{\mathcal{L}_A}\Big(m|\tau_j|^2\Big) \in W^{\infty,\infty}_{\mathcal{L}_A}(G)$, so that $m|\tau_j|^2$ equals $\mu_{\mathcal{L}_A}$-almost everywhere a continuous function. Now, by means of Propositions 1.73 and 3.7, and Corollary 1.80, we see that $\mathcal{K}_{\mathcal{L}_A}\Big(m|\tau_j|^2\Big)$ converges to $\mathcal{K}_{\mathcal{L}_A}(m)$ in $W^{-\infty,-\infty,1}(G)$, while clearly $m|\tau_j|^2$ converges locally uniformly to $\tau(0)m$. By means of Lemma 3.17 below, we see that $\tau(0) = 1$, so that m equals $\mu_{\mathcal{L}_A}$-almost everywhere a continuous function. □

In the following result we give conditions under which the space $\mathcal{M}(\mu_{\mathcal{L}_A}; F)$ is complete. Notice that the condition that F is continuously embedded into $W^{-\infty,0,2}(G)$ is rather natural in view of Lemma 3.3.

Proposition 3.15. *Let F be a Hausdorff locally convex space which is continuously embedded into $W^{-\infty,0,2}(G)$. Then, $F_{\mathcal{L}_A}$ is closed in F.*

In particular, the space $\mathcal{M}(\mu_{\mathcal{L}_A}; F)$ is complete (resp. semi-complete, quasi-complete) if F is complete (resp. semi-complete, quasi-complete).

In particular, this result applies to $F = L^p(G)$ for $p \in [1,2]$, $\mathcal{M}^1(G)$, $\mathcal{E}'^0(G)$, $\mathcal{D}(G)$, $\mathcal{S}(G)$, and many other spaces.

Proof. Take K_0 in the closure of $F_{\mathcal{L}_A}$ in F. Then, there is a filter $\mathfrak{F}$ on $\mathcal{M}(\mu_{\mathcal{L}_A}; F)$ such that $\mathcal{K}_{\mathcal{L}_A}(\mathfrak{F})$ converges to K_0 in F. Therefore, for every $\varphi \in \mathcal{D}(G)$ we have

$$\lim_{m,\mathfrak{F}-\mathfrak{F}} \int_{E_{\mathcal{L}_A}} |m|^2 \,\mathrm{d}\langle \mu_{\mathcal{L}_A} \cdot \varphi | \varphi \rangle = \lim_{m,\mathfrak{F}-\mathfrak{F}} \|\varphi * \mathcal{K}_{\mathcal{L}_A}(m)\|_2^2 = 0,$$

so that $\mathfrak{F}$ is a Cauchy filter in $\mathcal{L}^2(\langle \mu_{\mathcal{L}_A} \cdot \varphi | \varphi \rangle)$ for every $\varphi \in \mathcal{D}(G)$. Let S be a finite subset of $\mathcal{D}(G)$, and observe that $\mathfrak{F}$ converges to some universally measurable function m_S in $\mathcal{L}^2\Big(\sum_{\varphi \in S} \langle \mu_{\mathcal{L}_A} \cdot \varphi | \varphi \rangle\Big)$. Now, let (S_j) be an increasing sequence of finite subsets of $\mathcal{D}(G)$ such that $D := \bigcup_{j \in \mathbb{N}} S_j$ is dense in $\mathcal{D}(G)$. Then, $m_{S_{j_1}} = m_{S_{j_2}}$ $\langle \mu_{\mathcal{L}_A} \cdot \varphi | \varphi \rangle$-almost everywhere for every $j_1, j_2 \in \mathbb{N}$ such that $j_1 \leqslant j_2$ and for every $\varphi \in S_{j_1}$. Therefore, there is a universally measurable subset N of $E_{\mathcal{L}_A}$ such that N is $\langle \mu_{\mathcal{L}_A} \cdot \varphi | \varphi \rangle$-negligible for every $\varphi \in D$, and such that the sequence $m_{S_j}(x)$ is eventually constant for every $x \notin N$. Now, take $(\varepsilon_\varphi) \in \ell^1(D)$ so that $\varepsilon_\varphi > 0$ for every $\varphi \in D$, and define $\mu_D := \sum_{\varphi \in D} \varepsilon_\varphi \langle \mu_{\mathcal{L}_A} \cdot \varphi | \varphi \rangle$. Then, μ_D is a positive bounded Radon measure such that $\mu_D(N) = 0$ by [19, Proposition 1 of Chapter V, § 2, No. 2]. If we define

$$m_0(x) := \begin{cases} \lim\limits_{j \to \infty} m_{S_j}(x) & \text{if } x \in E_{\mathcal{L}_A} \setminus N \\ 0 & \text{if } x \in N, \end{cases}$$

[2] Here we denote by $\mathcal{L}_A^R$ the family of right-invariant differential operators which corresponds to $\mathcal{L}_A$.

then m_0 is universally measurable and $\mathfrak{F}$ converges to m_0 in $\mathcal{L}^2(\langle \mu_{\mathcal{L}_A} \cdot \varphi | \varphi \rangle)$ for every $\varphi \in D$. Therefore, $\mathfrak{F}(\mathcal{L}_A) \cdot \varphi$ converges to $m_0(\mathcal{L}_A) \cdot \varphi$ in $L^2(G)$ for every $\varphi \in D$, so that $m_0(\mathcal{L}_A) \cdot \varphi = \varphi * K_0$ for every $\varphi \in D$. Now, observe that the operator $\mathcal{D}(G) \ni \varphi \mapsto \varphi * K_0 \in L^2(G)$ is continuous, and that D is dense in $\mathcal{D}(G)$. Since the operator $m_0(\mathcal{L}_A)$ is closed, we have $\mathcal{D}(G) \subseteq \mathrm{dom}(m_0(\mathcal{L}_A))$ and $m_0(\mathcal{L}_A) \cdot \varphi = \varphi * K_0$ for every $\varphi \in \mathcal{D}(G)$. Therefore, $\mathcal{K}_{\mathcal{L}_A}(m_0) = K_0$, whence the result. □

Proposition 3.16. *Let F be a Hausdorff locally convex space which is continuously embedded into $\mathcal{D}'_w(G)$, and assume that the mapping $*\colon L^2(G) \times F \to L^2(G)$ is well-defined and continuous. Then, $F_{\mathcal{L}_A,0}$ is closed in F.*

Proof. For every $T \in F_{\mathcal{L}_A,0}$, denote by m_T a bounded continuous element of $\mathcal{M}(\mu_{\mathcal{L}_A}; F)$ such that $T = \mathcal{K}_{\mathcal{L}_A}(m_T)$; observe that the mapping $F_{\mathcal{L}_A,0} \ni T \mapsto m_T \in C_b(\sigma(\mathcal{L}_A))$ is linear and continuous. Let $\mathfrak{F}$ be a filter on $F_{\mathcal{L}_A,0}$ which converges to some T_0 in F, so that $m_{\mathfrak{F}}$ is a Cauchy filter on $C_b(\sigma(\mathcal{L}_A))$. Hence, $m_{\mathfrak{F}}$ converges uniformly to some m_0 on $\sigma(\mathcal{L}_A)$, so that $\mathfrak{F}$ converges to $\mathcal{K}_{\mathcal{L}_A}(m_0)$ in $\mathcal{D}'(G)$ by Theorem 1.62. Hence, $T = \mathcal{K}_{\mathcal{L}_A}(m_0)$ and the assertion follows. □

Lemma 3.17. *Let $\mathfrak{F}$ be a bounded filter on $\mathcal{L}^\infty(\mu_{\mathcal{L}_A})$ which converges in $\mu_{\mathcal{L}_A}$-measure to some m_0.[3] Then, $\mathcal{K}_{\mathcal{L}_A}(\mathfrak{F})$ converges to $\mathcal{K}_{\mathcal{L}_A}(m_0)$ in $W^{-\infty,0,2}(G)$.*

Proof. Notice first that $m_0 \in \mathcal{L}^\infty(\mu_{\mathcal{L}_A})$, so that $\mathcal{K}_{\mathcal{L}_A}(m_0)$ is defined and belongs to $W^{-\infty,0,2}(G)$ by Lemma 3.3; hence, we may assume that $m_0 = 0$. Take $\varphi \in \mathcal{D}(G)$. Then, [19, Proposition 21 of Chapter IV § 5, No. 11] implies that

$$\lim_{m,\mathfrak{F}} \|\varphi * \mathcal{K}_{\mathcal{L}_A}(m)\|_2^2 = \lim_{m,\mathfrak{F}} \int_{E_{\mathcal{L}_A}} |m|^2 \, \mathrm{d}\langle \mu_{\mathcal{L}_A} \cdot \varphi | \varphi \rangle = 0.$$

The assertion follows from Theorem 1.76. □

3.2 Rockland Families

In this section, we introduce a class of admissible families in which we are particularly interested. They are characterized by the following theorem, which enriches [58, Proposition 3.6.3].

Theorem 3.18. *Let $\mathcal{L}_A$ be a (not necessarily commutative) non-empty finite family of formally self-adjoint, homogeneous, left-invariant differential operators without constant terms on G. Then, the following conditions are equivalent:*

1. *$\mathcal{L}_A$ is jointly hypoelliptic;*
2. *there is a constant $C > 0$ such that, for every $j = 1, \dots, n$ and for every $\varphi \in C_b(G)$ such that $\mathcal{L}_A \varphi = 0$,*
$$|(X_j \varphi)(e)| \leqslant C \|\varphi\|_\infty;$$
3. *for every continuous non-trivial irreducible unitary representation π of G in a hilbertian space H, the family $\mathrm{d}\pi(\mathcal{L}_A)$ is jointly injective on $C^\infty(\pi)$;*
4. *the (non-unital) algebra generated by $\mathcal{L}_A$ contains a Rockland operator, possibly with respect to a different family of dilations on G;*
5. *the (non-unital) algebra generated by $\mathcal{L}_A$ contains a hypoelliptic operator.*

Assume, in addition, that the elements of $\mathcal{L}_A$ commute. Then, the preceding conditions are equivalent to the following ones:

6. *$\mathcal{L}_A$ is admissible and $\mathcal{K}_{\mathcal{L}_A}$ maps $\mathcal{S}(E_{\mathcal{L}_A})$ into $\mathcal{S}(G)$.*
7. *$\mathcal{L}_A$ is admissible and $\mathcal{K}_{\mathcal{L}_A}$ maps $\mathcal{S}(E_{\mathcal{L}_A})$ into $W^{0,\infty,1}(G)$.*

The proofs of the implications **1** $\implies$ **2** and **2** $\implies$ **3** are an adaptation of the proof of [7, Theorem 1].

[3]This means that $\mathfrak{F}$ converges in $\langle \mu_{\mathcal{L}_A} \cdot \varphi_1 | \varphi_2 \rangle$-measure to m_0 for every $\varphi_1, \varphi_2 \in L^2(G)$.

Proof. **1** $\implies$ **2.** Define V as the kernel of $\mathcal{L}_A$ in $C_b(G)$. Then, V is closed in $C_b(G)$, so that it is a Banach space. In addition, the inclusion $V \subseteq \mathcal{E}(G)$ has a closed graph, so that it is continuous. Therefore, there is a constant $C > 0$ such that, for every $\varphi \in V$ and for every $j = 1, \dots n$,

$$|(X_j\varphi)(e)| \leqslant C\|\varphi\|_\infty.$$

2 $\implies$ **3.** Let π be a continuous non-trivial irreducible unitary representation of G in a hilbertian space H; take $v \in C^\infty(\pi)$ and assume that $\mathrm{d}\pi(\mathcal{L}_A)\cdot v = 0$. Define $\varphi\colon g \mapsto \langle \pi(g)\cdot v | v\rangle$, so that $\varphi \in C^\infty(G) \cap C_b(G)$ and

$$(\mathcal{L}_A\varphi)(g) = \langle \mathrm{d}\pi(\mathcal{L}_A)\cdot \pi(g)\cdot v | v\rangle = \langle \pi(g)\cdot v | \mathrm{d}\pi(\mathcal{L}_A)\cdot v\rangle = 0$$

for every $g \in G$. Since $\mathcal{L}_A[(L_g\varphi)(r\,\cdot)] = 0$ for every $g \in G$ and for every $r > 0$,

$$r^{\mathrm{d}_j}|(X_j\varphi)(g)| = |X_j[(L_g\varphi)(r\,\cdot)](e)| \leqslant C\|(L_g\varphi)(r\,\cdot)\|_\infty = C\|\varphi\|_\infty$$

for every $j = 1, \dots, n$. By passing to the limit for $r \to +\infty$, we infer that $X_j\varphi = 0$ for $j = 1, \dots, n$, so that φ is constant. In particular,

$$\langle \pi(g)\cdot v | v\rangle = \langle \pi(e)\cdot v | v\rangle = \|v\|^2$$

for every $g \in G$; since $\|\pi(g)\cdot v\| = \|v\|$, this implies that $\pi(g)\cdot v = v$ for every $g \in G$. Since π is irreducible and non-trivial, this proves that $v = 0$.

3 $\implies$ **4.** This is the implication $(ii) \implies (i)$ of [58, Proposition 3.6.3].

4 $\implies$ **5.** This is trivial.

5 $\implies$ **1.** Take an open subset V of G and $T \in \mathcal{D}'(T)$ such that $\mathcal{L}_A T$ is C^∞ on V. Take $P \in \mathbb{C}[A]$ such that $P(0) = 0$ and $P(\mathcal{L}_A)$ is hypoelliptic. Then, it is clear that $P(\mathcal{L}_A)T$ is C^∞ on V, so that T is C^∞ on V.

Now, assume that the $\mathcal{L}_\alpha$ commute as differential operators.

4 $\implies$ **6.** This follows from [58, Propositions 1.4.4, 3.1.2, and 4.2.1]; indeed, if $P \in \mathbb{C}[A]$ is such that $P(\mathcal{L}_A)$ is Rockland with respect to a suitable family of dilations on G, then $(\overline{P}P)(\mathcal{L}_A)$ is a positive Rockland operator thanks to the characterization of Rockland operators provided in [47].

6 $\implies$ **7.** This is trivial. □

Before we prove the implication **7** $\implies$ **3**, we need an analogue of [58, Proposition 3.2.4], which is interesting in its own rights.

Theorem 3.19. *Assume that $\mathcal{L}_A$ is an admissible family, and that $\mathcal{K}_{\mathcal{L}_A}$ maps $\mathcal{S}(E_{\mathcal{L}_A})$ into $W^{0,\infty,1}(G)$. Let π be a continuous unitary representation of G in a hilbertian space H. Then, the following hold:*

- $\mathrm{d}\pi(\mathcal{L}_A)$ *is a commuting family of self-adjoint operators on H;*
- $\sigma(\mathrm{d}\pi(\mathcal{L}_A)) \subseteq \sigma(\mathcal{L}_A)$*;*
- *if m is an element of $\mathcal{M}(\mu_{\mathcal{L}_A}; W^{-\infty,0,1}(G))$ which is continuous on an open subset U of $E_{\mathcal{L}_A}$ which carries $\mu_{\mathrm{d}\pi(\mathcal{L}_A)}$, then*

$$\overline{\pi^*(\mathcal{K}_{\mathcal{L}_A}(m))} = m(\mathrm{d}\pi(\mathcal{L}_A)).$$

Proof. **1.** For what concerns the first assertion, it suffices to repeat the proof of [66, Corollary 2.4] with minor modifications.

2. Now, let us prove that $\mathcal{K}_{\mathcal{L}_A}$ maps $\mathcal{S}(E_{\mathcal{L}_A})$ continuously into $W^{\infty,\infty,1}(G)$. Observe first that Proposition 3.8 and Corollary 3.12 imply that

$$\mathcal{K}_{\mathcal{L}_A}(m_1 m_2) = \mathcal{K}_{\mathcal{L}_A}(m_1) * \mathcal{K}_{\mathcal{L}_A}(m_2) = \mathcal{K}_{\mathcal{L}_A}(\overline{m_1})^* * \mathcal{K}_{\mathcal{L}_A}(m_2) \in W^{\infty,\infty,1}(G)$$

for every $m_1, m_2 \in \mathcal{S}(E_{\mathcal{L}_A})$. Now, apply the Fourier transform to Theorem 1.61, applied with $F = \mathcal{S}(E_{\mathcal{L}_A})$. Then, every element of $\mathcal{S}(E_{\mathcal{L}_A})$ is a finite sum of products of elements of $\mathcal{S}(E_{\mathcal{L}_A})$, so that $\mathcal{K}_{\mathcal{L}_A}$ maps $\mathcal{S}(E_{\mathcal{L}_A})$ into $W^{\infty,\infty,1}(G)$; continuity follows from the closed graph theorem and Lemma 3.17.

3. Now, take $m \in \mathcal{D}(E_{\mathcal{L}_A})$ and $\tau \in \mathcal{D}(G)$ so that $\chi_{B(0,1)} \leqslant \tau \leqslant \chi_{B(0,2)}$; define $\tau_k \coloneqq \tau(2^{-k}\,\cdot\,)$ for every $k \in \mathbb{N}$. Then, it is easily seen that for every $k \in \mathbb{N}$ there is a sequence $(P_{k,h})_h$ of polynomials on $E_{\mathcal{L}_A}$ such that the sequence $(\tau_k P_{k,h})_h$ converges to $\tau_k m$ in $\mathcal{D}(E_{\mathcal{L}_A})$. Take $v \in H$. Then,

$$
\begin{aligned}
m(\mathrm{d}\pi(\mathcal{L}_A)) \cdot v &= \lim_{k\to\infty} (\tau_k m)(\mathrm{d}\pi(\mathcal{L}_A)) \cdot \pi^*(\mathcal{K}_{\mathcal{L}_A}(\tau_k)) \cdot v \\
&= \lim_{k\to\infty} \lim_{h\to\infty} (\tau_k P_{k,h})(\mathrm{d}\pi(\mathcal{L}_A)) \cdot \pi^*(\mathcal{K}_{\mathcal{L}_A}(\tau_k)) \cdot v \\
&= \lim_{k\to\infty} \lim_{h\to\infty} \tau_k(\mathrm{d}\pi(\mathcal{L}_A)) \cdot \pi^*(P_{k,h}(\mathcal{L}_A) \cdot \mathcal{K}_{\mathcal{L}_A}(\tau_k)) \cdot v \\
&= \lim_{k\to\infty} \tau_k(\mathrm{d}\pi(\mathcal{L}_A)) \cdot \pi^*(\mathcal{K}_{\mathcal{L}_A}(\tau_k m)) \cdot v \\
&= \pi^*(\mathcal{K}_{\mathcal{L}_A}(m)) \cdot v,
\end{aligned}
$$

where each equality holds since:

1. $\tau_k m = m$ from some k on, while $\pi^*(\mathcal{K}_{\mathcal{L}_A}(\tau_k)) \cdot v$ converges to v in H since $\mathcal{K}_{\mathcal{L}_A}(\tau_k)$ converges to δ_e in $\mathcal{M}^1_c(G)$ (cf. the proof of Corollary 3.14);

2. $\tau_k P_{k,h}$ converges uniformly to $\tau_k m$ as $h \to \infty$;

3. $(\tau_k P_{k,h})(\mathrm{d}\pi(\mathcal{L}_A)) = \tau_k(\mathrm{d}\pi(\mathcal{L}_A)) \cdot P_{k,h}(\mathrm{d}\pi(\mathcal{L}_A))$ and $\pi^*(\mathcal{K}_{\mathcal{L}_A}(\tau_k)) \cdot v$ belongs to $C^\infty(\pi)$ since $\mathcal{K}_{\mathcal{L}_A}(\tau_k) \in W^{\infty,\infty,1}(G)$; in addition, $P_{k,h}(\mathrm{d}\pi(\mathcal{L}_A)) \cdot \pi^*(\mathcal{K}_{\mathcal{L}_A}(\tau_k)) = \pi^*(P_{k,h}(\mathcal{L}_A) \cdot \mathcal{K}_{\mathcal{L}_A}(\tau_k))$;

4. $\tau_k P_{k,h}$ converges to $\tau_k m$ in $\mathcal{S}(E_{\mathcal{L}_A})$ as $h \to \infty$, so that $P_{k,h}(\mathcal{L}_A) \cdot \mathcal{K}_{\mathcal{L}_A}(\tau_k) = \mathcal{K}_{\mathcal{L}_A}(\tau_k P_{k,h})$ converges to $\mathcal{K}_{\mathcal{L}_A}(\tau_k m)$ in $W^{\infty,\infty,1}(G)$ as $h \to \infty$; this in turn implies that $\pi^*(P_{k,h}(\mathcal{L}_A) \cdot \mathcal{K}_{\mathcal{L}_A}(\tau_k)) \cdot v$ converges to $\pi^*(\mathcal{K}_{\mathcal{L}_A}(\tau_k m)) \cdot v$ in H, as $h \to \infty$;

5. $\tau_k m = m$, so that $\mathcal{K}_{\mathcal{L}_A}(\tau_k m) = \mathcal{K}_{\mathcal{L}_A}(m)$, for k sufficiently large. On the other hand, τ_k converges pointwise and boundedly to χ_E, so that $\tau_k(\mathrm{d}\pi(\mathcal{L}_A))$ converges to $\chi_E(\mathrm{d}\pi(\mathcal{L}_A)) = I_{\mathcal{L}(H)}$ pointwise. This implies that $\tau_k(\mathrm{d}\pi(\mathcal{L}_A)) \cdot \pi^*(\mathcal{K}_{\mathcal{L}_A}(\tau_k m)) \cdot v$ converges to $\pi^*(\mathcal{K}_{\mathcal{L}_A}(m)) \cdot v$ in H.

The arbitrariness of v then implies that

$$
\mathcal{K}_{\mathrm{d}\pi(\mathcal{L}_A)}(m) = \pi^*(\mathcal{K}_{\mathcal{L}_A}(m)).
$$

Therefore, $\mathcal{K}_{\mathrm{d}\pi(\mathcal{L}_A)}(m) = \pi^*(\mathcal{K}_{\mathcal{L}_A}(m)) = 0$ for every $m \in \mathcal{D}(E_{\mathcal{L}_A})$ which vanishes on $\sigma(\mathcal{L}_A)$, so that $\sigma(\mathrm{d}\pi(\mathcal{L}_A)) \subseteq \sigma(\mathcal{L}_A)$.

4. Next, take $m \in C_0(E_{\mathcal{L}_A})$ such that $\mathcal{K}_{\mathcal{L}_A}(m) \in \mathcal{M}^1(G)$. Then, there is a sequence (m_j) of elements of $\mathcal{D}(E_{\mathcal{L}_A})$ which converges to m in $C_0(E_{\mathcal{L}_A})$. For every $\mu \in \mathcal{M}^1(G)$ denote by $N_2(\mu)$ the norm of the mapping $L^2(G) \ni f \mapsto f * \mu \in L^2(G)$. Then,

$$
\begin{aligned}
\lim_{j\to\infty} N_2(\mathcal{K}_{\mathcal{L}_A}(m) - \mathcal{K}_{\mathcal{L}_A}(m_j)) &= \lim_{j\to\infty} \|(m - m_j)(\mathcal{L}_A)\|_{\mathcal{L}(L^2(G))} \\
&\leqslant \lim_{j\to\infty} \|m - m_j\|_{C_0(E_{\mathcal{L}_A})} = 0.
\end{aligned}
$$

By Theorem A.20,

$$
\begin{aligned}
\lim_{j\to\infty} \|\pi^*(\mathcal{K}_{\mathcal{L}_A}(m)) - m_j(\mathrm{d}\pi(\mathcal{L}_A))\|_{\mathcal{L}(H)} &= \lim_{j\to\infty} \|\pi^*(\mathcal{K}_{\mathcal{L}_A}(m) - \mathcal{K}_{\mathcal{L}_A}(m_j))\|_{\mathcal{L}(H)} \\
&\leqslant \lim_{j\to\infty} N_2(\mathcal{K}_{\mathcal{L}_A}(m) - \mathcal{K}_{\mathcal{L}_A}(m_j)) = 0.
\end{aligned}
$$

Analogously,

$$
\lim_{j\to\infty} \|m(\mathrm{d}\pi(\mathcal{L}_A)) - m_j(\mathrm{d}\pi(\mathcal{L}_A))\|_{\mathcal{L}(H)} \leqslant \lim_{j\to\infty} \|m - m_j\|_{C_0(E_{\mathcal{L}_A})} = 0.
$$

We then infer that $m(\mathrm{d}\pi(\mathcal{L}_A)) = \pi^*(\mathcal{K}_{\mathcal{L}_A}(m))$ as claimed.

5. Now, take $m \in C(E_{\mathcal{L}_A})$ such that $\mathcal{K}_{\mathcal{L}_A}(m) \in W^{-\infty,0,1}(G)$. Then, Corollary 3.12 implies that, with the notation of **3**,

$$
\mathcal{K}_{\mathcal{L}_A}(m\tau_k) = \mathcal{K}_{\mathcal{L}_A}(\tau_k) * \mathcal{K}_{\mathcal{L}_A}(m) \in W^{0,\infty,1}(G).
$$

Then, Proposition 1.73 and Corollary 1.80 imply that $\mathcal{K}_{\mathcal{L}_A}(m\tau_k)$ converges to $\mathcal{K}_{\mathcal{L}_A}(m)$ in $W_c^{-\infty,0,1}(G)$, so that $\pi^*(\mathcal{K}_{\mathcal{L}_A}(m\tau_k))$ converges to $\pi^*(\mathcal{K}_{\mathcal{L}_A}(m))$ in $\mathcal{L}_c(C^\infty(\pi); H)$. Further, **4** above shows that $\pi^*(\mathcal{K}_{\mathcal{L}_A}(m\tau_k)) = (m\tau_k)(\mathrm{d}\pi(\mathcal{L}_A))$ for every $k \in \mathbb{N}$. Then, take $v \in C^\infty(\pi)$, and observe that $\tau_k(\mathrm{d}\pi(\mathcal{L}_A)) \cdot v$ converges to v in H, while

$$m(\mathrm{d}\pi(\mathcal{L}_A)) \cdot \tau_k(\mathrm{d}\pi(\mathcal{L}_A)) \cdot v = (m\tau_k)(\mathrm{d}\pi(\mathcal{L}_A)) \cdot v = \pi^*(\mathcal{K}_{\mathcal{L}_A}(m\tau_k)) \cdot v$$

converges to $\pi^*(\mathcal{K}_{\mathcal{L}_A}(m)) \cdot v$ in H. Since $m(\mathrm{d}\pi(\mathcal{L}_A))$ is a closed operator, this implies that $v \in \mathrm{dom}(m(\mathrm{d}\pi(\mathcal{L}_A)))$, and that

$$m(\mathrm{d}\pi(\mathcal{L}_A)) \cdot v = \pi^*(\mathcal{K}_{\mathcal{L}_A}(m)) \cdot v.$$

Conversely, take $v \in \mathrm{dom}(m(\mathrm{d}\pi(\mathcal{L}_A)))$. Then, it is clear that $\tau_k(\mathrm{d}\pi(\mathcal{L}_A)) \cdot v$ converges to v in $\mathrm{dom}(m(\mathrm{d}\pi(\mathcal{L}_A)))$. Since $\tau_k(\mathrm{d}\pi(\mathcal{L}_A)) \cdot v = \pi^*(\mathcal{K}_{\mathcal{L}_A}(\tau_k)) \cdot v \in C^\infty(\pi)$, it follows that $C^\infty(\pi)$ is dense in $\mathrm{dom}(m(\mathrm{d}\pi(\mathcal{L}_A)))$, so that $m(\mathrm{d}\pi(\mathcal{L}_A))$ is the closure of $\pi^*(\mathcal{K}_{\mathcal{L}_A}(m))$.

6. Finally, take an open subset U of $E_{\mathcal{L}_A}$ whose complement is $\mu_{\mathrm{d}\pi(\mathcal{L}_A)}$-negligible, and take $m \in \mathcal{M}(\mu_{\mathcal{L}_A}; W^{-\infty,0,1}(G))$ so that m is continuous on U. Then, there is an increasing sequence (ψ_j) of elements of $\mathcal{D}(E_{\mathcal{L}_A})$ which converges pointwise to χ_U. Therefore, Corollary 3.12 implies that

$$\mathcal{K}_{\mathcal{L}_A}(m) * \mathcal{K}_{\mathcal{L}_A}(\psi_j) = \mathcal{K}_{\mathcal{L}_A}(m\psi_j) = \mathcal{K}_{\mathcal{L}_A}(\psi_j) * \mathcal{K}_{\mathcal{L}_A}(m) \in W^{\infty,0,1}(G).$$

In addition, $m\psi_j$ is continuous, so that **5** above and Proposition 1.85 imply that $\psi_j(\mathrm{d}\pi(\mathcal{L}_A)) = \pi^*(\mathcal{K}_{\mathcal{L}_A}(\psi_j))$ and that

$$\begin{aligned}
\psi_j(\mathrm{d}\pi(\mathcal{L}_A)) \cdot \overline{\pi^*(\mathcal{K}_{\mathcal{L}_A}(m))} &\subseteq \overline{\pi^*(\mathcal{K}_{\mathcal{L}_A}(m))} \cdot \psi_j(\mathrm{d}\pi(\mathcal{L}_A)) \\
&= \pi^*(\mathcal{K}_{\mathcal{L}_A}(m\psi_j)) \\
&= (m\psi_j)(\mathrm{d}\pi(\mathcal{L}_A)) \\
&= m(\mathrm{d}\pi(\mathcal{L}_A)) \cdot \psi_j(\mathrm{d}\pi(\mathcal{L}_A)) \\
&\supseteq \psi_j(\mathrm{d}\pi(\mathcal{L}_A)) \cdot m(\mathrm{d}\pi(\mathcal{L}_A)).
\end{aligned}$$

Now, $\psi_j(\mathrm{d}\pi(\mathcal{L}_A))$ converges pointwise to $\chi_U(\mathrm{d}\pi(\mathcal{L}_A)) = I_H$. Therefore,

$$\overline{\pi^*(\mathcal{K}_{\mathcal{L}_A}(m))} = m(\mathrm{d}\pi(\mathcal{L}_A)),$$

whence the result. □

Now we are able to conclude the proof of Theorem 3.18.

*Proof of the implication **7** $\implies$ **3** of Theorem 3.18.* Notice first that, by [60, Proposition 1.1], we may replace the family of dilations of G with another one in such a way that $\mathcal{L}_\alpha$ is homogeneous of homogeneous degree $\delta'_\alpha \in \mathbb{Z}_+^*$ for every $\alpha \in A$. Therefore, there is a polynomial $P \in \mathbb{R}[A]$ which induces on $E_{\mathcal{L}_A}$ a proper positive homogeneous polynomial mapping without constant terms. Take a continuous non-trivial irreducible unitary representation π of G in a hilbertian space H, and take $v \in C^\infty(\pi)$ so that $\mathrm{d}\pi(\mathcal{L}_A) \cdot v = 0$. Then, $\mathrm{d}\pi(P(\mathcal{L}_A)) \cdot v = 0$ since $P(0) = 0$.

Now, define $p_t \coloneqq \mathcal{K}_{\mathcal{L}_A}(e^{-tP})$, so that $p_t \in W^{0,\infty,1}(G)$ for every $t > 0$, and observe that $\pi^*(p_t) = e^{-t\mathrm{d}\pi(P(\mathcal{L}_A))}$ by Theorem 3.19, so that $\pi^*(p_t) \cdot v = v$ for every $t > 0$. Define $\varphi\colon g \mapsto \langle \pi(g) \cdot v | v \rangle$, and let us show that $\varphi * p_t = \varphi$ for every $t > 0$. Indeed,

$$\begin{aligned}
(\varphi * p_t)(g_1) &= \int_G \varphi(g_1 g_2) p_t(g_2^{-1}) \,\mathrm{d}g_2 \\
&= \left\langle \pi(g_1) \cdot \int_G p_t(g_2) \pi^*(g_2) \cdot v \,\mathrm{d}g_2 \middle| v \right\rangle \\
&= \langle \pi(g_1) \cdot \pi^*(p_t) \cdot v | v \rangle \\
&= \varphi(g_1)
\end{aligned}$$

for every $g_1 \in G$, since G is unimodular. Then, Lemma 1.81 implies that φ is constant. As in the proof of the implication **2** $\implies$ **3**, we then conclude that $v = 0$, whence the result. □

Definition 3.20. Let $\mathcal{L}_A$ be a commutative finite family of formally self-adjoint, homogeneous, left-invariant differential operators on G without constant terms. We shall say that $\mathcal{L}_A$ is a Rockland family if it satisfies the equivalent conditions of Theorem 3.18.

Definition 3.21. We say that a Rockland family $\mathcal{L}_A$ satisfies property:

$(S)_C$ if $\mathcal{S}(G, \mathcal{L}_A) \coloneqq \mathcal{K}_{\mathcal{L}_A}(\mathcal{S}(E_{\mathcal{L}_A}))$ is closed in $\mathcal{S}(G)$;

$(S)_0$ if $\mathcal{S}_{\mathcal{L}_A,0}(G) = \mathcal{S}(G, \mathcal{L}_A)$;

(S) if $\mathcal{S}_{\mathcal{L}_A}(G) = \mathcal{S}(G, \mathcal{L}_A)$.

We end this section with a simple extension of a result of [59]; we present a proof for the sake of completeness. Recall that $\nu_{\widehat{G}}$ denotes the Plancherel measure on $\widehat{G}$.

Proposition 3.22. *Assume that $\mathcal{L}_A$ is a Rockland family, and take $m \in \mathcal{M}(\mu_{\mathcal{L}_A}; W^{-\infty,0,2}(G))$. Then, for $\nu_{\widehat{G}}$-almost every $\pi \in \widehat{G}$, the function m is $\mu_{\mathrm{d}\pi(\mathcal{L}_A)}$-measurable and*

$$m(\mathrm{d}\pi(\mathcal{L}_A)) = \overline{(\mathcal{F}\mathcal{K}_{\mathcal{L}_A}(m))(\pi)}.$$

Proof. **1.** Define

$$\mu' \colon \mathcal{D}^0(E_{\mathcal{L}_A}) \ni \varphi \mapsto [(v_\pi) \mapsto (\varphi(\mathrm{d}\pi(\mathcal{L}_A)) \cdot v_\pi)] \in \mathcal{L}\left(\int_{\widehat{G}}^{\oplus} \mathcal{L}^2(H_\pi) \,\mathrm{d}\nu_{\widehat{G}}(\pi)\right);$$

we shall prove that μ' is a well-defined spectral measure and that $\int_{E_{\mathcal{L}_A}} \lambda \,\mathrm{d}\mu'(\lambda) = \mathcal{F}\mathcal{L}_A\mathcal{F}^{-1}$. As for what concerns the first assertion, observe that Theorem 3.19 implies that $\pi \mapsto \varphi(\mathrm{d}\pi(\mathcal{L}_A))$ is a measurable field of operators on $\widehat{G}$ whenever $\varphi \in \mathcal{D}(E_{\mathcal{L}_A})$, and then for every $\varphi \in \mathcal{D}^0(E_{\mathcal{L}_A})$ by approximation. As for what concerns the second one, take $\tau \in \mathcal{D}(E_{\mathcal{L}_A})$ so that $\chi_{V_1} \leqslant \tau \leqslant \chi_{V_2}$ for some relatively compact open neighbourhoods V_1, V_2 of 0 such that $\overline{V_1} \subseteq V_2$. Then, define $\tau_j \coloneqq \tau(2^{-j}\,\cdot\,)$ for every $j \in \mathbb{N}$, and take $\varphi, \psi \in \mathcal{D}(G)$, $\pi \in \widehat{G}$, and $\alpha \in A$. Let us prove that $(\tau_j \,\mathrm{pr}_\alpha)(\mathrm{d}\pi(\mathcal{L}_A))$ converges to $\mathrm{d}\pi(\mathcal{L}_\alpha)$ in $\mathcal{L}(C^\infty(\pi); H_\pi)$. Indeed, it suffice to observe that

$$\sum_{\alpha \in A} \mathrm{d}\pi(\mathcal{L}_\alpha)^2 + I_{H_\pi} \colon C^\infty(\pi) \to H_\pi$$

is continuous, and that $\frac{\tau_j \,\mathrm{pr}_\alpha}{\|\cdot\|^2 + \chi_{E_{\mathcal{L}_A}}}$ converges uniformly to $\frac{\mathrm{pr}_\alpha}{\|\cdot\|^2 + \chi_{E_{\mathcal{L}_A}}}$. Now, $\pi^*(\varphi) \colon H_\pi \to C^\infty(\pi)$ is continuous, so that $(\tau_j \,\mathrm{pr}_\alpha)(\mathrm{d}\pi(\mathcal{L}_A)) \cdot \pi^*(\varphi)$ converges to $\mathrm{d}\pi(\mathcal{L}_\alpha) \cdot \pi^*(\varphi)$ in $\mathcal{L}(H_\pi)$. Next, observe that ψ can be written as a finite sum of (binary) convolutions of elements of $\mathcal{D}(G)$ (cf. Theorem 1.61), so that by means of the Plancherel theorem we see that $\pi^*(\psi) \in \mathcal{L}^1(H_\pi)$ for $\nu_{\widehat{G}}$-almost every $\pi \in \widehat{G}$, and that $\int_{\widehat{G}}^* \|\pi^*(\psi)\|_{\mathcal{L}^1(H_\pi)} \,\mathrm{d}\nu_{\widehat{G}}(\pi) < \infty$. Therefore,

$$\lim_{j \to \infty} \mathrm{Tr}((\tau_j \,\mathrm{pr}_\alpha)(\mathrm{d}\pi(\mathcal{L}_A)) \cdot \pi^*(\varphi) \cdot \pi^*(\psi)^*) = \mathrm{Tr}(\mathrm{d}\pi(\mathcal{L}_\alpha) \cdot \pi^*(\varphi) \cdot \pi^*(\psi)^*).$$

Since,

$$|\mathrm{Tr}((\tau_j \,\mathrm{pr}_\alpha)(\mathrm{d}\pi(\mathcal{L}_A)) \cdot \pi^*(\varphi) \cdot \pi^*(\psi)^*)| \leqslant \|\mathrm{d}\pi(\mathcal{L}_\alpha) \cdot \pi^*(\varphi)\|_{\mathcal{L}(H_\pi)} \|\pi^*(\psi)^*\|_{\mathcal{L}^1(H_\pi)},$$

the dominated convergence theorem implies that

$$\lim_{j \to \infty} \langle \mu'(\tau_j \,\mathrm{pr}_\alpha) \cdot \mathcal{F}(\varphi) | \mathcal{F}(\psi) \rangle = \langle \mathrm{d}\pi(\mathcal{L}_\alpha) \cdot \mathcal{F}(\varphi) | \mathcal{F}(\psi) \rangle.$$

Taking into account the arbitrariness of φ and ψ, and the fact that $\mathcal{D}(G)$ is a core for $\mathcal{L}_\alpha$, the assertion follows. Therefore, $\mu'(\varphi) = \mathcal{F}\mu_{\mathcal{L}_A}(\varphi)\mathcal{F}^{-1}$ for every $\varphi \in \mathcal{D}^0(E_{\mathcal{L}_A})$.

Now, let us prove that, for every $\mu_{\mathcal{L}_A}$-measurable function $m \colon E_{\mathcal{L}_A} \to \mathbb{C}$, the function m is $\mu_{\mathrm{d}\pi(\mathcal{L}_A)}$-measurable for $\nu_{\widehat{G}}$-almost every $\pi \in \widehat{G}$, and that

$$\int_{E_{\mathcal{L}_A}} m \,\mathrm{d}\mu' \cdot v_\pi = m(\mathrm{d}\pi(\mathcal{L}_A)) \cdot v_\pi$$

for every $v \in \mathcal{F}(\mathrm{dom}(m(\mathcal{L}_A)))$ and for $\nu_{\widehat{G}}$-almost every $\pi \in \widehat{G}$.

Indeed, the assertion is clear if $m \in \mathcal{D}(E_{\mathcal{L}_A})$. If m is positive, bounded, and lower semi-continuous, then there is an increasing sequence of positive elements of $\mathcal{D}(E_{\mathcal{L}_A})$ which converges pointwise to m, so that the assertion holds for m.

Now, if $m = \chi_E$, where E is a $\mu_{\mathcal{L}_A}$-negligible set, then there is a decreasing sequence (m_k) of positive, bounded, and lower semi-continuous functions which are greater than m and converge $\mu_{\mathcal{L}_A}$-almost everywhere to m. If $\widetilde{m} = \inf_{k\in\mathbb{N}} m_k$, then it is clear that $\widetilde{m}(\mathrm{d}\pi(\mathcal{L}_A)) = 0$ for $\nu_{\widehat{G}}$-almost every $\pi \in \widehat{G}$; since $0 \leqslant m(\mathrm{d}\pi(\mathcal{L}_A)) \leqslant \widetilde{m}(\mathrm{d}\pi(\mathcal{L}_A))$, it follows that $m(\mathrm{d}\pi(\mathcal{L}_A)) = 0$. Although this does *not* prove that $\mu_{\mathrm{d}\pi(\mathcal{L}_A)}$ is of base $\mu_{\mathcal{L}_A}$ for $\nu_{\widehat{G}}$-almost every $\pi \in \widehat{G}$ (which, in general, is not true), this result enables us to deal with not necessarily Borel functions.

If m is positive and bounded, then we may take (m_k) as above and conclude that the assertion holds for m since the set where m differs from $\widetilde{m}$ is $\mu_{\mathcal{L}_A}$-negligible.

If m is bounded, then the assertion follows by reducing to the positive and negative parts of the real and imaginary parts of m.

Finally, consider the general case. Define $E_k \coloneqq \{\, \lambda \in E_{\mathcal{L}_A} \colon |m(\lambda)| \leqslant k \,\}$ for every $k \in \mathbb{N}$, and take $v \in \mathcal{F}(\mathrm{dom}(m(\mathcal{L}_A)))$. Up to a subsequence, we may then assume that $\int_{E_k} m \,\mathrm{d}\mu' \cdot v_\pi$ converges to $\int_{E_{\mathcal{L}_A}} m \,\mathrm{d}\mu' \cdot v_\pi$ for $\nu_{\widehat{G}}$-almost every $\pi \in \widehat{G}$. Now,

$$\int_{E_k} m \,\mathrm{d}\mu' \cdot v_\pi = (m\chi_{E_k})(\mathrm{d}\pi(\mathcal{L}_A)) \cdot v_\pi = m(\mathrm{d}\pi(\mathcal{L}_A)) \cdot \chi_{E_k}(\mathrm{d}\pi(\mathcal{L}_A)) \cdot v_\pi$$

for $\nu_{\widehat{G}}$-almost every $\pi \in \widehat{G}$. Since $\chi_{E_k}(\mathrm{d}\pi(\mathcal{L}_A)) \cdot v_\pi$ converges to v_π in H_π for every $\pi \in \widehat{G}$, and since $m(\mathrm{d}\pi(\mathcal{L}_A))$ is a well-defined closed operator on H_π for $\nu_{\widehat{G}}$-almost every $\pi \in \widehat{G}$, the assertion follows.

2. Take $m \in \mathcal{M}(\mu_{\mathcal{L}_A}; L^2(G))$ and $\varphi \in \mathcal{D}(G)$. Then, for $\nu_{\widehat{G}}$-almost every $\pi \in \widehat{G}$,

$$\begin{aligned} m(\mathrm{d}\pi(\mathcal{L}_A)) \cdot (\mathcal{F}\varphi)(\pi) &= \int_{E_{\mathcal{L}_A}} m \,\mathrm{d}\mu' \cdot (\mathcal{F}\varphi)(\pi) \\ &= \mathcal{F}(m(\mathcal{L}_A)\varphi)(\pi) \\ &= \mathcal{F}(\varphi * \mathcal{K}_{\mathcal{L}_A}(m))(\pi) \\ &= (\mathcal{F}\mathcal{K}_{\mathcal{L}_A}(m))(\pi) \cdot (\mathcal{F}\varphi)(\pi). \end{aligned}$$

Let D be a countable dense subset of $\mathcal{D}(G)$, so that D is also dense in $L^2(G)$. Let us prove that $\pi^*(D)$ is dense in $\mathcal{L}^2(H_\pi)$ for $\nu_{\widehat{G}}$-almost every $\pi \in \widehat{G}$. Indeed, let (Φ_j) be a sequence of measurable vector fields with values in the $\mathcal{L}^2(H_\pi)$ such that $(\Phi_{j,\pi})$ is total in $\mathcal{L}^2(H_\pi)$ for every $\pi \in \widehat{G}$. Since $\nu_{\widehat{G}}$ is σ-finite, we may assume that $\Phi_j \in \int_{\widehat{G}}^{\oplus} \mathcal{L}^2(H_\pi) \,\mathrm{d}\nu_{\widehat{G}}$ for every $j \in \mathbb{N}$. Then, for every $j \in \mathbb{N}$ there is a sequence $(\varphi_{j,k})_k$ in D such that $(\mathcal{F}\varphi_{j,k})_k$ converges to Φ_j in $\int_{\widehat{G}}^{\oplus} \mathcal{L}^2(H_\pi) \,\mathrm{d}\nu_{\widehat{G}}$; we may also assume that there is a $\nu_{\widehat{G}}$-negligible subset N of $\widehat{G}$ such that $(\mathcal{F}\varphi_{j,k})(\pi)$ converges to $\Phi_{j,\pi}$ for every $\pi \in \widehat{G} \setminus N$ and for every $j \in \mathbb{N}$. Hence, the family $(\pi^*(\varphi_{j,k}))_{j,k\in\mathbb{N}}$ is total in $\mathcal{L}^2(H_\pi)$ for every $\pi \in \widehat{G} \setminus N$, whence our assertion.

Then, fix a representative of $\mathcal{F}(\mathcal{K}_{\mathcal{L}_A}(m))$, and take a $\nu_{\widehat{G}}$-negligible subset N' of $\widehat{G}$ such that

$$m(\mathrm{d}\pi(\mathcal{L}_A)) \cdot \pi^*(\varphi) = (\mathcal{F}\mathcal{K}_{\mathcal{L}_A}(m))(\pi) \cdot \pi^*(\varphi)$$

for every $\pi \in \widehat{G} \setminus N'$ and for every $\varphi \in D$. Then,

$$m(\mathrm{d}\pi(\mathcal{L}_A)) = (\mathcal{F}\mathcal{K}_{\mathcal{L}_A}(m))(\pi)$$

for every $\pi \in \widehat{G} \setminus (N \cup N')$, whence the assertion in this case. The general case is established as in the proof of Theorem 3.19. Alternatively, it is an easy consequence of Proposition 3.29 below. ☐

3.3 Weighted Subcoercive Systems

In this section we briefly recall the definition of a weighted subcoercive operator (cf. [82]), as well as that of a weighted subcoercive system (cf. [58]), and compare the latter notion with that of a Rockland family.

For this reason, in this section we shall depart from the general notation of this chapter, and G will denote a general connected Lie group with Lie algebra $\mathfrak{g}$.

Definition 3.23. Let $(F_\lambda)_{\lambda>0}$ be an increasing family of vector subspaces of $\mathfrak{g}$ such that $[F_{\lambda_1}, F_{\lambda_2}] \subseteq F_{\lambda_1+\lambda_2}$ for every $\lambda_1, \lambda_2 > 0$, and such that $\bigcup_{\lambda>0} F_\lambda = \mathfrak{g}$ and $\bigcap_{\lambda>0} F_\lambda = 0$. We call (F_λ) a filtration of $\mathfrak{g}$ for short.

Then, a reduced weighted algebraic basis $A_1, \dots, A_n$ of $\mathfrak{g}$ adapted to the filtration (F_λ) is a linearly independent system of elements of $\mathfrak{g}$ such that the following holds: "let λ_j, for every $j = 1, \dots, n$, be the least $\lambda > 0$ such that $A_j \in F_\lambda$. Then, for every $\lambda > 0$, F_λ is the vector space generated by the commutators $(\operatorname{ad} A_{\alpha_1}) \cdots (\operatorname{ad} A_{\alpha_{k-1}}) A_{\alpha_k}$ as $k \in \mathbb{N}^*$ and $\lambda_{\alpha_1} + \cdots + \lambda_{\alpha_k} \leqslant \lambda$.[4] In addition, for every $\lambda > 0$, the space generated by the A_j with $\lambda_j = \lambda$ has null intersection with $F_{\lambda'}$ for every $\lambda' \in]0, \lambda[$." We then call $\lambda_1, \dots, \lambda_n$ the weights of $A_1, \dots, A_n$.

It is clear that to every filtration of $\mathfrak{g}$ one may associate a reduced weighted algebraic basis. In addition, it is clear that a reduced weighted algebraic basis generates the Lie algebra $\mathfrak{g}$.

Definition 3.24. Let $A_1, \dots, A_n$ be a reduced weighted algebraic basis adapted to a filtration (F_λ) of $\mathfrak{g}$. Then, a non-zero left-invariant differential operator $\mathcal{L}$ on G is weighted subcoercive if it can be written in the form $\sum_\alpha c_\alpha A_{\alpha_1} \cdots A_{\alpha_k}$, where $c_\alpha \in \mathbb{C}$ for every α, and $A_{\alpha_1} \cdots A_{\alpha_k} = I$ when $k = 0$, with

$$m := \max\{\, \lambda_{\alpha_1} + \cdots + \lambda_{\alpha_k} : c_\alpha \neq 0 \,\} \in \bigcap_{j=1}^{n} 2\lambda_j \mathbb{N}^*,$$

and if the following local Gårding inequality holds: there are $a > 0$, $b \in \mathbb{R}$, and an open neighbourhood V of e in G such that

$$\operatorname{Re} \int_G \varphi \, \overline{\mathcal{L}_C \varphi} \, \mathrm{d}\nu \geqslant a \sum_{2\lambda_{\alpha_1} + \cdots + 2\lambda_{\alpha_k} \leqslant m} \| A_{\alpha_1} \cdots A_{\alpha_k} \varphi \|_{L^2(\nu)}^2 + b \|\varphi\|_{L^2(\nu)}^2$$

for every $\varphi \in \mathcal{D}(V)$, where ν is a (fixed) *right* Haar measure on G.

See [82, 58] for several equivalent definitions of weighted subcoercive operators.

Observe that a *positive* Rockland operator on a homogeneous group and a (hypoelliptic) sub-Laplacian on a general connected Lie group are always weighted subcoercive (cf. [58, 1.4.4]).

Definition 3.25. Let $(\mathcal{L}_1, \dots, \mathcal{L}_k)$ be a finite commuting family of left-invariant differential operators on G which are formally self-adjoint with respect to the right Haar measure. Then, $(\mathcal{L}_1, \dots, \mathcal{L}_k)$ is a weighted subcoercive system if $P(\mathcal{L}_1, \dots, \mathcal{L}_k)$ is a weighted subcoercive operator for some real polynomial P in k indeterminates.

Notice that, if $\mathcal{L}$ is a formally self-adjoint operator on G, then the family $(\mathcal{L})$ can be a weighted subcoercive system even if $\mathcal{L}$ is not a weighted subcoercive operator. For example, if G is a homogeneous group and $\mathcal{L}$ is a formally self-adjoint Rockland operator thereon, then $(\mathcal{L})$ is a weighted subcoercive system since $\mathcal{L}^2$ is a *positive* Rockland operator, hence a weighted subcoercive operator. On the other hand, if $\mathcal{L}$ is a weighted subcoercive operator, then it is the generator of a continuous semigroup of endomorphisms of $L^2(G)$ by [58, Theorem 1.4.1 (a)], applied to the right regular representation of G in $L^2(G)$. In particular, since by homogeneity the $L^2(G)$-spectrum of $\mathcal{L}$ must be either $\mathbb{R}_-$, $\mathbb{R}_+$, or $\mathbb{R}$, it follows that $\mathcal{L}$ is a positive operator. In other words, a Rockland operator is weighted subcoercive if and only if it is positive.

Now, observe that, if G is a homogeneous group and $\mathcal{L}_1, \dots, \mathcal{L}_k$ are formally self-adjoint, *homogeneous*, commuting left-invariant differential operators without constant terms on G, then Theorem 3.18 and [58, Proposition 3.6.3] show that $(\mathcal{L}_1, \dots, \mathcal{L}_k)$ is a weighted subcoercive system if and only if it is a Rockland family. Therefore, *in our context*, there is no need to distinguish between the notions of a weighted subcoercive system and that of a Rockland family, so that we shall generally neglect the former. Nevertheless, several results of this chapter extend to weighted subcoercive systems on suitable classes of Lie groups, with only minor modifications in the proofs. Since we shall almost only work in the homogeneous setting, we shall not pursue these generalizations.

[4] Here we denote by $\operatorname{ad} X$, for $X \in \mathfrak{g}$, the endomorphism $Y \mapsto [X, Y]$ of the vector space $\mathfrak{g}$.

3.4 The Plancherel Measure

In this section, $\mathcal{L}_A$ is assumed to be a Rockland family; we shall introduce a relevant tool in the study of Rockland families, that is, the Plancherel measure. In the classical case, that is, when $G = \mathbb{R}^n$ and $\mathcal{L}_A = -i\partial = (-i\partial_j)_{j=1}^n$, the 'kernel transform' $\mathcal{K}_{-i\partial}$ is nothing but the inverse Fourier transform $\mathcal{F}^{-1}$. Let us denote here by E the dual of $\mathbb{R}^n$ with respect to the Fourier transform. It is well-known, then, that $m \in L^2(E)$ if and only if $\mathcal{K}_{-i\partial}(m) \in L^2(\mathbb{R}^n)$, and that the norms of m and $\mathcal{K}_{-i\partial}(m)$ are equal *if we endow E with the Plancherel measure β.* Here we look for something similar.

The existence of the Plancherel measure was first proved by M. Christ (cf. [27, Proposition 3]) for a homogeneous sub-Laplacian on a stratified group, and then generalized by A. Martini (cf. [58, Theorem 3.2.7]) to weighted subcoercive systems of differential operators on arbitrary connected Lie groups. We shall only sketch a different proof which is based on the classical equality $(\mathcal{F}^{-1}m)(0) = \int_E m \, \mathrm{d}\beta$; nevertheless, while A. Martini's proof holds even in non-unimodular groups, our method fails in that context.

Theorem 3.26. *There is a unique positive Radon measure $\beta_{\mathcal{L}_A}$ on $E_{\mathcal{L}_A}$ such that the following hold:*

1. $\mathcal{L}^2(\beta_{\mathcal{L}_A}) = \mathcal{M}(\mu_{\mathcal{L}_A}; L^2(G))$ *and*

$$\|m\|_{L^2(\beta_{\mathcal{L}_A})} = \|\mathcal{K}_{\mathcal{L}_A}(m)(g)\|_{L^2(G)}$$

for every $m \in \mathcal{L}^2(\beta_{\mathcal{L}_A})$; in addition, $\beta_{\mathcal{L}_A}$ and $\mu_{\mathcal{L}_A}$ share the same negligible sets;

2. $(r\,\cdot)_*(\beta_{\mathcal{L}_A}) = r^{-Q}\beta_{\mathcal{L}_A}$ *for every $r > 0$; furthermore, $\beta_{\mathcal{L}_A}(\{0\}) = 0$.*

Proof. Take a positive $m \in \mathcal{D}(G)$; then, for every $\psi \in \mathcal{D}(G)$,

$$\langle \psi * \mathcal{K}_{\mathcal{L}_A}(m) | \psi \rangle_{L^2(G)} = \langle m(\mathcal{L}_A) \cdot \psi | \psi \rangle \geqslant 0,$$

so that $\mathcal{K}_{\mathcal{L}_A}(m)$ is of positive type. Therefore, $\mathcal{K}_{\mathcal{L}_A}(m)(e) \geqslant 0$ and the mapping $\beta_{\mathcal{L}_A} : \mathcal{D}(E_{\mathcal{L}_A}) \ni m \mapsto \mathcal{K}_{\mathcal{L}_A}(m)(e) \in \mathbb{C}$ is a positive linear functional on $\mathcal{D}(E_{\mathcal{L}_A})$; hence, it defines a positive (Radon) measure on $E_{\mathcal{L}_A}$. Then, Corollary 3.12 implies that, for every $m \in \mathcal{D}(E_{\mathcal{L}_A})$,

$$\int_E |m|^2 \, \mathrm{d}\beta_{\mathcal{L}_A} = [\mathcal{K}_{\mathcal{L}_A}(m) * \mathcal{K}_{\mathcal{L}_A}(m)^*](e) = \int_G |\mathcal{K}_{\mathcal{L}_A}(m)(g)|^2 \, \mathrm{d}\nu_G(g).$$

Let $\mathcal{K}_{\mathcal{L}_A,2}$ be the unique isometry of $\mathcal{L}^2(\beta_{\mathcal{L}_A})$ into $L^2(G)$ which extends the isometry $\mathcal{D}(G) \ni m \mapsto \mathcal{K}_{\mathcal{L}_A}(m) \in L^2(G)$. Let us sketch the remaining steps of the proof.

- $\mathcal{K}_{\mathcal{L}_A}(m) = \mathcal{K}_{\mathcal{L}_A,2}(m)$ if m belongs to $\mathcal{L}^2_+(\beta_{\mathcal{L}_A})$ and is bounded and lower semi-continuous: take an increasing sequence of elements of $\mathcal{S}(E_{\mathcal{L}_A})$ which converges pointwise to m and use Lemma 3.17.

- $\beta_{\mathcal{L}_A}$-negligible sets are $\mu_{\mathcal{L}_A}$-negligible: since $\mu_{\mathcal{L}_A}$ is positive, it is sufficient to prove the assertion for countable intersections of $\beta_{\mathcal{L}_A}$-integrable open sets.

- m is $\mu_{\mathcal{L}_A}$-measurable and $\mathcal{K}_{\mathcal{L}_A}(m) = \mathcal{K}_{\mathcal{L}_A,2}(m)$ if m belongs to $\mathcal{L}^2_+(\beta_{\mathcal{L}_A})$ and is bounded: use Lemma 3.17.

- $\mu_{\mathcal{L}_A}$-negligible sets are $\beta_{\mathcal{L}_A}$-negligible: use the separability of $L^2(G)$ as in the proof of Proposition 3.15.

- $m \in \mathcal{M}(\mu_{\mathcal{L}_A}; L^2(G))$ and $\mathcal{K}_{\mathcal{L}_A}(m) = \mathcal{K}_{\mathcal{L}_A,2}(m)$ if $m \in \mathcal{L}^2(\beta_{\mathcal{L}_A})$: approximate m by bounded functions and use the fact that $m(\mathcal{L}_A)$ is closed as in the proof of Proposition 3.10.

- $\mathcal{M}(\mu_{\mathcal{L}_A}; L^2(G)) = \mathcal{L}^2(\beta_{\mathcal{L}_A})$: approximate the elements of $\mathcal{M}(\mu_{\mathcal{L}_A}; L^2(G))$ by bounded functions with compact support.

Finally, uniqueness is clear, while homogeneity is proved as in [58, Proposition 3.6.1]. □

Definition 3.27. We shall denote by $\beta_{\mathcal{L}_A}$ the positive (Radon) measure defined in Theorem 3.26.

Furthermore, we shall denote by $S_{\mathcal{L}_A}$ the unit sphere with respect to the *fixed* homogeneous norm $|\cdot|$ on $E_{\mathcal{L}_A}$, that is $\{ \lambda \in E_{\mathcal{L}_A} : |\lambda| = 1 \}$; we shall also denote by $\nu_{\mathbb{R}_+^*}$ the Haar measure on $\mathbb{R}_+^*$ such that $\nu_{\mathbb{R}_+^*}(\mathbb{R}_+^*)([1,e]) = 1$, and we shall denote by $\widetilde{\beta}_{\mathcal{L}_A}$ the unique measure on $S_{\mathcal{L}_A}$ such that (cf. Proposition 1.12)

$$\int_{E_{\mathcal{L}_A}} \varphi \,\mathrm{d}\beta_{\mathcal{L}_A} = \int_{\mathbb{R}_+^* \times S_{\mathcal{L}_A}} \varphi(r\cdot s) r^Q \,\mathrm{d}(\nu_{\mathbb{R}_+^*} \otimes \widetilde{\beta}_{\mathcal{L}_A})(r,s)$$

for every $\varphi \in \mathcal{D}^0(E_{\mathcal{L}_A})$.

Here we present an example which essentially covers the original result by M. Christ [27, Proposition 5]. The possibility to achieve such an explicit description of the Plancherel measure is, however, due to the properties of homogeneity of the operator involved and to the very simple geometry of $\mathbb{R}$. With more than one operator (or with not necessarily homogeneous operators) the situation is much more complicated.

Lemma 3.28. *Let $\mathcal{L}$ be a Rockland operator of homogeneous degree $\delta > 0$. Then, there are $C_{\mathcal{L},-}, C_{\mathcal{L},+} \geqslant 0$ such that*

$$\langle \beta_{\mathcal{L}}, \varphi \rangle = C_{\mathcal{L},-} \int_{\mathbb{R}_-} \varphi(\lambda)(-\lambda)^{\frac{Q}{\delta}-1} \,\mathrm{d}\lambda + C_{\mathcal{L},+} \int_{\mathbb{R}_+} \varphi(\lambda)\lambda^{\frac{Q}{\delta}-1} \,\mathrm{d}\lambda$$

for every $\varphi \in \mathcal{D}^0(\mathbb{R})$.

Proof. It suffices to take $C_{\mathcal{L},\pm} \coloneqq \widetilde{\beta}_{\mathcal{L}}(\{ \pm 1 \})$. ☐

Now we show how the set of multipliers which admit a kernel via spectral theory can be described more precisely.

Proposition 3.29. *The space $\mathcal{M}(\mu_{\mathcal{L}_A}; \mathcal{D}'_w(G))$ is the set of $\beta_{\mathcal{L}_A}$-measurable functions m defined on $E_{\mathcal{L}_A}$ such that*

$$\frac{m}{1+|\cdot|^k} \in \mathcal{L}^2(\beta_{\mathcal{L}_A})$$

for some $k \in \mathbb{N}$.

Proof. Observe first that, if $\frac{m}{1+|\cdot|^k} \in \mathcal{L}^2(\beta_{\mathcal{L}_A})$, then $\frac{m}{P} \in \mathcal{L}^2(\beta_{\mathcal{L}_A})$ for some proper polynomial P on $E_{\mathcal{L}_A}$. Therefore, Corollary 3.13 implies that $m \in \mathcal{M}(\mu_{\mathcal{L}_A}; \mathcal{D}'_w(G))$.

Conversely, assume that $m \in \mathcal{M}(\mu_{\mathcal{L}_A}; \mathcal{D}'_w(G))$ and take $r > 0$. Take, in addition, $\tau \in \mathcal{D}(E_{\mathcal{L}_A})$ so that $\tau(\lambda) = 1$ for every $\lambda \in V = \{ \lambda' \in E_{\mathcal{L}_A} : |\lambda'| \leqslant 1 \}$. Then, Proposition 3.7 and Corollary 3.12 imply that $m(r\cdot\,)\,\tau \in \mathcal{M}(\mu_{\mathcal{L}_A}; \mathcal{D}'_w(G))$ and that

$$\mathcal{K}_{\mathcal{L}_A}(m(r\cdot\,)\,\tau) = \mathcal{K}_{\mathcal{L}_A}(\tau) * [(r\cdot\,)_* \mathcal{K}_{\mathcal{L}_A}(m)] \in L^2(G),$$

so that $m(r\cdot\,)\,\tau \in \mathcal{L}^2(\beta_{\mathcal{L}_A})$ by Theorem 3.26. Now, Proposition 1.32 implies that there is a finite family $(f_\gamma)_{|\gamma|\leqslant h}$ of elements of $L^2(G)\cdot\beta$ such that

$$\mathcal{K}_{\mathcal{L}_A}(m) = \sum_{|\gamma|\leqslant h} \mathbf{Y}^\gamma f_\gamma.$$

Therefore,

$$\begin{aligned}\|m(r\cdot\,)\tau\|_{L^2(\beta_{\mathcal{L}_A})} &= \|\mathcal{K}_{\mathcal{L}_A}(\tau) * [(r\cdot\,)_* \mathcal{K}_{\mathcal{L}_A}(m)]\|_{L^2(G)} \\ &\leqslant \sum_{|\gamma|\leqslant h} r^{\mathrm{d}_\gamma - \frac{Q}{2}} \|X_n^{\gamma_n} \dots X_1^{\gamma_1} \mathcal{K}_{\mathcal{L}_A}(\tau)\|_{L^1(G)} \|f_\gamma\|_{L^2(G)}\end{aligned}$$

for every $r > 0$. Therefore, there are a constant $C > 0$ and $h' \in \mathbb{N}$ such that

$$\|m\chi_{r\cdot V}\|_{L^2(\beta_{\mathcal{L}_A})} \leqslant C(1 + r^{h'})$$

for every $r > 0$. Hence, it will suffice to take $k \coloneqq h' + 1$. ☐

By means of the Plancherel isometry, we are now able to extend $\mathcal{K}_{\mathcal{L}_A}$ to $\mathcal{L}^1(\beta_{\mathcal{L}_A})$. The image of this mapping may then be characterized as the set of convolutions of kernels corresponding to multipliers in $\mathcal{L}^2(\beta_{\mathcal{L}_A})$.

Proposition 3.30. *The mapping* $\mathcal{K}_{\mathcal{L}_A}\colon \mathcal{M}(\mu_{\mathcal{L}_A};\mathcal{D}'_w(G))\cap\mathcal{L}^1(\beta_{\mathcal{L}_A})\to\mathcal{D}'(G)$ *extends to a unique continuous linear mapping*

$$\mathcal{K}_{\mathcal{L}_A,1}\colon \mathcal{L}^1(\beta_{\mathcal{L}_A})\to C_0(G).$$

Furthermore $\mathcal{K}_{\mathcal{L}_A,1}$ *has norm* 1 *and maps* $\mathcal{L}^1_+(\beta_{\mathcal{L}_A})$ *isometrically into the set of continuous functions of positive type. Finally, if* $m_1,m_2\in\mathcal{L}^2(\beta_{\mathcal{L}_A})$, *then*

$$\mathcal{K}_{\mathcal{L}_A,1}(m_1m_2)=\mathcal{K}_{\mathcal{L}_A}(m_1)*\mathcal{K}_{\mathcal{L}_A}(m_2).$$

Proof. **1.** Take $m\in\mathcal{L}^1(\beta_{\mathcal{L}_A})$ and assume that m is bounded. Then, Theorem 3.26 and Proposition 3.11, the latter applied with $F_1=F_2=L^2(G)$ and $F_3=\{\, f\in L^2(G)\colon L^2(G)*f\subseteq L^2(G)\,\}$, imply that

$$\begin{aligned}\|\mathcal{K}_{\mathcal{L}_A}(m)\|_{C_0(G)}&=\left\|\mathcal{K}_{\mathcal{L}_A}\left(\sqrt{|m|}\right)*\mathcal{K}_{\mathcal{L}_A}\left(\operatorname{sgn}(m)\sqrt{|m|}\right)\right\|_{C_0(G)}\\&\leqslant\left\|\mathcal{K}_{\mathcal{L}_A}\left(\sqrt{|m|}\right)\right\|_{L^2(G)}\left\|\mathcal{K}_{\mathcal{L}_A}\left(\operatorname{sgn}(m)\sqrt{|m|}\right)\right\|_{L^2(G)}\\&=\left\|\sqrt{|m|}\right\|^2_{\mathcal{L}^2(\beta_{\mathcal{L}_A})}\\&=\|m\|_{\mathcal{L}^1(\beta_{\mathcal{L}_A})}.\end{aligned}$$

2. Take $m\in\mathcal{L}^1(\beta_{\mathcal{L}_A})\cap\mathcal{M}(\mu_{\mathcal{L}_A};\mathcal{D}'_w(G))$ and let, for every $k\in\mathbb{N}$, E_k be the set of $\lambda\in E_{\mathcal{L}_A}$ such that $|m(\lambda)|\leqslant k$. Then, for every $\psi\in\mathcal{D}(G)$,

$$\begin{aligned}\psi*\mathcal{K}_{\mathcal{L}_A}(m)&=\lim_{k\to\infty}\psi*\mathcal{K}_{\mathcal{L}_A}(m\chi_{E_k})\\&=\lim_{k\to\infty}\psi*\left(\mathcal{K}_{\mathcal{L}_A}\left(\sqrt{|m\chi_{E_k}|}\right)*\mathcal{K}_{\mathcal{L}_A}\left(\operatorname{sgn}(m\chi_{E_k})\sqrt{|m\chi_{E_k}|}\right)\right)\\&=\psi*\left(\mathcal{K}_{\mathcal{L}_A}\left(\sqrt{|m|}\right)*\mathcal{K}_{\mathcal{L}_A}\left(\operatorname{sgn}(m)\sqrt{|m|}\right)\right)\end{aligned}$$

where the first limit is in $L^2(G)$ and equality follows from spectral theory, while the second limit is in $C_0(G)$ and equality follows from Theorem 3.26, since the sequences $\left(\sqrt{|m\chi_{E_k}|}\right)$ and $\left(\operatorname{sgn}(m\chi_{E_k})\sqrt{|m\chi_{E_k}|}\right)$ converge to $\sqrt{|m|}$ and $\operatorname{sgn}(m)\sqrt{|m|}$, respectively, in $\mathcal{L}^2(\beta_{\mathcal{L}_A})$. Therefore, as in **1** we infer that $\|\mathcal{K}_{\mathcal{L}_A}(m)\|_{C_0(G)}\leqslant\|m\|_{\mathcal{L}^1(\beta_{\mathcal{L}_A})}$.

3. By **2**, there is a unique continuous linear mapping $\mathcal{K}_{\mathcal{L}_A,1}\colon\mathcal{L}^1(\beta_{\mathcal{L}_A})\to C_0(G)$ which extends $\mathcal{K}_{\mathcal{L}_A}$ on $\mathcal{L}^1(\beta_{\mathcal{L}_A})\cap\mathcal{M}(\mu_{\mathcal{L}_A};\mathcal{D}'_w(G))$; furthermore, $\mathcal{K}_{\mathcal{L}_A,1}$ has norm at most 1. By approximation it is then easily seen that $\mathcal{K}_{\mathcal{L}_A,1}(m_1m_2)=\mathcal{K}_{\mathcal{L}_A}(m_1)*\mathcal{K}_{\mathcal{L}_A}(m_2)$ for every $m_1,m_2\in\mathcal{L}^2(\beta_{\mathcal{L}_A})$.

4. Finally, take $m\in\mathcal{L}^1_+(\beta_{\mathcal{L}_A})$. Then, $\mathcal{K}_{\mathcal{L}_A,1}(m)=\mathcal{K}_{\mathcal{L}_A}(\sqrt{m})*\mathcal{K}_{\mathcal{L}_A}(\sqrt{m})^*$ thanks to **3** above and Proposition 3.8; hence, $\mathcal{K}_{\mathcal{L}_A,1}(m)$ is of positive type. In particular,

$$\begin{aligned}\|\mathcal{K}_{\mathcal{L}_A,1}(m)\|_{C_0(G)}&=\mathcal{K}_{\mathcal{L}_A,1}(m)(e)\\&=\|\mathcal{K}_{\mathcal{L}_A}(\sqrt{m})\|^2_{L^2(G)}\\&=\|\sqrt{m}\|^2_{\mathcal{L}^2(\beta_{\mathcal{L}_A})}\\&=\|m\|_{\mathcal{L}^1(\beta_{\mathcal{L}_A})}\end{aligned}$$

by Theorem 3.26. Hence, $\mathcal{K}_{\mathcal{L}_A,1}$ is an isometry on $\mathcal{L}^1_+(\beta_{\mathcal{L}_A})$, so that it has norm exactly 1. This completes the proof. □

Definition 3.31. We shall denote by $\mathcal{K}_{\mathcal{L}_A,1}$ the mapping defined in Proposition 3.30. We shall also denote by $\mathcal{K}_{\mathcal{L}_A,1}$ the corresponding mapping defined on $L^1(\beta_{\mathcal{L}_A})$.

Observe that, by interpolation, $\mathcal{K}_{\mathcal{L}_A}$ extends to a continuous linear mapping with norm at most 1 from $L^p(\beta_{\mathcal{L}_A})$ into $L^{p'}(G)$, for every $p \in [1,2]$. Nevertheless, we shall not need this extension.

In the next proposition we collect some properties of $\mathcal{K}_{\mathcal{L}_A,1}$ and $\mathcal{K}_{\mathcal{L}_A}$. The first and the third ones are the analogues of Propositions 3.7, 3.8, and 3.11, while the second property resembles a property of the classical (inverse) Fourier transform.

Proposition 3.32. *The following hold:*

1. *take $m \in \mathcal{L}^1(\beta_{\mathcal{L}_A})$ and $s > 0$. Then,*

$$\mathcal{K}_{\mathcal{L}_A,1}(m(s\,\cdot\,)) = s^{-Q}\mathcal{K}_{\mathcal{L}_A,1}(m)(s^{-1}\,\cdot\,)$$

and

$$\mathcal{K}_{\mathcal{L}_A,1}(\overline{m}) = \mathcal{K}_{\mathcal{L}_A,1}(m)^*;$$

2. *take $m \in \mathcal{L}^1(\beta_{\mathcal{L}_A})$. Then,*

$$\mathcal{K}_{\mathcal{L}_A,1}(m)(e) = \int_{E_{\mathcal{L}_A}} m \,\mathrm{d}\beta_{\mathcal{L}_A};$$

3. *take $m_1 \in \mathcal{M}(\mu_{\mathcal{L}_A}; \mathcal{M}^1(G))$ and $m_2 \in \mathcal{L}^1(\beta_{\mathcal{L}_A})$. Then,*

$$\mathcal{K}_{\mathcal{L}_A,1}(m_1 m_2) = \mathcal{K}_{\mathcal{L}_A}(m_1) * \mathcal{K}_{\mathcal{L}_A,1}(m_2) = \mathcal{K}_{\mathcal{L}_A,1}(m_2) * \mathcal{K}_{\mathcal{L}_A}(m_1).$$

Proof. **1.** This follows from Propositions 3.7 and 3.8 by approximation.

2. Notice first that Theorem 3.26 implies by polarization that, for every $m_1, m_2 \in \mathcal{L}^2(\beta_{\mathcal{L}_A})$,

$$[\mathcal{K}_{\mathcal{L}_A}(m_1) * \mathcal{K}_{\mathcal{L}_A}(m_2)^*](e) = \langle m_1 | m_2 \rangle_{\mathcal{L}^2(\beta_{\mathcal{L}_A})}.$$

Now, define $m_1 \coloneqq \sqrt{|m|}$ and $m_2 \coloneqq \operatorname{sgn}(m)\sqrt{|m|}$, so that $m_1, m_2 \in \mathcal{L}^2(\beta_{\mathcal{L}_A})$. Then, Proposition 3.30 implies that

$$\mathcal{K}_{\mathcal{L}_A,1}(m)(e) = [\mathcal{K}_{\mathcal{L}_A}(m_1) * \mathcal{K}_{\mathcal{L}_A}(\overline{m_2})^*](e) = \langle m_1 | \overline{m_2} \rangle_{\mathcal{L}^2(\beta_{\mathcal{L}_A})} = \int_{E_{\mathcal{L}_A}} m \,\mathrm{d}\beta_{\mathcal{L}_A}.$$

3. Let $(m_{2,j})$ be a sequence of elements of $\mathcal{D}(E_{\mathcal{L}_A})$ which converges to m_2 in $\mathcal{L}^1(\beta_{\mathcal{L}_A})$. Then, Proposition 3.11, applied with $F_1 = F_2 = \mathcal{M}^1(G)$ and $F_3 = L^2(G)$, implies that

$$\mathcal{K}_{\mathcal{L}_A}(m_1 m_{2,j}) = \mathcal{K}_{\mathcal{L}_A}(m_{2,j}) * \mathcal{K}_{\mathcal{L}_A}(m_1) = \mathcal{K}_{\mathcal{L}_A}(m_1) * \mathcal{K}_{\mathcal{L}_A}(m_{2,j}).$$

The assertion follows passing to the limit. □

3.5 The Integral Kernel

In this section, $\mathcal{L}_A$ is assumed to be a Rockland family; we prove the existence of an integral kernel for $\mathcal{K}_{\mathcal{L}_A,1}$. In the classical case, this is the mapping $(\xi, x) \mapsto e^{ix\xi}$.

Proposition 3.33. *There is a unique $\chi_{\mathcal{L}_A} \in L^\infty(\beta_{\mathcal{L}_A} \otimes \nu_G)$ such that, for every $m \in \mathcal{L}^1(\beta_{\mathcal{L}_A})$,*

$$\mathcal{K}_{\mathcal{L}_A,1}(m)(g) = \int_{E_{\mathcal{L}_A}} m(\lambda)\chi_{\mathcal{L}_A}(\lambda, g) \,\mathrm{d}\beta_{\mathcal{L}_A}(\lambda),$$

for ν_G-almost every $g \in G$. In addition, $\|\chi_{\mathcal{L}_A}\|_{L^\infty(\beta_{\mathcal{L}_A} \otimes \nu_G)} = 1$.

The following proof is a straightforward generalization of that of [83, Theorem 2.11].

Proof. Take $f \in \mathcal{L}^1(\nu_G; L^1(\beta_{\mathcal{L}_A}))$. Since the mapping

$$L^1(\beta_{\mathcal{L}_A}) \times G \ni (m, g) \mapsto \mathcal{K}_{\mathcal{L}_A,1}(m)(g) \in \mathbb{C}$$

is continuous by Proposition 3.30, it is clear that the mapping

$$G \ni g \mapsto \mathcal{K}_{\mathcal{L}_A,1}(f(g))(g) \in \mathbb{C}$$

is ν_G-measurable; in addition,

$$|\mathcal{K}_{\mathcal{L}_A,1}(f(g))(g)| \leqslant \|f(g)\|_{L^1(\beta_{\mathcal{L}_A})}$$

for every $g \in G$, so that the mapping $g \mapsto \mathcal{K}_{\mathcal{L}_A,1}(f(g))(g)$ is ν_G-integrable. Then, define

$$\langle \widetilde{\chi}, f \rangle := \int_G \mathcal{K}_{\mathcal{L}_A,1}(f(g))(g) \, \mathrm{d}\nu_G(g),$$

and observe that $\langle \widetilde{\chi}, f \rangle = 0$ if f vanishes ν_G-almost everywhere, so that $\widetilde{\chi}$ induces a continuous linear functional χ on $L^1(\nu_G; L^1(\beta_{\mathcal{L}_A}))$ with norm at most 1. Since $L^1(\nu_G; L^1(\beta_{\mathcal{L}_A}))$ is canonically isomorphic to $L^1(\beta_{\mathcal{L}_A} \otimes \nu_G)$, there is a unique $\chi_{\mathcal{L}_A} \in L^\infty(\beta_{\mathcal{L}_A} \otimes \nu_G)$ such that

$$\langle \chi, f \rangle = \int_G \int_{E_{\mathcal{L}_A}} \chi_{\mathcal{L}_A}(\lambda, g) f(g)(\lambda) \, \mathrm{d}\beta_{\mathcal{L}_A}(\lambda) \, \mathrm{d}\nu_G(g)$$

for every $f \in L^1(\nu_G; L^1(\beta_{\mathcal{L}_A}))$.

Now, take $\varphi \in \mathcal{D}(G)$ and $m \in L^1(\beta_{\mathcal{L}_A})$. Then, Fubini's theorem implies that

$$\begin{aligned} \int_G \varphi(g) \int_{E_{\mathcal{L}_A}} m(\lambda) \chi_{\mathcal{L}_A}(\lambda, g) \, \mathrm{d}\beta_{\mathcal{L}_A}(\lambda) \, \mathrm{d}\nu_G(g) &= \int_{E_{\mathcal{L}_A} \times G} (m \otimes \varphi) \chi_{\mathcal{L}_A} \, \mathrm{d}(\beta_{\mathcal{L}_A} \otimes \nu_G) \\ &= \int_G \varphi(g) \mathcal{K}_{\mathcal{L}_A,1}(m)(g) \, \mathrm{d}\nu_G(g). \end{aligned}$$

By the arbitrariness of φ, we infer that

$$\mathcal{K}_{\mathcal{L}_A,1}(m)(g) = \int_{E_{\mathcal{L}_A}} m(\lambda) \chi_{\mathcal{L}_A}(\lambda, g) \, \mathrm{d}\beta_{\mathcal{L}_A}(\lambda)$$

for ν_G-almost every $g \in G$. Since $\mathcal{K}_{\mathcal{L}_A,1}$ has norm 1 by Proposition 3.30, it follows that also $\chi_{\mathcal{L}_A}$ has norm 1. □

Definition 3.34. We shall denote by $\chi_{\mathcal{L}_A}$ the unique element of $L^\infty(\beta_{\mathcal{L}_A} \otimes \nu_G)$ such that

$$\mathcal{K}_{\mathcal{L}_A,1}(m)(g) = \int_{E_{\mathcal{L}_A}} m(\lambda) \chi_{\mathcal{L}_A}(\lambda, g) \, \mathrm{d}\beta_{\mathcal{L}_A}(\lambda)$$

for every $m \in \mathcal{L}^1(\beta_{\mathcal{L}_A})$ and for almost every $g \in G$.

Proposition 3.35. *For every $s > 0$ and for $(\beta_{\mathcal{L}_A} \otimes \nu_G)$-almost every $(\lambda, g) \in E_{\mathcal{L}_A} \times G$,*

$$\chi_{\mathcal{L}_A}(s \cdot \lambda, g) = \chi_{\mathcal{L}_A}(\lambda, s \cdot g).$$

In addition, there is a unique $\widetilde{\chi}_{\mathcal{L}_A} \in L^\infty(\widetilde{\beta}_{\mathcal{L}_A} \otimes \nu_G)$ such that

$$\chi_{\mathcal{L}_A}(\lambda, g) = \widetilde{\chi}_{\mathcal{L}_A}(|\lambda|^{-1} \cdot \lambda, |\lambda| \cdot g)$$

for $(\beta_{\mathcal{L}_A} \otimes \nu_G)$-almost every $(\lambda, g) \in E_{\mathcal{L}_A} \times G$;

Proof. **1.** Take $m \in \mathcal{D}(E_{\mathcal{L}_A})$ and $\varphi \in \mathcal{D}(G)$. Then, Fubini's theorem, Propositions 3.7 and 3.33, and **2** of Theorem 3.26 imply that

$$
\begin{aligned}
\int_{E_{\mathcal{L}_A}\times G} & m(\lambda)\varphi(g)\chi_{\mathcal{L}_A}(s\cdot\lambda, g)\,\mathrm{d}(\beta_{\mathcal{L}_A}\otimes\nu_G)(\lambda,g)\\
&= \int_G \varphi(g)\int_{E_{\mathcal{L}_A}} m(\lambda)\chi_{\mathcal{L}_A}(s\cdot\lambda,g)\,\mathrm{d}\beta_{\mathcal{L}_A}(\lambda)\,\mathrm{d}\nu_G(g)\\
&= s^{-Q}\int_G \varphi(g)\int_{E_{\mathcal{L}_A}} m(s^{-1}\cdot\lambda)\chi_{\mathcal{L}_A}(\lambda,g)\,\mathrm{d}\beta_{\mathcal{L}_A}(\lambda)\,\mathrm{d}\nu_G(g)\\
&= s^{-Q}\int_G \varphi\,\mathcal{K}_{\mathcal{L}_A}(m(s^{-1}\,\cdot\,))\,\mathrm{d}\nu_G\\
&= \int_G \varphi(g)\,\mathcal{K}_{\mathcal{L}_A}(m)(s\cdot g)\,\mathrm{d}\nu_G(g)\\
&= \int_G \varphi(g)\int_{E_{\mathcal{L}_A}} m(\lambda)\chi_{\mathcal{L}_A}(\lambda,s\cdot g)\,\mathrm{d}\beta_{\mathcal{L}_A}(\lambda)\,\mathrm{d}\nu_G(g)\\
&= \int_{E_{\mathcal{L}_A}\times G} m(\lambda)\varphi(g)\chi_{\mathcal{L}_A}(\lambda,s\cdot g)\,\mathrm{d}(\beta_{\mathcal{L}_A}\otimes\nu_G)(\lambda,g).
\end{aligned}
$$

The arbitrariness of m and φ then implies the claim.

2. Thanks to **2** of Theorem 3.26 and to **1** above, it will suffice to apply Corollary 1.13 with:

$$
\begin{aligned}
G &\coloneqq \mathbb{R}_+^* & X &\coloneqq E^*_{\mathcal{L}_A}\times G & r\cdot(\lambda,g) &\coloneqq (r\cdot\lambda, r^{-1}\cdot g)\\
\lvert(\lambda,g)\rvert &\coloneqq \lvert\lambda\rvert & \mu &\coloneqq [(\beta_{\mathcal{L}_A}\otimes\nu_G)]_{E^*_{\mathcal{L}_A}\times G} & f &\coloneqq \chi_{\mathbb{R}_+^*}\\
h &\coloneqq \chi_{\mathcal{L}_A}
\end{aligned}
$$

for every $r>0$ and for every $(\lambda,g)\in E^*_{\mathcal{L}_A}\times G$. Then, $S = S_{\mathcal{L}_A}\times G$ and clearly $\widetilde{\mu} = \widetilde{\beta}_{\mathcal{L}_A}\otimes\nu_G$, so that it suffices to set $\widetilde{\chi}_{\mathcal{L}_A} \coloneqq \widetilde{h}$. ◻

The following property is reminiscent of an analogous statement concerning Gelfand pairs. It extends [83, Proposition 2.14] to our setting; we shall nevertheless present an alternative proof.

Proposition 3.36. *Take $m\in\mathcal{M}(\mu_{\mathcal{L}_A};\mathcal{M}^1(G))$. Then,*

$$
\mathcal{K}_{\mathcal{L}_A}(m) * \chi_{\mathcal{L}_A}(\lambda,\,\cdot\,) = \chi_{\mathcal{L}_A}(\lambda,\,\cdot\,) * \mathcal{K}_{\mathcal{L}_A}(m) = m(\lambda)\chi_{\mathcal{L}_A}(\lambda,\,\cdot\,)
$$

for $\beta_{\mathcal{L}_A}$-almost every $\lambda\in E_{\mathcal{L}_A}$.

Proof. Notice first that, for every $\varphi_2\in\mathcal{D}(G)$, the linear functional

$$
L^\infty(G)\ni f\mapsto \langle \mathcal{K}_{\mathcal{L}_A}(m)*f,\varphi_2\rangle = \langle f, \mathcal{K}_{\mathcal{L}_A}(m)\check{} * \varphi_2\rangle \in \mathbb{C}
$$

is continuous with respect to the weak topology $\sigma(L^\infty(G), L^1(G))$. In addition, for every $\varphi_1\in\mathcal{D}(E_{\mathcal{L}_A})$,

$$
\mathcal{K}_{\mathcal{L}_A}(\varphi_1) = \int_{E_{\mathcal{L}_A}} \varphi_1(\lambda)\chi_{\mathcal{L}_A}(\lambda,\,\cdot\,)\,\mathrm{d}\beta_{\mathcal{L}_A}(\lambda)
$$

in $L^\infty(G)$, endowed with the weak topology $\sigma(L^\infty(G), L^1(G))$. Therefore,

$$
\begin{aligned}
\int_{E_{\mathcal{L}_A}} \langle \mathcal{K}_{\mathcal{L}_A}(m)*\chi_{\mathcal{L}_A}(\lambda,\,\cdot\,),\varphi_2\rangle\,\varphi_1(\lambda)\,\mathrm{d}\beta_{\mathcal{L}_A}(\lambda) &= \langle \mathcal{K}_{\mathcal{L}_A}(m)*\mathcal{K}_{\mathcal{L}_A}(\varphi_1),\varphi_2\rangle\\
&= \langle \mathcal{K}_{\mathcal{L}_A}(m\varphi_1),\varphi_2\rangle\\
&= \int_{E_{\mathcal{L}_A}} (m\varphi_1)(\lambda)\langle\chi_{\mathcal{L}_A}(\lambda,\,\cdot\,),\varphi_2\rangle\,\mathrm{d}\beta_{\mathcal{L}_A}(\lambda),
\end{aligned}
$$

whence the assertion by the arbitrariness of φ_2. The other equality is proved similarly. ◻

The following result shows that there are well-behaved representatives of $\chi_{\mathcal{L}_A}$. It extends [83, Lemmas 2.12 and 2.15 and Propositions 2.17 and 2.18] to the present setting.

Theorem 3.37. *There is a representative χ_0 of $\chi_{\mathcal{L}_A}$ such that the following hold:*

1. *$\chi_0(\lambda, \cdot\,)$ is a function of positive type of class C^∞ for every $\lambda \in E_{\mathcal{L}_A}$;*

2. *for every γ_1, γ_2 there is a constant $C_{\gamma_1,\gamma_2} > 0$ such that*

$$\|\mathbf{Y}^{\gamma_1}\mathbf{X}^{\gamma_2}\chi_0(\lambda, \cdot\,)\|_\infty \leqslant C_{\gamma_1,\gamma_2}|\lambda|^{\mathrm{d}_{\gamma_1}+\mathrm{d}_{\gamma_2}}$$

   for every $\lambda \in E_{\mathcal{L}_A}$;

3. *$\chi_0(\lambda, e) = 1$ for every $\lambda \in E_{\mathcal{L}_A}$;*

4. *$\chi_0(\lambda, \cdot\,)$ converges to $\chi_0(0, \cdot\,) = 1$ in $\mathcal{E}(G)$ as $\lambda \to 0$;*

5. *$\chi_0(\,\cdot\,, g)$ is $\beta_{\mathcal{L}_A}$-measurable for every $g \in G$.*

Proof. Take a representative χ_1 of $\chi_{\mathcal{L}_A}$ such that $\chi_1(\lambda, \cdot\,)$ belongs to $\mathcal{L}^\infty(G)$ for *every* $\lambda \in E_{\mathcal{L}_A}$. In addition, take $\tau \in \mathcal{S}(E_{\mathcal{L}_A})$ so that $\tau(0) = 1$ and $\tau(\lambda) > 0$ for every $\lambda \in E_{\mathcal{L}_A}$. Hence, Proposition 3.36 implies that, if we define

$$\chi_2(\lambda, g) \coloneqq \frac{1}{\tau(\lambda)}[\chi_1(\lambda, \cdot\,) * \mathcal{K}_{\mathcal{L}_A}(\tau)](g)$$

for *every* $(\lambda, g) \in E_{\mathcal{L}_A} \times G$, then χ_2 is a representative of $\chi_{\mathcal{L}_A}$ and $\chi_2(\lambda, \cdot\,)$ is of class C^∞ for every $\lambda \in E_{\mathcal{L}_A}$. Since $\int_{E_{\mathcal{L}_A}} m \,\mathrm{d}\beta_{\mathcal{L}_A} = \mathcal{K}_{\mathcal{L}_A}(m)(e)$ for every $m \in \mathcal{S}(E_{\mathcal{L}_A})$, we see that $\chi_2(\lambda, e) = 1$ for $\beta_{\mathcal{L}_A}$-almost every $\lambda \in E_{\mathcal{L}_A}$. Analogously, since $\langle \varphi * \mathcal{K}_{\mathcal{L}_A}(m)|\varphi\rangle \geqslant 0$ for every positive $m \in \mathcal{S}(E_{\mathcal{L}_A})$ and for every $\varphi \in C_c^\infty(G)$, we see that $\chi_2(\lambda, \cdot\,)$ is of positive type for $\beta_{\mathcal{L}_A}$-almost every $\lambda \in E_{\mathcal{L}_A}$. Therefore, we may assume that $\chi_2(\lambda, \cdot\,)$ is a C^∞ function of positive type which takes the value 1 at e for every $\lambda \in E_{\mathcal{L}_A}$.

Now, as before we see that for every $t > 0$ there is a $\beta_{\mathcal{L}_A}$-negligible subset N_t of $E_{\mathcal{L}_A}$ such that

$$\tau(t \cdot \lambda)^2\chi_2(\lambda, g) = [\mathcal{K}_{\mathcal{L}_A}(\tau(t\,\cdot\,)) * \chi_2(\lambda, \cdot\,) * \mathcal{K}_{\mathcal{L}_A}(\tau(t\,\cdot\,))](g)$$

for every $\lambda \in E_{\mathcal{L}_A} \setminus N_t$ and for every $g \in G$. Define $N \coloneqq \{\,0\,\} \cup \left(\bigcup_{t\in\mathbb{Q}_+^*} N_t\right)$, and take $\lambda \in E_{\mathcal{L}_A} \setminus N$, $g \in G$, $t > 0$, and a sequence (t_j) of elements of $\mathbb{Q}_+^*$ which converges to t. Then,

$$\lim_{j\to\infty} \mathcal{K}_{\mathcal{L}_A}(\tau(t_j\,\cdot\,)) = \lim_{j\to\infty} t_j^{-Q}\mathcal{K}_{\mathcal{L}_A}(\tau)\big(t_j^{-1}\,\cdot\,\big) = t^{-Q}\mathcal{K}_{\mathcal{L}_A}(\tau)\big(t^{-1}\,\cdot\,\big) = \mathcal{K}_{\mathcal{L}_A}(\tau(t\,\cdot\,))$$

in $L^1(G)$. Hence,

$$\lim_{j\to\infty} \mathcal{K}_{\mathcal{L}_A}(\tau(t_j\,\cdot\,)) * \chi_2(\lambda, \cdot\,) * \mathcal{K}_{\mathcal{L}_A}(\tau(t_j\,\cdot\,)) = \mathcal{K}_{\mathcal{L}_A}(\tau(t\,\cdot\,)) * \chi_2(\lambda, \cdot\,) * \mathcal{K}_{\mathcal{L}_A}(\tau(t\,\cdot\,))$$

uniformly on G, so that

$$|\tau(t \cdot \lambda)|^2\chi_2(\lambda, g) = [\mathcal{K}_{\mathcal{L}_A}(\tau(t\,\cdot\,)) * \chi_2(\lambda, \cdot\,) * \mathcal{K}_{\mathcal{L}_A}(\tau(t\,\cdot\,))](g)$$

for every $t > 0$, for every $\lambda \in E_{\mathcal{L}_A} \setminus N$, and for every $g \in G$. Define $\chi_0 \colon E_{\mathcal{L}_A} \times G \to \mathbb{C}$ so that $\chi_0(\lambda, g) \coloneqq \chi_2(\lambda, g)$ if $\lambda \notin N$, while $\chi_0(\lambda, g) \coloneqq 1$ otherwise, so that χ_0 is still a representative of $\chi_{\mathcal{L}_A}$. Now, set $t \coloneqq \frac{1}{|\lambda|}$ for $\lambda \notin N$ and take $\gamma_1, \gamma_2 \in \mathbb{N}^n$; then $\mathbf{Y}^{\gamma_1}\mathbf{X}^{\gamma_2}\chi_0(\lambda, \cdot\,)$ equals

$$\frac{1}{|\tau(|\lambda|^{-1} \cdot \lambda)|^2}\mathbf{Y}^{\gamma_1}\mathcal{K}_{\mathcal{L}_A}\Big(\tau(|\lambda|^{-1}\,\cdot\,)\Big) * \chi_0(\lambda, \cdot\,) * \mathbf{X}^{\gamma_2}\mathcal{K}_{\mathcal{L}_A}\Big(\tau(|\lambda|^{-1}\,\cdot\,)\Big).$$

Set

$$C_{\gamma_1,\gamma_2} \coloneqq \frac{1}{\min_{|\lambda|=1}|\tau(\lambda)|^2}\|\mathbf{Y}^{\gamma_1}\mathcal{K}_{\mathcal{L}_A}(\tau)\|_1\|\mathbf{X}^{\gamma_2}\mathcal{K}_{\mathcal{L}_A}(\tau)\|_1;$$

then,

$$\|\mathbf{Y}^{\gamma_1}\mathbf{X}^{\gamma_2}\chi_0(\lambda, \cdot\,)\| \leqslant C_{\gamma_1,\gamma_2}|\lambda|^{\mathrm{d}_{\gamma_1}+\mathrm{d}_{\gamma_2}}$$

for every $\lambda \in E_{\mathcal{L}_A} \setminus N$. Since $\chi_0(\lambda, \cdot\,) = \chi_G$ by definition for every $\lambda \in N$, the preceding estimate holds for every $\lambda \in E_{\mathcal{L}_A}$, up to replace $C_{0,0}$ with $\max(1, C_{0,0})$.

Now, take a sequence (λ_j) of non-zero elements of $E_{\mathcal{L}_A}$ which tends to 0. Then, the sequence $(\chi_0(\lambda_j, \cdot\,))$ is bounded in $\mathcal{E}(G)$, hence relatively compact. Therefore, there is a strictly increasing sequence (j_k) of elements of $\mathbb{N}$ such that the sequence $(\chi_0(\lambda_{j_k}, \cdot\,))$ converges to some $\widetilde{\chi}_0$ in $\mathcal{E}(G)$. If the set of $k \in \mathbb{N}$ such that $\lambda_{j_k} \in N$ is infinite, then clearly $\widetilde{\chi}_0 = \chi_G$. Otherwise, we may assume that $\lambda_{j_k} \notin N$ for every $k \in \mathbb{N}$. Since the sequence $(\chi_0(\lambda_{j_k}, \cdot\,))$ is also bounded in $C_b(G)$, it is easily seen that

$$\widetilde{\chi}_0 = \mathcal{K}_{\mathcal{L}_A}(\tau(t\,\cdot\,)) * \widetilde{\chi}_0 * \mathcal{K}_{\mathcal{L}_A}(\tau(t\,\cdot\,))$$

for every $t > 0$. Then, Lemma 1.81 implies that $\widetilde{\chi}_0$ is constant. Now, $\chi_0(\lambda, e) = 1$ for every $\lambda \in E_{\mathcal{L}_A}$, so that $\widetilde{\chi}_0(e) = 1$. Hence, $\widetilde{\chi}_0 = 1$. Since the sequence (λ_j) was arbitrary, $\chi_0(\lambda, \cdot\,)$ converges to $\chi_0(0, \cdot\,) = 1$ in $\mathcal{E}(G)$ as $\lambda \to 0$. □

Remark 3.38. Theorem 3.37 is optimal. In other words, without further assumptions on $\mathcal{L}_A$, the kernel $\chi_{\mathcal{L}_A}$ does not have more ‘regularity’ properties than those that Theorem 3.37 explicitly states, at least in terms of continuity.

Take $G = \mathbb{H}^3$, and let (X_1, X_2, Y_1, Y_2, T) be the standard basis of left-invariant vector fields on G; define $\mathcal{L}_j \coloneqq -(X_j^2 + Y_j^2)$ for $j = 1, 2$. Set $\mathcal{L} \coloneqq \theta_1 \mathcal{L}_1 + \theta_2 \mathcal{L}_2$, where $\theta_1, \theta_2 > 0$ and $\frac{\theta_1}{\theta_2}$ is irrational, and $\mathcal{L}_A \coloneqq (\mathcal{L}, -iT)$. Then, Proposition 7.16 shows that $\chi_{\mathcal{L}_A}$ has a representative which is C^∞ (even analytic) in a neighbourhood of every point of $[\sigma(\mathcal{L}_A) \setminus (\mathbb{R}_+ \times \{\,0\,\})] \times G$ and which is continuous at every point of $\{\,(0,0)\,\} \times G$, in accordance with Proposition 3.35 and Theorem 3.37; in other words, we may have deduced the existence of such a representative arguing only on the properties stated in Proposition 3.35 and Theorem 3.37 and on the form of $\sigma(\mathcal{L}_A)$. Nevertheless, Theorem 7.45 shows that $\chi_{\mathcal{L}_A}$ has no continuous representatives.

Lemma 3.39. *Take χ_0 as in Theorem 3.37. Then, the mapping*

$$\lambda \mapsto \chi_0(\lambda, \cdot\,) \in \mathcal{B}_c^{\infty,\infty}(G)$$

is $\beta_{\mathcal{L}_A}$-measurable.

Proof. Indeed, Lemma A.21 implies that the mapping $\lambda \mapsto \chi_0(\lambda, \cdot\,) \in \mathcal{E}(G)$ is $\beta_{\mathcal{L}_A}$-measurable. However, **2** of Theorem 3.37 implies that the set of $\chi_0(\lambda, \cdot)$, as λ stays in a fixed compact subset of $E_{\mathcal{L}_A}$, is bounded in $\mathcal{B}_c^{\infty,\infty}(G)$. Since the topologies induced by $\mathcal{E}(G)$ and $\mathcal{B}_c^{\infty,\infty}(G)$ on the bounded subsets of $\mathcal{B}_c^{\infty,\infty}(G)$ coincide, this completes the proof by the principle of localization. □

Proposition 3.40. *Take $m_1, m_2 \in \mathcal{M}(\mu_{\mathcal{L}_A}; W^{-\infty,-\infty,1}(G))$. Then,*

$$\mathcal{K}_{\mathcal{L}_A}(m_1) * \chi_{\mathcal{L}_A}(\lambda, \cdot\,) * \mathcal{K}_{\mathcal{L}_A}(m_2) = (m_1 m_2)(\lambda) \chi_{\mathcal{L}_A}(\lambda, \cdot\,)$$

for $\beta_{\mathcal{L}_A}$-almost every $\lambda \in E_{\mathcal{L}_A}$.

Proof. Observe first that we may choose a representative χ_0 of $\chi_{\mathcal{L}_A}$ as in Theorem 3.37, and prove the assertion with that representative. Next, notice that for every $\varphi_1 \in \mathcal{D}(E_{\mathcal{L}_A})$ we have

$$\mathcal{K}_{\mathcal{L}_A}(m_1 \varphi_1 m_2) = \mathcal{K}_{\mathcal{L}_A}(m_1) * \mathcal{K}_{\mathcal{L}_A}(\varphi_1) * \mathcal{K}_{\mathcal{L}_A}(m_2)$$

thanks to Corollary 3.12. In addition, the mapping

$$\mathcal{B}_c^{\infty,\infty}(G) \ni f \mapsto \langle \mathcal{K}_{\mathcal{L}_A}(m_1) * f * \mathcal{K}_{\mathcal{L}_A}(m_2), \varphi_2 \rangle$$

is continuous for every $\varphi_2 \in \mathcal{D}(G)$ thanks to Proposition 1.73. Finally, let us prove that

$$\mathcal{K}_{\mathcal{L}_A}(\varphi_1) = \int_{E_{\mathcal{L}_A}} \varphi_1(\lambda) \chi_0(\lambda, \cdot\,) \, \mathrm{d}\beta_{\mathcal{L}_A}$$

in $\mathcal{B}_c^{\infty,\infty}(G)$. Indeed, by Lemma 3.39 the mapping $\lambda \mapsto \chi_0(\lambda, \cdot\,) \in \mathcal{B}_c^{\infty,\infty}(G)$ is $\beta_{\mathcal{L}_A}$-measurable and locally bounded in measure. Since $\mathcal{B}_c^{\infty,\infty}(G)$ is quasi-complete, and since $\varphi_1 \in \mathcal{D}(E_{\mathcal{L}_A})$, [20, Proposition 8 of Chapter VI, § 1, No. 2] implies that the integral exists, so that it must coincide with $\mathcal{K}_{\mathcal{L}_A}(\varphi_1)$ since $\mathcal{B}_c^{\infty,\infty}(G)$ embeds continuously into $L^\infty(G)$. Then, we may proceed as in the proof of Proposition 3.36. □

3.6 The Multiplier Transform

In this section, $\mathcal{L}_A$ is assumed to be a Rockland family.

Definition 3.41. Let K be a compact subset of $E_{\mathcal{L}_A}$; denote temporarily by $L^2_K(\beta_{\mathcal{L}_A})$ the set of $f \in L^2(\beta_{\mathcal{L}_A})$ whose support lies in K, endowed with the topology induced by $L^2(\beta_{\mathcal{L}_A})$. Then, we shall denote by $L^2_{\mathrm{comp}}(\beta_{\mathcal{L}_A})$ the inductive limit of the $L^2_K(\beta_{\mathcal{L}_A})$, as K runs through the directed set of compact subsets of $E_{\mathcal{L}_A}$.

Lemma 3.42. *$\mathcal{K}_{\mathcal{L}_A}$ induces a continuous linear mapping from $L^2_{\mathrm{comp}}(\beta_{\mathcal{L}_A})$ into $W^{\infty,\infty,2}(G)$.*

Proof. Take $m \in L^2(\beta_{\mathcal{L}_A})$ with support contained in some compact subset K of $E_{\mathcal{L}_A}$, and take $\tau \in \mathcal{D}(E_{\mathcal{L}_A})$ so that $\tau = 1$ on K. Then, Corollary 3.12 implies that

$$\mathcal{K}_{\mathcal{L}_A}\left(|\tau|^2 m\right) = \mathcal{K}_{\mathcal{L}_A}(\tau) * \mathcal{K}_{\mathcal{L}_A}(m) * \mathcal{K}_{\mathcal{L}_A}(\tau)^* \in W^{\infty,\infty,2}(G).$$

The assertion then follows from the closed graph theorem. □

Definition 3.43. We shall define $\mathcal{M}_{\mathcal{L}_A} \colon W^{-\infty,-\infty,2}(G) \to L^2_{\mathrm{loc}}(G)$ as the transpose of the continuous linear mapping $L^2_{\mathrm{comp}}(\beta_{\mathcal{L}_A}) \ni m \mapsto \mathcal{K}_{\mathcal{L}_A}(m)\check{} \in W^{\infty,\infty,2}(G)$ (cf. Lemma 3.42).

Proposition 3.44. *$\mathcal{M}_{\mathcal{L}_A}$ induces the adjoint of $\mathcal{K}_{\mathcal{L}_A} \colon L^2(\beta_{\mathcal{L}_A}) \to L^2(G)$.*

Proof. The assertion follows from Proposition 3.8. □

Proposition 3.45. *Take $T \in W^{-\infty,-\infty,2}(G)$ and $r > 0$. Then,*

$$\mathcal{M}_{\mathcal{L}_A}((r\,\cdot)_* T) = \mathcal{M}_{\mathcal{L}_A}(T)(r\,\cdot) \qquad \textit{and} \qquad \mathcal{M}_{\mathcal{L}_A}(T^*) = \overline{\mathcal{M}_{\mathcal{L}_A}(T)}.$$

Proof. The assertion follows from Propositions 3.7 and 3.8 by transposition. □

Here we prove a prototype result about some sort of compatibility between $\mathcal{M}_{\mathcal{L}_A}$ and convolution with kernels. Many similar statements can be proved with analogous techniques.

Proposition 3.46. *Take $p_1, p_2, p_3 \in [1, \infty]$ such that $\frac{1}{p_1'} + \frac{1}{p_2'} + \frac{1}{p_3'} = \frac{1}{2}$, and take $m_1 \in \mathcal{M}(\mu_{\mathcal{L}_A}; W^{-\infty,\infty,p_1}(G))$, $m_2 \in \mathcal{M}(\mu_{\mathcal{L}_A}; W^{\infty,-\infty,p_2}(G))$, and $T \in W^{-\infty,-\infty,p_3}(G)$. Then,*

$$\mathcal{M}_{\mathcal{L}_A}(\mathcal{K}_{\mathcal{L}_A}(m_1) * T * \mathcal{K}_{\mathcal{L}_A}(m_2)) = m_1 m_2 \mathcal{M}_{\mathcal{L}_A}(T).$$

Proof. Observe first that $p_1, p_2, p_3 \leqslant 2$. In addition, $\frac{1}{p_1'} + \frac{1}{p_2'} = \frac{1}{2} - \frac{1}{p_3'} \in \left[0, \frac{1}{2}\right]$, so that $T \in W^{-\infty,-\infty,2}(G)$ and $\mathcal{K}_{\mathcal{L}_A}(m_2) * \mathcal{S}(G) * \mathcal{K}_{\mathcal{L}_A}(m_1) \subseteq W^{\infty,\infty,2}(G)$ thanks to Proposition 1.41. Then, take $m \in \mathcal{D}(E_{\mathcal{L}_A})$ and observe that Corollaries 1.71, 1.72, and 3.12, and Propositions 1.46, 1.55 and 3.8 imply that

$$\begin{aligned}
\langle \mathcal{M}_{\mathcal{L}_A}(\mathcal{K}_{\mathcal{L}_A}(m_1) * T * \mathcal{K}_{\mathcal{L}_A}(m_2)), m\rangle &= \langle \mathcal{K}_{\mathcal{L}_A}(m_1) * T * \mathcal{K}_{\mathcal{L}_A}(m_2), \mathcal{K}_{\mathcal{L}_A}(m)\check{}\,\rangle \\
&= \langle T, (\mathcal{K}_{\mathcal{L}_A}(m_2) * \mathcal{K}_{\mathcal{L}_A}(m) * \mathcal{K}_{\mathcal{L}_A}(m_1))\check{}\,\rangle \\
&= \langle T, \mathcal{K}_{\mathcal{L}_A}(m_1 m_2 m)\check{}\,\rangle \\
&= \langle m_1 m_2 \mathcal{M}_{\mathcal{L}_A}(T), m\rangle,
\end{aligned}$$

whence the result by the arbitrariness of m. □

Corollary 3.47. *If $m \in \mathcal{M}(\mu_{\mathcal{L}_A}; W^{-\infty,-\infty,2}(G))$, then $m = \mathcal{M}_{\mathcal{L}_A}(\mathcal{K}_{\mathcal{L}_A}(m))$.*

Proof. Take $p_1 = p_2 = 1$, $p_3 = 2$, and $m_1 = m_2 = \chi_{E_{\mathcal{L}_A}}$ in Proposition 3.46. □

Proposition 3.48. *Let χ_0 be a representative of $\chi_{\mathcal{L}_A}$ as in Theorem 3.37. Then, the following hold:*

1. *for every $T \in W^{-\infty,-\infty,1}(G)$,*

$$[\mathcal{M}_{\mathcal{L}_A}(T)](\lambda) \coloneqq \left\langle T, \overline{\chi_0(\lambda, \cdot\,)} \right\rangle$$

for $\beta_{\mathcal{L}_A}$-almost every $\lambda \in E_{\mathcal{L}_A}$;

2. *for every* $T \in W^{-\infty,-\infty,1}(G)$*, there are* $C, d > 0$ *such that*

$$|[\mathcal{M}_{\mathcal{L}_A}(T)](\lambda)| \leqslant C(1+|\lambda|)^d$$

for $\beta_{\mathcal{L}_A}$*-almost every* $\lambda \in E_{\mathcal{L}_A}$*;*

3. $\mathcal{M}_{\mathcal{L}_A}$ *induces a continuous linear mapping from* $W^{-\infty,-\infty,1}(G)$ *into* $L^\infty_{\mathrm{loc}}(\beta_{\mathcal{L}_A})$*;*

4. $\mathcal{M}_{\mathcal{L}_A}$ *induces a continuous linear mapping from* $\mathcal{M}^1(G)$ *into* $L^\infty(\beta_{\mathcal{L}_A})$*.*

By interpolation, $\mathcal{M}_{\mathcal{L}_A}$ then induces a continuous linear mapping from $L^p(G)$ into $L^{p'}(\beta_{\mathcal{L}_A})$ for every $p \in [1,2]$.

Proof. **1.** Indeed, take $T \in W^{-\infty,-\infty,1}(G)$ and $m \in \mathcal{D}(E_{\mathcal{L}_A})$. As in the proof of Proposition 3.40, we see that the mapping $\lambda \mapsto m(\lambda)\chi_0(\lambda, \cdot\,) \in \mathcal{B}^{\infty,\infty}_c(G)$ has weak integral $\mathcal{K}_{\mathcal{L}_A}(m)$, so that

$$\langle T, \mathcal{K}_{\mathcal{L}_A}(m)\check{}\,\rangle = \int_{E_{\mathcal{L}_A}} \langle T, \chi_0(\lambda, \cdot^{-1})\rangle m(\lambda)\,\mathrm{d}\beta_{\mathcal{L}_A}(\lambda).$$

Now, $\chi_0(\lambda, \cdot^{-1}) = \overline{\chi_0(\lambda, \cdot\,)}$ for every $\lambda \in E_{\mathcal{L}_A}$, $\mathcal{D}(E_{\mathcal{L}_A})$ is dense in $L^2_{\mathrm{comp}}(\beta_{\mathcal{L}_A})$, and the mapping $\lambda \mapsto \left\langle T, \overline{\chi_0(\lambda, \cdot\,)}\right\rangle$ is locally bounded (cf. **2** below). Hence, the assertion follows.

2. Observe first that there are $k \in \mathbb{N}$ and a constant $C > 0$ such that

$$|\langle T, \varphi\rangle| \leqslant C\|\varphi\|_{W^{k,k,\infty}(G)}$$

for every $\varphi \in \mathcal{B}^{\infty,\infty}(G)$ (cf. Proposition 1.32). Therefore, the assertion follows easily from **2** of Theorem 3.37.

3. The third assertion follows from the fact that the set of $\overline{\chi_0(\lambda, \cdot\,)}$, as λ stays in a compact subset of $E_{\mathcal{L}_A}$, is bounded in $\mathcal{B}^{\infty,\infty}_c(G)$.

4. The fourth assertion follows easily from **1**. ☐

Corollary 3.49. *Take* $m \in L^\infty_{\mathrm{loc}}(\beta_{\mathcal{L}_A})$ *such that* $\mathcal{K}_{\mathcal{L}_A}(m) \in W^{-\infty,-\infty,1}(G)$*. Then,* m *is continuous at* 0 *and*

$$m(0) = \langle \mathcal{K}_{\mathcal{L}_A}(m), 1\rangle.$$

Proof. Take a representative χ_0 of $\chi_{\mathcal{L}_A}$ as in Theorem 3.37, and define

$$m_0(\lambda) \coloneqq \left\langle \mathcal{K}_{\mathcal{L}_A}(m), \overline{\chi_0(\lambda, g)}\right\rangle,$$

for every $\lambda \in E_{\mathcal{L}_A}$, so that m_0 is a representative of m thanks to Corollary 3.47 and Proposition 3.48. The assertion follows from the properties of χ_0. ☐

Proposition 3.50. *The following conditions are equivalent:*

1. $\chi_{\mathcal{L}_A}$ *has a representative* χ_0 *such that* $\chi_0(\,\cdot\,, g)$ *is continuous on* $\sigma(\mathcal{L}_A)$ *for almost every* $g \in G$*;*

2. $\mathcal{M}_{\mathcal{L}_A}$ *induces a continuous linear mapping from* $L^1(G)$ *into* $C_0(\sigma(\mathcal{L}_A))$*;*

3. $\mathcal{M}_{\mathcal{L}_A}$ *induces a continuous linear mapping from* $W^{-\infty,-\infty,1}(G)$ *into* $\mathcal{E}^0(\sigma(\mathcal{L}_A))$*;*

4. $\chi_{\mathcal{L}_A}$ *has a continuous representative.*

Proof. **1** $\implies$ **2.** Observe first that we may assume that $\chi_0(\,\cdot\,, g)$ is continuous on $\sigma(\mathcal{L}_A)$ for every $g \in G$, and take $\varphi \in L^1(G)$. Then, it is easily seen that

$$[\mathcal{M}_{\mathcal{L}_A}(\varphi)](\lambda) = \int_G \overline{\chi_0(\lambda, \cdot\,)}\,\varphi\,\mathrm{d}\nu_G$$

for $\beta_{\mathcal{L}_A}$-almost every $\lambda \in E_{\mathcal{L}_A}$. Now, it is easily seen that there is a negligible subset N of G such that $|\chi_0(\lambda, g)| \leqslant 1$ for every $(\lambda, g) \in \sigma(\mathcal{L}_A) \times (G \setminus N)$. Take $\lambda_0 \in \sigma(\mathcal{L}_A)$; then the dominated convergence theorem implies that

$$\lim_{\substack{\lambda \to \lambda_0\\ \lambda \in \sigma(\mathcal{L}_A)}} \int_G \overline{\chi_0(\lambda, g)}\varphi(g)\,\mathrm{d}g = \int_G \overline{\chi_0(\lambda_0, g)}\varphi(g)\,\mathrm{d}g.$$

Hence, $\mathcal{M}_{\mathcal{L}_A}(\varphi)$ has a representative m_0 which is continuous and bounded on $\sigma(\mathcal{L}_A)$, and m_0 is uniquely determined. Therefore, $\mathcal{M}_{\mathcal{L}_A}$ induces a continuous linear mapping from $L^1(G)$ into $C_b(\sigma(\mathcal{L}_A))$. Now, take $\tau \in \mathcal{D}(E_{\mathcal{L}_A})$ such that $\tau(0) = 1$, and define $\tau_j := \tau(2^{-j}\,\cdot\,)$ for every $j \in \mathbb{N}$. Then, Proposition 3.46 implies that

$$\mathcal{M}_{\mathcal{L}_A}(\varphi * \mathcal{K}_{\mathcal{L}_A}(\tau_j)) = m_0 \tau_j$$

on $\sigma(\mathcal{L}_A)$. Since $\varphi * \mathcal{K}_{\mathcal{L}_A}(\tau_j)$ converges to φ in $L^1(G)$ by Proposition 3.7 and Corollary 3.49, the assertion follows.

2 $\implies$ 4. Take $\tau \in \mathcal{S}(E_{\mathcal{L}_A})$ so that $\tau(\lambda) > 0$ for every $\lambda \in E_{\mathcal{L}_A}$. Observe that the mapping $G \ni g \mapsto \mathcal{K}_{\mathcal{L}_A}(\tau)(g\,\cdot\,) \in L^1(G)$ is continuous, so that also the mapping $G \ni g \mapsto \mathcal{M}_{\mathcal{L}_A}(\mathcal{K}_{\mathcal{L}_A}(\tau)(g\,\cdot\,)) \in C_0(\sigma(\mathcal{L}_A))$ is continuous. Therefore, the mapping

$$\sigma(\mathcal{L}_A) \times G \ni (\lambda, g) \mapsto \mathcal{M}_{\mathcal{L}_A}(\mathcal{K}_{\mathcal{L}_A}(\tau)(g\,\cdot\,))(\lambda) \in \mathbb{C}$$

is continuous. Now, let χ_0 be a representative of $\chi_{\mathcal{L}_A}$ as in Theorem 3.37. Then, Proposition 3.40 implies that

$$\begin{aligned}\mathcal{M}_{\mathcal{L}_A}(\mathcal{K}_{\mathcal{L}_A}(\tau)(g\,\cdot\,))(\lambda) &= \int_G \mathcal{K}_{\mathcal{L}_A}(\tau)(gg')\chi_0(\lambda, g'^{-1})\,\mathrm{d}g' \\ &= [\mathcal{K}_{\mathcal{L}_A}(\tau) * \chi_0(\lambda, \,\cdot\,)](g) \\ &= \tau(\lambda)\chi_0(\lambda, g)\end{aligned}$$

for $(\beta_{\mathcal{L}_A} \otimes \nu_G)$-almost every $(\lambda, g) \in E_{\mathcal{L}_A} \times G$. In particular, $\chi_{\mathcal{L}_A}$ has a representative which is continuous on $\sigma(\mathcal{L}_A) \times G$. By [17, Corollary to Theorem 2 of Chapter IX, § 4, No. 3], $\chi_{\mathcal{L}_A}$ has a continuous representative.

4 $\implies$ 1. Obvious.

2 $\implies$ 3. Since $\mathcal{D}(G)$ is dense in $W^{-\infty,-\infty,1}(G)$, the assertion follows from Proposition 3.48.

3 $\implies$ 2. Clearly, $\mathcal{M}_{\mathcal{L}_A}$ induces a continuous linear mapping from $L^1(G)$ into $C_b(\sigma(\mathcal{L}_A))$. Arguing as in the proof of the implication **1 $\implies$ 2**, we see that $\mathcal{M}_{\mathcal{L}_A}$ induces a continuous linear mapping from $L^1(G)$ into $C_0(\sigma(\mathcal{L}_A))$. □

Corollary 3.51. *Assume that $\chi_{\mathcal{L}_A}$ has a continuous representative. Then, $\chi_{\mathcal{L}_A}$ has a representative χ_0 such that the following hold:*

1. *χ_0 is continuous on $E_{\mathcal{L}_A} \times G$;*
2. *$\chi_0(\lambda, \,\cdot\,)$ is a function of positive type of class C^∞ for every $\lambda \in \sigma(\mathcal{L}_A)$;*
3. *the mapping $\sigma(\mathcal{L}_A) \ni \lambda \mapsto \chi_0(\lambda, \,\cdot\,) \in \mathcal{E}(G)$ is continuous;*
4. *$\chi_0(\lambda, e) = 1$ for every $\lambda \in \sigma(\mathcal{L}_A)$;*
5. *for every γ_1, γ_2 there is a constant $C_{\gamma_1,\gamma_2} > 0$ such that*

$$\|\mathbf{Y}^{\gamma_1}\mathbf{X}^{\gamma_2}\chi_0(\lambda, \,\cdot\,)\|_\infty \leqslant C_{\gamma_1,\gamma_2}|\lambda|^{\mathrm{d}_{\gamma_1}+\mathrm{d}_{\gamma_2}}$$

for every $\lambda \in \sigma(\mathcal{L}_A)$.

Proof. Repeat the proof of Theorem 3.37 with minor modifications. □

Remark 3.52. Take a sub-Laplacian $\mathcal{L}$ on $G = \mathbb{H}^1 \times \mathbb{R}$ and a basis T of the derived algebra of the Lie algebra of G. Then, Proposition 7.16 shows that $\chi_{(\mathcal{L}, iT)}$ has no continuous representatives, even though $(\mathcal{L}, iT)$ satisfies property (RL) by Theorem 7.20.

3.7 The Banach Algebra $L^1_{\mathcal{L}_A}(G)$

In this section, $\mathcal{L}_A$ is assumed to be a Rockland family; we study the Gelfand spectrum of the commutative Banach algebra $L^1_{\mathcal{L}_A}(G)$, and relate it to $\sigma(\mathcal{L}_A)$. This will provide an abstract characterization of property (RL).

Before starting, let us observe that G is hermitian, that is, $L^1(G)$ is symmetric (cf. [68, 12.5.17]); therefore, every closed $*$-subalgebra of $L^1(G)$ is symmetric (cf. [68, 9.8.3]).

Proposition 3.53. *Define, for every* $\lambda \in \sigma(\mathcal{L}_A)$,

$$\Phi_{\mathcal{L}_A}(\lambda)\colon L^1_{\mathcal{L}_A,0}(G) \ni f \mapsto \mathcal{M}_{\mathcal{L}_A}(f)(\lambda) \in \mathbb{C},$$

where $\mathcal{M}_{\mathcal{L}_A}(f)$ *is identified with its unique representative in* $C_0(\sigma(\mathcal{L}_A))$*. Then,* $\Phi_{\mathcal{L}_A}$ *defines a homeomorphism between* $\sigma(\mathcal{L}_A)$ *and* $\Delta(L^1_{\mathcal{L}_A,0}(G))$.

Proof. The proof follows the lines of that of [58, Proposition 3.3.13], where $\Gamma^1_{\mathcal{L}_A}$ is replaced throughout by $L^1_{\mathcal{L}_A,0}(G)$. □

More precisely, we have the following result.

Proposition 3.54. *Let* $\varphi\colon G \to \mathbb{C}$ *be a* C^∞ *function of positive type such that* $\mathcal{L}_A\varphi = \lambda\varphi$ *for some* $\lambda \in E_{\mathcal{L}_A}$*, and take* $m \in \mathcal{M}(\mu_{\mathcal{L}_A}; W^{-\infty,-\infty,1}(G))$ *so that* m *is continuous on a neighbourhood of* λ*. Then,*

$$\mathcal{K}_{\mathcal{L}_A}(m) * \varphi = m(\lambda)\varphi.$$

Proof. Indeed, [58, Proposition 3.3.3] implies that there are a continuous unitary representation π of G in a hilbertian space H and a cyclic vector v in $C^\infty(\pi)$ such that $\mathrm{d}\pi(\mathcal{L}_A)\cdot v = \lambda v$ and $\varphi(g) = \langle \pi(g)\cdot v|v\rangle$ for every $g \in G$. Assume first that m is continuous on $E_{\mathcal{L}_A}$ and that $\mathcal{K}_{\mathcal{L}_A}(m) \in W^{-\infty,0,1}(G)$. Then, Theorem 3.19 implies that, for every $g \in G$,

$$\begin{aligned} m(\lambda)\varphi(g) &= m(\lambda)\langle \pi(g)\cdot v|v\rangle \\ &= \langle m(\mathrm{d}\pi(\mathcal{L}_A))\cdot \pi(g)\cdot v|v\rangle \\ &= \langle \pi^*(\mathcal{K}_{\mathcal{L}_A}(m))\cdot \pi(g)\cdot v|v\rangle \\ &= \big\langle \mathcal{K}_{\mathcal{L}_A}(m), \langle \pi(\,\cdot\,^{-1}g)\cdot v|v\rangle\big\rangle \\ &= (\mathcal{K}_{\mathcal{L}_A}(m) * \varphi)(g), \end{aligned}$$

whence the result in this case.

Now, assume that m is continuous on some open neighbourhood U of λ and that $\mathcal{K}_{\mathcal{L}_A}(m) \in W^{-\infty,-\infty,1}(G)$, and take $\tau \in \mathcal{D}(U)$ such that $\tau(\lambda) = 1$. Then, Corollary 3.12 implies that

$$\mathcal{K}_{\mathcal{L}_A}(m\tau) = \mathcal{K}_{\mathcal{L}_A}(m) * \mathcal{K}_{\mathcal{L}_A}(\tau) \in W^{-\infty,0,1}(G),$$

so that

$$m(\lambda)\varphi = (m\tau)(\lambda)\varphi = \mathcal{K}_{\mathcal{L}_A}(m\tau) * \varphi = \mathcal{K}_{\mathcal{L}_A}(m) * \mathcal{K}_{\mathcal{L}_A}(\tau) * \varphi = \mathcal{K}_{\mathcal{L}_A}(m) * \varphi,$$

whence the result. □

Lemma 3.55. *The space* $W^{\infty,\infty,1}_{\mathcal{L}_A}(G)$ *is dense in* $L^1_{\mathcal{L}_A}(G)$ *and in* $W^{-\infty,-\infty,1}_{\mathcal{L}_A}(G)$.

Proof. Indeed, take $T \in W^{-\infty,-\infty,1}_{\mathcal{L}_A}(G)$ and $\tau \in \mathcal{D}(E_{\mathcal{L}_A})$ such that $\tau(0) = 1$; define $\tau_j := \tau(2^{-j}\,\cdot\,)$ for every $j \in \mathbb{N}$. Then, $\mathcal{K}_{\mathcal{L}_A}(\tau_j) * T * \mathcal{K}_{\mathcal{L}_A}(\tau_j)^*$ belongs to $W^{\infty,\infty,1}_{\mathcal{L}_A}(G)$ thanks to Corollary 3.12; in addition, it converges to T in $W^{-\infty,-\infty,1}_{\mathcal{L}_A}(G)$. The other assertion is proved similarly. □

Proposition 3.56. *Let* Φ *be a character of* $L^1_{\mathcal{L}_A}(G)$*. Then,* Φ *extends to a unique continuous linear functional on* $W^{-\infty,-\infty,1}_{\mathcal{L}_A}(G)$*. In addition, there is* $\varphi \in \mathcal{B}^{\infty,\infty}(G)$ *such that*

$$\Phi(T) = \langle T, \varphi\rangle$$

for every $T \in W^{-\infty,-\infty,1}_{\mathcal{L}_A}(G)$.

Proof. Observe first that there is $\varphi_0 \in L^\infty(G)$ such that

$$\Phi(f) = \langle f, \varphi_0\rangle$$

for every $f \in L^1_{\mathcal{L}_A}(G)$. If $\Phi \neq 0$, then Lemma 3.55 implies that there is $h \in W^{\infty,\infty,1}_{\mathcal{L}_A}(G)$ such that $\Phi(h) \neq 0$. Then,

$$\Phi(f) = \Phi(h)^{-2}\Phi(h * f * h) = \Phi(h)^{-2}\big\langle f, \check{h} * \varphi_0 * \check{h}\big\rangle$$

for every $f \in L^1_{\mathcal{L}_A}(G)$. Since $\check{h} * \varphi_0 * \check{h} \in \mathcal{B}^{\infty,\infty}(G)$, this proves that Φ is continuous with respect to the topology of $W^{-\infty,-\infty,1}(G)$, so that the assertion follows from Lemma 3.55. □

Proposition 3.57. *Take $T \in W^{-\infty,-\infty,1}_{\mathcal{L}_A}(G)$. Then, the mapping*

$$\theta_T \colon \Delta(L^1_{\mathcal{L}_A}(G)) \ni \Phi \mapsto \Phi(T) \in \mathbb{C}$$

is continuous. In addition, $\overline{\theta_T(\Phi)} = \theta_{T^*}(\Phi)$ for every $\Phi \in \Delta(L^1_{\mathcal{L}_A}(G))$, so that θ_T is real-valued if $T = T^*$.

Proof. Let us first show that θ_T is continuous. Let (Φ_j) be a sequence which converges to some Φ in $\Delta(L^1_{\mathcal{L}_A}(G))$. Since $\Phi \neq 0$, Lemma 3.55 implies that there is $h \in W^{\infty,\infty,1}_{\mathcal{L}_A}(G)$ such that $\Phi(h) \neq 0$, so that $\Phi_j(h) \neq 0$ for all but finitely many $j \in \mathbb{N}$. Now, $L^1_{\mathcal{L}_A}(G)$ is a closed $*$-subalgebra of $L^1(G)$ by Proposition 3.15, so that it is symmetric by the remarks at the beginning of this section. Therefore,

$$\lim_{j\to\infty} \Phi_j(T) = \lim_{j\to\infty} |\Phi_j(h)|^{-2} \Phi_j(h^* * T * h) = |\Phi(h)|^{-2} \Phi(h^* * T * h) = \Phi(T)$$

since $h^* * T * h \in L^1_{\mathcal{L}_A}(G)$. Since $\Delta(L^1_{\mathcal{L}_A}(G))$ is metrizable, this proves that θ_T is continuous. In addition,

$$\theta_{T^*}(\Phi) = |\Phi(h)|^{-2} \Phi(h^* * T^* * h) = \overline{|\Phi(h)|^{-2} \Phi(h^* * T * h)} = \overline{\theta_T(\Phi)}$$

since $h^* * T^* * h = (h^* * T * h)^*$. □

Theorem 3.58. *Let $\Delta_+(L^1_{\mathcal{L}_A}(G))$ be the set of $\Phi \in \Delta(L^1_{\mathcal{L}_A}(G))$ such that there is a function of positive type $\varphi \in C^\infty(G)$ such that $\mathcal{L}_A \varphi = \lambda \varphi$ for some $\lambda \in \mathbb{C}^A$, and*

$$\Phi(f) = \langle f, \overline{\varphi} \rangle$$

for every $f \in L^1_{\mathcal{L}_A}(G)$. Then, the following hold:

1. *if Φ and λ are as above, then $\lambda = \Phi(\mathcal{L}_A \delta_e) \in \sigma(\mathcal{L}_A)$;*

2. *$\Delta_+(L^1_{\mathcal{L}_A}(G))$ is a closed subspace of $\Delta(L^1_{\mathcal{L}_A}(G))$;*

3. *the mapping*
$$\theta_{\mathcal{L}_A} \colon \Delta(L^1_{\mathcal{L}_A}(G)) \ni \Phi \mapsto \Phi(\mathcal{L}_A \delta_e) \in \sigma(\mathcal{L}_A)$$
is proper and onto, as well as its restriction to $\Delta_+(L^1_{\mathcal{L}_A}(G))$.

Proof. **1.** Indeed, by approximation we see that

$$\Phi(\mathcal{L}_A \delta_e) = \langle \mathcal{L}_A \delta_e, \overline{\varphi} \rangle = \langle \delta_e, \overline{\mathcal{L}_A \varphi} \rangle = \overline{\lambda \varphi(e)}.$$

However,

$$1 = \Phi(\delta_e) = \langle \delta_e, \overline{\varphi} \rangle,$$

so that $\overline{\lambda} = \Phi(\mathcal{L}_A \delta_e)$. Finally, [58, Corollary 3.3.11] shows that $\overline{\lambda} \in \sigma(\mathcal{L}_A) \subseteq \mathbb{R}^A$, so that $\lambda = \Phi(\mathcal{L}_A \delta_e)$.

2. Let (Φ_j) be a sequence of elements of $\Delta_+(L^1_{\mathcal{L}_A}(G))$ which converges to some Φ in $\Delta(L^1_{\mathcal{L}_A}(G))$. Take, for every $j \in \mathbb{N}$, a function of positive type $\varphi_j \in C^\infty(G)$ such that $\mathcal{L}_A \varphi_j = \Phi_j(\mathcal{L}_A \delta_e) \varphi_j$ and

$$\Phi_j(T) = \langle T, \overline{\varphi_j} \rangle$$

for every $T \in W^{-\infty,-\infty,1}_{\mathcal{L}_A}(G)$. As in **1**, we see that $\|\varphi_j\|_\infty = \varphi_j(e) = 1$ for every $j \in \mathbb{N}$, so that we may assume that the sequence (φ_j) converges to some continuous function of positive type φ in the weak topology $\sigma(L^\infty(G), L^1(G))$. Therefore,

$$\mathcal{L}_A \varphi = \lim_{j\to\infty} \mathcal{L}_A \varphi_j = \lim_{j\to\infty} \Phi_j(\mathcal{L}_A \delta_e) \varphi_j = \Phi(\mathcal{L}_A \delta_e) \varphi$$

by Proposition 3.57, while clearly

$$\Phi(f) = \lim_{j\to\infty} \Phi_j(f) = \lim_{j\to\infty} \langle f, \overline{\varphi_j} \rangle = \langle f, \overline{\varphi} \rangle$$

for every $f \in L^1_{\mathcal{L}_A}(G)$. Then, [58, Proposition 3.3.1] shows that $\varphi \in C^\infty(G)$, so that $\Phi \in \Delta_+(L^1_{\mathcal{L}_A}(G))$.

3. Take $\Phi \in \Delta(L^1_{\mathcal{L}_A}(G))$, and let us prove that Φ does not vanish identically on $L^1_{\mathcal{L}_A,0}(G)$. Indeed, assume by contradiction that Φ vanishes on $L^1_{\mathcal{L}_A,0}(G)$, and take $f \in L^1_{\mathcal{L}_A}(G)$. Take $\tau \in \mathcal{S}(E_{\mathcal{L}_A})$ such that $\tau(0) = 1$, and define $\tau_j := \tau(2^{-j}\,\cdot\,)$ for every $j \in \mathbb{N}$. Then, $f * \mathcal{K}_{\mathcal{L}_A}(\tau_j)$ belongs to $L^1_{\mathcal{L}_A}(G)$ by Corollary 3.12, and converges to f in $L^1(G)$ thanks to Proposition 3.7 and Corollary 3.49. Therefore,

$$\Phi(f) = \lim_{j\to\infty} \Phi(f * \mathcal{K}_{\mathcal{L}_A}(\tau_j)) = \lim_{j\to\infty} \Phi(f)\Phi(\mathcal{K}_{\mathcal{L}_A}(\tau_j)) = 0,$$

so that $\Phi = 0$: contradiction.

Then, Φ induces a non-zero character of $L^1_{\mathcal{L}_A,0}(G)$, so that Proposition 3.53 shows that $\theta_{\mathcal{L}_A}(\Phi) \in \sigma(\mathcal{L}_A)$. Now, let us prove that $\theta_{\mathcal{L}_A}$ is proper. Let K be a compact subset of $\sigma(\mathcal{L}_A)$, and let (Φ_j) be a sequence in $\theta_{\mathcal{L}_A}^{-1}(K)$. Then, Proposition 3.53 shows that we may assume that the sequence (Φ_j) converges to some $\widetilde{\Phi}$ in $\Delta(L^1_{\mathcal{L}_A,0}(G))$. In addition, since $\Delta'(L^1_{\mathcal{L}_A}(G))$ is compact and metrizable, we may also assume that (Φ_j) converges to some Φ in $\Delta'(L^1_{\mathcal{L}_A}(G))$. Then, $\widetilde{\Phi}$ is the restriction of Φ to $L^1_{\mathcal{L}_A,0}(G)$, so that $\Phi \neq 0$ and then $\Phi \in \Delta(L^1_{\mathcal{L}_A}(G))$. Hence, $\theta_{\mathcal{L}_A}$ is proper on $\Delta(L^1_{\mathcal{L}_A}(G))$. By **2** above, it follows that $\theta_{\mathcal{L}_A}$ is proper on $\Delta_+(L^1_{\mathcal{L}_A}(G))$.

Finally, let us prove that $\theta_{\mathcal{L}_A}(\Delta_+(L^1_{\mathcal{L}_A}(G))) = \sigma(\mathcal{L}_A)$. Let D be a countable dense subset of $L^1_{\mathcal{L}_A}(G)$, and take a representative χ_0 of $\chi_{\mathcal{L}_A}$ as in Theorem 3.37. Then, Propositions 3.40 and 3.48, and Corollary 3.47 imply that there is a $\beta_{\mathcal{L}_A}$-negligible subset N of $E_{\mathcal{L}_A}$ such that

$$\chi_0(\lambda,\,\cdot\,) * f = \left\langle f, \overline{\chi_0(\lambda,\,\cdot\,)}\right\rangle \chi_0(\lambda,\,\cdot\,)$$

for every $\lambda \in E_{\mathcal{L}_A} \setminus N$ and for every $f \in D$. Thanks to Proposition 3.40, we may further assume that $\mathcal{L}_A \chi_0(\lambda,\,\cdot\,) = \lambda\chi_0(\lambda,\,\cdot\,)$ for every $\lambda \in \sigma(\mathcal{L}_A) \setminus N$. Then, fix $\lambda_0 \in \sigma(\mathcal{L}_A) \setminus N$, and observe that $\chi_0(\lambda_0,\,\cdot\,)$ is a function of positive type and of class C^∞; in addition, if $f_1 \in L^1_{\mathcal{L}_A}(G)$ and $f_2^* \in D$, then

$$\left\langle f_1 * f_2, \overline{\chi_0(\lambda_0,\,\cdot\,)}\right\rangle = \left\langle f_1, \overline{\chi_0(\lambda_0,\,\cdot\,) * f_2^*}\right\rangle = \left\langle f_1, \overline{\chi_0(\lambda_0,\,\cdot\,)}\right\rangle \left\langle f_2, \overline{\chi_0(\lambda_0,\,\cdot\,)}\right\rangle.$$

By density, the same assertion holds for every $f_1, f_2 \in L^1_{\mathcal{L}_A}(G)$. Therefore,

$$\Phi\colon L^1_{\mathcal{L}_A}(G) \ni f \mapsto \left\langle f, \overline{\chi_0(\lambda_0,\,\cdot\,)}\right\rangle \in \mathbb{C}$$

is an element of $\Delta_+(L^1_{\mathcal{L}_A}(G))$ and $\theta_{\mathcal{L}_A}(\Phi) = \lambda_0$. Hence, the image of $\theta_{\mathcal{L}_A}$ is a closed subset of $\sigma(\mathcal{L}_A)$ which contains $\sigma(\mathcal{L}_A) \setminus N$. Since N is $\beta_{\mathcal{L}_A}$-negligible and $\sigma(\mathcal{L}_A)$ is the support of $\beta_{\mathcal{L}_A}$, it follows that $\theta_{\mathcal{L}_A}$ is onto. □

Corollary 3.59. *Keep the notation of Theorem 3.58. Then, the following conditions are equivalent:*

1. *$\mathcal{L}_A$ satisfies property (RL);*
2. *$\theta_{\mathcal{L}_A}$ is one-to-one;*
3. *$\theta_{\mathcal{L}_A}$ is one-to-one on $\Delta_+(L^1_{\mathcal{L}_A}(G))$.*

In this case, $\Delta_+(L^1_{\mathcal{L}_A}(G)) = \Delta(L^1_{\mathcal{L}_A}(G))$.

Proof. Assume first that $\mathcal{L}_A$ satisfies property (*RL*). Then, $\theta_{\mathcal{L}_A}$ is a homeomorphism of $\Delta(L^1_{\mathcal{L}_A}(G))$ onto $\sigma(\mathcal{L}_A)$ by Proposition 3.53.

Conversely, assume that $\theta_{\mathcal{L}_A}$ is a homeomorphism of $\Delta_+(L^1_{\mathcal{L}_A}(G))$ onto $\sigma(\mathcal{L}_A)$. Take $f \in L^1_{\mathcal{L}_A}(G)$, and let m be its Gelfand transform. It will suffice to show that $\mathcal{K}_{\mathcal{L}_A}(m \circ \theta_{\mathcal{L}_A}^{-1}) = f$. Take χ_0 and N as in the last part of the proof of Theorem 3.58. Then,

$$m(\theta_{\mathcal{L}_A}^{-1}(\lambda)) = \left\langle f, \overline{\chi_0(\lambda,\,\cdot\,)}\right\rangle$$

for every $\lambda \in \sigma(\mathcal{L}_A) \setminus N$. Therefore, $m \circ \theta_{\mathcal{L}_A}^{-1}$ is a representative of $\mathcal{M}_{\mathcal{L}_A}(f)$, whence the result. □

Chapter 4

Quotients, Products, Image Families

In this chapter we develop some tools in order to deduce some properties of the families into consideration from other known families.

4.1 Quotients

Theorem 4.1. *Let G and G' be two homogeneous groups, $\mathcal{L}_A$ a Rockland family on G, and π a homogeneous homomorphism of G* onto *G'. Then, the following hold:*

1. $\mathrm{d}\pi(\mathcal{L}_A)$ *is a Rockland family;*
2. $\sigma(\mathrm{d}\pi(\mathcal{L}_A)) \subseteq \sigma(\mathcal{L}_A)$*;*
3. *if m is an element of $\mathcal{M}(\mu_{\mathcal{L}_A}; W^{-\infty,-\infty,1}(G))$ which is continuous on an open subset U of $E_{\mathcal{L}_A}$ which carries $\beta_{\mathrm{d}\pi(\mathcal{L}_A)}$, then*

$$\pi_*(\mathcal{K}_{\mathcal{L}_A}(m)) = \mathcal{K}_{\mathrm{d}\pi(\mathcal{L}_A)}(m).$$

Proof. In order to prove that $\mathrm{d}\pi(\mathcal{L}_A)$ is a Rockland family, observe that if ϖ is a non-trivial continuous irreducible unitary representation of G', then $\varpi \circ \pi$ is a non-trivial continuous irreducible unitary representation of G, and that $C^\infty(\varpi) = C^\infty(\varpi \circ \pi)$ since π is a submersion.

Then, it suffices to apply Theorem 3.19 to the right quasi-regular representation ϖ of G in G', observing that $\mathcal{D}(G')$ is dense in $\mathcal{E}(\varpi) = W_0^{0,\infty,2}(G')$. The general form of **3** is proved by approximation, taking into account Proposition 1.73 and Corollary 1.80. ☐

Remark 4.2. Theorem 4.1 may fail if m is not continuous, even if $\mathcal{K}_{\mathcal{L}_A}(m) \in \mathcal{S}(G)$. See the proofs Proposition 7.44 and Theorem 7.45 for an example. More precisely, it may happen that $\pi_*(\mathcal{K}_{\mathcal{L}_A}(m))$ does not even correspond to *any* multiplier of $\mathrm{d}\pi(\mathcal{L}_A)$.

4.2 Products

Unlike quotients, products are very well-behaved, and (almost) all the properties in which we are interested transfer easily from the factors to the product. Nevertheless, this is not surprising, since the structure of a product is much simpler than that of a quotient. As a matter of fact, even though we prove only the relevant implications, also the converse ones are true for (almost) every statement. We leave the details to the reader.

In this section, $(G_A)_{A\in\mathcal{A}}$ denotes a finite family of homogeneous groups; we define $G := \prod_{A\in\mathcal{A}} G_A$.[1] For every $A \in \mathcal{A}$, $\mathcal{L}_A = (\mathcal{L}_\alpha)_{\alpha\in A}$ denotes a Rockland family on G_A; we denote by $\mathcal{L}'_A = (\mathcal{L}'_\alpha)_{\alpha\in A}$ the corresponding family of operators on G. Finally, define $\mathcal{L}'_{\mathcal{A}} = ((\mathcal{L}'_\alpha)_{\alpha\in A})_{A\in\mathcal{A}}$.

Proposition 4.3. *$\mathcal{L}'_{\mathcal{A}}$ is a Rockland family.*

[1] To avoid notational issues, we assume that the elements of $\mathcal{A}$ are pairwise disjoint.

Proof. The assertion follows from [58, Propositions 3.4.2 and 3.6.3]. □

Proposition 4.4. *Take a $\mu_{\mathcal{L}_A}$-measurable function $m_A\colon E_{\mathcal{L}_A}\to\mathbb{C}$ for every $A\in\mathcal{A}$. Then, $\bigotimes_{A\in\mathcal{A}} m_A$ is $\mu_{\mathcal{L}'_\mathcal{A}}$-measurable and $\big(\bigotimes_{A\in\mathcal{A}} m_A\big)(\mathcal{L}'_\mathcal{A})=\bigotimes_{A\in\mathcal{A}} m_A(\mathcal{L}_A)$.*

Further, if $m_A\in\mathcal{M}(\mu_{\mathcal{L}_A};\mathcal{D}'_w(G_A))$ for every $A\in\mathcal{A}$, then $\bigotimes_{A\in\mathcal{A}} m_A\in\mathcal{M}(\mu_{\mathcal{L}'_\mathcal{A}};\mathcal{D}'_w(G))$ and

$$\mathcal{K}_{\mathcal{L}'_A}\left(\bigotimes_{A\in\mathcal{A}} m_A\right)=\bigotimes_{\alpha\in A}\mathcal{K}_{\mathcal{L}_A}(m_A).$$

Proof. The first assertion follows easily by spectral theory, while the second assertion is a trivial consequence of the first one when each m_A is bounded. The general case is an easy consequence of Proposition 3.29 and [58, Proposition 3.3.4]. □

Proposition 4.5. *We have*

$$\beta_{\mathcal{L}'_\mathcal{A}}=\bigotimes_{A\in\mathcal{A}}\beta_{\mathcal{L}_A}$$

and

$$\chi_{\mathcal{L}'_\mathcal{A}}((\lambda_A),(g_A))=\prod_{A\in\mathcal{A}}\chi_{\mathcal{L}_A}(\lambda_A,g_A)$$

for $(\beta_{\mathcal{L}'_\mathcal{A}}\otimes\nu_G)$-almost every $((\lambda_A),(g_A))$.

Proof. The first assertion follows from [58, Proposition 3.3.4]. The second assertion is an easy consequence of the first one. □

We now pass to properties (RL) and (S). We begin with an elementary lemma.

Lemma 4.6. *Assume that $\mathcal{A}=\{A_1,A_2\}$, and take $m\in L^1(\beta_{\mathcal{L}'_\mathcal{A}})$ and $\mu\in\mathcal{M}^1(G_{A_2})$. Then, there is $m_\mu\in L^1(\beta_{\mathcal{L}_{A_1}})$ such that*

$$\int_{G_{A_2}}\mathcal{K}_{\mathcal{L}'_\mathcal{A},1}(m)(\,\cdot\,,g_2)\,\mathrm{d}\mu(g_2)=\mathcal{K}_{\mathcal{L}_{A_1},1}(m_\mu).$$

Proof. Observe first that $L^1(\beta_{\mathcal{L}'_\mathcal{A}})\cong L^1(\beta_{\mathcal{L}_{A_1}})\widehat{\otimes}L^1(\beta_{\mathcal{L}_{A_2}})$ thanks to Proposition 4.5 and [84, Exercise 46.5]. Therefore, [84, Theorem 45.1] implies that there are $(c_j)\in\ell^1$ and two bounded sequences $(m_{j,1}),(m_{j,2})$ in $L^1(\beta_{\mathcal{L}_{A_1}})$ and $L^1(\beta_{\mathcal{L}_{A_2}})$, respectively, such that

$$m=\sum_{j\in\mathbb{N}}c_j(m_{j,1}\otimes m_{j,2})$$

in $L^1(\beta_{\mathcal{L}'_\mathcal{A}})$. Therefore,

$$m_\mu:=\sum_{j\in\mathbb{N}}c_j\int_{G_2}\mathcal{K}_{\mathcal{L}_{A_2},1}(m_{j,2})(g_2)\,\mathrm{d}\mu(g_2)\,m_{j,1}$$

is a well-defined element of $L^1(\beta_{\mathcal{L}_{A_1}})$ which satisfies

$$\int_{G_{A_2}}\mathcal{K}_{\mathcal{L}'_\mathcal{A},1}(m)(\,\cdot\,,g_2)\,\mathrm{d}\mu(g_2)=\mathcal{K}_{\mathcal{L}_{A_1},1}(m_\mu),$$

whence the result. □

Corollary 4.7. *Assume that $\mathcal{A}=\{A_1,A_2\}$, and take $m\in\mathcal{M}(\mu_{\mathcal{L}'_\mathcal{A}};L^1(G_\mathcal{A}))$ and $f\in L^\infty(G_{A_2})$. Then,*

$$\int_{G_{A_2}}\mathcal{K}_{\mathcal{L}'_\mathcal{A}}(m)(\,\cdot\,,g_2)f(g_2)\,\mathrm{d}\nu_{G_{A_2}}(g_2)\in L^1_{\mathcal{L}_{A_1}}(G_{A_1}).$$

In addition, $\mathcal{K}_{\mathcal{L}'_\mathcal{A}}(m)(\,\cdot\,,g_2)\in L^1_{\mathcal{L}_{A_1}}(G_{A_1})$ for almost every $g_2\in G_{A_2}$.

Proof. **1.** Assume first that m is compactly supported. Let (K_j) be an increasing sequence of compact subsets of G_{A_2} whose union is G_{A_2}. Then,

$$\lim_{j\to\infty}\int_{K_j}\mathcal{K}_{\mathcal{L}'_\mathcal{A}}(m)(\,\cdot\,,g_2)f(g_2)\,\mathrm{d}\nu_{G_{A_2}}(g_2)=\int_{G_{A_2}}\mathcal{K}_{\mathcal{L}'_\mathcal{A}}(m)(\,\cdot\,,g_2)f(g_2)\,\mathrm{d}\nu_{G_{A_2}}(g_2)$$

in $L^1(G_{A_1})$ by dominated convergence. Now, clearly $m\in L^1(\beta_{\mathcal{L}'_\mathcal{A}})$, so that Lemma 4.6 implies that

$$\int_{K_j}\mathcal{K}_{\mathcal{L}'_\mathcal{A}}(m)(\,\cdot\,,g_2)f(g_2)\,\mathrm{d}\nu_{G_{A_2}}(g_2)\in L^1_{\mathcal{L}_{A_1}}(G_{A_1})$$

for every $j\in\mathbb{N}$, whence the first assertion thanks to Proposition 3.15. The second assertion follows from Lemma 4.6 taking Dirac deltas.

2. Now, take $\tau\in\mathcal{D}(E_{\mathcal{L}_\mathcal{A}})$ such that $\tau(0)=1$, and define $\tau_j:=\tau(2^{-j}\,\cdot\,)$ for every $j\in\mathbb{N}$. Then, Corollary 3.12 implies that

$$\mathcal{K}_{\mathcal{L}'_\mathcal{A}}(m\tau_j)=\mathcal{K}_{\mathcal{L}'_\mathcal{A}}(\tau_j)*\mathcal{K}_{\mathcal{L}'_\mathcal{A}}(m)\in L^1(G_\mathcal{A})$$

for every $j\in\mathbb{N}$, so that it converges to $\mathcal{K}_{\mathcal{L}'_\mathcal{A}}(m)$ in $L^1(G_\mathcal{A})$. Then, both assertions follow from **1** above and Proposition 3.15. ☐

Theorem 4.8. *If $\mathcal{L}_A$ satisfies property (RL) for every $A\in\mathcal{A}$, then $\mathcal{L}'_\mathcal{A}$ satisfies property (RL).*

Proof. **1.** Proceeding by induction, we may reduce to the case in which $\mathcal{A}=\{A_1,A_2\}$. In order to simplify the notation, we shall simply write G_j instead of G_{A_j} for $j=1,2$. Now, take $f\in L^1_{\mathcal{L}'_\mathcal{A}}(G)$ and let m be a representative of $\mathcal{M}_{\mathcal{L}'_\mathcal{A}}(f)$. Then, Corollary 3.47, Proposition 4.5 and Fubini's theorem imply that

$$\mathcal{M}_{\mathcal{L}_{A_1}}[g_1\mapsto\mathcal{M}_{\mathcal{L}_{A_2}}[f(g_1,\,\cdot\,)](\lambda_2)](\lambda_1)=m(\lambda_1,\lambda_2)$$

for $\beta_{\mathcal{L}_{A_1}}$-almost every $\lambda_1\in E_{\mathcal{L}_{A_1}}$ and for $\beta_{\mathcal{L}_{A_2}}$-almost every $\lambda_2\in E_{\mathcal{L}_{A_2}}$. Observe that Corollary 4.7 implies that $f(g_1,\,\cdot\,)\in L^1_{\mathcal{L}_{A_2}}(G_2)$ for almost every $g_1\in G_1$, and that by assumption $\mathcal{M}_{\mathcal{L}_{A_2}}$ induces a continuous linear mapping from $L^1_{\mathcal{L}_{A_2}}(G_2)$ into $C_0(\sigma(\mathcal{L}_{A_2}))$. Therefore, the mapping $g_1\mapsto\mathcal{M}_{\mathcal{L}_{A_2}}[f(g_1,\,\cdot\,)]$ defines an element of $L^1(G_1;C_0(\sigma(\mathcal{L}_{A_2})))$.

2. Let us prove that, for every $\mu\in\mathcal{M}^1(\sigma(\mathcal{L}_{A_2}))$, the mapping $g_1\mapsto(\mu\cdot\mathcal{M}_{\mathcal{L}_{A_2}})[f(g_1,\,\cdot\,)]$ belongs to $L^1_{\mathcal{L}_{A_1}}(G_1)$. Indeed, the preceding considerations show that $\mu\cdot\mathcal{M}_{\mathcal{L}_{A_2}}$ defines an element of $L^1_{\mathcal{L}_{A_2}}(G_2)'$, so that $\mu\cdot\mathcal{M}_{\mathcal{L}_{A_2}}$ is represented by an element of $L^\infty(G_2)$; hence, the assertion follows from Corollary 4.7.

Now, let us prove that the mapping

$$\mathcal{M}^1_c(\sigma(\mathcal{L}_{A_1}))\ni\mu\mapsto\big[g_1\mapsto(\mu\cdot\mathcal{M}_{\mathcal{L}_{A_2}})[f(g_1,\,\cdot\,)]\big]\in L^1_{\mathcal{L}_{A_1}}(G_1)$$

is continuous. By the preceding computations, it will suffice to prove continuity with respect to the codomain $L^1(G_1)$. Now, $L^1(G_1;C_0(\sigma(\mathcal{L}_{A_2})))\cong L^1(G_1)\widehat{\otimes}C_0(\sigma(\mathcal{L}_{A_2}))$ thanks to [84, Theorem 46.2]. Therefore, [84, Theorem 45.1] implies that there are $(c_j)\in\ell^1$ and two infinitesimal sequences $(h_j),(\varphi_j)$ in $L^1(G_1)$ and $C_0(\sigma(\mathcal{L}_{A_2}))$, respectively, such that

$$\big[g_1\mapsto\mathcal{M}_{\mathcal{L}_{A_2}}[f(g_1,\,\cdot\,)]\big]=\sum_{j\in\mathbb{N}}c_j(h_j\otimes\varphi_j)$$

in $L^1(G_1;C_0(\sigma(\mathcal{L}_{A_2})))$. Since

$$\big[g_1\mapsto(\mu\cdot\mathcal{M}_{\mathcal{L}_{A_2}})[f(g_1,\,\cdot\,)]\big]=\sum_{j\in\mathbb{N}}c_j\langle\mu,\varphi_j\rangle h_j$$

in $L^1_{\mathcal{L}_{A_1}}(G_1)$, and since the mappings $\mathcal{M}^1_c(\sigma(\mathcal{L}_{A_1}))\ni\mu\mapsto\langle\mu,\varphi_j\rangle\in\mathbb{C}$, as j runs through $\mathbb{N}$, are equicontinuous, the assertion follows.

3. Now, $\mathcal{M}_{\mathcal{L}_{A_1}}$ induces a continuous linear mapping from $L^1_{\mathcal{L}_{A_1}}(G_1)$ into $C_0(\sigma(\mathcal{L}_{A_1}))$, so that **2** above implies that the mapping

$$\sigma(\mathcal{L}_{A_2})\ni\lambda_2\mapsto\mathcal{M}_{\mathcal{L}_{A_1}}\big(g_1\mapsto\mathcal{M}_{\mathcal{L}_{A_2}}[f(g_1,\,\cdot\,)](\lambda_2)\big)\in C_0(\sigma(\mathcal{L}_{A_1}))$$

is continuous. Therefore, the mapping

$$\sigma(\mathcal{L}'_\mathcal{A}) \ni (\lambda_1, \lambda_2) \mapsto \mathcal{M}_{\mathcal{L}_{A_1}}\big(g_1 \mapsto \mathcal{M}_{\mathcal{L}_{A_2}}[f(g_1, \cdot\,)](\lambda_2)\big)(\lambda_1) \in \mathbb{C}$$

is continuous, so that it extends to a continuous mapping m_0 on $E_{\mathcal{L}'_\mathcal{A}}$ by [17, Corollary to Theorem 2 of Chapter IX, § 4, No. 3]. Now, **1** implies that $m_0(\lambda_1, \lambda_2) = m(\lambda_1, \lambda_2)$ for $\beta_{\mathcal{L}_{A_1}}$-almost every $\lambda_1 \in E_{\mathcal{L}_{A_1}}$ and for $\beta_{\mathcal{L}_{A_2}}$-almost every $\lambda_2 \in E_{\mathcal{L}_{A_2}}$. Since both m and m_0 are $\beta_{\mathcal{L}'_\mathcal{A}}$-measurable, Tonelli's theorem implies that $m = m_0$ $\beta_{\mathcal{L}'_\mathcal{A}}$-almost everywhere, whence the result. □

Proposition 4.9. *If $\mathcal{L}_A$ satisfies property $(S)_C$ for every $A \in \mathcal{A}$, then $\mathcal{L}'_\mathcal{A}$ satisfies property $(S)_C$.*

Proof. Indeed, Lemma A.26 implies that $\mathcal{K}_{\mathcal{L}'_\mathcal{A}} = \widehat{\bigotimes}_{A\in\mathcal{A}} \mathcal{K}_{\mathcal{L}_A}$ induces a strict morphism of $\mathcal{S}(E_{\mathcal{L}'_\mathcal{A}}) \cong \widehat{\bigotimes\limits_{A\in\mathcal{A}}} \mathcal{S}(E_{\mathcal{L}_A})$ into $\widehat{\bigotimes\limits_{A\in\mathcal{A}}} \mathcal{S}(G_A) \cong \mathcal{S}(G)$. □

We now repeat, with some relevant modifications, the arguments that led to Theorem 4.8 in order to deal with property (S). We begin with another corollary of Lemma 4.6.

Corollary 4.10. *Assume that $\mathcal{A} = \{ A_1, A_2 \}$, and take $m \in \mathcal{M}(\mu_{\mathcal{L}'_\mathcal{A}}; \mathcal{S}(G_\mathcal{A}))$ and $T \in \mathcal{S}'(G_{A_2})$. Then,*

$$\big\langle T, g_2 \mapsto \mathcal{K}_{\mathcal{L}'_\mathcal{A}}(m)(\,\cdot\,, g_2)\big\rangle \in \mathcal{S}_{\mathcal{L}_{A_1}}(G_{A_1}).$$

Proof. Assume first that m is compactly supported. Let (μ_j) be a sequence of measures with compact support which converges to T in $\mathcal{S}'(G_{A_2})$. Then

$$\lim_{j\to\infty} \int_{G_{A_2}} \mathcal{K}_{\mathcal{L}'_\mathcal{A}}(m)(\,\cdot\,, g_2)\,\mathrm{d}\mu_j(g_2) = \big\langle T, g_2 \mapsto \mathcal{K}_{\mathcal{L}'_\mathcal{A}}(m)(\,\cdot\,, g_2)\big\rangle$$

in $\mathcal{S}(G_{A_1})$. Since $\int_{G_{A_2}} \mathcal{K}_{\mathcal{L}'_\mathcal{A}}(m)(\,\cdot\,, g_2)\,\mathrm{d}\mu_j(g_2) \in \mathcal{S}_{\mathcal{L}_{A_1}}(G_{A_1})$ by Lemma 4.6, and since $\mathcal{S}_{\mathcal{L}_{A_1}}(G_{A_1})$ is closed in $\mathcal{S}(G_{A_1})$ by Proposition 3.15, the assertion follows in this case.

The general case is then established as in the proof of Corollary 4.7. □

Theorem 4.11. *If $\mathcal{L}_A$ satisfies property (S) for every $A \in \mathcal{A}$, then $\mathcal{L}'_\mathcal{A}$ satisfies property (S).*

Proof. **1.** Proceeding by induction, we may reduce to the case in which $\mathcal{A} = \{ A_1, A_2 \}$. In order to simplify the notation, we shall simply write G_j and $\mathcal{S}(\sigma(\mathcal{L}_{A_j}))$ instead of G_{A_j} and $\mathcal{S}_{E_{\mathcal{L}_{A_j}},0}(\sigma(\mathcal{L}_{A_j}))$, respectively, for $j = 1, 2$. Now, take $f \in \mathcal{S}_{\mathcal{L}'_\mathcal{A}}(G)$ and let m be a representative of $\mathcal{M}_{\mathcal{L}'_\mathcal{A}}(f)$. As in the proof of Theorem 4.8, we see that

$$\mathcal{M}_{\mathcal{L}_{A_1}}[g_1 \mapsto \mathcal{M}_{\mathcal{L}_{A_2}}[f(g_1, \cdot\,)](\lambda_2)](\lambda_1) = m(\lambda_1, \lambda_2)$$

for $\beta_{\mathcal{L}_{A_1}}$-almost every $\lambda_1 \in E_{\mathcal{L}_{A_1}}$ and for $\beta_{\mathcal{L}_{A_2}}$-almost every $\lambda_2 \in E_{\mathcal{L}_{A_2}}$. Observe that Corollary 4.10 implies that $f(g_1, \cdot\,) \in \mathcal{S}_{\mathcal{L}_{A_2}}(G_2)$ for every $g_1 \in G_1$, and that by assumption $\mathcal{M}_{\mathcal{L}_{A_2}}$ induces an isomorphism of $\mathcal{S}_{\mathcal{L}_{A_2}}(G_2)$ onto $\mathcal{S}(\sigma(\mathcal{L}_{A_2}))$. Therefore, the mapping $g_1 \mapsto \mathcal{M}_{\mathcal{L}_{A_2}}[f(g_1, \cdot\,)]$ defines an element of $\mathcal{S}(G_1; \mathcal{S}(\sigma(\mathcal{L}_{A_2})))$.

2. Let us prove that the mapping $g_1 \mapsto \mathcal{M}_{\mathcal{L}_{A_2}}[f(g_1, \cdot\,)]$ canonically corresponds to an element of $\mathcal{S}_{\mathcal{L}_{A_1}}(G_1)\widehat{\otimes}\mathcal{S}(\sigma(\mathcal{L}_{A_2}))$. Indeed, take $T \in \mathcal{S}(\sigma(\mathcal{L}_{A_2}))'$; then Corollary 4.10 implies that

$$[g_1 \mapsto (T \cdot \mathcal{M}_{\mathcal{L}_{A_2}})[f(g_1, \cdot\,)]] \in \mathcal{S}_{\mathcal{L}_{A_1}}(G_1),$$

since $T \cdot \mathcal{M}_{\mathcal{L}_{A_2}}$ defines an element of $\mathcal{S}_{\mathcal{L}_{A_2}}(G_2)'$, which extends to an element of $\mathcal{S}'(G_2)$. Next, observe that [84, Proposition 50.4] implies that

$$\mathcal{S}_{\mathcal{L}_{A_1}}(G_1)\widehat{\otimes}\mathcal{S}(\sigma(\mathcal{L}_{A_2})) \cong \mathcal{L}(\mathcal{S}(\sigma(\mathcal{L}_{A_2}))'; \mathcal{S}_{\mathcal{L}_{A_1}}(G_1))$$

since $\mathcal{S}(\sigma(\mathcal{L}_{A_2}))$ is nuclear thanks to [84, Proposition 50.1]. Now, it is clear that the mapping $g_1 \mapsto \mathcal{M}_{\mathcal{L}_{A_2}}[f(g_1, \cdot\,)]$ belongs to $\mathcal{S}(G_1; \mathcal{S}(\sigma(\mathcal{L}_{A_2})))$, which is the canonical image of $\mathcal{S}(G_1)\widehat{\otimes}\mathcal{S}(\sigma(\mathcal{L}_{A_2})) \cong \mathcal{L}(\mathcal{S}(\sigma(\mathcal{L}_{A_2}))'; \mathcal{S}(G_1))$ (reason as above). Then, the preceding arguments imply our claim.

3. Now, $\mathcal{M}_{\mathcal{L}_{A_1}}$ induces an isomorphism of $\mathcal{S}_{\mathcal{L}_{A_1}}(G_1)$ onto $\mathcal{S}(\sigma(\mathcal{L}_{A_1}))$, so that the linear mapping

$$\mathcal{M}_{\mathcal{L}_{A_1}}\widehat{\otimes}I_{\mathcal{S}(\sigma(\mathcal{L}_{A_2}))}\colon \mathcal{S}_{\mathcal{L}_{A_1}}(G_1)\widehat{\otimes}\mathcal{S}(\sigma(\mathcal{L}_{A_2}))\to\mathcal{S}(\sigma(\mathcal{L}_{A_1}))\widehat{\otimes}\mathcal{S}(\sigma(\mathcal{L}_{A_2}))$$

is an isomorphism. In addition, for every $T\in\mathcal{S}(\sigma(\mathcal{L}_{A_2}))'$ and for every $\lambda_1\in\sigma(\mathcal{L}_{A_1})$,

$$\left\langle\delta_{\lambda_1}\widehat{\otimes}T,\left(\mathcal{M}_{\mathcal{L}_{A_1}}\widehat{\otimes}I_{\mathcal{S}(\sigma(\mathcal{L}_{A_2}))}\right)(g_1\mapsto\mathcal{M}_{\mathcal{L}_{A_2}}[f(g_1,\,\cdot\,)])\right\rangle=\mathcal{M}_{\mathcal{L}_{A_1}}[g_1\mapsto(T\cdot\mathcal{M}_{\mathcal{L}_{A_2}})[f(g_1,\,\cdot\,)]](\lambda_1)$$

(reason as in **2**). Choosing $T=\delta_{\lambda_2}$ for $\lambda_2\in\sigma(\mathcal{L}_{A_2})$, and taking into account Corollary 2.9, we see that the mapping

$$\sigma(\mathcal{L}'_{\mathcal{A}})\ni(\lambda_1,\lambda_2)\mapsto\mathcal{M}_{\mathcal{L}_{A_1}}(g_1\mapsto\mathcal{M}_{\mathcal{L}_{A_2}}[f(g_1,\,\cdot\,)](\lambda_2))(\lambda_1)$$

extends to an element m_0 of $\mathcal{S}(E_{\mathcal{L}'_{\mathcal{A}}})$. Now, **1** implies that $m_0(\lambda_1,\lambda_2)=m(\lambda_1,\lambda_2)$ for $\beta_{\mathcal{L}_{A_1}}$-almost every $\lambda_1\in E_{\mathcal{L}_{A_1}}$ and for $\beta_{\mathcal{L}_{A_2}}$-almost every $\lambda_2\in E_{\mathcal{L}_{A_2}}$. Since both m and m_0 are $\beta_{\mathcal{L}'_{\mathcal{A}}}$-measurable, Tonelli's theorem implies that $m=m_0$ $\beta_{\mathcal{L}'_{\mathcal{A}}}$-almost everywhere. Hence, $\mathcal{L}'_{\mathcal{A}}$ satisfies property (S). □

4.3 Image Families

In this section, $\mathcal{L}_A$ denotes a Rockland family on a homogeneous group G, Γ denotes a non-empty finite set, and $P\colon E_{\mathcal{L}_A}\to\mathbb{R}^\Gamma$ denotes a polynomial mapping with homogeneous components.

Proposition 4.12. *The following hold:*

1. *$P(\mathcal{L}_A)$ is an admissible family;*
2. *$\mu_{P(\mathcal{L}_A)}=P_*(\mu_{\mathcal{L}_A})$ and $\sigma(P(\mathcal{L}_A))=\overline{P(\sigma(\mathcal{L}_A))}$;*
3. *a function $m\colon E_{P(\mathcal{L}_A)}\to\mathbb{C}$ belongs to $\mathcal{M}(\mu_{P(\mathcal{L}_A)};\mathcal{D}'_w(G))$ if and only if $m\circ P$ belongs to $\mathcal{M}(\mu_{\mathcal{L}_A};\mathcal{D}'_w(G))$; in this case,*

$$\mathcal{K}_{P(\mathcal{L}_A)}(m)=\mathcal{K}_{\mathcal{L}_A}(m\circ P).$$

Observe that the same assertions hold, except the one referring to the spectrum, if we only assume that P is a $\beta_{\mathcal{L}_A}$-measurable mapping with homogeneous components and that $P_\gamma(\mathcal{L}_A)$ is a (left-invariant) differential operator for every $\gamma\in\Gamma$.

Proof. The first assertion follows from Theorem 3.19, while the second assertion follows from spectral theory. The third assertion is an easy consequence of the second one. □

Proposition 4.13. *The following statements are equivalent:*

1. *$P(\mathcal{L}_A)$ is Rockland;*
2. *$P(\lambda)\neq 0$ for every $\lambda\in\sigma(\mathcal{L}_A)\cap S_{\mathcal{L}_A}$;*
3. *P is proper on $\sigma(\mathcal{L}_A)$.*

Proof. **1** $\implies$ **2.** Take $m\in\mathcal{S}(E_{P(\mathcal{L}_A)})$ such that $m(0)\neq 0$. Then, [58, Proposition 3.2.11] implies that $m\circ P\in C_0(\sigma(\mathcal{L}_A))$. If $P(\lambda)=0$ for some $\lambda\in\sigma(\mathcal{L}_A)\cap S_{\mathcal{L}_A}$, then $(m\circ P)(r\cdot\lambda)=m(r\cdot P(\lambda))=m(0)\neq 0$ for every $r>0$, so that $m\circ P\notin C_0(\sigma(\mathcal{L}_A))$: contradiction.

2 $\implies$ **1.** This follows from Proposition 2.13.

2 $\iff$ **3.** Just observe that the closed set

$$P^{-1}(\overline{B}(0,R))\cap\sigma(\mathcal{L}_A)=\left\{\lambda\in\sigma(\mathcal{L}_A)\setminus\{\,0\,\}\colon|\lambda|\leqslant\frac{R}{|P(|\lambda|^{-1}\cdot\lambda)|}\right\}\cup\{\,0\,\}$$

is bounded if and only if $P(\lambda)\neq 0$ for every $\lambda\in\sigma(\mathcal{L}_A)\cap S_{\mathcal{L}_A}$. □

Proposition 4.14. *Assume that $P(\mathcal{L}_A)$ is a Rockland family. Then,*

$$\beta_{P(\mathcal{L}_A)} = P_*(\beta_{\mathcal{L}_A}).$$

In addition, let $(\beta_{\lambda'})_{\lambda' \in E_{P(\mathcal{L}_A)}}$ be a disintegration of $\beta_{\mathcal{L}_A}$ relative to P. Then,

$$\chi_{P(\mathcal{L}_A)}(\lambda', g) = \int_{E_{\mathcal{L}_A}} \chi_{\mathcal{L}_A}(\lambda, g) \, \mathrm{d}\beta_{\lambda'}(\lambda)$$

for $(\beta_{P(\mathcal{L}_A)} \otimes \nu_G)$-almost every $(\lambda', g) \in E_{P(\mathcal{L}_A)} \times G$.

Observe that the existence of a disintegration follows from [19, Theorem 1 of Chapter VI, § 3, No. 1].

Proof. The first assertion follows easily from Propositions 4.12 and 4.13. Then, take a representative χ_0 of $\chi_{\mathcal{L}_A}$. Let us first observe that the family $(\beta_{\lambda'} \otimes \nu_G)$ is $\beta_{P(\mathcal{L}_A)}$-adequate and that $\beta_{\mathcal{L}_A} \otimes \nu_G = \int_{E_{P(\mathcal{L}_A)}} (\beta_{\lambda'} \otimes \nu_G) \, \mathrm{d}\beta_{P(\mathcal{L}_A)}$, so that χ_0 is $(\beta_{\lambda'} \otimes \nu_G)$-measurable for $\beta_{P(\mathcal{L}_A)}$-almost every $\lambda' \in E_{P(\mathcal{L}_A)}$ by [19, Proposition 4 of Chapter V, § 3, No. 2]. Now, take $m \in \mathcal{D}(E_{P(\mathcal{L}_A)})$ and $\varphi \in \mathcal{D}(G)$. Then, Fubini's theorem implies that

$$\begin{aligned}
\int_{E_{P(\mathcal{L}_A)} \times G} (m \otimes \varphi) \chi_{P(\mathcal{L}_A)} \, \mathrm{d}(\beta_{P(\mathcal{L}_A)} \otimes \nu_G) &= \int_G \varphi \, \mathcal{K}_{P(\mathcal{L}_A)}(m) \, \mathrm{d}\nu_G \\
&= \int_G \varphi \, \mathcal{K}_{\mathcal{L}_A}(m \circ P) \, \mathrm{d}\nu_G \\
&= \int_{E_{\mathcal{L}_A} \times G} [(m \circ P) \otimes \varphi] \chi_0 \, \mathrm{d}(\beta_{\mathcal{L}_A} \otimes \nu_G) \\
&= \int_{E_{P(\mathcal{L}_A)} \times G} \int_{E_{\mathcal{L}_A}} (m \circ P)(\lambda) \varphi(g) \chi_0(\lambda, g) \, \mathrm{d}\beta_{\lambda'}(\lambda) \, \mathrm{d}(\beta_{P(\mathcal{L}_A)} \otimes \nu_G)(\lambda', g).
\end{aligned}$$

Since $\beta_{\lambda'}$ is concentrated on $P^{-1}(\lambda')$ for $\beta_{P(\mathcal{L}_A)}$-almost every $\lambda' \in E_{P(\mathcal{L}_A)}$, the last expression equals

$$\int_{E_{P(\mathcal{L}_A)} \times G} m(\lambda') \varphi(g) \int_{E_{\mathcal{L}_A}} \chi_0(\lambda, g) \, \mathrm{d}\beta_{\lambda'}(\lambda) \, \mathrm{d}(\beta_{P(\mathcal{L}_A)} \otimes \nu_G)(\lambda', g),$$

whence the result by the arbitrariness of m and φ. □

Proposition 4.15. *Assume that the following conditions hold:*

1. *$\sigma(\mathcal{L}_A)$ is subanalytic;*
2. *$\sigma(P(\mathcal{L}_A))$ is Nash subanalytic;*
3. *$P(\mathcal{L}_A)$ is a Rockland family;*
4. *$\mathcal{L}_A$ satisfies property $(S)_C$.*

Then, $P(\mathcal{L}_A)$ satisfies property $(S)_C$.

Proof. This follows from Proposition 4.13 and Theorem 2.29. □

The following result deals with a rather simple case in which Proposition 4.15 may not be applicable.

Proposition 4.16. *Assume that the following conditions hold:*

1. $\operatorname{Card} \Gamma = 1$ *and P is linear;*
2. *the closure of $\sigma(\mathcal{L}_A)$ with respect to the Zariski topology is $E_{\mathcal{L}_A}$;*
3. *$P(\mathcal{L}_A)$ is a Rockland family.*

Then, the following hold:

(i) if $\mathcal{L}_A$ satisfies property $(S)_C$, then $P(\mathcal{L}_A)$ satisfies property $(S)_C$;

(ii) if $\mathcal{L}_A$ satisfies property (S), then $P(\mathcal{L}_A)$ satisfies property (S).

As an application, one may prove that, if $\mathcal{L}$ is a sub-Laplacian on a Heisenberg group $\mathbb{H}^n$, and if T is an element of the centre of the Lie algebra of $\mathbb{H}^n$, then the family $(\mathcal{L} + iT)$ satisfies property (S) whenever it is Rockland.

Proof. Both assertions are consequences of the following statement:

'If $m \in C_b(E_{P(\mathcal{L}_A)})$ and $m \circ P$ equals an element of $\mathcal{S}(E_{\mathcal{L}_A})$ on $\sigma(\mathcal{L}_A)$, then m equals an element of $\mathcal{S}(E_{P(\mathcal{L}_A)})$ on $\sigma(P(\mathcal{L}_A))$.'

Then, take $m \in C_b(E_{P(\mathcal{L}_A)})$, and assume that there is $m_1 \in \mathcal{S}(E_{\mathcal{L}_A})$ such that $m \circ P = m_1$ on $\sigma(\mathcal{L}_A)$. Notice that we may take $\Sigma \subseteq \sigma(\mathcal{L}_A)$ so that $\sigma(\mathcal{L}_A) \subseteq \bigcup_{r \geqslant 0} r\Sigma$,[2] and $P(x) = \pm 1$ for every $x \in \Sigma$. Define $\Sigma_\pm$ as the set of $x \in \Sigma$ such that $P(x) = \pm 1$, so that Σ is the disjoint union of Σ_- and Σ_+. Take x in $\Sigma_\pm$ (if not empty). Then, $m_1(\lambda x) = m(\pm\lambda)$ for every $\lambda \geqslant 0$, so that m coincides with a Schwartz function on $\mathbb{R}_\pm$. If either Σ_- or Σ_+ is empty, then the assertion follows. Now, assume that $\Sigma_-, \Sigma_+ \neq \emptyset$. Notice that there is $\varepsilon \in \{ -, + \}$ such that the closure of $\bigcup_{r \geqslant 0} r\Sigma_\varepsilon$ with respect to the Zariski topology is $E_{\mathcal{L}_A}$. Then, take $k \in \mathbb{N}$ and let P_k be the Taylor polynomial of order k of m_1 about 0; in addition, let $P_{-,k}$ and $P_{+,k}$ be the left and right Taylor polynomials of order k, respectively, of m about 0. Then, the polynomial $P_k - (P_{\pm,k} \circ P)$ vanishes on $\bigcup_{r \geqslant 0} r\Sigma_\pm$, so that $P_k = P_{\varepsilon,k} \circ P$. Hence, $m_-^{(k)}(0) = m_+^{(k)}(0)$ for every $k \in \mathbb{N}$, so that we may take $m_2 \coloneqq m$ by the arbitrariness of k, and we are done. □

Further results concerning properties (RL) and (S) may be proved making use of the results of Sections 2.3 and 2.4.

4.4 Functional Equivalence and Completeness

In this section, $\mathcal{L}_A$ and $\mathcal{L}'_{A'}$ denote two Rockland families on a homogeneous group G.

Definition 4.17. $\mathcal{L}_A$ and $\mathcal{L}'_{A'}$ are equivalent if they generate the same algebra.

In other words, $\mathcal{L}_A$ and $\mathcal{L}'_{A'}$ are equivalent if and only if there are two polynomial mappings with homogeneous components $P\colon E_{\mathcal{L}_A} \to E_{\mathcal{L}'_{A'}}$ and $Q\colon E_{\mathcal{L}'_{A'}} \to E_{\mathcal{L}_A}$ such that $P(\mathcal{L}_A) = \mathcal{L}'_{A'}$ and $Q(\mathcal{L}'_{A'}) = \mathcal{L}_A$. Therefore, the results of Section 4.3 show that P and Q induce two isomorphisms between $\mathcal{S}_{E_{\mathcal{L}_A},0}(\sigma(\mathcal{L}_A))$ and $\mathcal{S}_{E_{\mathcal{L}'_{A'}},0}(\sigma(\mathcal{L}'_{A'}))$ which are inverse of one another, and that $\mathcal{L}_A$ satisfies property (RL), (S), $(S)_C$, etc., if and only if $\mathcal{L}'_{A'}$ satisfies property (RL), (S), $(S)_C$, etc., respectively.

Therefore, our analysis truly depends only on the algebra generated by the chosen Rockland family; nevertheless, we shall not pursue this approach any further. Instead, let us state the following definition

Definition 4.18. $\mathcal{L}_A$ and $\mathcal{L}'_{A'}$ are functionally equivalent if $\mathcal{D}'_{\mathcal{L}_A}(G) = \mathcal{D}'_{\mathcal{L}_A}(G)$.

In other words, $\mathcal{L}_A$ and $\mathcal{L}'_{A'}$ are functionally equivalent if and only if there are two measurable mappings $m_1\colon E_{\mathcal{L}_A} \to E_{\mathcal{L}'_{A'}}$ and $m_2\colon E_{\mathcal{L}'_{A'}} \to E_{\mathcal{L}_A}$ such that $m_1(\mathcal{L}_A) = \mathcal{L}'_{A'}$ and $m_2(\mathcal{L}'_{A'}) = \mathcal{L}_A$.

Notice that this kind of equivalence is much weaker than the preceding one; for example, properties (RL) and (S) are *not* preserved by functional equivalence.

Definition 4.19. $\mathcal{L}_A$ is complete if it generates the unital algebra $(\mathfrak{U}_\mathbb{C}(\mathfrak{g}))_{\mathcal{L}_A}$.

Proposition 4.20. *$\mathcal{L}_A$ is complete if and only if every Rockland family which is functionally equivalent to $\mathcal{L}_A$ is actually an image family of $\mathcal{L}_A$.*

Proof. One implication is clear. Then, assume that every Rockland family which is functionally equivalent to $\mathcal{L}_A$ is actually an image family of $\mathcal{L}_A$, and take $T \in \mathcal{A} \coloneqq (\mathfrak{U}_\mathbb{C}(\mathfrak{g}))_{\mathcal{L}_A}$. Let $T_1, \dots, T_k$ be the homogeneous components of T, and let $\delta_1, \dots, \delta_k$ be their (positive) homogeneous degrees. Now, take $0 < r_1 < \dots < r_k$, and observe that the matrix $(r_{j_1}^{\delta_{j_2}})_{j_1, j_2 = 1, \dots, k}$ is invertible by [73]. Since $(r_{j_1} \cdot)_* T = \sum_{j_2=1}^k r_{j_1}^{\delta_{j_2}} T_{j_2} \in \mathcal{A}$ for every $j_1 = 1, \dots, k$ (cf. Proposition 3.7), it follows that $T_1, \dots, T_k \in \mathcal{A}$. Then, the family obtained by adding to $\mathcal{L}_A$ the

[2] Notice that here we consider scalar multiplication, and *not* the dilations of $E_{\mathcal{L}_A}$.

left-invariant differential operators associated with the T_j such that $\delta_j > 0$ is a Rockland family, and is functionally equivalent to $\mathcal{L}_A$. Therefore, the operators $T_1, \dots, T_k$ belong to the unital algebra generated by $\mathcal{L}_A$, so that also T does. The assertion follows. □

Observe, in addition, that property (S) implies completeness, even though the converse fails in general (cf. Proposition 7.52).

Proposition 4.21. *If $\mathcal{L}_A$ satisfies property (S), then it is complete.*

Proof. Assume that $\mathcal{L}_A$ is not complete, so that Proposition 4.20 implies that there is $m \in \mathcal{M}(\mu_{\mathcal{L}_A}; \mathfrak{U}_\mathbb{C}(\mathfrak{g}))$ such that $\mathcal{K}_{\mathcal{L}_A}(m)$ is homogeneous of homogeneous degree $\delta \geqslant 0$, and such that m is not equal $\beta_{\mathcal{L}_A}$-almost everywhere to any polynomials.

Set $k := \min\{\, h \in \mathbb{N} \colon \delta \leqslant h \min_{\alpha \in A} \delta_\alpha \,\}$,[3] and assume by contradiction that $m \in C^k(E_{\mathcal{L}_A})$. Then [43, Theorem 1.37] implies that there is a family with finite support $(P_{\delta'})_{0 \leqslant \delta' \leqslant \delta}$, where $P_{\delta'}$ is a homogeneous polynomial of homogeneous degree δ' for every $\delta' \in [0, \delta]$, such that

$$m(\lambda) = \sum_{0 \leqslant \delta' \leqslant \delta} P_{\delta'}(\lambda) + o\Big(|\lambda|^\delta\Big)$$

as $\lambda \to 0$. Now, recall that we have $m(r \cdot \lambda) = r^\delta m(\lambda)$ for every $r > 0$ and for $\beta_{\mathcal{L}_A}$-almost every $\lambda \in E_{\mathcal{L}_A}$. Since m is continuous, the preceding equality holds for *every* $\lambda \in \sigma(\mathcal{L}_A)$. Now, fix a non-zero $\lambda \in \sigma(\mathcal{L}_A)$. Then, we have

$$r^\delta m(\lambda) = m(r \cdot \lambda) = \sum_{0 \leqslant \delta' \leqslant \delta} P_{\delta'}(r \cdot \lambda) + o\Big(|r \cdot \lambda|^\delta\Big) = \sum_{0 \leqslant \delta' \leqslant \delta} r^{\delta'} P_{\delta'}(\lambda) + o\big(r^\delta\big)$$

for $r \to 0^+$, so that $P_{\delta'}(\lambda) = 0$ for every $\delta' \in [0, \delta[$ and $P_\delta(\lambda) = m(\lambda)$. Therefore, we have $m = P_\delta$ on $\sigma(\mathcal{L}_A)$: contradiction.

Hence, if we take $\tau \in \mathcal{S}(E_{\mathcal{L}_A})$ so that $\tau(\lambda) > 0$ for every $\lambda \in E_{\mathcal{L}_A}$, then $m\tau$ is not equal $\beta_{\mathcal{L}_A}$-almost everywhere to any elements of $\mathcal{S}(E_{\mathcal{L}_A})$, but

$$\mathcal{K}_{\mathcal{L}_A}(m\tau) = \mathcal{K}_{\mathcal{L}_A}(\tau) * \mathcal{K}_{\mathcal{L}_A}(m) \in \mathcal{S}(G)$$

by Corollary 3.12, so that $\mathcal{L}_A$ does not satisfy property (S). □

It therefore seems reasonable to only work with complete families; nevertheless we have not been able to prove that any Rockland family has a 'completion,' that is, that $(\mathfrak{U}_\mathbb{C}(\mathfrak{g}))_{\mathcal{L}_A}$ is a finitely generated algebra for every Rockland family $\mathcal{L}_A$, so that the notion of complete families is of limited use.

[3] Recall that δ_α is the homogeneous degree of $\mathcal{L}_\alpha$; we assume, for the sake of convenience, that no $\mathcal{L}_\alpha$ is zero.

Chapter 5

Mihlin Multipliers and Calderón-Zygmund Kernels

In this chapter we shall generalize to Rockland families a classical result about the correspondence between kernels which satisfy 'Calderón-Zygmund conditions of order ∞' and multipliers which satisfy 'Mihlin conditions of order ∞' (cf. [64, Theorems 2.1.11 and 2.2.1]).

We identify G with $\mathbb{R}^n$ by means of a homogeneous basis $X_1, \dots, X_n$ of left-invariant vector fields (and the group exponential map); $Y_1, \dots, Y_n$ denotes the corresponding basis of right-invariant vector fields. In addition to that, we shall assume that the homogeneous degrees $\mathrm{d}_1, \cdots, \mathrm{d}_n$ of $X_1, \dots, X_n$ are ordered so that $\mathrm{d}_1 \leqslant \dots \leqslant \mathrm{d}_n$. If $\gamma \in \mathbb{N}^n$, we define $\mathrm{d}_\gamma := \gamma_1 \mathrm{d}_1 + \cdots + \gamma_n \mathrm{d}_n$. In this chapter we shall write $K^{(j)}$ instead of $2^{jQ} K(2^j \cdot)$ for $K \in L^1_{\mathrm{loc}}(G)$.

Let us now define what we mean by 'Calderón-Zygmund kernels.'

Definition 5.1. Define $\mathcal{KZ}(G)$ as the space of tempered distributions K on G which coincide with a C^∞ function on $G^* := G \setminus \{ e \}$ and satisfy the following conditions:

1. **('Size conditions')** for every $\gamma \in \mathbb{N}^n$, there is $C_\gamma > 0$ such that
$$|(\mathbf{X}^\gamma K)(x)| \leqslant \frac{C_\gamma}{|x|^{Q + \mathrm{d}_\gamma}}$$
for every $x \neq e$;

2. **('Cancellation conditions')** there is $C > 0$ such that
$$|\langle K, \eta(\delta \cdot) \rangle| \leqslant C$$
for every $\delta > 0$ and for every $\eta \in \mathcal{D}(G)$ such that $\mathrm{Supp}(\eta) \subseteq \overline{B(0,1)}$ and $\|\eta\|_\infty$, $\|X_1 \eta\|_\infty, \dots, \|X_n \eta\|_\infty \leqslant 1$ ('normalized bump functions').

We shall endow $\mathcal{KZ}(G)$ with the norms

$$K \mapsto \sup_{|\gamma| \leqslant k} \sup_{x \in G^*} |x|^{Q + \mathrm{d}_\gamma} |(\mathbf{X}^\gamma K)(x)| + \sup_{\substack{\eta \in \mathcal{D}_{\overline{B}(0,1)}(G) \\ \|\eta\|_\infty, \|X_1 \eta\|_\infty, \dots, \|X_n \eta\|_\infty \leqslant 1}} \sup_{\delta > 0} |\langle K, \eta(\delta \cdot) \rangle|,$$

as k runs through $\mathbb{N}$.

Then, let us define what we mean by 'Mihlin-Hörmander multipliers.' Recall that δ_α is the homogeneous degree of $\mathcal{L}_\alpha$ for every $\alpha \in A$, 1 if $\mathcal{L}_\alpha = 0$; define $\delta_\gamma := \sum_{\alpha \in A} \gamma_\alpha \delta_\alpha$ for every $\gamma \in \mathbb{N}^A$.

Definition 5.2. Define $\mathcal{MH}(E_{\mathcal{L}_A})$ as the space of $m \in C^\infty(E^*_{\mathcal{L}_A})$, where $E^*_{\mathcal{L}_A} = E_{\mathcal{L}_A} \setminus \{ 0 \}$, such that for every $\gamma \in \mathbb{N}^n$ there is $C_\gamma > 0$ such that

$$|(\partial^\gamma m)(\lambda)| \leqslant \frac{C_\gamma}{|\lambda|^{\delta_\gamma}}$$

for every $\lambda \in E^*_{\mathcal{L}_A}$. We shall endow $\mathcal{MH}(E_{\mathcal{L}_A})$ with the norms

$$m \mapsto \sup_{|\gamma|\leqslant k} \sup_{\lambda\in E^*} |\lambda|^{\delta_\gamma} |(\partial^\gamma m)(\lambda)|,$$

as k runs through $\mathbb{N}$.

Then, with routine arguments one may prove the following results (cf., for example, [64, Theorem 2.2.1]).

Lemma 5.3. *Let $\mathfrak{K}$ be a bounded subset of $\mathcal{S}(G)^\mathbb{Z}$ such that K_j has integral 0 for every $(K_j) \in \mathfrak{K}$ for every $j \in \mathbb{Z}$.*

Then, for each $(K_j) \in \mathfrak{K}$ the sum $\sum_{j\in\mathbb{Z}} K_j^{(j)}$ converges absolutely in $\mathcal{E}(G^*)$ and in $\mathcal{S}'(G)$. In addition, the set of $\sum_{j\in\mathbb{Z}} K_j^{(j)}$, as (K_j) runs through $\mathfrak{K}$, is bounded in $\mathcal{KZ}(G)$.

Lemma 5.4. *Let $\mathfrak{M}$ be a bounded subset of $\mathcal{S}(E_{\mathcal{L}_A})^\mathbb{Z}$ such that $m_j(0) = 0$ for every $(m_j) \in \mathfrak{M}$ and for every $j \in \mathbb{Z}$.*

Then, for each $(m_j) \in \mathfrak{M}$ the sum $\sum_{j\in\mathbb{Z}} m_j(2^j\,\cdot\,)$ converges absolutely in $\mathcal{E}(E^*)$. In addition, the set of $\sum_{j\in\mathbb{Z}} m_j(2^j\,\cdot\,)$, as (m_j) runs through $\mathfrak{M}$, is bounded in $\mathcal{MH}(E_{\mathcal{L}_A})$.

Taking into account the fact that $\mathcal{K}_{\mathcal{L}_A}$ maps $\mathcal{S}(E_{\mathcal{L}_A})$ into $\mathcal{S}(G)$ continuously, we deduce the following result.

Corollary 5.5. *$\mathcal{K}_{\mathcal{L}_A}$ maps $\mathcal{MH}(E_{\mathcal{L}_A})$ into $\mathcal{KZ}(G)$ continuously.*

We now want to prove a converse of the preceding statement. In order to do that, we first need to define in a reasonable way the set of elements of $\mathcal{KZ}(G)$ which ought to be kernels corresponding to multipliers in $\mathcal{MH}(E_{\mathcal{L}_A})$. On the one hand, the space $\mathcal{KZ}_{\mathcal{L}_A}(G)$ of kernels belonging to $\mathcal{KZ}(G)$ is too large for our purposes, unless $\mathcal{L}_A$ satisfies property (S). On the other hand, the closure of $\mathcal{K}_{\mathcal{L}_A}(\mathcal{S}(E_{\mathcal{L}_A}))$ in $\mathcal{KZ}(G)$ is too small; since we have not been able to describe its weak closure, we do not know if it would be a reasonable choice. Therefore, we shall make use of the following definition.

Definition 5.6. Define $\mathcal{KZ}(G, \mathcal{L}_A)$ as the union of the closures in $\mathcal{S}'_w(G)$ of the subsets of $\mathcal{S}(G, \mathcal{L}_A)$ which are bounded in $\mathcal{KZ}(G)$.

Then, we can state the announced converse.

Theorem 5.7. *Assume $\mathcal{L}_A$ satisfies property $(S)_C$, and let $\mathfrak{K}$ be a subset of $\mathcal{S}'(G)$. Then, the following conditions are equivalent:*

(i) $\mathfrak{K}$ is a bounded subset of $\mathcal{KZ}(G, \mathcal{L}_A)$;

(ii) there is a bounded family $(m_K)_{K\in\mathfrak{K}}$ of elements of $\mathcal{MH}(E_{\mathcal{L}_A})$ such that $K = \mathcal{K}_{\mathcal{L}_A}(m_K)$ for every $K \in \mathfrak{K}$;

(iii) there is a bounded family $(m_{K,j})_{K\in\mathfrak{K},j\in\mathbb{Z}}$ of elements of $\mathcal{S}(E_{\mathcal{L}_A})$ which vanish at 0 such that $K = \sum_{j\in\mathbb{Z}} \mathcal{K}_{\mathcal{L}_A}(m_{K,j})^{(j)}$ in $\mathcal{S}'_w(G)$ for every $K \in \mathfrak{K}$;

(iv) there is a bounded family $(H_{K,j})_{K\in\mathfrak{K},j\in\mathbb{Z}}$ of elements of $\mathcal{K}_{\mathcal{L}_A}(\mathcal{S}(E_{\mathcal{L}_A}))$ with integral 0 such that $K = \sum_{j\in\mathbb{Z}} H_{K,j}^{(j)}$ in $\mathcal{S}'_w(G)$ for every $K \in \mathfrak{K}$.

Furthermore, $\mathcal{K}_{\mathcal{L}_A}$ induces a strict morphism of $\mathcal{MH}(E_{\mathcal{L}_A})$ onto $\mathcal{KZ}(G, \mathcal{L}_A)$.

In particular, $\mathcal{KZ}(G, \mathcal{L}_A)$ is closed in $\mathcal{KZ}(G)$.

In order to prove Theorem 5.7, we need the following proposition.

Proposition 5.8. *Let $\mathfrak{K}$ be a bounded subset of $\mathcal{KZ}(G)$, $\varphi \in \mathcal{S}(G)$, and $k \in \mathbf{d}\cdot\mathbb{N}^n$. Assume that $\langle\varphi|P\rangle = 0$ for every polynomial of homogeneous degree $< k$. Then, for every homogeneous differential operator X with continuous coefficients on G^* and with homogeneous degree d, there is a constant $C_X > 0$ such that*

$$|X(K * \varphi)(x)| \leqslant \frac{C_X}{|x|^{Q+k+\mathrm{Re}\, d}}$$

for every $x \in G^*$ and for every $K \in \mathfrak{K}$.

In particular, for $k=0$ we see that $\mathcal{KZ}(G)$ embeds continuously into $W^{\infty,-\infty,p}(G)$ for every $p\in]1,\infty]$ and, analogously, into $W^{-\infty,\infty,p}(G)$ (cf. Corollary 1.77).

Proof. Fix $\eta\in\mathbb{N}^n$ and take $\tau\in\mathcal{D}(G)$ such that $\chi_{B(0,1)}\leqslant\tau\leqslant\chi_{B(0,2)}$. Notice that by Proposition 2.18 there is a family with finite support $(\varphi_{k,\gamma})_{\mathrm{d}_\gamma\geqslant k}$ of elements of $\mathcal{S}(G)$ such that

$$\varphi=\sum_{\mathrm{d}_\gamma\geqslant k}\mathbf{Y}^\gamma\varphi_{k,\gamma}.$$

Then, for every $K\in\mathfrak{K}$,

$$\mathbf{Y}^\eta(K*\varphi)=\sum_\gamma \mathbf{Y}^\eta(K*(\mathbf{Y}^\gamma\varphi_{k,\gamma}))=\sum_\gamma (X_n^{\gamma_n}\dots X_1^{\gamma_1}\mathbf{Y}^\eta K)*\varphi_{k,\gamma}$$

For every $K\in\mathfrak{K}$ and for every γ, define $K_{0,\gamma,\eta}\coloneqq\tau(X_n^{\gamma_n}\dots X_1^{\gamma_1}\mathbf{Y}^\eta K)$ and $K_{\infty,\gamma,\eta}\coloneqq(1-\tau)(X_n^{\gamma_n}\dots X_1^{\gamma_1}\mathbf{Y}^\eta K)$. Set $\mathrm{d}_{\gamma,\eta}\coloneqq Q+\mathrm{d}_\gamma+\mathrm{d}_\eta$. Then, for every γ there is a constant $C_{\gamma,\eta}>0$ such that

$$|K_{\infty,\gamma,\eta}(x)|\leqslant\frac{C_{\gamma,\eta}}{|x|^{\mathrm{d}_{\gamma,\eta}}}$$

for every $x\in G$. In particular, this implies that $|K_{\infty,\gamma,\eta}(x)|\leqslant C_{\gamma,\eta}$ for every $x\in G$, so that we may assume that

$$|K_{\infty,\gamma,\eta}(x)|\leqslant\frac{C_{\gamma,\eta}}{(1+|x|)^{\mathrm{d}_{\gamma,\eta}}}$$

for every $x\in G$. Take $C>0$ such that $|xy|\leqslant C(|x|+|y|)$ for every $x,y\in G$. Then,

$$\begin{aligned}\left\|(1+|\cdot|)^{d_{\gamma,\eta}}(K_{\infty,\gamma,\eta}*\varphi_{k,\gamma})\right\|_\infty&\leqslant C^{\mathrm{d}_{\gamma,\eta}}\left\|\left((1+|\cdot|)^{\mathrm{d}_{\gamma,\eta}}|K_{\infty,\gamma,\eta}|\right)*\left((1+|\cdot|)^{\mathrm{d}_{\gamma,\eta}}|\varphi_{k,\gamma}|\right)\right\|_\infty\\&\leqslant C^{\mathrm{d}_{\gamma,\eta}}C_{\gamma,\eta}\left\|(1+|\cdot|)^{\mathrm{d}_{\gamma,\eta}}\varphi_{k,\gamma}\right\|_1\end{aligned}$$

for every $K\in\mathfrak{K}$. Therefore, there is constant $C'_{\gamma,\eta}$ such that

$$|(K_{\infty,\gamma,\eta}*\varphi_{k,\gamma})(x)|\leqslant\frac{C'_{\gamma,\eta}}{(1+|x|)^{Q+\mathrm{d}_\gamma+\mathrm{d}_\eta}}$$

for every $x\in G$.

Next, notice that

$$(K_{0,\gamma,\eta}*\varphi_{k,\gamma})(x)=\langle K_{0,\gamma,\eta},L_x\check{\varphi}_{k,\gamma}\rangle=(-1)^{|\gamma|+|\eta|}\langle K,Y_n^{\eta_n}\dots Y_1^{\eta_1}\mathbf{X}^\gamma(\tau L_x\check{\varphi}_{k,\gamma})\rangle.$$

Now, the set of

$$(1+|x|)^{\mathrm{d}_{\gamma,\eta}}\tau L_x\check{\varphi}_{k,\gamma},$$

as x runs through G, is bounded in $\mathcal{D}_{B(0,2)}(G)$. By the cancellation conditions, this proves that there is a constant $C''_{\gamma,\eta}>0$ such that

$$|(K_{0,\gamma,\eta}*\varphi_{k,\gamma})(x)|\leqslant\frac{C''_{\gamma,\eta}}{(1+|x|)^{Q+\mathrm{d}_\gamma+\mathrm{d}_\eta}}$$

for every $x\in G$. Therefore,

$$|\mathbf{Y}^\eta(K*\varphi)(x)|\leqslant\frac{1}{(1+|x|)^{Q+k+\mathrm{d}_\eta}}\sum_{\gamma\colon\varphi_{k,\gamma}\neq0}(C'_{\gamma,\eta}+C''_{\gamma,\eta}).$$

This proves the first assertion when X has the form $\mathbf{Y}^\eta$ for some η. Now, let X be arbitrary. Then, there is a unique family with finite support of continuous functions (f_η) such that

$$X=\sum_\eta f_\eta\mathbf{Y}^\eta.$$

Moreover, by uniqueness and homogeneity, f_η is homogeneous of degree $\mathrm{d}_\eta-d$ for every η. Therefore, it is easily seen that there is $C'>0$ such that

$$|X(K*\varphi)(x)|\leqslant\frac{C'}{|x|^{Q+k+\mathrm{Re}\,d}}$$

for every $x\in G^*$. □

Proof of Theorem 5.7. **1.** Let us first prove the implication **(ii)** $\implies$ **(iii)**. Take a positive function $\varphi \in \mathcal{D}(E^*_{\mathcal{L}_A})$ such that $\sum_{j\in\mathbb{Z}} \varphi(2^j\,\cdot\,) = \chi_{E^*}$; define $\psi := \mathcal{K}_{\mathcal{L}_A}(\varphi)$ and $m_{K,j} := m_K(2^{-j}\,\cdot\,)\varphi$. It is easily seen that the family $(m_{K,j})_{K,j}$ is bounded in $\mathcal{S}(E_{\mathcal{L}_A})$. In addition, $m_{K,j}(0) = 0$, and, since the sum $\sum_{j\in\mathbb{Z}} m_{K,j}(2^j\,\cdot\,)$ converges pointwise and boundedly to m_K on $E^*_{\mathcal{L}_A}$, the sum $\sum_{j\in\mathbb{Z}} m_{K,j}(2^j\mathcal{L}_A)$ converges pointwise to $m_K(\mathcal{L}_A)$. Hence, $K = \mathcal{K}_{\mathcal{L}_A}(m_K)$.

2. Now, consider the implications **(iii)** $\implies$ **(iv)** and **(iv)** $\implies$ **(i)**. By hypothesis, the set of $H_{K,j} := \mathcal{K}_{\mathcal{L}_A}(m_{K,j})$, as K runs through $\mathfrak{K}$ and j runs through $\mathbb{Z}$, is bounded in $\mathcal{S}(G,\mathcal{L}_A)$; furthermore, its elements have integral 0 by Corollary 3.49. This proves the implication **(iii)** $\implies$ **(iv)**. Now, Lemma 5.3 implies that the set of $\sum_{j\in J} H^{(j)}_{K,j}$, as K runs through $\mathfrak{K}$ and J runs through the set of finite subsets of $\mathbb{Z}$, is bounded in $\mathcal{KZ}(G)$ and contained in $\mathcal{S}(G,\mathcal{L}_A)$. Hence, $\mathfrak{K}$ is a bounded subset of $\mathcal{KZ}(G)$ contained in $\mathcal{KZ}(G,\mathcal{L}_A)$. This proves the implication **(iv)** $\implies$ **(i)**.

3. Notice that, by the way, we have proved that that $\mathcal{K}_{\mathcal{L}_A}$ maps $\mathcal{MH}(E_{\mathcal{L}_A})$ into $\mathcal{KZ}(G,\mathcal{L}_A)$ continuously. Indeed, the implication **(ii)** $\implies$ **(i)** shows that $\mathcal{K}_{\mathcal{L}_A}$ maps $\mathcal{MH}(E_{\mathcal{L}_A})$ into $\mathcal{KZ}(G,\mathcal{L}_A)$ boundedly, and the assertion follows from the fact that $\mathcal{MH}(E_{\mathcal{L}_A})$ is a Fréchet space.

4. Let us now prove the implication **(iii)** $\implies$ **(ii)**. Indeed, define $m_K := \sum_{j\in\mathbb{Z}} m_{K,j}(2^j\,\cdot\,)$ in $\mathcal{E}(E^*_{\mathcal{L}_A})$; by Lemma 5.4, this definition is well-posed; furthermore, the set of m_K, as K runs through $\mathfrak{K}$, is bounded in $\mathcal{MH}(E_{\mathcal{L}_A})$. Lemma 5.4 also implies that the set of $\sum_{j\in J} m_{K,j}(2^j\,\cdot\,)$, as K runs through $\mathfrak{K}$ and J runs through the set of finite subsets of $\mathbb{Z}$, is bounded in $\mathcal{MH}(E_{\mathcal{L}_A})$. Hence, the sum $\sum_{j\in\mathbb{Z}} m_{K,j}(2^j\,\cdot\,)$ converges pointwise and boundedly to m_K on $E^*_{\mathcal{L}_A}$. This proves that $K = \sum_{j\in\mathbb{Z}} \mathcal{K}_{\mathcal{L}_A}(m_{K,j})^{(j)} = \mathcal{K}_{\mathcal{L}_A}(m_K)$ in $\mathcal{S}'(G)$ by Lemma 3.17.

5. Assume that $\mathfrak{K}$ is a bounded subset of $\mathcal{KZ}(G,\mathcal{L}_A)$. Take some $\varphi \in \mathcal{D}(E^*_{\mathcal{L}_A})$ such that $\sum_{j\in\mathbb{Z}} \varphi(2^j\,\cdot\,)^2 = \chi_{E^*_{\mathcal{L}_A}}$, and define $\psi := \mathcal{K}_{\mathcal{L}_A}(\varphi)$. We shall prove that there is a bounded family $(m_{K,j})_{K\in\mathfrak{K},j\in\mathbb{Z}}$ of elements of $\mathcal{D}(E^*_{\mathcal{L}_A})$ such that $\mathcal{K}_{\mathcal{L}_A}(m_{K,j}) = K^{(-j)} * \psi * \psi$.[1] Indeed, define $\mathcal{L} := \sum_{\alpha\in A} \mathcal{L}^2_\alpha$. Then $\mathcal{L}$ is a left-invariant differential operator without constant term. Further, for every $k \in \mathbb{N}$ the mapping $\lambda \mapsto \frac{1}{\|\lambda\|^{2k}}\varphi(\lambda)$ belongs to $\mathcal{S}(E_{\mathcal{L}_A})$, so that there is $\psi_k \in \mathcal{S}(G,\mathcal{L}_A)$ such that $\psi = \mathcal{L}^k\psi_k$. Since the set of $K^{(-j)}$, as K runs through $\mathfrak{K}$ and j runs through $\mathbb{Z}$, is bounded in $\mathcal{KZ}(G)$, Proposition 5.8 implies that the set of $K^{(-j)} * \psi$, as K runs through $\mathfrak{K}$ and j runs through $\mathbb{Z}$, is bounded in $\mathcal{S}(G,\mathcal{L}_A)$ (cf. **6** below). Hence, there is a bounded family $(\widetilde{m}_{K,j})_{K\in\mathfrak{K},j\in\mathbb{Z}}$ of elements of $\mathcal{S}(E_{\mathcal{L}_A})$ such that $\mathcal{K}_{\mathcal{L}_A}(\widetilde{m}_{K,j}) = K^{(-j)} * \psi$ for every $K \in \mathcal{K}_{\mathcal{L}_A}$ and for every $j \in \mathbb{Z}$. Define $m_{K,j} := \widetilde{m}_{K,j}\varphi$ for every $K \in \mathfrak{K}$ and $j \in \mathbb{Z}$. Then $(m_{K,j})$ is a bounded family of elements of $\mathcal{D}(E^*_{\mathcal{L}_A})$, and $\mathcal{K}_{\mathcal{L}_A}(m_{K,j}) = K^{(-j)} * \psi * \psi$.

6. Now we prove that $\mathcal{K}_{\mathcal{L}_A}$ maps $\mathcal{MH}(E_{\mathcal{L}_A})$ *onto* $\mathcal{KZ}(G,\mathcal{L}_A)$. Indeed, let $K \in \mathcal{KZ}(G,\mathcal{L}_A)$. Then there is a bounded subset B of $\mathcal{KZ}(G)$ contained in $\mathcal{S}(G,\mathcal{L}_A)$ such that K belongs to the closure $\overline{B}$ of B with respect to the topology induced by $\mathcal{S}'_w(G)$. Since $\mathcal{S}(G)$ is barrelled, and since B is pointwise bounded, it follows that B is equicontinuous, so that also $\overline{B}$ is. Now, since $\mathcal{S}(G)$ is a Fréchet-Montel space, it is separable, so that the uniformity induced on $\overline{B}$ by $\mathcal{S}'_w(G)$ is metrizable. Hence, there is a sequence (K_k) of elements of B which converges to K in $\mathcal{S}'_w(G)$. By **4** and **5**, there is a bounded sequence (m_k) in $\mathcal{MH}(E_{\mathcal{L}_A})$, whose elements belong to $\mathcal{S}(E_{\mathcal{L}_A})$, such that $\mathcal{K}_{\mathcal{L}_A}(m_k) = K_k$ for every $k \in \mathbb{N}$. Now, the sequence (m_k) is obviously bounded in $\mathcal{E}(E^*_{\mathcal{L}_A})$; since this latter space is a Fréchet-Montel space, we may assume that (m_k) converges to some m in $\mathcal{E}(E^*_{\mathcal{L}_A})$. It is then easily verified that m belongs to $\mathcal{MH}(E_{\mathcal{L}_A})$. Notice that then (m_k) converges pointwise and boundedly to m on $E^*_{\mathcal{L}_A}$, so that $\mathcal{K}_{\mathcal{L}_A}(m) = K$.

7. Now, the implication **(i)** $\implies$ **(iii)** is an easy consequence of **5** and **6** (use Lemma 5.3 to prove that $K = \sum_{j\in\mathbb{Z}} \mathcal{K}_{\mathcal{L}_A}(m_{K,j})^{(j)}$ in $\mathcal{S}'_w(G)$). Finally, $\mathcal{K}_{\mathcal{L}_A}$ is a strict morphism thanks to the implication **(i)** $\implies$ **(ii)** and to the fact that $\mathcal{KZ}(G,\mathcal{L}_A)$ is a metrizable space. □

Corollary 5.9. *If $\mathcal{L}_A$ satisfies property (S), then $\mathcal{KZ}_{\mathcal{L}_A}(G) = \mathcal{KZ}(G,\mathcal{L}_A)$. In other words, if an element of $\mathcal{KZ}(G)$ corresponds to some multiplier, then this multiplier can be taken in $\mathcal{MH}(E_{\mathcal{L}_A})$.*

Proof. Observe first that property (S) implies property $(S)_C$ thanks to Proposition 3.15, so that Theorem 5.7 applies; in particular, $\mathcal{KZ}(G,\mathcal{L}_A) \subseteq \mathcal{KZ}_{\mathcal{L}_A}(G)$. Now, take $K \in \mathcal{KZ}_{\mathcal{L}_A}(G)$,

[1] Notice that, since convolution of Schwartz functions is associative, the formula $K^{(-j)}*\psi*\psi$ is not ambiguous.

and take a positive function $\tau \in \mathcal{D}(E_{\mathcal{L}_A})$ such that τ vanishes in a neighbourhood of 0 and

$$\sum_{j\in\mathbb{Z}} \tau(2^j\,\cdot\,) = \chi_{E^*_{\mathcal{L}_A}}$$

pointwise, and define $\varphi \coloneqq \mathcal{K}_{\mathcal{L}_A}(\tau)$. Reasoning as in the proof of Theorem 5.7 and taking Proposition 5.8 into account, we see that $(K*\varphi^{(j)})_{j\in\mathbb{Z}}$ is a bounded family in $\mathcal{S}(G)$ with integral 0. In addition, $K*\varphi^{(j)} \in \mathcal{S}_{\mathcal{L}_A}(G) = \mathcal{S}(G,\mathcal{L}_A)$ for every $j\in\mathbb{Z}$, thanks to Corollary 3.12. Since $K = \sum_{j\in\mathbb{Z}} K*\varphi^{(j)}$ in $\mathcal{S}'_w(G)$, Theorem 5.7 implies that $K \in \mathcal{KZ}(G,\mathcal{L}_A)$, whence the result. □

Now we consider a converse of Proposition 5.8. Namely, we want to show that, if a Schwartz function φ convolves *all* Calderón-Zygmund kernels into the Schwartz space, then all the moments of φ must vanish. Actually, we shall prove a somewhat more precise version of this result, since we shall show how much faster than a Calderón-Zygmund kernel should $K*\varphi$ decay, for *all* K, in order that the moments of φ, up to a certain homogeneous degree, should vanish.

The following lemma allows us to find, for every Schwartz function ψ, a differential operator X such that ψ and $X\varphi$ have the same moments up to a fixed homogeneous degree. Here, φ is a function whose moments, except for the 0-th, vanish. This will allow us to reason by induction in the result which follows.

Lemma 5.10. *Let φ be an element of $\mathcal{S}(G)$ such that $\int_G \varphi(x)x^\beta\,\mathrm{d}x = \delta_{\beta,0}$ for every β. Then, for every $\psi\in\mathcal{S}(G)$ and for every $k\in\mathbf{d}\cdot\mathbb{N}^n$, there is a finite family $(c_\beta)_{\mathrm{d}_\beta\leqslant k}$ such that*

$$\int_G \psi(x)x^\gamma\,\mathrm{d}x = \sum_{\mathrm{d}_\beta\leqslant k} c_\beta \int_G (\mathbf{X}^\beta\varphi)(x)x^\gamma\,\mathrm{d}x$$

for every γ such that $\mathrm{d}_\gamma\leqslant k$. Further, if there is some $h<k$ such that $\int_G \psi(x)x^\gamma\,\mathrm{d}x = 0$ whenever $\mathrm{d}_\gamma\leqslant h$, then we may take (c_β) so that $c_\beta=0$ whenever $\mathrm{d}_\beta\leqslant h$.

Proof. Notice that the hypothesis on φ means that

$$\int_G \varphi\,P\,\mathrm{d}\nu_G = P(e)$$

for every polynomial mapping P on G. Now, Corollary 2.15 implies that for every β there is a polynomial P_β of homogeneous degree d_β such that

$$(X_n^{\gamma_n}\dots X_1^{\gamma_1}P_\beta)(e) = \delta_{\beta,\gamma}$$

for every γ. Notice that this implies that the family $(P_\beta)_{\mathrm{d}_\beta\leqslant k}$ is a basis of the space of polynomials of homogeneous degree at most k. Therefore, for every β such that $\mathrm{d}_\beta\leqslant k$ there is a finite family $(c_{\beta,\beta'})_{\mathrm{d}_{\beta'}\leqslant k}$ of real numbers such that

$$x^\beta = \sum_{\mathrm{d}_{\beta'}\leqslant k} c_{\beta,\beta'}P_{\beta'}(x)$$

for every $x\in G$.

Set $c_\beta \coloneqq (-1)^{|\beta|}\int_G \psi\,P_\beta\,\mathrm{d}\nu_G$ for every β such that $\mathrm{d}_\beta\leqslant k$. Then,

$$\begin{aligned}
\int_G \psi(x)x^\gamma\,\mathrm{d}x &= \sum_{\mathrm{d}_{\beta'}\leqslant k} c_{\gamma,\beta'}(-1)^{|\beta'|}c_{\beta'}\\
&= \sum_{\mathrm{d}_\beta,\mathrm{d}_{\beta'}\leqslant k} c_{\gamma,\beta'}(-1)^{|\beta'|}c_{\beta'}(X_n^{\beta_n}\dots X_1^{\beta_1}P_{\beta'})(e)\\
&= \sum_{\mathrm{d}_\beta,\mathrm{d}_{\beta'}\leqslant k} c_{\gamma,\beta'}(-1)^{|\beta|}c_\beta \int_G \varphi\,X_n^{\beta_n}\dots X_1^{\beta_1}P_{\beta'}\,\mathrm{d}\nu_G\\
&= \sum_{\mathrm{d}_\beta,\mathrm{d}_{\beta'}\leqslant k} c_{\gamma,\beta'}c_\beta \int_G (\mathbf{X}^\beta\varphi)P_{\beta'}\,\mathrm{d}\nu_G\\
&= \sum_{\mathrm{d}_\beta\leqslant k} c_\beta \int_G (\mathbf{X}^\beta\varphi)(x)x^\gamma\,\mathrm{d}x,
\end{aligned}$$

where the third equality holds since $\int_G \varphi\,X_n^{\beta_n}\dots X_1^{\beta_1}P_{\beta'}\,\mathrm{d}\nu_G = (X_n^{\beta_n}\dots X_1^{\beta_1}P_{\beta'})(e) = \delta_{\beta,\beta'}$. The last assertion follows easily from the definition of (c_β). □

Here we state the 'converse' of Proposition 5.8. Notice that the conditions on the decay of the convolution $K*\varphi$ are not the same as those we found in Proposition 5.8. Nevertheless, this had to be expected, since there K was fixed and φ varied, while now φ is fixed and K varies; this asymmetry is reflected by the asymmetry of the statements.

The main strategy of the proof is to essentially reduce to the case in which $k=0$, which is relatively easy to handle. This is accomplished by several manipulations which employ both Lemma 5.10 and Proposition 5.8.

Proposition 5.11. *Let $k \in \mathbf{d}\cdot\mathbb{N}^n$, $\varepsilon>0$, and $\varphi\in\mathcal{S}(G)$. Assume that for every $K\in\mathcal{KZ}(G)$ there is a constant $C_K>0$ such that*

$$|(\mathbf{X}^\beta(K*\varphi))(x)| \leqslant \frac{C_K}{|x|^{Q+\mathrm{d}_\beta+\varepsilon}}$$

for every β such that $\mathrm{d}_\beta\leqslant k$ and for every $x\in G^*$. Then, $\int_G \varphi\, P\,\mathrm{d}\nu_G=0$ for every polynomial of homogeneous degree at most k.

The same conclusion holds if $K*\varphi$ is replaced by $\varphi*K$ in the above hypothesis.

Proof. **1.** Let us prove that for every $K\in\mathcal{KZ}(G)$ and for every continuous homogeneous differential operator X of homogeneous degree $d\leqslant k$, there is a constant $C_{K,X}>0$ such that

$$|(X(K*\varphi))(x)| \leqslant \frac{C_{K,X}}{(1+|x|)^{Q+d+\varepsilon}}$$

for every $x\in G$. Indeed, notice first that there is a unique family with finite support (f_β) of continuous functions such that

$$X=\sum_\beta f_\beta \mathbf{X}^\beta.$$

By uniqueness and homogeneity we then infer that f_β is homogeneous of degree $\mathrm{d}_\beta-d$ for every β. Further, there is a constant $C_X>0$ such that $\sup_{|x|=1}|f_\beta(x)|\leqslant C_X$ for every β, so that,

$$|(X(K*\varphi))(x)| \leqslant \frac{N_X C_X C_K}{|x|^{Q+d+\varepsilon}}$$

for every $x\in G^*$, where N_X is the cardinality of the support of (f_β). Next, notice that Proposition 5.8 implies that $X(K*\varphi)$ is bounded. Hence, we may find a constant $C_{K,X}>0$ such that

$$|(X(K*\varphi))(x)| \leqslant \frac{C_{K,X}}{(1+|x|)^{Q+d+\varepsilon}}$$

as claimed.

It is now clear that the second assertion of the statement follows from the first one, applied to the opposite group of G.

2. Now, assume by contradiction that $\int_G \varphi(x)x^\beta\neq 0$ for some β such that $\mathrm{d}_\beta\leqslant k$. Without loss of generality, we may assume that $\int_G\varphi(x)x^\beta=0$ for every β such that $\mathrm{d}_\beta<k$. We shall prove that, for every $\psi\in\mathcal{S}(G)$ with integral 1 and for every $K\in\mathcal{KZ}(G)$, there is a constant $C_{X,\psi,K}>0$ such that

$$|(X(K*\psi))(x)| \leqslant \frac{C_{X,\psi,K}}{(1+|x|)^{Q+k'}}$$

for every $x\in G$. Here, k' is the minimum between $k+\varepsilon$ and the least element of $\mathbf{d}\cdot\mathbb{N}^n$ which is strictly greater than k.

Take $\psi_1\in\mathcal{S}(G)$ such that $\int_G\psi_1\,P\,\mathrm{d}\nu_G=P(e)$ for every polynomial P.[2] Then, there is a finite family $(c_\beta)_{\mathrm{d}_\beta=k}$ of real numbers such that, once we define $X:=\sum_{\mathrm{d}_\beta=k}c_\beta\mathbf{X}^\beta$, we have $\int_G(\varphi-X\psi_1)P\,\mathrm{d}\nu_G=0$ for every polynomial of homogeneous degree at most k (cf. Lemma 5.10). Hence, Proposition 5.8 implies that for every $K\in\mathcal{KZ}(G)$ there is $C'_{X,K}>0$ such that

$$|(K*(\varphi-X\psi_1))(x)| \leqslant \frac{C'_{X,K}}{(1+|x|)^{Q+k'}}$$

[2]It suffices to take ψ_1 so that its euclidean Fourier transform is identically 1 in a neighbourhood of 0.

for every $x \in G$. Take any $\psi \in \mathcal{S}(G)$ with integral 1. Then, Proposition 5.8 implies that there is $C'_{X,\psi,K} > 0$ such that

$$|K * (X\psi_1 - X\psi))(x)| = |X(K * (\psi_1 - \psi))(x)| \leqslant \frac{C'_{X,\psi,K}}{(1+|x|)^{Q+k'}}$$

for every $x \in G$. Combining these inequalities with **1**, we infer that there is $C_{X,\psi,K} > 0$ such that

$$|(X(K * \psi))(x)| \leqslant \frac{C_{X,\psi,K}}{(1+|x|)^{Q+k'}}$$

for every $x \in G$.

3. Now, Lemma 2.14 implies that there are a non-zero homogeneous right-invariant differential operator Y of homogeneous degree k and a family with finite support (P_β) of homogeneous polynomials such that

$$X = Y + \sum_{\mathrm{d}_\beta \geqslant k} P_\beta \mathbf{Y}^\beta.$$

Moreover, P_β has homogeneous degree $\mathrm{d}_\beta - k$ and does not depend on the x_j of greatest homogeneous degree. Therefore,

$$|((YK) * \psi)(x)| = |(X(K * \psi))(x)| \leqslant \frac{C_{X,\psi,K}}{(1+|x|)^{Q+k'}}$$

for every $K \in \mathcal{KZ}(G)$, for every $\psi \in \mathcal{S}(G)$ and for every $x \in G$ such that $x_j = 0$ if $\mathrm{d}_j < \mathrm{d}_n$.[3] In particular, this holds for $x = (0, \dots, 0, x_n)$ for every $x_n \in \mathbb{R}$.

Now, fix $\widetilde{x} = \exp_G(tX_n)$, where t is fixed so that $|\widetilde{x}| = 1$. Then, there is a polynomial P such that $(YP)(\widetilde{x}) = 2$. Take $\tau_1, \tau_2 \in \mathcal{D}(G)$ such that the following hold:

1. τ_1 and τ_2 are supported on the set C of $x \in G$ such that $\frac{3}{4} < |x| < \frac{3}{2}$;
2. $\tau_1 \tau_2 = 0$;
3. $P\tau_1 - \tau_2$ has zero integral;
4. τ_1 is identically 1 on $B(\widetilde{x}, 2r)$ for some $r > 0$ such that $(YP)(x) \geqslant 1$ for every $x \in B(\widetilde{x}, r)$ and $\overline{B}(\widetilde{x}, 2r)$ is contained in C;[4]

Define $K = \sum_{j \in \mathbb{Z}} (P\tau_1 - \tau_2)^{(j)}$, so that $K^{(j)} = K$ and $K \in \mathcal{KZ}(G)$ for every $j \in \mathbb{Z}$ by Lemma 5.3. Furthermore,

$$(YK)(x) = 2^{j(Q+k)}(YP)(2^j \cdot x) \geqslant 2^{j(Q+k)}$$

for every $x \in 2^{-j} \cdot B(\widetilde{x}, r)$ and for every $j \in \mathbb{Z}$. Now, take a positive $\psi \in \mathcal{D}(G)$ which is supported on $B(e, r)$ and has integral 1. Then, for every $j \in \mathbb{N}$,

$$(2^j \cdot \widetilde{x})B(e, r) = B(2^j \cdot \widetilde{x}, r) \subseteq B(2^j \cdot \widetilde{x}, 2^j r) = 2^j \cdot B(\widetilde{x}, r),$$

so that

$$((YK) * \psi)(2^j \cdot \widetilde{x}) = \int_G (YK)((2^j \cdot \widetilde{x})x)\psi(x^{-1}) \, \mathrm{d}x \geqslant 2^{-j(Q+k)} = \frac{1}{|2^j \cdot \widetilde{x}|^{Q+k}}.$$

This contradicts the estimate $|((YK) * \psi)(x)| \leqslant \frac{C_{X,\psi,K}}{(1+|x|)^{Q+k'}}$ of **2**. The proof is complete. ☐

Finally, we state as a corollary the cleaner result which one obtains by considering the limiting case $k \to \infty$ of Propositions 5.11 and 5.8.

Corollary 5.12. *Let $\varphi \in \mathcal{S}(G)$. Then, the following conditions are equivalent:*

1. $\int_G \varphi \, P \, \mathrm{d}\nu_G = 0$ for every polynomial P;

[3] Indeed, for such x, $P_\beta(x) = P_\beta(0) = 0$ for every β such that $\mathrm{d}_\beta > k$.

[4] Here, the balls are relative to the left-invariant quasi-distance induced by $|\cdot|$; in other words, $B(x, r)$ is the set of $y \in G$ such that $|x^{-1}y| < r$. Then, $B(x, r) = xB(e, r)$.

2. *for every* $k \in \mathbb{N}$ *there is a finite family* $(\varphi_{k,\beta})$, *with* $\mathrm{d}_\beta \geqslant k$, *of elements of* $\mathcal{S}(G)$ *such that* $\varphi = \sum_{\mathrm{d}_\beta \geqslant k} \mathbf{X}^\beta \varphi_{k,\beta}$;

3. *for every* $k \in \mathbb{N}$ *there is a finite family* $(\widetilde{\varphi}_{k,\beta})$, *with* $\mathrm{d}_\beta \geqslant k$, *of elements of* $\mathcal{S}(G)$ *such that* $\varphi = \sum_{\mathrm{d}_\beta \geqslant k} \mathbf{Y}^\beta \widetilde{\varphi}_{k,\beta}$;

4. *the mapping* $K \mapsto K * \varphi$ *induces a continuous homomorphism of* $\mathcal{KZ}(G)$ *into* $\mathcal{S}(G)$;

5. *the mapping* $K \mapsto \varphi * K$ *induces a continuous homomorphism of* $\mathcal{KZ}(G)$ *into* $\mathcal{S}(G)$.

Proof. **1** $\iff$ **2** $\iff$ **3.** This follows from Corollary 2.18.

2 $\implies$ **4.** Since $\mathcal{KZ}(G)$ is a Fréchet space, the result follows from Proposition 5.8.

4 $\implies$ **2.** This follows from Proposition 5.11.

3 $\iff$ **5.** This is the equivalence **2** $\iff$ **4**, applied to the opposite group of G. □

Chapter 6

Abelian Groups

6.1 General Properties

Here, we consider the case of a general family of operators on an abelian group. We present first the easy consequences of the results of Section 4.3. Then, we proceed to finer results in some particular cases.

Proposition 6.1. *Let G be the group $\mathbb{R}^n$ endowed with a family of dilations, denote by ∂ the family $(\partial_1, \dots, \partial_n)$ of partial derivatives, and define $\mathcal{L}_A = P(-i\partial)$ for some finite family of polynomials $P = (P_\alpha)_{\alpha \in A}$ in n variables. Then, the following hold:*

1. *$\mathcal{L}_A$ is an admissible family;*
2. *$\sigma(P(\mathcal{L}_A)) = \overline{P(E_{-i\partial})}$;*
3. *$m \in \mathcal{M}(\mu_{\mathcal{L}_A}; \mathcal{D}'_w(G))$ if and only if $m \circ P \in \mathcal{F}(W^{-\infty,2}(G))$; in this case,*[1]

$$\mathcal{K}_{\mathcal{L}_A}(m) = \mathcal{F}^{-1}(m \circ P).$$

Proof. All statements follow easily from Proposition 4.12 and from the properties of the Fourier transform. □

Proposition 6.2. *Keep the notation of Proposition 6.1. Then, $\mathcal{L}_A$ is a Rockland family if and only if P is proper; in this case, $\mathcal{L}_A$ satisfies property $(S)_C$.*

Next, let r be the maximum rank of P' and set $c_\lambda := \int_{P^{-1}(\lambda)} \frac{1}{J_r P(\lambda')} \mathrm{d}\mathcal{H}^{n-r}(\lambda')$ for $\mathcal{H}^r$-almost every $\lambda \in \sigma(\mathcal{L}_A)$. Then,

$$\beta_{\mathcal{L}_A} = \frac{1}{(2\pi)^n} c \, \chi_{\sigma(\mathcal{L}_A)} \cdot \mathcal{H}^r,$$

and

$$\chi_{\mathcal{L}_A}(\lambda, g) = \frac{1}{c_\lambda} \int_{P^{-1}(\lambda)} \frac{1}{J_r P(\lambda')} e^{i\lambda' g} \, \mathrm{d}\mathcal{H}^{n-r}(\lambda')$$

for $(\beta_{\mathcal{L}_A} \otimes \nu_G)$-almost every (λ, g).

Proof. The first assertion follows from Propositions 4.13 and 4.15, since $\sigma(\mathcal{L}_A)$ is semi-algebraic by [29, Corollary 2.4], hence Nash subanalytic.

The second assertion then follows from Propositions 4.14 and 2.26. □

Proposition 6.3. *Keep the notation of Proposition 6.1. If P is proper and $\beta_{-i\partial}$ is P-connected, then $\mathcal{L}_A$ satisfies property (RL).*

Proof. Keep the notation of Proposition 6.2. Notice that [75, Theorem 1] implies that $P^{-1}(\lambda)$ is an analytic submanifold of $E_{-i\partial}$ for $\mathcal{H}^r$-almost every $\lambda \in \sigma(\mathcal{L}_A)$, hence for $\beta_{\mathcal{L}_A}$-almost every $\lambda \in E_{\mathcal{L}_A}$. In addition, if $P^{-1}(\lambda)$ is an analytic submanifold of $E_{-i\partial}$, then $\chi_{P^{-1}(\lambda)} \cdot \mathcal{H}^{n-r}$ has support $P^{-1}(\lambda)$. Then, the assertion follows from Proposition 2.21. □

[1]Here, the Fourier transform is computed with respect to the identification of G and $E_{-i\partial}$ with $\mathbb{R}^n$ by means of the chosen basis ∂.

Remark 6.4. Here we show a proper polynomial mapping $P\colon \mathbb{R}^2 \to \mathbb{R}^2$ with homogeneous components such that there is a $\beta_{P(-i\partial)}$-measurable non-continuous function $(m_1, m_2)\colon \mathbb{R}^2 \to \mathbb{R}^2$ such that $m_1(P(-i\partial)) = -\partial_1^2$ and $m_2(P(-i\partial)) = -\partial_2^2$. This example was communicated to the author by J. M. Gamboa.

Define $P^{(1)}\colon \mathbb{R}^2 \ni (x,y) \mapsto (x^2, y^2) \in \mathbb{R}^2$ and let $P^{(2)}$ be the polynomial mapping from $\mathbb{R}^2$ into $\mathbb{R}^2$ which corresponds to $\mathbb{C} \ni z \mapsto z^4 \in \mathbb{C}$ under the identification $\mathbb{R}^2 \ni (x,y) \mapsto x+iy \in \mathbb{C}$; define $P \coloneqq P^{(2)} \circ P^{(1)}$. It is easily seen that P is a proper polynomial mapping with homogeneous components. In addition, the image of $P^{(1)}$ is $\mathbb{R}_+^2$, while $P^{(2)}$ induces a bijection between $\left(\mathbb{R}_+ \times \mathbb{R}_+^*\right) \cup \{\,(0,0)\,\}$ and $\mathbb{R}^2$. Let (m_1, m_2) be the inverse bijection. Then, it is clear that $(m_1, m_2) \circ P = P^{(1)}$ on $\mathbb{R} \times \mathbb{R}^*$, hence $\beta_{-i\partial}$-almost everywhere. The assertion follows.

Notice, in addition, that (m_1, m_2) does not equal $\beta_{P(-i\partial)}$-almost everywhere a continuous function, so that $P(-i\partial)$ does not satisfy properties (RL) and (S).

6.2 The Case of One Operator

In this and the following section, we consider some special cases. The first special case is that of one *positive* operator with arbitrary order. Before we do that, we need to establish a few technical results.

Lemma 6.5. *Take $k, d \in \mathbb{N}^*$ and define $P(x) = x^d$ for every $x \in \mathbb{R}$. Take a function $m\colon P(\mathbb{R}) \to \mathbb{C}$ and assume that the following conditions hold:*

1. *$m \circ P$ is of class C^{kd} on $\mathbb{R}$;*
2. *$(m \circ P)^{(h)}(0) = 0$ if $h \leqslant kd$ and d does not divide h.*

Then m extends to an element of $C^k(\mathbb{R})$.

The proof is simple and left to the reader.

Lemma 6.6. *Let A be a non-empty finite set and endow $E \coloneqq \mathbb{R}^A$ with the structure of a homogeneous group. Take a positive, non-constant, homogeneous polynomial P in $\mathbb{R}[A]$ and assume that there is a homogeneous element x of E such that $P(x) \neq 0$. Then, the following statements are equivalent:*

1. *there are no positive polynomials $Q \in \mathbb{R}[A]$ and no $k \in \mathbb{N}$ such that $k \geqslant 2$ and $P = Q^k$;*
2. *if m is a complex-valued function defined on $\mathbb{R}_+$ such that $m \circ P$ is C^∞ on E, then m may be extended to an element of $C^\infty(\mathbb{R})$.*

Notice that, if P is not required to be positive, but vanishes only at 0, then a similar assertion still holds. Indeed, if P is negative, then it suffices to consider $-P$, while if P does not keep a constant sign, then A has one element and the assertion is trivial.

Proof. **1 $\implies$ 2.** Take $m\colon \mathbb{R}_+ \to \mathbb{C}$ and assume that $m \circ P$ is C^∞ on E. Notice that there is a homogeneous polynomial $P_x \in \mathbb{R}[X]$ such that $P(\lambda x) = P_x(\lambda)$ for every $\lambda \in \mathbb{R}$.[2] In particular, $m \circ P_x$ is of class C^∞. In addition, $P_x(X) = a_x X^{d_x}$ for some $a_x > 0$ and $d_x \in 2\mathbb{Z}_+^*$, so that m is of class C^∞ on $\mathbb{R}_+^*$. Further, $m \circ P_x$ admits a Taylor development $\sum_{j \in \mathbb{N}} a_j X^j$ at 0, so that m admits the asymptotic development $\sum_{j\in\mathbb{N}} a_{x,j} \lambda^{\frac{j}{d_x}}$ for $\lambda \to 0^+$, where $a_{x,j} \coloneqq \frac{a_j}{a_x^{1/d_x}}$ for every $j \in \mathbb{N}$. Suppose that there are some $j \in \mathbb{N} \setminus (d_x \mathbb{N})$ such that $a_{x,j} \neq 0$, and let j_x be the least of them. Let q_x, r_x be the quotient and the remainder, respectively, of the division of j_x by d_x.

Let us show that $\partial_x^{j_x} P^{\frac{j_x}{d_x}}$ is continuous. Define $\widetilde{m} \coloneqq m - \sum_{j=0}^{j_x d_x} a_{x,j} (\,\cdot\,)^{\frac{j}{d_x}}$. Then, $\widetilde{m} \circ P_x$ is C^∞ and $(\widetilde{m} \circ P_x)(\lambda) = o\Big(|\lambda|^{j_x d_x}\Big)$.[3] Hence, Lemma 6.5 implies that $\widetilde{m}$ may be extended to an element of $C^{j_x}(\mathbb{R})$. Let us then prove that

$$a_{x,j_x} \partial_x^{j_x} P^{\frac{j_x}{d_x}} = \partial_x^{j_x}(m \circ P) - \sum_{j=0}^{q_x} a_{x, d_x j} \partial_x^{j_x} P^j - \sum_{j=j_x+1}^{j_x d_x} a_{x,j} \partial_x^{j_x} P^{\frac{j}{d_x}} - \partial_x^{j_x}(\widetilde{m} \circ P)$$

[2] Notice that λx denotes the scalar multiplication of x by λ, *not* the dilate $\lambda \cdot x$ of x by λ, which by the way is meaningful only for $\lambda > 0$.

[3] Here, $|\lambda|$ denotes the usual absolute value of $\lambda \in \mathbb{R}$.

extends to a continuous function on $E' \coloneqq \{ x' \in E \colon P(x) \neq 0 \} \cup \{ 0 \}$. Indeed, this is clear for the first two terms, and follows from the above remarks for the fourth one. Let us then consider the third term. Notice that both ∂_x and P are homogeneous, and that $\partial_x^{j_x} P^{\frac{j_x}{d_x}}$ is homogeneous of homogeneous degree 0 on the x axis, hence on E. Hence, $\partial_x^{j_x}$ must be homogeneous of homogeneous degree $d\frac{j_x}{d_x}$, where d is the homogeneous degree of P. Then, $\partial_x^{j_x} P^{\frac{j}{d_x}}$ is homogeneous of degree $d\frac{j-j_x}{d_x} > 0$ for every $j = j_x+1, \dots, j_x d_x$, so that it may be extended by continuity at 0.

Therefore, $\partial_x^{j_x} P^{\frac{j_x}{d_x}}$ extends to a continuous function on E' which is homogeneous of homogeneous degree 0, so that it is constant. In addition, its constant value is clearly $C \coloneqq a_x^{j_x/d_x} j_x! \neq 0$. Now, Faà di Bruno's formula shows that

$$P^{1-\frac{r_x}{d_x}} = \frac{1}{C} P^{1-\frac{r_x}{d_x}} \partial_x^{j_x} P^{\frac{j_x}{d_x}} = \frac{1}{C} \sum_{\sum_{\ell=1}^{j_x} \ell \beta_\ell = j_x} \frac{j_x!}{\beta!} \left(\frac{j_x}{d_x} \right)_{|\beta|} P^{1+q_x-|\beta|} \prod_{\ell=1}^{j_x} \left(\frac{\partial_x^\ell P}{\ell!} \right)^{\beta_\ell}$$

on E', where $\left(\frac{j_x}{d_x} \right)_{|\beta|} \coloneqq \frac{j_x}{d_x}\left(\frac{j_x}{d_x} - 1 \right) \cdots \left(\frac{j_x}{d_x} - |\beta| + 1 \right)$ is the Pochhammer symbol. Let $\widetilde{Q}$ be the right hand side of the preceding formula. Then, $\widetilde{Q}$ is a rational function, so that there are $N, D \in \mathbb{R}[A]$, with $D \neq 0$, such that $\widetilde{Q} = \frac{N}{D}$. Further, $P^{d_x - r_x} = \widetilde{Q}^{d_x} = \frac{N^{d_x}}{D^{d_x}}$ on E'. Since $P^{d_x - r_x}$ is a polynomial, this proves that D^{d_x} divides N^{d_x}. However, $\mathbb{R}[A]$ is factorial, so that D divides N. Hence, $\widetilde{Q}$ is a polynomial, and $\widetilde{Q}$ is positive since P is. Next, let g be the greatest common divisor of d_x and $d_x - r_x$, and take $d', r' \in \mathbb{Z}_+^*$ such that $d_x = g\, d'$ and $d_x - r_x = g\, r'$. Hence,

$$\widetilde{Q}^{d'} = P^{r'}.$$

Since $\mathbb{R}[A]$ is factorial, this proves that there is a polynomial $Q \in \mathbb{R}[A]$ such that $Q^{r'} = \widetilde{Q}$ and $Q^{d'} = P$.[4] Now, $P = Q^{d'}$ and $d' > r' > 0$ since $r_x > 0$, so that $d' \geqslant 2$. Therefore, our assumption **1** cannot hold. It follows that $a_{x,j} = 0$ for every $j \notin d_x \mathbb{N}$, so that m has the asymptotic development $\sum_{j \in \mathbb{N}} a_{x, d_x j} \lambda^j$ for $\lambda \to 0^+$. The conclusion follows from Lemma 6.5.

2 $\implies$ **1.** Suppose by contradiction that there are a positive polynomial $Q \in \mathbb{R}[A]$ and $k \geqslant 2$ such that $P = Q^k$, and define $m \colon \lambda \mapsto \lambda^{\frac{1}{k}}$ on $\mathbb{R}_+$. Then, m is C^∞ on $\mathbb{R}_+^*$ and $\lim\limits_{\lambda \to 0^+} \frac{m(\lambda) - m(0)}{\lambda} = +\infty$. However, $m \circ P = Q$ since Q is positive, so that $m \circ P$ is of class C^∞: this contradicts our assumption **2**. □

We give the main result in the case of one operator in a slightly more general situation, since we do not restrict to abelian groups. In the case of abelian group, this result provides a complete characterization of the (positive) Rockland operators which satisfy property (S).

Theorem 6.7. *Let $\mathcal{L} = \lambda_G(P)$ be a positive Rockland operator on a non-trivial homogeneous group G for some $P \in \mathrm{Pol}(\mathfrak{g}^*; \mathbb{R})$. Then, the following hold:*

1. *$\mathcal{L}$ satisfies property $(S)_C$;*

2. *$\chi_{\mathcal{L}}$ has a continuous representative and $\widetilde{\chi}_{\mathcal{L}}$ has a C^∞ representative;*

3. *let k be the greatest $k' \in \mathbb{N}^*$ such that $P^{\frac{1}{k'}}$ is a polynomial mapping on $[\mathfrak{g}, \mathfrak{g}]^\circ$, and $\mathfrak{K}$ a bounded subset of $\mathcal{S}_{\mathcal{L}}(G)$. Then, there are k bounded families $(m_{0,\varphi})_{\varphi \in \mathfrak{K}}, \dots, (m_{k-1,\varphi})_{\varphi \in \mathfrak{K}}$ of elements of $\mathcal{S}(\mathbb{R})$ such that*[5]

$$\varphi = \mathcal{K}_{\mathcal{L}} \left(\sum_{h=0}^{k-1} (\,\cdot\,)^{\frac{h}{k}} m_{h,\varphi} \right)$$

for every $\varphi \in \mathfrak{K}$;

4. *take $m \in L^\infty(\beta_{\mathcal{L}})$; then, $\mathcal{K}_{\mathcal{L}}(m) \in \mathcal{KZ}(G)$ if and only if m has a representative in $\mathcal{MH}(E_{\mathcal{L}})$.*

[4] Indeed, take an irreducible polynomial T, and let h be the greatest integer such that T^h divides Q. Then, hd' is the greatest integer h' such that $T^{h'}$ divides $P^{r'}$; hence, r' divides hd'. Since r' and d' have no common divisors, this proves that r' divides h. By the arbitrariness of T we are then able to construct such Q.

[5] Notice, however, that a multiplier of this form need not have a Schwartz kernel, unless G is abelian.

Notice that **4** does *not* hold, in general, for more than one operator.

Proof. **1.** The abelian case follows immediately from Proposition 6.2. Then, let G' be the abelianization of G, and let $\pi\colon G \to G'$ be the canonical projection. Since clearly $\sigma(\mathcal{L}) = \sigma(\pi(\mathcal{L})) = \mathbb{R}_+$, by means of Theorem 4.1 we see that $\mathcal{S}(G;\mathcal{L})$ is closed in $\mathcal{S}(G)$.

2. The second assertion follows easily from Propositions 3.35 and 3.48.

3. Since $\chi_{\mathcal{L}}$ has a continuous representative by **2** above, Corollary 3.47, Proposition 3.50 and Theorem 4.1 imply that $\pi_*(\varphi) \in \mathcal{S}_{\mathrm{d}\pi(\mathcal{L})}(G')$ for every $\varphi \in \mathfrak{K}$. Since $\mathrm{d}\pi(\mathcal{L}) = \lambda_{G'}(P \circ {}^t\mathrm{d}\pi)$ by Proposition 1.16, the preceding analysis shows that there is a bounded family $(m_\varphi)_{\varphi\in\mathfrak{K}}$ of elements of $C_0(\mathbb{R}_+)$ such that $(m_\varphi \circ P \circ {}^t\mathrm{d}\pi)$ is a bounded family of elements of $\mathcal{S}([\mathfrak{g},\mathfrak{g}]^\circ)$, and such that

$$\varphi = \mathcal{K}_{\mathcal{L}}(m_\varphi)$$

for every $\varphi \in \mathfrak{K}$. Now, take $Q \in \mathrm{Pol}([\mathfrak{g},\mathfrak{g}]^\circ)$ so that $P \circ {}^t\mathrm{d}\pi = Q^k$. Observe that Lemma 6.6 implies that the mapping

$$\mathcal{S}_{\mathbb{R}}(\mathbb{R}_+) \ni \varphi \mapsto \varphi \circ Q \in \mathcal{S}([\mathfrak{g},\mathfrak{g}]^\circ)$$

is a strict morphism, so that $(m_\varphi \circ (\,\cdot\,)^k)$ is a bounded family of elements of $\mathcal{S}_{\mathbb{R}}(\mathbb{R}_+)$. Next, let $\sum_{\ell\in\mathbb{N}} a_{\ell,\varphi}\lambda^\ell$ be the Taylor development of $m_\varphi \circ (\,\cdot\,)^k$ at 0, for every $\varphi \in \mathfrak{K}$. Take $h \in \{1,\dots,k-1\}$; then, $((a_{h+k\ell,\varphi})_\ell)_{\varphi\in\mathfrak{K}}$ is a bounded family of elements of $\mathbb{C}^{\mathbb{N}}$. Since the linear mapping $\mathcal{S}(\mathbb{R}) \ni \varphi \mapsto (\varphi^{(\ell)}(0))_\ell \in \mathbb{C}^{\mathbb{N}}$ is onto by [50, Theorem 1.2.6], and since $\mathbb{C}^{\mathbb{N}}$ is a Fréchet-Montel space, [18, Corollary to Proposition 12 of Chapter II, § 4, No. 7] implies that there is a bounded family $(m_{h,\varphi})_{\varphi\in\mathfrak{K}}$ of elements of $\mathcal{S}(\mathbb{R})$ such that $m_{h,\varphi}$ has Taylor development $\sum_{\ell\in\mathbb{N}} a_{h+k\ell}\lambda^\ell$ at 0. Reasoning as before (but this time using the surjection $\mathcal{S}(\mathbb{R}) \to \mathcal{S}_{\mathbb{R}}(\mathbb{R}_+)$), we see that there is a bounded family $(m_{0,\varphi})$ of elements of $\mathcal{S}(\mathbb{R})$ such that

$$m_\varphi = \sum_{h=0}^{k-1} (\,\cdot\,)^{\frac{h}{k}} m_{h,\varphi}$$

on $\mathbb{R}_+$, for every $\varphi \in \mathfrak{K}$. The assertion follows.

4. By **1** above and Theorem 5.7, we only need to prove that, if $m \in \mathcal{M}(\mu_{\mathcal{L}}; \mathcal{K}\mathcal{Z}(G))$, then there is a bounded family $(K_j)_{j\in\mathbb{Z}}$ of elements of $\mathcal{S}(G;\mathcal{L})$ such that $\mathcal{K}_{\mathcal{L}}(m) = \sum_{j\in\mathbb{Z}} K_j^{(j)}$ in $\mathcal{S}'_w(G)$. Then, take $\varphi \in \mathcal{D}_+(\mathbb{R}^*)$ so that $\sum_{j\in\mathbb{Z}} \varphi(2^{-j}\,\cdot\,) = \chi_{\mathbb{R}^*}$, and define $m_j := m(2^j\,\cdot\,)\varphi$. Now, arguing as in the proof of Theorem 5.7 we see that the family $(\mathcal{K}_{\mathcal{L}}(m_j))$ is bounded in $\mathcal{S}(G)$. Next, **3** above implies that there are k bounded families $(m_{j,0})_j,\dots,(m_{j,k-1})_j$ of elements of $\mathcal{S}(\mathbb{R})$ such that

$$m_j = \sum_{h=0}^{k-1} (\,\cdot\,)^{\frac{h}{k}} m_{j,h}$$

$\beta_{\mathcal{L}}$-almost everywhere, for every $j \in \mathbb{N}$. Therefore, it is clear that m_j has a representative in $\mathcal{D}(\mathbb{R}^*)$. Therefore, the family $(\mathcal{K}_{\mathcal{L}}(m_j))$ is bounded in $\mathcal{S}(G;\mathcal{L})$. Finally, arguing as in the proof of Theorem 5.7 we see that $\mathcal{K}_{\mathcal{L}}(m) = \sum_{j\in\mathbb{Z}} \mathcal{K}_{\mathcal{L}}(m_j)^{(j)}$ in $\mathcal{S}'_w(G)$. □

6.3 Gelfand Pairs; Quadratic Operators

Proposition 6.8. *Assume that $\mathcal{L}_A$ is a Rockland family and that G is abelian. Suppose further that there is a compact Lie group G' which acts on G in such a way that $\mathcal{L}_A$ generates the unital algebra of translation-invariant G'-invariant differential operators. Then, the following hold:*

1. *the mapping $\Phi_F\colon f \mapsto \fint_{G'} f(g'\,\cdot\,)\,\mathrm{d}\nu_{G'}(g')$ is a continuous projector of F, where F is any one of the spaces $\mathcal{D}(G)$, $\mathcal{S}(G)$, $\mathcal{E}(G)$, $L^p(G)$ ($p \in [1,\infty]$), $\mathcal{E}'(G)$, $\mathcal{S}'(G)$, $\mathcal{D}'(G)$ (the last three spaces being endowed with either the weak or strong dual topology);*

2. *if $F = L^p(G)$ for some $p \in [1,\infty]$, then Φ_F is a contraction;*

3. *$\mathcal{K}_{\mathcal{L}_A}\mathcal{M}_{\mathcal{L}_A}$ equals $\Phi_{L^2(G)}$ on $L^2(G)$;*

4. *$\mathcal{L}_A$ satisfies property (S) and $\chi_{\mathcal{L}_A}$ has a continuous representative.*

Proof. **1–2.** Notice first that Φ_F is well defined. Indeed, if $F \neq L^\infty(G)$, then the mapping $G' \ni g' \mapsto f(g' \cdot\,) \in F$ is continuous; since G' is compact and since F is quasi-complete, it follows that the closed balanced convex envelope of the set of $f(g' \cdot\,)$, as g' runs through G', is compact in F. Hence, [19, Proposition 8 of Chapter VI, § 1, No. 2] implies that $\fint_{G'} f(g' \cdot\,)\, \mathrm{d}\nu_{G'}(g') \in F$. If $F = L^\infty(G)$, then the preceding remarks apply as well if we endow $L^\infty(G)$ with the weak topology $\sigma(L^\infty(G), L^1(G))$.

Now, if $F = L^p(G)$ for some $p \in [1, \infty]$, then $\|f\|_F = \|f(g' \cdot\,)\|_F$ for every $g' \in G'$, so that it is clear that Φ_F is a contraction, hence continuous. Next, if F is either $\mathcal{D}(G)$ or $\mathcal{S}(G)$, then the closed graph theorem implies that Φ_F is continuous. If, otherwise, F is either $\mathcal{D}'(G)$ or $\mathcal{S}'(G)$, endowed with either the weak or the strong dual topology, then it is easily seen that

$$\langle \Phi_F \cdot T, \varphi \rangle = \fint_{G'} \big\langle T, \varphi(g'^{-1} \cdot\,) \big\rangle \, \mathrm{d}\nu_{G'}(g') = \langle T, \Phi_{F'} \cdot \varphi \rangle$$

for every $T \in F$ and for every $\varphi \in F'$, so that Φ_F is continuous by transposition. By the closed graph theorem and transposition again, one proves that Φ_F is continuous for $F = \mathcal{E}(G)$ and $\mathcal{E}'(G)$, endowed with either the weak or the strong topology. In the same way we see that $\Phi_{L^2(G)}$ is self-adjoint, while it is clear that Φ_F is a projector.

3. Denote by P the restriction of $\mathcal{K}_{\mathcal{L}_A}\mathcal{M}_{\mathcal{L}_A}$ to $L^2(G)$. Then, the image of P is contained in the set of G'-invariant elements of $L^2(G)$, which is the image of $\Phi_{L^2(G)}$; conversely, every element in the image of $\Phi_{\mathcal{S}(G)}$ belongs to $\mathcal{S}_{\mathcal{L}_A}(G) \subseteq P(L^2(G))$ by [5, Theorem 6.1]. Now, $\mathcal{S}(G)$ is dense in $L^2(G)$, so that $\Phi_{L^2(G)}(\mathcal{S}(G))$ is dense in $\Phi_{L^2(G)}(L^2(G))$; therefore, P and $\Phi_{L^2(G)}$ are two self-adjoint projectors of $L^2(G)$ with the same image, so that they are equal.

4. By [5, Theorem 6.1], $\mathcal{L}_A$ satisfies property (S); since $\mathcal{K}_{\mathcal{L}_A}\mathcal{M}_{\mathcal{L}_A}$ induces a contraction of $L^1(G)$ by **1**, **2** and **3** above, it follows that $\chi_{\mathcal{L}_A}$ has a continuous representative. □

Corollary 6.9. *Take a finite family $(G_\alpha)_{\alpha \in A}$ of abelian homogeneous groups with the standard dilations, and set $G \coloneqq \prod_{\alpha \in A} G_\alpha$. For every $\alpha \in A$, let n_α be the dimension of G_α, and endow G_α with a scalar product and the corresponding Laplacian $\mathcal{L}_\alpha$.*

Consider the family $\mathcal{L}_A = (\mathcal{L}_\alpha)_{\alpha \in A}$. Then, $\mathcal{L}_A$ is a Rockland family and satisfies property (S). In addition, $\mathcal{K}_{\mathcal{L}_A}\mathcal{M}_{\mathcal{L}_A}$ maps $\mathcal{S}(G)$ onto itself, and

$$\beta_{\mathcal{L}_A} = \bigotimes_{\alpha \in A} \frac{1}{(4\pi)^{n_\alpha/2}\Gamma\big(\frac{n_\alpha}{2}\big)} (\,\cdot\,)^{\frac{n_\alpha}{2}-1} \chi_{\mathbb{R}_+} \mathcal{H}^1$$

while

$$\chi_{\mathcal{L}_A}(\lambda, x) = \prod_{\alpha \in A} \Gamma\Big(\frac{n_\alpha}{2}\Big) \frac{J_{\frac{n_\alpha}{2}-1}\big(\sqrt{\lambda_\alpha}|x_\alpha|\big)}{\Big(\frac{\sqrt{\lambda_\alpha}|x_\alpha|}{2}\Big)^{\frac{n_\alpha}{2}-1}}$$

for $(\beta_{\mathcal{L}_A} \otimes \nu_G)$-almost every (λ, x). In particular, $\chi_{\mathcal{L}_A}$ admits a representative of class C^ω.

Proof. By Proposition 4.3 to Theorem 4.11, we may reduce to the case in which $\operatorname{Card} A = 1$, and then drop the unnecessary indices. Then, Proposition 6.8, applied with G' equal to the group of linear isometries of G, implies that $\mathcal{K}_{\mathcal{L}_A}\mathcal{M}_{\mathcal{L}_A}$ maps $\mathcal{S}(G)$ onto $\mathcal{S}_{\mathcal{L}_A}(G)$, which is the set of radial Schwartz functions. The formulae for $\beta_{\mathcal{L}_A}$ and $\chi_{\mathcal{L}_A}$ are then obtained by means of Proposition 6.2 and the classical theory of Bessel functions (cf., for example, [80, Chapter IV] or [83, Corollary 3.4]). Analyticity follows from the fact that the mapping $\mathbb{R}_+^* \ni x \mapsto x^{1-\frac{n}{2}} J_{\frac{n}{2}-1}(x)$ equals $\sum_{k \in \mathbb{N}} \frac{(-1)^k x^{2k}}{2^{2k+\frac{n}{2}-1} k! \Gamma\left(k+\frac{n}{2}\right)}$, so that it extends to an entire function. □

Corollary 6.10. *Keep the hypotheses and the notation of Corollary 6.9. Let A' be a finite set and L a linear mapping of $\mathbb{R}^n$ into $\mathbb{R}^{A'}$ such that $\ker L \cap \mathbb{R}^n_+ = \{0\}$. Then, $L(\mathcal{L}_A)$ satisfies properties (RL) and (S).*

Proof. By Corollaries 2.30 and 6.9, and by Propositions 2.21 and 2.22, it will suffice to show that $\mathbb{R}^n_+ = \sigma(\mathcal{L}_A)$ is a subanalytic closed convex cone. However, it is clearly semi-algebraic (cf. [29, Corollary 2.4]), so that it is subanalytic by [9, Proposition 2.3]. □

Corollary 6.11. *Let $\mathcal{L}_A$ be a Rockland family with at most two elements on $\mathbb{R}^n$, $n \geqslant 3$. Assume that each element of $\mathcal{L}_A$ has order two. Then, $\mathcal{L}_A$ satisfies properties (RL) and (S).*

On the contrary, observe that the Rockland family $(\partial_1^2-\partial_2^2,\partial_1\partial_2)$ on $\mathbb{R}^2$ is functionally equivalent to the Rockland family $(\partial_1^2,\partial_2^2,\partial_1\partial_2)$, which satisfies properties (RL) and (S); therefore, it satisfies property (RL) but not property (S).

Indeed, functional equivalence follows once we observe that

$$-(\partial_1^2+\partial_2^2)=\sqrt{-(\partial_1^2-\partial_2^2)^2+4(\partial_1\partial_2)^2}$$

as positive self-adjoint operators on $L^2(\mathbb{R}^2)$. In addition, the family $(\partial_1^2,\partial_2^2,\partial_1\partial_2)$ generates the algebra of translation-invariant differential operators on $\mathbb{R}^2$ which are invariant under the symmetry $x\mapsto -x$, so that it satisfies properties (RL) and (S).

Proof. Indeed, this is clear if A has one element. Otherwise, take Q_1,Q_2 so that $\mathcal{L}_A=(Q_1(-i\partial),Q_2(-i\partial))$. Then, by means of [25] we see that we may diagonalize Q_1 and Q_2 simultaneously, so that $Q_1(-i\partial)$ and $Q_2(-i\partial)$ must be linear combinations of $\partial_1^2,\dots,\partial_n^2$. The result follows from Corollary 6.10. □

Chapter 7

2-Step Groups

In this chapter, G denotes a 2-step stratified group of dimension n; we assume that G is *not* abelian, since the case of abelian groups has already been treated in Chapter 6. We shall denote by $\mathfrak{g}_1$ and $\mathfrak{g}_2 = [\mathfrak{g}, \mathfrak{g}]$ the first and the second layer, respectively, of the stratification of the Lie algebra $\mathfrak{g}$ of G. For every $\omega \in \mathfrak{g}_2^*$, we define

$$B_\omega \colon \mathfrak{g}_1 \times \mathfrak{g}_1 \ni (X, Y) \mapsto \langle \omega, [X, Y] \rangle,$$

so that B_ω is a skew-symmetric bilinear form. For our purposes, it is worthwhile to consider 2-step stratified groups which satisfy a slight strengthening of the Moore-Wolf condition (cf. [61], and also [62]). We then say that G satisfies condition MW^+, or that G is an MW^+-group if B_ω is non-degenerate for some $\omega \in \mathfrak{g}_2^*$. Then, a 2-step stratified group satisfies the Moore-Wolf condition if and only if it is the direct sum of an MW^+-group and an abelian group.

We say that G is a Métivier group if B_ω is non-degenerate for every $\omega \neq 0$. Observe that, in this case, $\mathfrak{g}_2$ is the centre of the Lie algebra of G. We say that G is a group of Heisenberg type, or simply an H-type group, if its Lie algebra is endowed with a scalar product such that $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are orthogonal, and such that, for every $\omega \in \mathfrak{g}_2$, the skew-symmetric operator $J_\omega \colon \mathfrak{g}_1 \to \mathfrak{g}_1$ defined by

$$\langle [X, Y] | \omega \rangle = \langle J_\omega(X) | Y \rangle \qquad \forall X, Y \in \mathfrak{g}_1$$

satisfies $|J_\omega| = |\omega| I_{\mathfrak{g}_1}$. In particular, an H-type group is a Métivier group.

A Heisenberg group is a non-commutative Métivier group with one-dimensional centre.

The free 2-step stratified Lie group on d generators is the simply-connected Lie group associated with the quotient of the free Lie algebra $\mathfrak{g}$ on d generators by its ideal $[\mathfrak{g}, [\mathfrak{g}, \mathfrak{g}]]$.

We shall consider some Rockland families on G whose elements are sub-Laplacians and elements of the centre of $\mathfrak{g}$, and prove some sufficient conditions for the validity of properties (RL) and (S).

7.1 Quadratic Operators

Take a symmetric bilinear form Q on $\mathfrak{g}_1^*$, and observe that Q induces a quadratic form $Q^0 \colon \mathfrak{g}_1^* \oplus \mathfrak{g}_2^* \ni x_1^* + x_2^* \mapsto Q(x_1^*, x_1^*)$ on $\mathfrak{g}^*$. We shall say that the operator $\lambda_G(Q^0)$ on G associated with Q^0 as in Proposition 1.16 is the operator associated with Q.

Lemma 7.1. *Let Q be a symmetric bilinear form on $\mathfrak{g}_1^*$, and let $\mathcal{L}$ be the associated operator. Then, $\mathcal{L}$ is formally self-adjoint if and only if Q is real. In addition, $\mathcal{L}$ is formally self-adjoint and hypoelliptic if and only if Q is non-degenerate and either positive or negative.*

Proof. The first assertion follows from the fact that the formal adjoint of $\mathcal{L}$ is associated with $\overline{Q}$. The last assertion then follows from [49]. □

We recall the following definition from the theory of bilinear forms (cf. [15, Chapter IX, § 1, No. 1]).

Definition 7.2. Let V be a vector space and Φ a bilinear form on V. Then, define the left and right linear mappings $\mathrm{s}_\Phi, \mathrm{d}_\Phi \colon V \to V^*$ associated with Φ so that

$$\langle \mathrm{s}_\Phi(w), v \rangle = \langle \mathrm{d}_\Phi(v), w \rangle = \Phi(w, v),$$

for every $v, w \in V$. If s_Φ and d_Φ are bijective, then we denote by $\widehat{\Phi}$ the inverse of Φ on V^*, that is, $\Phi \circ \left(\mathrm{s}_\Phi^{-1} \times \mathrm{d}_\Phi^{-1}\right)$.

Proposition 7.3. *Let Q_1 and Q_2 be two symmetric bilinear forms on $\mathfrak{g}_1^*$, and let $\mathcal{L}_1$ and $\mathcal{L}_2$ be the associated operators. Then, $\mathcal{L}_1$ and $\mathcal{L}_2$ commute if and only if*

$$\mathrm{d}_{Q_1} \circ \mathrm{d}_{B_\omega} \circ \mathrm{d}_{Q_2} = \mathrm{d}_{Q_2} \circ \mathrm{d}_{B_\omega} \circ \mathrm{d}_{Q_1}$$

for every $\omega \in \mathfrak{g}_2^*$.

Proof. Choose a basis $(X_j)_{j\in J}$ of $\mathfrak{g}_1$ and a basis $(T_k)_{k\in K}$ of $\mathfrak{g}_2$, and let $(X_j^*)_{j\in J}$ and $(T_k^*)_{k\in K}$ be the corresponding dual bases. Define $a_{h,j_1,j_2} := Q_h(X_{j_1}^*, X_{j_2}^*)$ for every $h = 1, 2$ and for every $j_1, j_2 \in J$, so that d_{Q_h} is identified with the matrix $A_h := (a_{h,j_1,j_2})_{j_1,j_2\in J}$ for $h = 1, 2$. Analogously, define $b_{k,j_1,j_2} := B_{T_k^*}(X_{j_1}, X_{j_2})$ for every $k \in K$ and for every $j_1, j_2 \in J$, so that $\mathrm{d}_{B_{T_k^*}}$ is identified with the matrix $B_k := (b_{k,j_1,j_2})_{j_1,j_2\in J}$ for every $k \in K$. Now, define $Y_{j_1,j_2} := \frac{1}{2}(X_{j_1}X_{j_2} + X_{j_2}X_{j_1})$ for every $j_1, j_2 \in J$. Then,

$$\mathcal{L}_h = \sum_{j_1,j_2\in J} a_{h,j_1,j_2} Y_{j_1,j_2}$$

since Q_h is symmetric. In addition, for every $j_1, j_2, j_3, j_4 \in J$,

$$[Y_{j_1,j_2}, Y_{j_3,j_4}] = Y_{j_2,j_4}[X_{j_1}, X_{j_3}] + Y_{j_2,j_3}[X_{j_1}, X_{j_4}] + Y_{j_1,j_4}[X_{j_2}, X_{j_3}] + Y_{j_1,j_3}[X_{j_2}, X_{j_4}]$$

since the elements of $\mathfrak{g}_2 = [\mathfrak{g}_1, \mathfrak{g}_1]$ lie in the centre of $\mathfrak{U}(\mathfrak{g})$. Next, observe that, for every $j_1, j_2 \in J$,

$$[X_{j_1}, X_{j_2}] = \sum_{k\in K} b_{k,j_1,j_2} T_k.$$

Therefore,

$$\begin{aligned}
[\mathcal{L}_1, \mathcal{L}_2] &= \sum_{\substack{j_1,j_2,j_3,j_4\in J\\ k\in K}} a_{1,j_1,j_2} a_{2,j_3,j_4} [b_{k,j_1,j_3} Y_{j_2,j_4} + b_{k,j_1,j_4} Y_{j_2,j_3} + b_{k,j_2,j_3} Y_{j_1,j_4} + b_{k,j_2,j_4} Y_{j_1,j_3}] T_k \\
&= 2 \sum_{j_1,j_2\in J} \sum_{k\in K} c_{k,j_1,j_2} Y_{j_1,j_2} T_k,
\end{aligned}$$

where

$$c_{k,j_1,j_2} = \sum_{j_3,j_4\in J} (a_{1,j_1,j_3} a_{2,j_2,j_4} + a_{1,j_2,j_3} a_{2,j_1,j_4}) b_{k,j_3,j_4}$$

for every $k \in K$ and for every $j_1, j_2 \in J$. Now, it is clear that the distinct monomials in the family of the $Y_{j_1,j_2} T_k$, as $j_1, j_2 \in J$ and $k \in K$, are linearly independent (cf., for example, [21, Theorem 1 of Chapter I, § 2, No. 7]). In addition, denote by C_k the matrix $(c_{k,j_1,j_2})_{j_1,j_2\in J}$ for every $k \in K$. Now, since A_1 and A_2 are symmetric and since B_k is skew-symmetric,

$$C_k = A_1 B_k A_2 + {A_2}^t B_k A_1 = A_1 B_k A_2 - A_2 B_k A_1$$

for every $k \in K$. The assertion follows easily. □

It is worthwhile to consider some results on the simultaneous diagonalization of quadratic forms with respect to some skew-symmetric form.

Proposition 7.4. *Let V be a finite-dimensional vector space over $\mathbb{R}$, and let σ be a skew-symmetric bilinear form on V. In addition, let $(Q_\alpha)_{\alpha\in A}$ be a family of positive, non-degenerate bilinear forms on V such that the $\mathrm{d}_{Q_\alpha}^{-1} \circ \mathrm{d}_\sigma$, as α runs through A, commute.*

Then, there is a finite family $(P_\gamma)_{\gamma\in\Gamma}$ of projectors of V such that the following hold:

- *there is $\gamma_0 \in \Gamma$ such that $P_{\gamma_0}(V) = \ker \mathrm{d}_\sigma$;*
- *P_γ is σ- and Q_α-self-adjoint for every $\alpha \in A$ and for every $\gamma \in \Gamma$;*
- *$I_V = \sum_{\gamma\in\Gamma} P_\gamma$ and $P_\gamma P_{\gamma'} = 0$ for $\gamma, \gamma' \in \Gamma$, $\gamma \neq \gamma'$;*

- *the bilinear forms* $Q_\alpha(P_\gamma\,\cdot\,, P_\gamma\,\cdot\,)$*, as* $\alpha \in A$*, are all multiples of one another for every* $\gamma \in \Gamma$*,* $\gamma \neq \gamma_0$*.*

Observe that this also implies that, if R is the absolute value of $\mathrm{d}_{Q_\alpha}^{-1} \circ \mathrm{d}_\sigma$ relative to Q_α, then R is the absolute value of $\mathrm{d}_{Q_\alpha}^{-1} \circ \mathrm{d}_\sigma$ relative to $Q_{\alpha'}$ for every $\alpha' \in A$. Therefore, we may denote R by $|\mathrm{d}_{Q_\alpha}^{-1} \circ \mathrm{d}_\sigma|$ unambiguously.

Proof. Define $J_\alpha \coloneqq \mathrm{d}_{Q_\alpha}^{-1} \circ \mathrm{d}_\sigma$ for every $\alpha \in A$. Then, for every $x, y \in V$,

$$Q_\alpha(x, J_\alpha(y)) = \langle x, \mathrm{d}_\sigma(y)\rangle = \sigma(x, y),$$

so that

$$Q_\alpha(x, J_\alpha(y)) = \sigma(x, y) = -\sigma(y, x) = -Q_\alpha(y, J_\alpha(x)) = Q_\alpha(-J_\alpha(x), y).$$

Hence, J_α is Q_α-skew-adjoint, and then $-J_\alpha^2$ is Q_α-positive and self-adjoint, hence diagonalizable. In addition, for every $x, y \in V$,

$$\sigma(x, J_\alpha(y)) = Q_\alpha(x, J_\alpha^2(y)) = -Q_\alpha(J_\alpha(x), J_\alpha(y)) = \sigma(-J_\alpha(x), y),$$

so that J_α is also σ-skew-adjoint; hence, $-J_\alpha^2$ is σ-self-adjoint. Therefore, if Γ is the set of joint eigenvalues of the family $(-J_\alpha^2)_{\alpha\in A}$, and $(V_\gamma)_{\gamma\in\Gamma}$ is the corresponding family of joint eigenspaces, then the V_γ are pairwise σ-orthogonal. Now, let $U_\alpha R_\alpha$ be the polar decomposition of J_α relative to Q_α; in other words, R_α is the absolute value of J_α, while U_α is an isometry which restrict to the identity on $\ker \mathrm{d}_\sigma$. Next, take $\alpha_1, \alpha_2 \in A$. Since $[J_{\alpha_1}, J_{\alpha_2}] = 0$, we infer that R_{α_1}, R_{α_2}, U_{α_1}, and U_{α_2} all commute. In particular, define $V' \coloneqq \bigoplus_{\gamma\in\Gamma\setminus\{0\}} V_\gamma$, so that V' is an algebraic complement of the radical V_0 of σ, and is the orthogonal complement of V_0 with respect to Q_α for every $\alpha \in A$. Then, U_α is σ- and Q_α-skew-symmetric on V' for every $\alpha \in A$, so that $-U_\alpha^2$ is the identity on V'. Take $\alpha_1, \alpha_2 \in A$; then, for every $x, y \in V'$,

$$Q_{\alpha_2}(x, (U_{\alpha_1} U_{\alpha_2} R_{\alpha_2})(y)) = \sigma(x, U_{\alpha_1}(y)) = -\sigma(U_{\alpha_1}(x), y) = Q_{\alpha_2}((U_{\alpha_1} U_{\alpha_2})(x), R_{\alpha_2}(y))$$

and

$$Q_{\alpha_2}(x, (U_{\alpha_1} U_{\alpha_2} R_{\alpha_2})(y)) = \sigma(x, U_{\alpha_1}(y)) = Q_{\alpha_1}(x, -R_{\alpha_1}(y)),$$

so that $-U_{\alpha_1} U_{\alpha_2}$ is self-adjoint and positive with respect to $Q_{\alpha_2}(\,\cdot\,, R_{\alpha_2}\,\cdot\,)$ on V'. Since $(U_{\alpha_1} U_{\alpha_2})^2$ is the identity on V', it follows that $U_{\alpha_1} = U_{\alpha_2}$ on V'. Therefore, $U_{\alpha_1} = U_{\alpha_2}$ on V, so that there is U such that $U_\alpha = U$ for every $\alpha \in A$. In particular,

$$Q_{\alpha_1}(x, R_{\alpha_1}(y)) = \sigma(x, U^{-1}(y)) = Q_{\alpha_2}(x, R_{\alpha_2}(y))$$

for every $x, y \in V$, so that the V_γ, as γ runs through Γ, are Q_α-orthogonal for every $\alpha \in A$.

Now, take a non-zero $\gamma \in \Gamma$. Then, $\gamma_\alpha \neq 0$ for every $\alpha \in A$, so that the Q_α are multiple of one another on V_γ. The assertion follows. ☐

Choosing a symplectic basis of σ on each V_γ, $\gamma \neq \gamma_0$, as in [1, Corollary 5.6.3], we get the following corollary. Observe that $Q_\alpha(v_j, v_j)$ is the eigenvalue of $|\mathrm{d}_{Q_\alpha}^{-1} \circ \mathrm{d}_\sigma|$ on the V_γ to which v_j belongs.

Corollary 7.5. *Keep the hypotheses and the notation of Proposition 7.4, and let* m *be the dimension of* V*. Then, we may find a positive integer* $n \leqslant \frac{m}{2}$ *and a basis* $(v_j)_{j=1,\dots,m}$ *of* V *such that the following hold:*

- $Q_\alpha(v_j, v_j) = Q_\alpha(v_{n+j}, v_{n+j}) > 0$ *for every* $\alpha \in A$ *and for every* $j = 1, \dots, n$*;*
- $Q_\alpha(v_j, v_k) = 0$ *for every* $\alpha \in A$ *and for every* $j, k \in \{1, \dots, m\}$ *such that* $j \neq k$ *and either* $j \leqslant 2n$ *or* $k \leqslant 2n$*;*
- *for every* $j, k = 1, \dots, m$*,*

$$\sigma(v_j, v_k) = \begin{cases} 1 & \text{if } j \in \{1, \dots, n\} \text{ and } k = n + j; \\ -1 & \text{if } j \in \{n+1, \dots, 2n\} \text{ and } k = j - n; \\ 0 & \text{otherwise.} \end{cases}$$

Finally, let us show that the existence of commuting sub-Laplacians implies that the group is reducible in some sense. We consider only the case of MW^+-groups.

Proposition 7.6. *Take a finite family $(\mathcal{L}_\alpha)_{\alpha\in A}$ of commuting sub-Laplacians on an MW^+-group G, and let $(Q_\alpha)_{\alpha\in A}$ be the corresponding family of non-degenerate positive bilinear forms on $\mathfrak{g}_1^*$. Then, there is a finite family $(P_\gamma)_{\gamma\in\Gamma}$ of non-zero projectors of $\mathfrak{g}_1$ such that the following hold:*

- $I_{\mathfrak{g}_1} = \sum_{\gamma\in\Gamma} P_\gamma$ *and* $P_{\gamma_1}P_{\gamma_2} = 0$ *for every* $\gamma_1, \gamma_2 \in \Gamma$ *such that* $\gamma_1 \neq \gamma_2$*;*
- P_γ *is* B_ω*- and* $\widehat{Q}_\alpha$*-self-adjoint for every* $\gamma \in \Gamma$*, for every* $\omega \in \mathfrak{g}_2^*$*, and for every* $\alpha \in A$*;*
- *for every* $\gamma \in \Gamma$*, the symmetric bilinear forms* $Q_\alpha\big({}^tP_\gamma\,\cdot\,, {}^tP_\gamma\,\cdot\big)$*, as* α *runs through* A*, are mutually proportional.*

Proof. **1.** Fix $\omega_0 \in \mathfrak{g}_2^*$ such that B_{ω_0} is non-degenerate. Then, Corollary 7.5 implies that we may find a basis $X_1, \dots, X_{2n}$ of $\mathfrak{g}_1$ such that $\mathrm{d}_{B_{\omega_0}}$ is represented by the matrix

$$\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix},$$

while, for every $\alpha \in A$, d_{Q_α} is represented by the matrix

$$\begin{pmatrix} D_\alpha & 0 \\ 0 & D_\alpha \end{pmatrix}$$

for some diagonal matrix D_α. Denote by $d_{\alpha,1}, \dots, d_{\alpha,n}$ the diagonal elements of D_α, and denote by $(a_{\omega,j,k})$ the matrix associated with d_{B_ω}, for every non-zero $\omega \in \mathfrak{g}_2^*$.

2. Assume that A has exactly two elements α_1, α_2, and define

$$\Gamma := \left\{ \frac{d_{\alpha_1,j}}{d_{\alpha_2,j}} : j \in \{\,1, \dots, n\,\} \right\}$$

and, for every $\gamma \in \Gamma$, let V_γ be the vector subspace of $\mathfrak{g}_1$ generated by the set

$$\left\{ X_j, X_{n+j} : \frac{d_{\alpha_1,j}}{d_{\alpha_2,j}} = \gamma \right\}.$$

Next, take $j, k \in \{\,1, \dots, n\,\}$ such that $\frac{d_{\alpha_1,j}}{d_{\alpha_2,j}} \neq \frac{d_{\alpha_1,k}}{d_{\alpha_2,k}}$. Apply Proposition 7.3, and observe that the (j,k)-th components of (the matrices representing) the equality

$$\mathrm{d}_{Q_{\alpha_1}} \circ \mathrm{d}_{B_\omega} \circ \mathrm{d}_{Q_{\alpha_2}} = \mathrm{d}_{Q_{\alpha_2}} \circ \mathrm{d}_{B_\omega} \circ \mathrm{d}_{Q_{\alpha_1}}$$

give

$$d_{\alpha_1,j} a_{\omega,j,k} d_{\alpha_2,k} = d_{\alpha_2,j} a_{\omega,j,k} d_{\alpha_1,k},$$

so that $a_{\omega,j,k} = 0$. Considering the components $(n+j,k), (j,n+k)$ and $(n+j,n+k)$, we see that $a_{\omega,n+j,k} = a_{\omega,j,n+k} = a_{\omega,n+j,n+k} = 0$. Therefore, $B_\omega(V_{\gamma_1}, V_{\gamma_2}) = \{\,0\,\}$ for every non-zero $\omega \in \mathfrak{g}_2^*$ and for every $\gamma_1, \gamma_2 \in \Gamma$ such that $\gamma_1 \neq \gamma_2$. Then, define P_γ as the projector of $\mathfrak{g}_1$ onto V_γ with kernel $\bigoplus_{\gamma'\neq\gamma} V_{\gamma'}$.

3. Consider the general case. Take $\alpha_1, \alpha_2 \in A$ such that $\alpha_1 \neq \alpha_2$, and define $\Gamma_{\alpha_1,\alpha_2} := \left\{ \frac{d_{\alpha_1,j}}{d_{\alpha_2,j}} : j \in \{\,1, \dots, n\,\} \right\}$ and the corresponding projectors $P_{\alpha_1,\alpha_2,\gamma}$ as in **2** above. Define, for every $\gamma \in \prod_{\alpha_1\neq\alpha_2} \Gamma_{\alpha_1,\alpha_2}$,

$$P_\gamma := \prod_{\alpha_1\neq\alpha_2} P_{\alpha_1,\alpha_2,\gamma_{\alpha_1,\alpha_2}};$$

notice that P_γ is a projector since the $P_{\alpha_1,\alpha_2,\gamma_{\alpha_1,\alpha_2}}$ commute. Then, it suffice to take Γ as the set of $\gamma \in \prod_{\alpha_1\neq\alpha_2} \Gamma_{\alpha_1,\alpha_2}$ such that $P_\gamma \neq 0$. □

7.2 Plancherel Measure and Integral Kernel

In this section, $(Q_\eta)_{\eta\in H}$ denotes a finite family of *positive* symmetric bilinear forms on $\mathfrak{g}_1^*$ and $(T_1,\dots,T_{n_2})$ a basis of $\mathfrak{g}_2$. We shall denote by $\mathcal{L}_\eta$ the sub-Laplacian induced by Q_η, and we shall assume that $\mathcal{L}_A \coloneqq (\mathcal{L}_H, (-iT_k)_{k=1,\dots,n_2})$ is a Rockland family. Observe that this condition is equivalent to the fact that the sum of the $\mathcal{L}_\eta$ is hypoelliptic. Indeed, if π_0 is the projection of G onto its abelianization, then $\mathrm{d}\pi_0(\mathcal{L}_A)$ is a Rockland family, so that $\mathcal{F}(\mathrm{d}\pi_0(\mathcal{L}_A)\delta_0)$ vanishes only at 0. Since $\mathcal{F}(\mathrm{d}\pi_0(\mathcal{L}_\eta)\delta_0)\geqslant 0$ and $\mathrm{d}\pi_0(T_k)=0$ for every $\eta\in H$ and for every $k=1,\dots,n_2$, this implies that $\sum_{\eta\in H}\mathcal{F}(\mathrm{d}\pi_0(\mathcal{L}_\eta)\delta_0)$ vanishes only at 0, so that $\sum_{\eta\in H} Q_\eta$ is positive and non-degenerate, so that $\sum_{\eta\in H}\mathcal{L}_\eta$ is hypoelliptic. We may therefore assume that Q_η is positive and non-degenerate for every $\eta\in H$.

We shall assume that either one of the following conditions hold:

- G is an MW^+-group;
- $\mathrm{Card}(H)=1$; in this case, we shall omit unnecessary indices.

We shall also endow $\mathfrak{g}$ with a scalar product for which $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are orthogonal, and which induces $\widehat{Q}_{\eta_0}$ on $\mathfrak{g}_1$ for some *fixed* $\eta_0\in H$. If n is the dimension of G, then we may define $\mathcal{H}^n$ with respect to the induced distance, and $\exp_*(\mathcal{H}^n)$ is a Haar measure on G. Up to a normalization, *we may then assume that* $(\exp_G)_*(\mathcal{H}^n)$ *is the chosen Haar measure on* G. We shall endow $\mathfrak{g}_2^*$ with the scalar product induced by that of $\mathfrak{g}_2$, and then with the corresponding Hausdorff measure.

We define

$$J_{Q_\eta,\omega}\coloneqq \mathrm{d}_{Q_\eta}\circ \mathrm{d}_{B_\omega}\colon \mathfrak{g}_1\to\mathfrak{g}_1$$

for every $\eta\in H$ and for every $\omega\in\mathfrak{g}_2^*$. In addition, define $d\coloneqq \min_{\omega\in\mathfrak{g}_2^*}\dim\ker \mathrm{d}_{B_\omega}$, so that G is an MW^+-group if and only if $d=0$. We shall denote by W the set of $\omega\in\mathfrak{g}_2^*$ such that $\dim\ker\mathrm{d}_{B_\omega}>d$, so that G is a Métivier group if and only if $d=0$ and $W=\{0\}$.

We shall denote by Ω the set of $\omega\in\mathfrak{g}_2^*\setminus W$ where $\mathrm{Card}(\sigma(|J_{Q_H,\omega}|))$ attains its maximum $\overline{h}$.[1] Notice that Ω is open and dense, as the next lemma shows.

Lemma 7.7. *W and $\mathfrak{g}_2^*\setminus\Omega$ are algebraic varieties.*

As the proof shows, the multiplicities of the eigenvalues are constant on Ω.

Proof. Choose $\eta\in H$ and define $C_{\eta,\omega}$ so that $X^d C_{\eta,\omega}(X^2)$ is the characteristic polynomial of $-J^2_{Q_\eta,\omega}$. Then, it is clear that W is the zero locus of the polynomial mapping $\omega\mapsto C_{\eta,\omega}(0)$, so that it is an algebraic variety.

Next, take $k\in\{1,\dots,n_1\}$ and let $\mathfrak{P}_k$ be the set of partitions of $\{1,\dots,n_1\}$ into k non-empty sets. Define

$$N_k((X_{\eta,1},\dots,X_{\eta,n_1})_{\eta\in H})\coloneqq \prod_{\mathcal{K}\in\mathfrak{P}_k}\sum_{K\in\mathcal{K}}\sum_{\eta\in H}\sum_{k_1,k_2\in K}(X_{\eta,k_1}-X_{\eta,k_2})^2,$$

so that N_k is a $\mathfrak{S}_{n_1}^H$-invariant polynomial. Take $\widetilde{\mu}_{\eta,1,\omega},\dots,\widetilde{\mu}_{\eta,n_1,\omega}\in\mathbb{R}_+$ so that

$$0,\dots,0,\pm i\widetilde{\mu}_{\eta,1,\omega},\dots,\pm i\widetilde{\mu}_{\eta,n_1,\omega}$$

are the eigenvalues of $J_{Q_\eta,\omega}$ for every $\omega\in\mathfrak{g}_2^*$ and for every $\eta\in H$. Then, the $\mathfrak{S}_{n_1}^H$-invariant function $\omega\mapsto N_k((\widetilde{\mu}^2_{\eta,1,\omega},\dots,\widetilde{\mu}^2_{\eta,n_1,\omega})_{\eta\in H})$ in the roots of the polynomials $(C_{\eta,\omega})_{\eta\in H}$ is a polynomial mapping (cf. [14, Theorem 1 of Chapter IV, § 6, No. 1] for the case where H has one element). Therefore, the set of $\omega\in\mathfrak{g}_2^*$ such that $N_k((\widetilde{\mu}_{\eta,1,\omega},\dots,\widetilde{\mu}_{\eta,n_1,\omega})_{\eta\in H})=0$ is an algebraic variety W_k. In addition, it is clear that Ω is the complement of $W\cup W_{\overline{h}-1}$, so that it is open in the Zariski topology. ∎

Proposition 7.8. *There are four analytic mappings*

$$\mu\colon\Omega\to((\mathbb{R}_+^*)^{\overline{h}})^H$$
$$P\colon\Omega\to\mathcal{L}(\mathfrak{g}_1)^{\overline{h}}$$
$$P_0\colon\mathfrak{g}_2^*\setminus W\to\mathcal{L}(\mathfrak{g}_1)$$
$$\rho\colon\Omega\to\{1,\dots,\overline{h}\}^{n_1}$$

[1] By an abuse of notation, we denote by $|J_{Q_H,\omega}|$ the family $(|J_{Q_\eta,\omega}|)_{\eta\in H}$.

such that the following hold:

- *the mapping*

$$\Omega \ni \omega \mapsto \mu_{\eta,\rho_{k,\omega},\omega} \in \mathbb{R}_+$$

  extends to a continuous mapping $\omega \mapsto \widetilde{\mu}_{\eta,k,\omega}$ *on* $\mathfrak{g}_2^*$ *for every* $k = 1, \dots, n_1$ *and for every* $\eta \in H$*;*

- *for every* $h = 0, \dots, \overline{h}$ *and for every* $\omega \in \Omega$ *(for every* $\omega \in \mathfrak{g}_2^* \setminus W$ *if* $h = 0$*),* $P_{h,\omega}$ *is a* B_ω*- and* $\widehat{Q}_H$*-self-adjoint projector of* $\mathfrak{g}_1$*;*

- *if* $h = 1, \dots, \overline{h}$ *and* $\omega \in \Omega$*, then* $\operatorname{Tr} P_{h,\omega} = 2 \operatorname{Card}(\{ k \in \{ 1, \dots, n_1 \} \colon \rho_{k,\omega} = h \})$*;*

- $\sum_{h=0}^{\overline{h}} P_{h,\omega} = I_{\mathfrak{g}_1}$ *and* $\sum_{h=1}^{\overline{h}} \mu_{\eta,h,\omega} P_{h,\omega} = |J_{Q_\eta,\omega}|$ *for every* $\omega \in \Omega$ *and for every* $\eta \in H$*;*

- $P_{0,\omega}(\mathfrak{g}_1) = \ker \mathrm{d}_{B_\omega}$ *for every* $\omega \in \mathfrak{g}_2^* \setminus W$*.*

The proof is based on [54, § 1.3–4 and § 5.1 of Chapter II] when $\operatorname{Card}(H) = 1$. The extension to the case $\operatorname{Card}(H) > 1$ follows by means of Proposition 7.6. We leave the details to the reader.

Definition 7.9. We shall define $\mu, \widetilde{\mu}, P$ and P_0 as in Proposition 7.8. In addition, we define $\mathbf{n_1} \colon \Omega \to (\mathbb{N}^*)^{\overline{h}}$ so that $n_{1,h,\omega} = \frac{1}{2} \operatorname{Tr} P_{h,\omega}$ for every $h = 1, \dots, \overline{h}$ and for every $\omega \in \Omega$.

Furthermore, we shall sometimes identify μ_ω with the linear mapping

$$\mathbb{R}^{\overline{h}} \ni \lambda \mapsto \left(\sum_{h=1}^{\overline{h}} \mu_{\eta,h,\omega} \lambda_h \right)_{\eta \in H} \in \mathbb{R}^H$$

for every $\omega \in \Omega$. Analogous notation for $\widetilde{\mu}_\omega$.

With the above notation, we have $\mu_\omega(\mathbf{n_1}_{,\omega}) = \left(\sum_{h=1}^{\overline{h}} \mu_{\eta,h,\omega} n_{1,h,\omega} \right)_{\eta \in H}$. Observe, in addition, that the index 1 in $\mathbf{n_1}$ refers to the first layer $\mathfrak{g}_1$, just as the index 2 in n_2 refers to the second layer $\mathfrak{g}_2$.

Corollary 7.10. *The function* $\omega \mapsto \mu_{\eta,\omega}(\mathbf{n_1}_{,\omega}) = \widetilde{\mu}_{\eta,\omega}(\mathbf{1}_{n_1})$ *is a norm on* $\mathfrak{g}_2^*$ *which is analytic on* $\mathfrak{g}_2^* \setminus W$ *for every* $\eta \in H$*.*

Proof. Observe that

$$2\mu_{\eta,\omega}(\mathbf{n_1}_{,\omega}) = \| J_{Q_\eta,\omega} \|_1 = \| J_{Q_\eta,\omega} + P_{0,\omega} \|_1 - d$$

for every $\omega \in \mathfrak{g}_2^*$, and that the linear mapping $\omega \mapsto J_{Q_\eta,\omega}$ is one-to-one since G is stratified. The assertion follows. □

Definition 7.11. By an abuse of notation, we shall denote by (x,t) the elements of G, where $x \in \mathfrak{g}_1$ and $t \in \mathfrak{g}_2$, thus identifying (x,t) with $\exp_G(x,t)$. For every $x \in \mathfrak{g}_1$ and for every $\omega \in \mathfrak{g}_2^* \setminus W$, we shall define

$$x_{0,\omega} \coloneqq P_{0,\omega}(x),$$

while, for every $\omega \in \Omega$ and for every $h = 1, \dots, \overline{h}$,

$$x_{h,\omega} \coloneqq \sqrt{\mu_{\eta_0,h,\omega}} P_{h,\omega}(x).$$

By an abuse of notation, we write x_ω instead of $(x_{h,\omega})_{h=1,\dots,\overline{h}}$, so that $|x_\omega| = \left(\sum_{h=1}^{\overline{h}} |x_{h,\omega}|^2 \right)^{1/2}$.

Proposition 7.12. *The mapping*

$$\mathfrak{g}_1 \times \Omega \ni (x, \omega) \mapsto \sum_{h=1}^{\overline{h}} x_{h,\omega}$$

extends uniquely to a continuous function on $\mathfrak{g}_1 \times \mathfrak{g}_2^*$ *which is analytic on* $\mathfrak{g}_1 \times (\mathfrak{g}_2^* \setminus W)$*.*

Proof. Observe that, for every $\omega \in \mathfrak{g}_2^*$, $-J^2_{Q_{\eta_0},\omega} = J^*_{Q_{\eta_0},\omega} J_{Q_{\eta_0},\omega}$ is positive, and that

$$-J^2_{Q_{\eta_0},\omega} + P_{0,\omega}$$

is positive and non-degenerate as long as $\omega \notin W$. Therefore, the mapping

$$\omega \mapsto \sqrt[4]{-J^2_{Q_{\eta_0},\omega}} = \sqrt[4]{-J^2_{Q_{\eta_0},\omega} + P_{0,\omega}} - P_{0,\omega} \in \mathcal{L}(\mathfrak{g}_1)$$

is continuous on $\mathfrak{g}_2^*$ and analytic on $\mathfrak{g}_2^* \setminus W$ thanks to [23, Proposition 10 of Chapter I, § 4, No. 8].[2] Then, it suffices to observe that

$$\sqrt[4]{-J^2_{Q_{\eta_0},\omega}}(x) = \sum_{h=1}^{\overline{h}} x_{h,\omega}$$

for every $\omega \in \Omega$ and for every $x \in \mathfrak{g}_1$. □

Definition 7.13. Define G_ω, for every $\omega \in \mathfrak{g}_2^*$, as the quotient of G by its normal subgroup $\exp_G(\ker \omega)$.

Then, G_0 is the abelianization of G, and we identify it with $\mathfrak{g}_1$. If $\omega \neq 0$, then we shall identify G_ω with $\mathfrak{g}_1 \oplus \mathbb{R}$, endowed with the product

$$(x_1, t_1)(x_2, t_2) \coloneqq \left(x_1 + x_2, t_1 + t_2 + \frac{1}{2} B_\omega(x_1, x_2) \right)$$

for every $x_1, x_2 \in \mathfrak{g}_1$ and for every $t_1, t_2 \in \mathbb{R}$. Hence,

$$\pi_\omega(x, t) = (x, \omega(t))$$

for every $(x, t) \in G$.

Proposition 7.14. *Define*

$$\varpi \colon \bigcup_{\omega \in \Omega} (\{ \omega \} \times G_\omega) \ni (\omega, (x, t)) \mapsto \omega \in \Omega,$$

and identify the domain of ϖ with $\Omega \times (\mathfrak{g}_1 \oplus \mathbb{R})$ as an analytic manifold, so that ϖ becomes an analytic submersion.

Then, ϖ defines a fibre bundle with base Ω and fibres isomorphic to $G' \coloneqq \mathbb{H}^{n_1} \oplus \mathbb{R}^d$. More precisely, for every $\omega_0 \in \Omega$, there is an analytic trivialization (U, ψ) of ϖ such that the following hold:

- *U is an open neighbourhood of ω_0 in Ω;*
- *$\psi \colon \varpi^{-1}(U) \to U \times G'$ is an analytic diffeomorphism such that $\mathrm{pr}_1 \circ \psi = \varpi$ and such that $\psi_\omega \coloneqq \mathrm{pr}_2 \circ \psi \colon \varpi^{-1}(\omega) \to G'$ is a group isomorphism for every $\omega \in U$;*
- *if $(X_1, \dots, X_{2n_1}, T, Y_1, \dots, Y_d)$ is a basis of left-invariant vector fields on G' which at the origin induce the partial derivatives along the coordinate axes, then*

$$\mathrm{d}(\psi_\omega \circ \pi_\omega)(\mathcal{L}_\eta) = -\sum_{k=1}^{n_1} \widetilde{\mu}_{\eta,k,\omega} (X_k^2 + X_{n_1+k}^2) - \sum_{k=1}^{d} Y_k^2$$

  and

$$\mathrm{d}(\psi_\omega \circ \pi_\omega)(T_\ell) = \omega(T_\ell) T$$

  for every $\eta \in H$, for every $\ell = 1, \dots, n_2$, and for every $\omega \in U$.

The proof is omitted. It basically consists in using the projectors P_h to propagate locally a given basis of eigenvectors and then in 'symplectifying' the new basis in order to meet the requirements.

[2] For what concerns continuity, just observe that $\sqrt[4]{\cdot}$ is continuous on the cone of positive endomorphisms of $\mathfrak{g}_1$, which is the closure of the cone of non-degenerate positive endomorphisms of $\mathfrak{g}_1$, as in [50, p. 85].

Definition 7.15. For every $\omega \in \mathfrak{g}_2^* \setminus W$, define $|\mathrm{Pf}(\omega)| := \prod_{h=1}^{\overline{h}} \mu_{\eta_0,h,\omega}^{n_{1,h,\omega}}$, the Pfaffian of ω (cf. [3]).

Furthermore, take $m, \gamma \in \mathbb{N}$. Then, we shall denote by Λ_m^γ the m-th Laguerre polynomial of order γ. In other words, $\Lambda_m^\gamma(X) = \sum_{j=0}^m \binom{m+\gamma}{m-j} \frac{(-X)^j}{j!}$.

Before we state the next result, where we find relatively explicit formulae for the Plancherel measure and the integral kernel associated with $\mathcal{L}_A$, let us comment briefly our techniques. Thanks to the form of the Plancherel formula for G (see [3]), we basically reduce to study $\mathrm{d}\pi_\omega(\mathcal{L}_A)$ for $\omega \neq 0$, or only for $\omega \in \Omega$. Therefore, the analysis of $\mathcal{L}_A$ is reduced to the case in which $n_2 = 1$. If G is actually a Heisenberg group, then the Plancherel formula involves only the Bargmann-Fock representations π_λ ($\lambda \neq 0$), and it is well-known that $\mathrm{d}\pi_\lambda(\mathcal{L}_A)$ has an orthonormal basis of eigenfunctions which consists of suitably normalized monomials; the corresponding functions of positive type on G are then Laguerre functions (cf. [53]). If G has higher-dimensional centre, then it splits into the product of a Heisenberg group and an abelian group, and the results are somewhat similar, even though the presence of the abelian factor causes a 'superposition' of different 'layers' of the Plancherel measure associated with $\mathcal{L}_A$.

Proposition 7.16. *Define, for every $\gamma_1, \gamma_2 \in \mathbb{N}^{\overline{h}}$,*

$$\varphi_{\gamma_1,\gamma_2} \colon \mathfrak{g}_1^{\overline{h}} \times \mathbb{R} \ni (x,t) := e^{it} \prod_{h=1}^{\overline{h}} e^{-\frac{1}{4}|x_h|^2} \Lambda_{\gamma_{2,h}}^{\gamma_{1,h}-1}\left(\frac{1}{2}|x_h|^2\right)$$

and

$$\Phi_d \colon \mathbb{R}_+ \ni x \mapsto \Gamma\left(\frac{d}{2}\right) \frac{J_{\frac{d}{2}-1}(x)}{\left(\frac{x}{2}\right)^{\frac{d}{2}-1}}.$$

Then, the following hold:

1. *suppose that $d > 0$, and define for every $(\lambda, \omega) \in \mathbb{R} \times (\mathfrak{g}_2^* \setminus W)$ such that $\lambda > \mu_\omega(\mathbf{n}_{\mathbf{1},\omega})$,*[3]

$$c_{\lambda,\omega} := \sum_{\substack{\gamma \in \mathbb{N}^{\overline{h}} \\ \mu_\omega(\mathbf{n}_{\mathbf{1},\omega}+2\gamma) < \lambda}} \binom{\mathbf{n}_{\mathbf{1},\omega} + \gamma - \mathbf{1}_{\overline{h}}}{\gamma} (\lambda - \mu_\omega(\mathbf{n}_{\mathbf{1},\omega} + 2\gamma))^{\frac{d}{2}-1}.$$

Then, for every $\varphi \in \mathcal{D}^0(E_{\mathcal{L}_A})$,

$$\int_{E_{\mathcal{L}_A}} \varphi \,\mathrm{d}\beta_{\mathcal{L}_A} = \frac{\pi^{\frac{d}{2}}}{(2\pi)^{n_1+n_2+d}\Gamma\left(\frac{d}{2}\right)} \int_{\lambda > \mu_\omega(\mathbf{n}_{\mathbf{1},\omega})} \varphi(\lambda, \omega(\mathbf{T})) c_{\lambda,\omega} |\mathrm{Pf}(\omega)| \,\mathrm{d}(\lambda,\omega);$$

2. *suppose that $d = 0$, and define $\Sigma_\omega := \mu_\omega(\mathbf{n}_{\mathbf{1},\omega} + 2\mathbb{N}^{\overline{h}})$ for every $\omega \in \Omega$; then, for every $\varphi \in \mathcal{D}^0(E_{\mathcal{L}_A})$,*

$$\int_{E_{\mathcal{L}_A}} \varphi \,\mathrm{d}\beta_{\mathcal{L}_A} = \frac{1}{(2\pi)^{n_1+n_2}} \int_{\mathfrak{g}_2^*} \sum_{\gamma \in \Sigma_\omega} c'_{\gamma,\omega} \varphi(\gamma, \omega(\mathbf{T})) |\mathrm{Pf}(\omega)| \,\mathrm{d}\omega,$$

where

$$c'_{\gamma,\omega} := \sum_{\substack{\gamma' \in \mathbb{N}^{\overline{h}} \\ \mu_\omega(\mathbf{n}_{\mathbf{1},\omega}+2\gamma') = \gamma}} \binom{\mathbf{n}_{\mathbf{1},\omega} + \gamma' - \mathbf{1}_{\overline{h}}}{\gamma'}$$

for every $\omega \in \Omega$ and for every $\gamma \in \Sigma_\omega$;

3. *suppose that $d > 0$; then, for $(\beta_{\mathcal{L}_A} \otimes \nu_G)$-almost every $((\lambda, \omega(\mathbf{T})), (x,t))$,*

$$\chi_{\mathcal{L}_A}((\lambda, \omega(\mathbf{T})), (x,t)) = \frac{1}{c_{\lambda,\omega}} \sum_{\substack{\gamma \in \mathbb{N}^{\overline{h}} \\ \mu_\omega(\mathbf{n}_{\mathbf{1},\omega}+2\gamma) < \lambda}} (\lambda - \mu_\omega(\mathbf{n}_{\mathbf{1},\omega} + 2\gamma))^{\frac{d}{2}-1} \times$$
$$\times \varphi_{\mathbf{n}_{\mathbf{1},\omega},\gamma}(x_\omega, \omega(t)) \Phi_d\left(\sqrt{\lambda - \mu_\omega(\mathbf{n}_{\mathbf{1},\omega} + 2\gamma)}\,|x_{0,\omega}|\right)$$

[3] Recall that, in this case, we assumed that $\mathrm{Card}(H) = 1$.

4. *suppose that $d=0$; then, for $(\beta_{\mathcal{L}_A}\otimes\nu_G)$-almost every $((\gamma,\omega(\mathbf{T})),(x,t))$,*

$$\chi_{\mathcal{L}_A}((\gamma,\omega(\mathbf{T})),(x,t))=\frac{1}{c'_{\gamma,\omega}}\sum_{\mu_\omega(\mathbf{n}_{\mathbf{1},\omega}+2\gamma')=\gamma}\varphi_{\mathbf{n}_{\mathbf{1},\omega},\gamma'}(x_\omega,\omega(t)).$$

Proof. **1–2.** We follow the construction of the Plancherel measure of [3] as in [58, 4.4.1].

Now, let π be an irreducible unitary representation of G into a infinite dimensional hilbertian space H. Then, Schur's lemma (cf. [42, (3.5)]) implies that there is a unique $\omega\in\mathfrak{g}_2^*$ such that $\pi(0,t)=e^{i\omega(t)}$ for every $t\in\mathfrak{g}_2$. Hence, there is a unique irreducible representation $\widetilde{\pi}$ of G_ω such that $\pi=\widetilde{\pi}\circ\pi_\omega$. Since G_0 is abelian and H is infinite-dimensional, we have $\omega\neq 0$. Then, Schur's lemma again implies that there is a unique $\tau\in P_{0,\omega}(\mathfrak{g}_1)^*$ such that $\widetilde{\pi}(x,t)=e^{it+i\tau(x)}$ for every $(x,t)\in P_{0,\omega}(\mathfrak{g}_1)\times\mathbb{R}$. Since the quotient of G_ω by its normal subgroup $P_{0,\omega}(\mathfrak{g}_1)$ is a Heisenberg group, the Stone–Von Neumann theorem (cf. [42, (6.49)]) implies that there is one and only one such $\widetilde{\pi}$, once ω and τ are fixed; we denote by $\pi_{\omega,\tau}$ the corresponding representation π of G, and by $H_{\omega,\tau}$ the corresponding hilbertian space.

Now, fix $\omega\in\mathfrak{g}_2^*\setminus W$ and $\tau\in P_{0,\omega}(\mathfrak{g}_1)^*$. Then, for every $f\in L^2(G)$,

$$\|f\|_2^2=\frac{1}{(2\pi)^{n_1+n_2+d}}\int_{\mathfrak{g}_2^*}\int_{P_{0,\omega}(\mathfrak{g}_1)^*}\|\pi_{\omega,\tau}(f)\|_2^2|\mathrm{Pf}(\omega)|\,\mathrm{d}\tau\,\mathrm{d}\omega.$$

Now, it is well-known that there is a commutative family $(P_{\omega,\tau,\gamma})_{\gamma\in\mathbb{N}^{\overline{h}}}$ of self-adjoint projectors of $H_{\omega,\tau}$ such that $I_{H_{\omega,\tau}}=\sum_{\gamma\in\mathbb{N}^{\overline{h}}}P_{\omega,\tau,\gamma}$ pointwise, and such that for every $\gamma\in\mathbb{N}^{\overline{h}}$ we have $\operatorname{Tr}P_{\omega,\tau,\gamma}=\binom{\mathbf{n}_{\mathbf{1},\omega}+\gamma-\mathbf{1}_{\overline{h}}}{\gamma}$ and

$$\mathrm{d}\pi_{\omega,\tau}(\mathcal{L}_A)\cdot P_{\omega,\tau,\gamma}=(|\tau|^2+\mu_\omega(\mathbf{n}_{\mathbf{1},\omega}+2\gamma),\omega(\mathbf{T}))P_{\omega,\tau,\gamma}.$$

Therefore, for every $\varphi\in\mathcal{D}(E_{\mathcal{L}_A})$,

$$\|\mathcal{K}_{\mathcal{L}_A}(\varphi)\|_2^2=\frac{1}{(2\pi)^{n_1+n_2+d}}\int_{\mathfrak{g}_2^*}\int_{P_{0,\omega}(\mathfrak{g}_1)^*}\sum_{\gamma\in\mathbb{N}^{\overline{h}}}\binom{\mathbf{n}_{\mathbf{1},\omega}+\gamma-\mathbf{1}_{\overline{h}}}{\gamma}\times$$
$$\times\left|\varphi\Big(|\tau|^2+\mu_\omega(\mathbf{n}_{\mathbf{1},\omega}+2\gamma),\omega(\mathbf{T})\Big)\right|^2|\mathrm{Pf}(\omega)|\,\mathrm{d}\tau\,\mathrm{d}\omega.$$

The stated formulae for $\beta_{\mathcal{L}_A}$ follow.

3–4. Now, we shall deal with the integral kernel. Observe first that from the stated Plancherel formula for G we deduce the following inversion formula:

$$f(x,t)=\frac{1}{(2\pi)^{n_1+n_2+d}}\int_{\mathfrak{g}_2^*}\int_{P_{0,\omega}(\mathfrak{g}_1)^*}\operatorname{Tr}(\pi_{\omega,\tau}(x,t)^*\pi_{\omega,\tau}(f))|\mathrm{Pf}(\omega)|\,\mathrm{d}\tau\,\mathrm{d}\omega$$

for every $f\in\mathcal{S}(G)$. If $\varphi\in\mathcal{D}(E_{\mathcal{L}_A})$, then

$$\mathcal{K}_{\mathcal{L}_A}(\varphi)(x,t)=\frac{1}{(2\pi)^{n_1+n_2+d}}\int_{\mathfrak{g}_2^*}\int_{P_{0,\omega}(\mathfrak{g}_1)^*}\sum_{\gamma\in\mathbb{N}^{\overline{h}}}\varphi\Big(|\tau|^2+\mu_\omega(\mathbf{n}_{\mathbf{1},\omega}+2\gamma),\omega(\mathbf{T})\Big)\times$$
$$\times\operatorname{Tr}(\pi_{\omega,\tau}(x,t)^*P_{\omega,\tau,\gamma})|\mathrm{Pf}(\omega)|\,\mathrm{d}\tau\,\mathrm{d}\omega.$$

Next, observe that

$$\operatorname{Tr}(\pi_{\omega,\tau}(x,t)^*P_{\omega,\tau,\gamma})=e^{i\tau(x_{0,\omega})}\varphi_{\mathbf{n}_{\mathbf{1},\omega},\gamma}(x_\omega,\omega(t))$$

by [53, Proposition 2] and [38, 10.12 (41)], while

$$⨍_{\partial B(0,1)\cap P_{0,\omega}(\mathfrak{g}_1)^*}e^{i\tau(x_{0,\omega})}\,\mathrm{d}\mathcal{H}^{d-1}(\tau)=\Gamma\left(\frac{d}{2}\right)\frac{J_{\frac{d}{2}-1}(|x_{0,\omega}|)}{\left(\frac{|x_{0,\omega}|}{2}\right)^{\frac{d}{2}-1}}=\Phi_d(|x_{0,\omega}|).$$

The asserted formulae for $\chi_{\mathcal{L}_A}$ follow. □

Proposition 7.17. *Take $m\colon E_{\mathcal{L}_A}\to\mathbb{C}$ so that $\mathcal{K}_{\mathcal{L}_A}(m)\in\mathcal{M}^1(G)$, and $k\in\mathbb{N}$ such that $k<n_2$. For every element X of the Grassmannian $G_k(\mathfrak{g}_2)$ of k-dimensional subspaces of $\mathfrak{g}_2$, let P_X be the canonical projection of G onto $G/\exp_G(X)$. Then, for almost every $X\in G_k(\mathfrak{g}_2)$,*

$$(P_X)_*(\mathcal{K}_{\mathcal{L}_A}(m))=m(\mathrm{d}P_X(\mathcal{L}_A)).$$

Observe that the assertion with $k = n_2$ need not hold, even if we take m carefully (cf. Theorem 7.45 below).

Proof. Keep the notation of the proof of Proposition 7.16. By Proposition 3.22, there is a negligible subset N_1 of $\mathfrak{g}_2^*$ such that $W \subseteq N_1$ and such that for every $\omega \in \mathfrak{g}_2^* \setminus N_1$ there is a negligible subset $N_{2,\omega}$ of $P_{0,\omega}(\mathfrak{g}_1)^*$ such that

$$\pi^*_{\omega,\tau}(\mathcal{K}_{\mathcal{L}_A}(m)) = m(\mathrm{d}\pi_{\omega,\tau}(\mathcal{L}_A))$$

for every $\tau \in P_{0,\omega}(\mathfrak{g}_1)^* \setminus N_{2,\omega}$. Now, by means of [20, Ex. 11 of Chapter VIII, § 3] and [19, Proposition 2 and Corollary 1 to Proposition 3 of Chapter V, § 3], we see that there is a negligible subset N_1' of $G_k(\mathfrak{g}_2)$ such that, for every $X \in G_k(\mathfrak{g}_2) \setminus N_1'$, the set $X^\circ \cap N_1$ is negligible.

Fix $X \in G_k(\mathfrak{g}_2) \setminus N_1'$. Then, for every $\omega \in X^\circ \setminus \{ 0 \}$ and for every $\tau \in P_{0,\omega}(\mathfrak{g}_1)^*$ there is a unique representation $\pi_{X,\omega,\tau}$ of $P_X(G)$ into $H_{\omega,\tau}$ such that

$$\pi_{\omega,\tau} = \pi_{X,\omega,\tau} \circ P_X.$$

Therefore,

$$\pi^*_{X,\omega,\tau}(P_X(\mathcal{K}_{\mathcal{L}_A}(m))) = m(\mathrm{d}\pi_{X,\omega,\tau}(\mathrm{d}P_X(\mathcal{L}_A)))$$

for every $\omega \in X^\circ \setminus N_1$ and for every $\tau \in P_{0,\omega}(\mathfrak{g}_1)^* \setminus N_{2,\omega}$. Now, Proposition 3.22 again implies that there are a negligible subset $N_{1,X}$ of X° which contains $X^\circ \cap N_1$ and, for every $\omega \in X^\circ \setminus N_{1,X}$, a negligible subset $N_{2,X,\omega}$ of $P_{0,\omega}(\mathfrak{g}_1)^*$ which contains $N_{2,\omega}$ such that

$$\pi^*_{X,\omega,\tau}(\mathcal{K}_{\mathrm{d}P_X(\mathcal{L}_A)}(m)) = m(\mathrm{d}\pi_{X,\omega,\tau}(\mathrm{d}P_X(\mathcal{L}_A)))$$

for every $\tau \in P_{0,\omega}(\mathfrak{g}_1)^* \setminus N_{2,X,\omega}$. Since the representations $\pi_{X,\omega,\tau}$, as ω runs through $X^\circ \setminus N_{1,X}$ and τ runs through $P_{0,\omega}(\mathfrak{g}_1)^* \setminus N_{2,X,\omega}$, form a co-negligible subset of the dual of $P_X(G)$, it follows that $(P_X)_*(\mathcal{K}_{\mathcal{L}_A}(m)) = \mathcal{K}_{\mathrm{d}P_X(\mathcal{L}_A)}(m)$. □

Before we pass to the following sections, let us summarize in a table the most important results which we shall prove for the family $(\mathcal{L}_H, (-iT_1, \dots, -iT_{n_2'}))$, $n_2' \leqslant n_2$.

	$d = 0$	$d > 0$
$W = \{ 0 \}$	Examples with $\neg[(RL) \vee (S)]$	(RL) and (S)
$n_2' < n_2$		(RL)
$n_2' < n_2$, $W = \{ 0 \}$, and $\mathrm{Card}(H) = 1$	(RL) and (S) a.e.	(RL) and (S)

Here, 'a.e.' means 'for almost every choice of $(T_1, \dots, T_{n_2'})$ in $\mathfrak{g}_2^{n_2'}$.' Nevertheless, we do not know of any examples where properties (RL) and (S) do not hold, $n_2' < n_2$, and $W = \{ 0 \}$.

Remark 7.18. Let $T_1', \dots, T_k'$ be k homogeneous elements of the centre $\mathfrak{z}$ of $\mathfrak{g}$. We wish to show that the study of the family $(\mathcal{L}_H, -iT_1', \dots, -iT_k')$ is a consequence of the study of the families of the form $(\mathcal{L}_H, -iT_1, \dots, -iT_{n_2'})$, where $T_1, \dots, T_{n_2'} \in \mathfrak{g}_2$. Notice that, since $\mathfrak{z} = \mathfrak{g}_2$ if $d = 0$, we may assume that $d > 0$. Recall that, in this case, $\mathrm{Card}(H) = 1$.

Notice that we may assume that there is $k' \in \{ 0, \dots, k \}$ such that $T_1', \dots, T_{k'}' \in \mathfrak{g}_2$ while $T_{k'+1}', \dots, T_k' \notin \mathfrak{g}_2$; let $\mathfrak{g}''$ be the vector subspace of $\mathfrak{g}$ generated by $T_{k'+1}', \dots, T_k'$. Observe that, since $T_{k'+1}', \dots, T_k'$ are homogeneous and do not belong to $\mathfrak{g}_2$, clearly $\mathfrak{g}'' \subseteq \mathfrak{g}_1$. Let $\mathfrak{g}_1'$ be the $\widehat{Q}$-orthogonal complement of $\mathfrak{g}''$ in $\mathfrak{g}_1$, and define $\mathfrak{g}' \coloneqq \mathfrak{g}_1' \oplus \mathfrak{g}_2$. Then, $\mathfrak{g}_1 = \mathfrak{g}_1' \oplus \mathfrak{g}''$ and $\mathfrak{g}$ is the direct sum of its ideals $\mathfrak{g}'$ and $\mathfrak{g}''$.

Let G' and G'' be the Lie subgroups of G corresponding to $\mathfrak{g}'$ and $\mathfrak{g}''$, and let $\mathcal{L}'$ and $\mathcal{L}''$ be the sub-Laplacians on G' and G'', respectively, corresponding to the restriction of Q to $\mathfrak{g}_1'^* \cong \mathfrak{g}''^\circ \cap \mathfrak{g}_2^\circ$ and $\mathfrak{g}''^* \cong \mathfrak{g}'^\circ$. By an abuse of notation, then, $\mathcal{L} = \mathcal{L}' + \mathcal{L}''$, so that the family $(\mathcal{L}, -iT_1', \dots, -iT_k')$ is equivalent to the family $(\mathcal{L}', -iT_1', \dots, -iT_k')$. Now, the family $(-iT_{k'+1}', \dots, -iT_k')$ on G'' satisfies property (RL) by classical Fourier analysis. Therefore, Theorem 4.8 and its easy converse imply that the family $(\mathcal{L}, -iT_1', \dots, -iT_k')$ on G satisfies property (RL) if and only if the family $(\mathcal{L}', -iT_1', \dots, -iT_{k'}')$ on G' satisfies property (RL).

Similar arguments apply to property (S) and the continuity of the integral kernel.

7.3 Property (RL)

Keep the notation of Section 7.2.

In this section we shall present several sufficient conditions for the validity of property (RL). We begin with the case in which $d > 0$, which is much simpler than the case in which $d = 0$. First of all, when $d > 0$ the spectrum of $\mathcal{L}_A$ is a convex cone, whereas when $d = 0$ the spectrum of $\mathcal{L}_A$ is a countable union of semianalytic sets. In addition, when $d > 0$, we can basically ignore the Laguerre polynomials of higher order which appear in the integral kernel, thanks to Proposition 3.22. Indeed, with reference to the proof of Proposition 7.16, the 'ground state,' that is, the first eigenvalue of $\mathrm{d}\pi_{\omega,\tau}(\mathcal{L}_A)$, is sufficient to cover the whole of $\sigma(\mathcal{L}_A)$, as ω and τ vary. This fact leads to significant simplifications, as the basic Lemma 7.19 shows.

Concerning the case in which $d > 0$, then, we need to distinguish between the 'full' family $\mathcal{L}_A$, for which we can prove continuity of the multipliers only on a dense subset of the spectrum *in full generality* (cf. Lemma 7.19), and the 'partial' families $(\mathcal{L}, (-iT_1, \dots, -iT_{n_2'}))$ for $n_2' < n_2$, for which by means of a deeper analysis we are able to prove property (RL) in full generality (cf. Theorem 7.22).

When $d > 0$ and we deal with the 'full' family $\mathcal{L}_A$, we observed above that we can prove in full generality that every integrable kernel corresponds to a multiplier which is continuous on a dense subset of the spectrum. Nevertheless, we can prove that property (RL) holds for the 'full' family $\mathcal{L}_A$ in the following situations: when P_0 extends to a continuous function on $\mathfrak{g}_2^* \setminus \{ 0 \}$, for example when $W = \{ 0 \}$, or when G is the product of an MW^+-group and a non-trivial abelian group (cf. Theorem 7.20); when G is a free 2-step stratified group on an odd number of generators (cf. Theorem 7.21). In the first case, we employ directly the simplified 'inversion formula' for $\mathcal{K}_{\mathcal{L}_A}$ which is available in this case, while in the second case we employ the simple structure of free groups to prove that the L^1 kernels are invariant under sufficiently many linear transformations in order that the above-mentioned inversion formula give rise to a continuous multiplier.

The case in which $d = 0$ is much more complicated. Even though in this case we are able to prove continuity results for $\chi_{\mathcal{L}_A}$ under rather strong assumptions (cf. Theorem 7.24) and then deduce property (RL) under slightly weaker assumptions (cf. Theorem 7.25), the techniques employed are much more involved, and the generality of the obtained results is much narrower. Let us comment a little more on the assumptions of Theorem 7.25. Besides the conditions that Ω is $\mathfrak{g}_2^* \setminus \{ 0 \}$ and that μ is constant on the unit sphere associated with the norm $\mu_{\eta_0}(\mathbf{n_1})$, we need to add the condition that $\dim_{\mathbb{R}} \mu_\omega(\mathbb{R}^{\overline{h}}) = \dim_{\mathbb{Q}} \mu_\omega(\mathbb{Q}^{\overline{h}})$ for some, and then all, $\omega \in \Omega$. Even though this condition may appear peculiar, we cannot get rid of it without running into counterexamples, as Theorem 7.45 shows. Furthermore, observe that, even though Theorem 7.45 is the main application of Theorems 7.24 and 7.25, the latter result can be applied to more general sub-Laplacians on H-type groups, as we show in Section 7.7.

Our last results in the case in which $d = 0$ concern families of the form $(\mathcal{L}, (-iT_1, \dots, -iT_{n_2'}))$ for $n_2' < n_2$. Notice that, in this case, we do not only reduce the number of elements of $\mathfrak{g}_2$, but we assume that $\mathrm{Card}(H) = 1$. In this case, indeed, the spectrum of $(\mathcal{L}, (-iT_1, \dots, -iT_{n_2'}))$ is no longer a countable union of semianalytic sets, but a convex cone, so that things are somewhat easier and we can prove more general results than for the 'full family' $\mathcal{L}_A$. In Theorem 7.28, we show that property (RL) holds if $W = \{ 0 \}$ and $\beta_{(\mathcal{L}, (-iT_1, \dots, -iT_{n_2'}))}$ is a measure with base $\mathcal{H}^{n_2+1}$; this is the generic case as Corollary 7.29 shows; this happens, in particular, if $\Omega = \mathfrak{g}_2^* \setminus \{ 0 \}$. Some applications are shown in Section 7.7.

Our last result concerns the case of general MW^+-groups (cf. Theorem 7.30); in this case, however, the hypotheses are much more restrictive than in the preceding ones. It nonetheless applies when G is a product of Heisenberg groups and $\mathcal{L}$ is a sum of sub-Laplacians on each factor (cf. Theorem 7.48).

Let us begin with a simple lemma.

Lemma 7.19. *Take $f \in L^1_{\mathcal{L}_A}(G)$. Then, $\mathcal{M}_{\mathcal{L}_A}(f)$ has a representative which is continuous on*

$$\{ (\mu_\omega(\mathbf{n}_{\mathbf{1},\omega}), \omega(\mathbf{T}))\colon \omega \in \mathfrak{g}_2^* \}.$$

If, in addition, $d > 0$, then $\mathcal{M}_{\mathcal{L}_A}(f)$ has a representative which is continuous on

$$\{ (\mu_\omega(\mathbf{n}_{\mathbf{1},\omega}), \omega(\mathbf{T}))\colon \omega \in \mathfrak{g}_2^* \} \cup \{ (\lambda, \omega(\mathbf{T}))\colon \omega \in \mathfrak{g}_2^* \setminus W, \lambda \geqslant \mu_\omega(\mathbf{n}_{\mathbf{1},\omega}) \}.$$

Proof. The first assertion follows from Propositions 7.12 and 7.16.

Then, assume that $d > 0$ and keep the notation of the proof of Proposition 7.16, and fix a multiplier m of f. By Proposition 3.22, there is a negligible subset N_1 of $\mathfrak{g}_2^*$ such that for every $\omega \in \mathfrak{g}_2^* \setminus N_1$ there is negligible subset $N_{2,\omega}$ of $P_{0,\omega}(\mathfrak{g}_1)^*$ such that

$$\pi^*_{\omega,\tau}(f) = m(\mathrm{d}\pi_{\omega,\tau}(\mathcal{L}_A))$$

for every $\tau \in P_{0,\omega}(\mathfrak{g}_1)^* \setminus N_{2,\omega}$. Notice that we may assume that $W \subseteq N_1$. Therefore, for every $\omega \in \mathfrak{g}_2^* \setminus N_1$ and for every $\tau \in P_{0,\omega}(\mathfrak{g}_1)^* \setminus N_{2,\omega}$,

$$\begin{aligned} m(\mu_\omega(\mathbf{n_{1,\omega}}) + |\tau|^2, \omega(\mathbf{T})) &= \frac{1}{\operatorname{Tr} P_{\omega,\tau,0}} \operatorname{Tr}(m(\mathrm{d}\pi_{\omega,\tau}(\mathcal{L}_A))P_{\omega,\tau,0}) \\ &= \int_G f(x,t) e^{-\frac{1}{4}|x_\omega|^2 + i\omega(t) + i\tau(x_{0,\omega})} \,\mathrm{d}\nu_G(x,t). \end{aligned}$$

Now, for every $\omega \in \mathfrak{g}_2^* \setminus N_1$ there is negligible subset $N_{3,\omega}$ of $\mathbb{R}_+^*$ such that, for every $\lambda \in \mathbb{R}_+^* \setminus N_{3,\omega}$, we have $\mathcal{H}^{d-1}\big(\partial B_{P_{0,\omega}(\mathfrak{g}_1)^*}\big(0,\sqrt{\lambda}\big) \cap N_{2,\omega}\big) = 0$. Therefore, for every $\omega \in \mathfrak{g}_2^* \setminus N_1$ and for every $\lambda \in \mathbb{R}_+^* \setminus N_{3,\omega}$,

$$\begin{aligned} m(\mu_\omega(\mathbf{n_{1,\omega}}) + \lambda, \omega(\mathbf{T})) &= \fint_{\partial B(0,\sqrt{\lambda})} \int_G f(x,t) e^{-\frac{1}{4}|x_\omega|^2 + i\omega(t) + i\tau(x_{0,\omega})} \,\mathrm{d}\nu_G(x,t) \,\mathrm{d}\mathcal{H}^{d-1}(\tau) \\ &= \int_G f(x,t) e^{-\frac{1}{4}|x_\omega|^2 + i\omega(t)} \Phi_d\Big(\sqrt{\lambda}|x_{0,\omega}|\Big) \,\mathrm{d}\nu_G(x,t). \end{aligned}$$

Now, the mapping

$$(\omega,\lambda) \mapsto \int_G f(x,t) e^{-\frac{1}{4}|x_\omega|^2 + i\omega(t)} \Phi_d\Big(\sqrt{\lambda}|x_{0,\omega}|\Big) \,\mathrm{d}\nu_G(x,t)$$

is continuous on $[(\mathfrak{g}_2^* \setminus W) \times \mathbb{R}_+] \cup [\mathfrak{g}_2^* \times \{0\}]$ by Proposition 7.12, so that by means of Tonelli's theorem we see that it induces a representative of m which satisfies the conditions of the statement. □

7.3.1 The Case $d > 0$ and $n_2' = n_2$

Theorem 7.20. *Assume that $d > 0$ and that P_0 can be extended to a continuous function on $\mathfrak{g}_2^* \setminus \{0\}$. Then, $\mathcal{L}_A$ satisfies property (RL).*

Notice that, by polarization, P_0 has a continuous extension to $\mathfrak{g}_2^* \setminus \{0\}$ if and only if $|P_0(x)|$ has a continuous extension to $\mathfrak{g}_2^* \setminus \{0\}$ for every $x \in \mathfrak{g}_1$.

In addition, observe that the hypotheses of the theorem hold in the following situations:

- when $d > 0$ and $W = \{0\}$, for example when G is the free 2-step nilpotent group on three generators;
- when $d > 0$ and P_0 is constant on $\mathfrak{g}_2^* \setminus W$, for example when $G = G' \times \mathbb{R}^d$ for some MW^+-group G', such as a product of Heisenberg groups;

Proof. **1.** Keep the notation of the proof of Lemma 7.19. Assume first that $n_2 = 1$, so that $W = \{0\}$. In addition, $\ker \mathrm{d}_{\sigma_\omega} = \ker \mathrm{d}_{\sigma_{-\omega}}$ for every $\omega \in \mathfrak{g}_2^*$, so that P_0 is constant on $\mathfrak{g}_2^* \setminus \{0\}$. The computations of the proof of Lemma 7.19 then lead to the conclusion.

2. Denote by $\widetilde{P}_0$ the continuous extension of P_0 to $\mathfrak{g}_2^* \setminus \{0\}$; observe that $\widetilde{P}_{0,\omega}$ is a self-adjoint projector of $\mathfrak{g}_1$ of rank d for every non-zero $\omega \in \mathfrak{g}_2^*$. Take $f \in L^1_{\mathcal{L}_A}(G)$ and define, for every non-zero $\omega \in \mathfrak{g}_2^*$ and for every $\lambda \geqslant 0$,

$$m(\mu_\omega(\mathbf{n_{1,\omega}}) + \lambda, \omega(\mathbf{T})) \coloneqq \int_G f(x,t) e^{-\frac{1}{4}|x_\omega|^2 + i\omega(t)} \Phi_d\Big(\sqrt{\lambda}|\widetilde{P}_{0,\omega}(x)|\Big) \,\mathrm{d}\nu_G(x,t),$$

so that $f = \mathcal{K}_{\mathcal{L}_A}(m)$. Then, m is clearly continuous con $\sigma(\mathcal{L}_A) \setminus (\mathbb{R} \times \{0\}^{n_2})$, and $m(\mu_{r\omega}(\mathbf{n_{1,\omega}}) + \lambda, r\omega(\mathbf{T}))$ converges to

$$\int_G f(x,t) \Phi_d\Big(\sqrt{\lambda}|\widetilde{P}_{0,\omega}(x)|\Big) \,\mathrm{d}\nu_G(x,t)$$

as $r \to 0^+$, uniformly as ω runs through the unit sphere S of $\mathfrak{g}_2^*$. Therefore, it will suffice to prove that the above integrals do not depend on $\omega \in S$ for every $\lambda \geqslant 0$. Indeed, Theorem 4.1 implies that, for every $\omega \in S$,

$$(\pi_\omega)_*(f) = \mathcal{K}_{\mathrm{d}\pi_\omega(\mathcal{L}_A)}(m).$$

Now, **1** above implies that the family $\mathrm{d}\pi_\omega(\mathcal{L}_A)$ satisfies property (RL). Then, Theorem 4.1 implies that

$$(\pi_0)_*(f) \in L^1_{\mathrm{d}\pi_0(\mathcal{L}_A)}(G_0);$$

in addition, $\mathrm{d}\pi_0(\mathcal{L}_A)$ is identified with $(-\Delta, 0, \dots, 0)$, where Δ is the Laplacian associated with the scalar product $\widehat{Q}$ on $\mathfrak{g}_1$. Then,

$$\int_G f(x,t)\Phi_d\Big(\sqrt{\lambda}|\widetilde{P}_{0,\omega}(x)|\Big)\,\mathrm{d}\nu_G(x,t) = \int_{\mathfrak{g}_1} (\pi_0)_*(f)(x)\Phi_d\Big(\sqrt{\lambda}|\widetilde{P}_{0,\omega}(x)|\Big)\,\mathrm{d}x,$$

so that the assertion follows since $(\pi_0)_*(f)$ is rotationally invariant. □

Notice that the proof of the preceding theorem is based on the following observation: "For $\mathcal{L}_A$ to satisfy property (RL), it is (necessary and) sufficient that, for every $f \in L^1_{\mathcal{L}_A}(G)$, for every $\lambda > 0$, and for every $\omega_0 \in W$, the integral

$$\int_G f(x,t)e^{-\frac{1}{4}|x_\omega|^2 + i\omega_0(t)}\Phi_d\Big(\sqrt{\lambda}|\widetilde{P}(x)|\Big)\,\mathrm{d}\nu_G(x,t)$$

be independent of the limit point $\widetilde{P}$ of $P_{0,\omega}$ as $\omega \to \omega_0$, $\omega \notin W$."

Under the hypotheses of Theorem 7.20, the preceding condition is trivially satisfied for every non-zero $\omega_0 \in W$; we established it also for $\omega_0 = 0$ with a different technique. When G is a free group (on an odd number of generators), we can develop further the technique used to establish the case $\omega_0 = 0$ in order to prove that property (RL) holds without further assumptions.

Theorem 7.21. *Assume that G is a free 2-step stratified group on an odd number of generators. Then, $\mathcal{L}_A$ satisfies property (RL).*

Proof. Take $f \in L^1_{\mathcal{L}_A}(G)$; by Lemma 7.19, f has a multiplier m which is continuous on $\sigma(\mathcal{L}_A) \setminus (\mathbb{R} \times W(\mathbf{T}))$. Now,

$$(\pi_\omega)_*(f)(x,t) = \int_{\omega(t')=t} f(x,t')\,\mathrm{d}t'$$

for almost every $(x,t) \in G_\omega$. Then, Theorem 4.1 implies that $(\pi_\omega)_*(f)$ is invariant under the isometries which restrict to the identity on $(\ker \mathrm{d}_{\sigma_\omega})^\perp$, for every $\omega \in \mathfrak{g}_2^* \setminus W$; indeed, $\mathrm{d}\pi_\omega(\mathcal{L}_A)$ is invariant under such isometries, and these isometries are group automorphisms. Next, take $\omega \in W$ and an isometry U of G_ω which restricts to the identity on $(\ker \mathrm{d}_{\sigma_\omega})^\perp$. Since $\dim \ker \mathrm{d}_{\sigma_\omega}$ is odd, there must be some $v \in \ker \mathrm{d}_{\sigma_\omega}$ such that $U \cdot v = \pm v$. Let V be the orthogonal complement of $\mathbb{R}v$ in $\ker \mathrm{d}_{\sigma_\omega}$, so that V is U-invariant. Now, let σ_V be a standard symplectic form on the hilbertian space V,[4] and define ω_p, for every $p \in \mathbb{N}$, so that

$$\sigma_{\omega_p} = \sigma_\omega + 2^{-p}\sigma_V;$$

this is possible since G is a free 2-step stratified group. Then, ω_p belongs to $\mathfrak{g}_2^* \setminus W$ and converges to ω. In addition, $(\pi_{\omega_p})_*(f)$ is U-invariant thanks to Theorem 4.1 (cf. also the results of Section 7.5 below). Now, it is easily seen that $(\pi_{\omega_p})_*(f)$ converges to $(\pi_\omega)_*(f)$ in $L^1(\mathfrak{g}_1 \oplus \mathbb{R})$, so that $(\pi_\omega)_*(f)$ is U-invariant.

Then, the mapping

$$m_1 \colon \mathbb{R}_+ \times (\mathfrak{g}_2^* \setminus W) \ni (\lambda, \omega) \mapsto \int_G f(x,t)e^{-\frac{1}{4}|x_\omega|^2 + i\omega(t)}\Phi_1\Big(\sqrt{\lambda}|x_{0,\omega}|\Big)\,\mathrm{d}\nu_G(x,t),$$

extends to a continuous function on $\mathbb{R}_+ \times \mathfrak{g}_2^*$. Since $m(\lambda, \omega(\mathbf{T})) = m_1(\lambda - \mu_\omega(\mathbf{n}_{\mathbf{1},\omega}), \omega)$ for every $(\lambda, \omega(\mathbf{T})) \in \sigma(\mathcal{L}_A)$, with $\omega \notin W$, the assertion follows. □

[4] That is, choose a symplectic form σ_V on V so that V admits an orthonormal basis (relative to the scalar product) which is also a symplectic basis (relative to σ_V).

7.3.2 The Case $d > 0$ and $n_2' < n_2$

Theorem 7.22. *Assume that $d > 0$, and take $n_2' < n_2$. Then, the family $(\mathcal{L}, (-iT_j)_{j=1,\dots,n_2'})$ satisfies property (RL).*

Proof. Define $\mathcal{L}'_{A'} = (\mathcal{L}, (-iT_j)_{j=1,\dots,n_2'})$, and let $L\colon E_{\mathcal{L}_A} \to E_{\mathcal{L}'_{A'}}$ be the unique linear mapping such that $\mathcal{L}'_{A'} = L(\mathcal{L}_A)$. Until the end of the proof, we shall identify $\mathfrak{g}_2^*$ with $\mathbb{R}^{n_2}$ by means of the mapping $\omega \mapsto \omega(\mathbf{T})$. In addition, define and $X \coloneqq (\sigma(\mathcal{L}_A) \setminus W) \cup \partial\sigma(\mathcal{L}_A)$, so that X is a Polish space by [17, Theorem 1 of Chapter IX, § 6, No. 1]. Let β be the (Radon) measure induced by $\beta_{\mathcal{L}_A}$ on X, so that $\operatorname{Supp}(\beta) = X$. Let L' be the restriction of L to X. Since $\sigma(\mathcal{L}_A)$ is a convex cone by Corollary 7.10 and since W is $\beta_{\mathcal{L}_A}$-negligible, Proposition 2.22 implies that β is L'-connected. In addition, $L'(X) = \sigma(\mathcal{L}'_{A'})$.

Now, Proposition 2.26 implies that β has a disintegration $(\beta_{\lambda'})_{\lambda' \in E_{\mathcal{L}'_{A'}}}$ such that $\beta_{\lambda'}$ is equivalent to $\chi_{L'^{-1}(\lambda')} \cdot \mathcal{H}^{n_2 - n_2'}$ for $\beta_{\mathcal{L}'_{A'}}$-almost every $\lambda' \in E_{\mathcal{L}'_{A'}}$. Observe that $L^{-1}(\lambda') \cap \sigma(\mathcal{L}_A)$ is a convex set of dimension $n_2 - n_2'$ for $\beta_{\mathcal{L}'_{A'}}$-almost every $\lambda' \in E_{\mathcal{L}'_{A'}}$. In addition, $W \cap L^{-1}(\lambda')$ is an algebraic variety of dimension at most $n_2 - n_2' - 1$ for $\beta_{\mathcal{L}'_{A'}}$-almost every $\lambda' \in E_{\mathcal{L}'_{A'}}$, for otherwise $\mathcal{H}^{n_2+1}(W)$ would be non-zero, which is absurd. Therefore, $\operatorname{Supp}(\beta_{\lambda'}) = L'^{-1}(\lambda')$ for $\beta_{\mathcal{L}'_{A'}}$-almost every $\lambda' \in E_{\mathcal{L}'_{A'}}$.

Now, take $m_0 \in L^\infty(\beta_{\mathcal{L}_A})$ so that $\mathcal{K}_{\mathcal{L}'_{A'}}(m_0) \in L^1(G)$. Let us prove that m_0 has a continuous representative. Indeed, Lemma 7.19 implies that there is a continuous function m_1 on X such that $m_0 \circ L' = m_1$ β-almost everywhere. Hence, Proposition 2.21 implies that there is a function $m_2\colon \sigma(\mathcal{L}'_{A'}) \to \mathbb{C}$ such that $m_2 \circ L' = m_1$. Since the mapping $L\colon \partial\sigma(\mathcal{L}_A) \to \sigma(\mathcal{L}'_{A'})$ is proper and onto, and since $\partial\sigma(\mathcal{L}_A) \subseteq X$, it follows that m_2 is continuous. The assertion follows (cf. [17, Corollary to Theorem 2 of Chapter IX, § 4, No. 2]). ☐

7.3.3 The Case $d = 0$ and $n_2' = n_2$

We begin with a technical lemma.

Lemma 7.23. *Let $V, \widetilde{V}$ be two finite-dimensional vector spaces over $\mathbb{R}$, L a discrete subgroup of V, C the convex envelope of $\mathbb{R}_+ F$ for some finite subset F of L which generates V, and $\mu\colon V \to \widetilde{V}$ a linear mapping which is proper on C. Assume that $L \cap \ker\mu$ generates $\ker\mu$, and take $\xi \in \mu(C)$. Define*

$$V_\xi \coloneqq \mu^{-1}(\xi) \qquad S_\xi \coloneqq V_\xi \cap C$$
$$n_\xi \coloneqq \dim_{\mathbb{R}} S_\xi \qquad \nu_\xi \coloneqq \frac{1}{\mathcal{H}^{n_\xi}(S_\xi)} \chi_{S_\xi} \cdot \mathcal{H}^{n_\xi}.$$

Take $x_0 \in C$ and define, for every $\lambda \in \mathbb{R}_+^*$ and for every $\gamma \in \mu(x_0 + L \cap C)$,

$$\nu_{\lambda,\gamma} = \frac{1}{c_\gamma} \sum_{\substack{\gamma' \in L\cap C\\ \gamma = \mu(x_0+\gamma')}} \delta_{\lambda(x_0+\gamma')},$$

where $c_\gamma = \operatorname{Card}\big(\mu^{-1}(\gamma) \cap (x_0 + L \cap C)\big)$. Then,

$$\lim_{\substack{(\lambda\gamma,\lambda)\to(\xi,0)\\ \gamma\in\mu(x_0+L\cap C)}} \nu_{\lambda,\gamma} = \nu_\xi$$

in $\mathcal{E}_c'^0(V)$.

Proof. **1.** Define $\Sigma \coloneqq \mu(x_0 + L\cap C)$ and define $\mathfrak{F}_\xi$ as the filter '$(\lambda,\gamma) \in \mathbb{R}_+^* \times \Sigma, (\lambda\gamma, \lambda) \to (\xi, 0)$.' Observe that it will suffice to prove that $\nu_{\lambda,\gamma}$ converges vaguely to ν_ξ along $\mathfrak{F}_\xi$. Indeed, the $\nu_{\lambda,\gamma}$ are probability measures supported in

$$S_{\lambda\gamma} \subseteq C \cap \mu^{-1}(K) \tag{1}$$

eventually along $\mathfrak{F}_\xi$, where K is a compact neighbourhood of ξ in $\widetilde{V}$. Since μ is proper on C, the assertion follows.

Now, let us prove that we may reduce to the case in which $x_0 = 0$. Indeed, define

$$\nu^0_{\lambda,\gamma} \coloneqq \frac{1}{c_\gamma} \sum_{\substack{\gamma' \in L\cap C\\ \gamma=\mu(x_0+\gamma')}} \delta_{\lambda\gamma'}.$$

It will then suffice to prove that $\nu_{\lambda,\gamma} - \nu^0_{\lambda,\gamma}$ converges vaguely to 0 along $\mathfrak{F}_\xi$. However, take $\varphi \in C_c(V)$ and $\varepsilon > 0$. Then, there is a neighbourhood U of 0 in V such that $|\varphi(x_1)-\varphi(x_2)| < \varepsilon$ for every $x_1, x_2 \in V$ such that $x_1-x_2 \in U$. Therefore, $\left|\left\langle \nu_{\lambda,\gamma} - \nu^0_{\lambda,\gamma}, \varphi\right\rangle\right| < \varepsilon$ as long as $\lambda x_0 \in U$, hence eventually along $\mathfrak{F}_\xi$. The assertion follows.

2. Observe that C is a polyhedral convex cone. In addition, let n be the dimension of V, and let $(F_h)_{h\in H}$ be the (finite) family of $(n-1)$-dimensional facets of C; observe that F_h is a convex cone for every $h \in H$, so that $0 \in F_h$. Take, for every $h \in H$, some $p_h \in V^*$ such that $F_h = \ker p_h \cap C$ and $p_h(C) \subseteq \mathbb{R}_+$. Then, C is the set of $x \in V$ such that $p_h(x) \geqslant 0$ for every $h \in H$, and $L \cap \ker p_h$ generates $\ker p_h$ for every $h \in H$.

In addition, let H_ξ be the set of $h \in H$ such that $p_h(S_\xi) = \{0\}$, and let H'_ξ be its complement in H. We shall write p_{H_ξ} and $p_{H'_\xi}$ instead of $(p_h)_{h\in H_\xi}$ and $(p_h)_{h\in H'_\xi}$, respectively. Define $V'_\xi \coloneqq V_\xi \cap \ker p_{H_\xi}$. Then, $V'_\xi \cap p^{-1}_{H'_\xi}\left((\mathbb{R}^*_+)^{H'_\xi}\right)$ is the interior of S_ξ in V'_ξ; now, by convexity $V'_\xi \cap p^{-1}_{H'_\xi}\left((\mathbb{R}^*_+)^{H'_\xi}\right)$ is not empty, so that V'_ξ is the affine space generated by S_ξ.

3. Define $W_\xi \coloneqq V'_\xi - V'_\xi$, and observe that $L \cap W_\xi$ generates W_ξ. Indeed, the linear mapping $(\mu, p_{H_\xi})\colon V \to \widetilde{V} \times \mathbb{R}^{H_\xi}$ maps L into the discrete subgroup $\mu(L) \times \prod_{h\in H_\xi} p_h(L)$ of $\widetilde{V} \times \mathbb{R}^{H_\xi}$, and W_ξ is the kernel of (μ, p_{H_ξ}), whence the assertion.

Therefore, there are two subspaces W'_ξ and W''_ξ of V such that the following hold (cf. [17, Exercises 2 and 3 of Chapter VII, § 1]):

- $W_\xi \oplus W'_\xi = V_0$ and $V_0 \oplus W''_\xi = V$;
- $L \cap W'_\xi$ and $L \cap W''_\xi$ generate W'_ξ and W''_ξ, respectively, over $\mathbb{R}$;
- $(L \cap W_\xi) \oplus (L \cap W'_\xi) \oplus (L \cap W''_\xi) = L$ as abelian groups.

Therefore, we may endow V and $\widetilde{V}$ with two scalar products such that W_ξ, W'_ξ, and W''_ξ are orthogonal, and μ induces an isometry of W''_ξ into $\widetilde{V}$. We may further assume that $\|p_h\| \leqslant 1$ for every $h \in H$.

4. Define, for $\lambda > 0$ and $\gamma \in \Sigma$,

$$r_{\xi,\lambda,\gamma} \coloneqq \inf\{\, r > 0 \colon S_{\lambda\gamma} \subseteq B(S_\xi, r)\,\} + \lambda,$$

so that $S_{\lambda\gamma} \subseteq B(S_\xi, r_{\xi,\lambda,\gamma})$. Let us prove that $r_{\xi,\lambda,\gamma}$ converges to 0 along $\mathfrak{F}_\xi$.

Indeed, let $\mathfrak{U}$ be an ultrafilter finer than $\mathfrak{F}_\xi$. Denote by $\mathcal{K}$ the space of non-empty compact subsets of V endowed with the Hausdorff distance d_H. By (1), [16, Proposition 10 of Chapter I, § 6, No. 7] and [2, Theorem 6.1], we see that $S_{\lambda\gamma}$ has a (unique) limit S in $\mathcal{K}$ along $\mathfrak{U}$. Now, for every closed neighbourhood K of ξ in $\widetilde{V}$,

$$S_{\lambda\gamma} \subseteq C \cap \mu^{-1}(K)$$

as long as $\lambda\gamma \in K$, so that, by passing to the limit along $\mathfrak{U}$,

$$S \subseteq C \cap \mu^{-1}(K).$$

By the arbitrariness of K, it follows that $S \subseteq S_\xi$. Therefore,

$$r_{\xi,\lambda,\gamma} \leqslant d_H(S, S_{\lambda\gamma}) + \lambda,$$

so that $r_{\xi,\lambda,\gamma}$ converges to 0 along $\mathfrak{U}$. Thanks to [16, Proposition 2 of Chapter I, § 7, No. 2], the arbitrariness of $\mathfrak{U}$ implies that $r_{\xi,\lambda,\gamma}$ converges to 0 along $\mathfrak{F}_\xi$.

5. Now, let π_ξ be the affine projection of V onto V'_ξ with fibres parallel to $W'_\xi \oplus W''_\xi$. Reasoning as in **1** and taking **4** into account, we see that $\nu_{\lambda,\gamma} - (\pi_\xi)_*(\nu_{\lambda,\gamma})$ converges vaguely to 0 along $\mathfrak{F}_\xi$, so that it will suffice to prove that $(\pi_\xi)_*(\nu_{\lambda,\gamma})$ converges vaguely to ν_ξ along $\mathfrak{F}_\xi$.

Now, if $n_\xi = 0$, then $(\pi_\xi)_*(\nu_{\lambda,\gamma}) = \delta_{\xi'} = \nu_\xi$, where ξ' is the unique element of S_ξ. Therefore, we may assume that $n_\xi > 0$.

Next, take $\varepsilon > 0$ and $x, y \in S_{\xi,\lambda,\gamma} := \mathrm{Supp}((\pi_\xi)_*(\nu_{\lambda,\gamma}))$. Assume that $B(x,\varepsilon)\cap p_{H_\xi}^{-1}\left(\mathbb{R}_+^{H_\xi}\right) \subseteq C$, and that $r_{\xi,\lambda,\gamma} < \varepsilon$. Take $y' \in \mathrm{Supp}(\nu_{\lambda,\gamma})$ such that $\pi_\xi(y') = y$, and let us prove that $y' + x - y \in \mathrm{Supp}(\nu_{\lambda,\gamma})$. Indeed, it is clear that $x - y \in \lambda L \cap W_\xi$, so that $y' + x - y \in \lambda L$. Hence, it will suffice to prove that $y' + x - y \in C$. Now, since $y' \in S_{\lambda\gamma} \subseteq B(S_\xi, \varepsilon)$, it follows that there is $x' \in S_\xi$ such that $|y' - x'| < \varepsilon$, so that

$$\varepsilon^2 > |y' - x'|^2 = |y - x'|^2 + |y' - y|^2$$

since $y - x' \in W_\xi$ and $y' - y \in W'_\xi \oplus W''_\xi$. Therefore, $|y' - y| < \varepsilon$; since, in addition, $p_h(y' + x - y) = p_h(y') \geqslant 0$ for every $h \in H_\xi$, it follows that $y' + x - y \in B(x,\varepsilon) \cap p_{H_\xi}^{-1}\left(\mathbb{R}_+^{H_\xi}\right) \subseteq C$.

6. By the arguments of **5** above, we see that there is a function $c_{\xi,\lambda,\gamma}$ on $S_{\xi,\lambda,\gamma}$ such that

$$(\pi_\xi)_*(\nu_{\lambda,\gamma}) = \sum_{x \in S_{\xi,\lambda,\gamma}} c_{\xi,\lambda,\gamma}(x)\delta_x,$$

and such that $c_{\xi,\lambda,\gamma}(x) \geqslant c_{\xi,\lambda,\gamma}(y)$ eventually along $\mathfrak{F}_\xi$ whenever $x, y \in S_{\xi,\lambda,\gamma}$ and $B(x,\varepsilon) \cap p_{H_\xi}^{-1}\left(\mathbb{R}_+^{H_\xi}\right) \subseteq C$ for some fixed $\varepsilon > 0$. In particular, $c_{\xi,\lambda,\gamma}$ is constant on the set of $x \in S_{\xi,\lambda,\gamma}$ such that $B(x,\varepsilon) \cap p_{H_\xi}^{-1}\left(\mathbb{R}_+^{H_\xi}\right) \subseteq C$.

Now, let us prove that, if $\varepsilon \leqslant \min_{h \in H'_\xi} \min_{S_\xi} p_h$ and if $x \in V'_\xi$ and $B(x,\varepsilon) \cap V'_\xi \subseteq C$, then $B(x,\varepsilon) \cap p_{H_\xi}^{-1}\left(\mathbb{R}_+^{H_\xi}\right) \subseteq C$. Indeed, take $x' \in B(x,\varepsilon)$, and assume that $p_h(y) \geqslant 0$ for every $h \in H_\xi$. Take $h \in H'_\xi$, and observe that $|p_h(y - x)| \leqslant |y - x| < \varepsilon$, so that $p_h(y) = p_h(x) + p_h(y - x) \geqslant p_h(x) - \varepsilon \geqslant 0$ by our choice of ε. By the arbitrariness of h, it follows that $y \in C$.

7. Finally, take a fundamental parallelotope P_ξ of $L \cap W_\xi$, and extend $c_{\xi,\lambda,\gamma}$ to a function on V which is constant on $x + \lambda P_\xi$ for every $x \in \pi_\xi(\lambda L)$, and vanishes on the complement of $S_{\xi,\lambda,\gamma} + \lambda P_\xi$. Then, $\nu_{\xi,\lambda,\gamma} := \frac{1}{\mathcal{H}^{n_\xi}(\lambda P_\xi)} c_{\xi,\lambda,\gamma} \cdot \mathcal{H}^{n'_\xi}$ is a probability measure; in addition, as in **1** we see that $(\pi_\xi)_*(\nu_{\lambda,\gamma}) - \nu_{\xi,\lambda,\gamma}$ converges vaguely to 0 along $\mathfrak{F}_\xi$, so that it will suffices to show that $\nu_{\xi,\lambda,\gamma}$ converges vaguely to ν_ξ along $\mathfrak{F}_\xi$. However, if S'_ξ denotes the boundary of S_ξ in V'_ξ, then **4** and **6** imply that $\frac{1}{\mathcal{H}^{n_\xi}(\lambda P_\xi)} c_{\xi,\lambda,\gamma}$ is uniformly bounded eventually along $\mathfrak{F}_\xi$, and converges on $V \setminus S'_\xi$ to a function g which is 0 on the complement of S_ξ, and is constant on $S_\xi \setminus S'_\xi$. The assertion follows by dominated convergence. □

Theorem 7.24. *Assume that $d = 0$, that $\Omega = \mathfrak{g}_2^* \setminus \{0\}$, that $\dim_\mathbb{Q} \mu_\omega(\mathbb{Q}^{\overline{h}}) = \dim_\mathbb{R} \mu_\omega(\mathbb{R}^{\overline{h}})$ for every $\omega \in \Omega$, and that P_h is constant on Ω. Then, $\chi_{\mathcal{L}_A}$ has a continuous representative.*

Proof. **1.** We shall simply write P_h and $\mathbf{n_1}$ to denote the constant values of the functions $\omega \mapsto P_{h,\omega}$ and $\omega \mapsto \mathbf{n}_{\mathbf{1},\omega}$, respectively. In addition, we shall denote by $|\cdot|'$ the norm $\mu_{\eta_0}(\mathbf{n_1})$, and by S' the corresponding unit sphere. Choose $\omega_0 \in S'$ and define, for every $x \in \mathfrak{g}_1$, $\mu_{\eta,h} := \mu_{\eta,h,\omega_0}$ and $x_h := x_{h,\omega_0}$. Then, $\mu_{\eta,h,\omega} = |\omega|'\mu_{\eta,h}$ and $x_{h,\omega} = \sqrt{|\omega|'}x_h$ for every $\omega \in \Omega$.

For every $\xi \in \mu(\mathbb{R}_+^{\overline{h}})$, denote by $\mathfrak{F}_\xi$ the filter '$(\lambda,\gamma) \in \mathbb{R}_+^* \times \Sigma, (\lambda\gamma, \lambda) \to (\xi, 0)$,' where $\Sigma := \mu(\mathbf{n_1} + 2\mathbb{N}^H)$. In addition, define, for every $\lambda \in \mathbb{R}_+^*$ and for every $\gamma \in \Sigma$,

$$\nu'_{\lambda,\gamma} = \sum_{\gamma = \mu(\mathbf{n_1} + 2\gamma')} \delta_{\lambda(\mathbf{n_1} + 2\gamma')},$$

and $\nu_{\lambda,\gamma} := \frac{1}{\nu'_{\lambda,\gamma}(\mathbb{R}^{\overline{h}})} \cdot \nu'_{\lambda,\gamma}$, so that $\nu_{\lambda,\gamma}$ is a probability measure. Then, Lemma 7.23 implies that $\nu_{\lambda,\gamma}$ converges to some probability measure ν_ξ in $\mathcal{E}'^0_c(\mathbb{R}^{\overline{h}})$ along $\mathfrak{F}_\xi$.

2. Define, for every $(x,t) \in \mathbb{R}^H \times \mathbb{R}$,

$$\chi_0(\lambda(\mathbf{n_1} + 2\gamma'), \lambda, x, t) = e^{-\frac{1}{4}\lambda|x|^2 + i\lambda t} \frac{1}{\binom{\mathbf{n_1} + \gamma' - \mathbf{1}_{\overline{h}}}{\gamma'}} \prod_{h=1}^{\overline{h}} \Lambda_{\gamma'_h}^{n_{1,h}-1}\left(\frac{1}{2}\lambda|x_h|^2\right)$$

for every $\lambda \in \mathbb{R}_+^*$ and for every $\gamma' \in \mathbb{N}^{\overline{h}}$, and

$$\chi_0(\xi', 0, x, t) := \prod_{h=1}^{\overline{h}} \frac{2^{n_{1,h}-1}(n_{1,h}-1)!}{\left(\sqrt{\xi'_h}|x_h|\right)^{n_{1,h}-1}} J_{n_{1,h}-1}\left(\sqrt{\xi'_h}|x_h|\right)$$

for every $\xi' \in \mathbb{R}_+^{\overline{h}}$. Then, χ_0 extends to a continuous function on $\mathbb{R}^{\overline{h}} \times \mathbb{R} \times \mathbb{R}^{\overline{h}} \times \mathbb{R}$.

Next, define, for every $\lambda \in \mathbb{R}_+$,

$$f_\lambda \colon \mathbb{R}^{\overline{h}} \ni x \mapsto (2\lambda)^{|\mathbf{n_1} - \mathbf{1}_{\overline{h}}|} \binom{\frac{x}{2\lambda} + \frac{\mathbf{n_1}}{2} - \mathbf{1}_{\overline{h}}}{\mathbf{n_1} - \mathbf{1}_{\overline{h}}} = \prod_{h=1}^{\overline{h}} \frac{(x_h + \lambda n_{1,h} - 2\lambda) \cdots (x_h - \lambda n_{1,h} + 2\lambda)}{(n_{1,h}-1)!},$$

and observe that f_λ converges locally uniformly to f_0 as $\lambda \to 0^+$. Therefore, $f_\lambda \cdot \nu_{\lambda,\gamma}$ converges to $f_0 \cdot \nu_\xi$ in $\mathcal{E}_c'^0(\mathbb{R}^{\overline{h}})$ along $\mathfrak{F}_\xi$. If we define $\nu'_{\lambda,\gamma} := \frac{1}{\nu_{\lambda,\gamma}(f_\lambda)} f_\lambda \cdot \nu_{\lambda,\gamma}$ and $\nu'_\xi := \frac{1}{\nu_\xi(f_0)} f_0 \cdot \nu_\xi$, then $\nu'_{\lambda,\gamma}$ converges to ν'_ξ in $\mathcal{E}_c'^0(\mathbb{R}^{\overline{h}})$ along $\mathfrak{F}_\xi$.

Define, for every $\omega \in \Omega$ and for every $\gamma \in \Sigma$,

$$\chi_1((|\omega|'\gamma, \omega(\mathbf{T})), (x,t)) := \left\langle \nu'_{|\omega|',\gamma}, \chi_0(\,\cdot\,, |\omega|', (|x_h|)_{h=1}^{\overline{h}}, \tfrac{\omega(t)}{|\omega|'}) \right\rangle,$$

so that χ_1 is a representative of $\chi_{\mathcal{L}_A}$ by Proposition 7.16. Now,

$$\lim_{(\lambda,\gamma),\mathfrak{F}_\xi} \chi_1((\lambda\gamma, \lambda\omega(\mathbf{T})), (x,t)) = \left\langle \nu'_\xi, \chi_0(\,\cdot\,, 0, (|x_h|)_{h=1}^{\overline{h}}, \omega(t)) \right\rangle$$

uniformly as ξ runs through $\mu(\mathbb{R}_+^{\overline{h}})$, as ω runs through S', and as (x,t) runs through a compact subset of G. Since $\left\langle \nu'_\xi, \chi_0(\,\cdot\,, 0, (|x_h|)_{h=1}^{\overline{h}}, \omega(t)) \right\rangle$ does *not* depend on ω, it follows that χ_1 is continuous on $\sigma(\mathcal{L}_A) \times G$. The assertion follows from [17, Corollary to Theorem 2 of Chapter IX, § 4, No. 2]. ☐

Theorem 7.25. *Assume that $d = 0$, that $\Omega = \mathfrak{g}_2^* \setminus \{0\}$, that $\dim_\mathbb{Q} \mu_\omega(\mathbb{Q}^{\overline{h}}) = \dim_\mathbb{R} \mu_\omega(\mathbb{R}^{\overline{h}})$ for every $\omega \in \Omega$, and that μ is constant where $\mu_{\eta_0}(\mathbf{n_1})$ is constant. Then, $\mathcal{L}_A$ satisfies property (RL).*

Proof. Take $\varphi \in L^1_{\mathcal{L}_A}(G)$. Let S' be the unit sphere associated with the homogeneous norm $|\cdot|' \colon \omega \mapsto \mu_{\eta_0,\omega}(\mathbf{n_1})$. Then, Proposition 7.17 implies that there is a negligible subset N of S' such that $(\pi_\omega)_*(\varphi) \in L^1_{\mathrm{d}\pi_\omega(\mathcal{L}_A)}(G_\omega)$ for every $\omega \in S' \setminus N$. Observe, in addition, that the mapping

$$\omega \mapsto (\pi_\omega)_*(\varphi) \in L^1(\mathfrak{g}_1 \oplus \mathbb{R})$$

is continuous on Ω, hence on S'. Now, fix $\omega_0 \in S'$, and take (U, ψ) as in Proposition 7.14. Then, it is easily seen that the mapping

$$U \cap S' \ni \omega \mapsto (\psi_\omega \circ \pi_\omega)_*(\varphi) \in L^1(G')$$

is continuous. Furthermore, observe that our assumptions imply that, with the notation of Proposition 7.14,

$$\mathcal{L}'_H := \mathrm{d}(\psi_\omega \circ \pi_\omega)(\mathcal{L}_H) = \left(-\sum_{k=1}^{d} Y_k^2 - \sum_{k=1}^{n_1} \widetilde{\mu}_{\eta,k}(X_k^2 + X_{n_1+k}^2) \right)_{\eta \in H}$$

does *not* depend on ω, while

$$\mathrm{d}(\psi_\omega \circ \pi_\omega)(\mathbf{T}) = \omega(\mathbf{T})T.$$

Observe that $(\mathcal{L}'_H, -iT)$ satisfies property (RL) by Theorem 7.24, and that $(\psi_\omega \circ \pi_\omega)_*(\varphi) \in L^1_{(\mathcal{L}'_H,-iT)}(G')$ for every $\omega \in S' \setminus N$, hence for every $\omega \in S'$ by continuity, since $L^1_{(\mathcal{L}'_H,-iT)}(G')$ is closed in $L^1(G')$ by Proposition 3.15. Therefore, the mapping

$$U \cap S' \ni \omega \mapsto \mathcal{M}_{(\mathcal{L}'_H,-iT)}((\psi_\omega \circ \pi_\omega)_*(\varphi)) \in C_0(\sigma(\mathcal{L}'_H, -iT))$$

is continuous. Now, take $\omega \in U \cap S' \setminus N$. Then, Theorem 4.1 implies that

$$\mathcal{M}_{(\mathcal{L}'_H, -iT)}((\psi_\omega \circ \pi_\omega)_*(\varphi))(\lambda, 0) = \mathcal{M}_{\mathrm{d}\pi_0(\mathcal{L}_H)}((\pi_0)_*(\varphi))(\lambda)$$

for every $\lambda \in \mathbb{R}^H$ such that $(\lambda, 0) \in \sigma(\mathcal{L}'_H, -iT)$, that is, for every $\lambda \in \sigma(\mathrm{d}\pi_0(\mathcal{L}_H))$. By continuity, this proves that the mapping $U \cap S' \ni \omega \mapsto \mathcal{M}_{(\mathcal{L}'_H, -iT)}((\psi_\omega \circ \pi_\omega)_*(\varphi))(\lambda, 0)$ is constant for every $\lambda \in \sigma(\mathrm{d}\pi_0(\mathcal{L}_H))$. Taking into account the arbitrariness of U, we infer that there is a unique $m \in C_0(\sigma(\mathcal{L}_A))$ such that

$$m(\lambda, \omega(\mathbf{T})) = \mathcal{M}_{(\mathcal{L}'_H, -iT)}((\psi_{U, \omega/|\omega|'} \circ \pi_{\omega/|\omega|'})_*(\varphi))(\lambda, |\omega|')$$

for every $(\lambda, \omega(\mathbf{T})) \in \sigma(\mathcal{L}_A)$ such that $\frac{\omega}{|\omega|'} \in U \cap S'$, where U runs through a finite covering of S', and ψ_U is the associated local trivialization as above. Hence, $\varphi = \mathcal{K}_{\mathcal{L}_A}(m)$ and the assertion follows. □

Here we prove a negative result.

Proposition 7.26. *Assume that G is the product of $k \geqslant 2$ MW^+-groups $G_1, \dots, G_k$, and assume that each G_j is endowed with a sub-Laplacian $\mathcal{L}_j$. Assume that* $\mathrm{Card}(H) = 1$ *and that $\mathcal{L} = \mathcal{L}_1 + \cdots + \mathcal{L}_k$. Then, $\mathcal{L}_A$ does not satisfy properties (RL) and (S).*

Proof. Take, for every $j = 1, \dots, k$, a basis $\mathbf{T}_j$ of the centre $\mathfrak{g}_{j,2}$ of the Lie algebra of G_j. Then, we may assume that $\mathcal{L}_A = (\mathcal{L}, -i\mathbf{T}_1, \dots, -i\mathbf{T}_k)$; define $\mathcal{L}'_{A'} \coloneqq (\mathcal{L}_1, \dots, \mathcal{L}_k, -i\mathbf{T}_1, \dots, -i\mathbf{T}_k)$. Then, there is a unique linear mapping $L\colon E_{\mathcal{L}'_{A'}} \to E_{\mathcal{L}_A}$ such that $\mathcal{L}_A = L(\mathcal{L}'_{A'})$. Now, take $j \in \{1, \dots, k\}$ and $\gamma \in \mathbb{N}^{\overline{h}_j}$, and define

$$C_{j,\gamma} \coloneqq \{ (\mu_{j,\omega}(\mathbf{n}_{\mathbf{1},j,\omega} + 2\gamma), \omega(\mathbf{T}_j)) \colon \omega \in \Omega_j \}.$$

Define

$$C \coloneqq \bigcup_{\gamma \in \prod_{j=1}^k \mathbb{N}^{\overline{h}_j}} \prod_{j=1}^k C_{j,\gamma_j},$$

and observe that $\beta_{\mathcal{L}'_{A'}}$ is equivalent to $\chi_C \cdot \mathcal{H}^{n_2}$. Now, define

$$N \coloneqq \mathbb{R}^k \times \bigcup_{\substack{\gamma \in \prod_{j=1}^k \mathbb{Z}^{\overline{h}_j} \\ \gamma \neq 0}} \left\{ \omega(\mathbf{T}) \colon \omega \in \prod_{j=1}^k \Omega_j, \quad \sum_{j=1}^k \mu_{j,\omega_j}(\gamma_j) = 0 \right\},$$

and observe that L is one-to-one on $\sigma(\mathcal{L}'_{A'}) \setminus N$. In addition, the μ_j are analytic and homogeneous of homogeneous degree 1, and the components of the Ω_j are unbounded; therefore, it is easily seen that N is $\beta_{\mathcal{L}'_{A'}}$-negligible. Therefore, there is a unique $m\colon E_{\mathcal{L}_A} \to E_{\mathcal{L}'_{A'}}$ such that $m \circ L$ is the identity of $\sigma(\mathcal{L}'_{A'}) \setminus N$, while m equals 0 on the complement of $L(N)$. Then, m is $\beta_{\mathcal{L}_A}$-measurable, and $\mathcal{K}_{\mathcal{L}_A}(m) = \mathcal{L}'_{A'}\delta_e$. Now, let us prove that m is not equal $\beta_{\mathcal{L}_A}$-almost everywhere to a continuous function. Assume by contradiction that $\mathcal{L}'_{A'}\delta_e = \mathcal{K}_{\mathcal{L}_A}(m')$ for some continuous function m', and let π_0 be the projection of G onto its abelianization. Then, Theorem 4.1 implies that the operators $\mathrm{d}\pi_0(\mathcal{L}_1), \dots, \mathrm{d}\pi_0(\mathcal{L}_k)$ belong to the functional calculus of $\pi_0(\mathcal{L})$, which is clearly absurd.

To conclude, simply take $\tau \in \mathcal{S}(E_{\mathcal{L}_A})$ such that $\tau(\lambda) \neq 0$ for every $\lambda \in E_{\mathcal{L}_A}$, and observe that $\mathcal{K}_{\mathcal{L}_A}(m\tau) = \mathcal{L}'_{A'}\mathcal{K}_{\mathcal{L}_A}(\tau)$ is a family of elements of $\mathcal{S}(G)$, while $m\tau$ is not equal $\beta_{\mathcal{L}_A}$-almost everywhere to a continuous function. □

7.3.4 The Case $d = 0$ and $n'_2 < n_2$

Lemma 7.27. *Let E_1, E_2 be two finite-dimensional vector spaces, C a convex subset of E_1 with non-empty interior, and $L\colon E_1 \to E_2$ a linear mapping which is proper on ∂C. Assume that for every $x \in \partial C$ either $L^{-1}(L(x)) \cap \partial C = \{x\}$ or ∂C is an analytic hypersurface of E_1 in a neighbourhood of x. Then, L induces an* open *mapping $L'\colon \partial C \to L(\partial C)$.*

Proof. Take $x \in \partial C$, and assume that $L^{-1}(L(x)) \cap \partial C = \{ x \}$. Define, for every $k \in \mathbb{N}$, $U_{x,k} := L^{-1}(\overline{B}(L(x), 2^{-k})) \cap \partial C$. Since L is proper on ∂C, $U_{x,k}$ is a compact neighbourhood of x for every $k \in \mathbb{N}$. In addition, $\bigcap_{k\in\mathbb{N}} U_{x,k} = \{ x \}$; hence, [16, Proposition 1 of Chapter 1, § 9, No. 3] implies that $(U_{x,k})$ is a fundamental system of neighbourhoods of x in ∂C, so that L' is open at x.

Now, assume that $L^{-1}(L(x)) \cap \partial C \neq \{ x \}$. Then, there is a convex neighbourhood U of $L^{-1}(L(x)) \cap \partial C$ such that $\partial C \cap U$ is an analytic hypersurface of E_1. Assume by contradiction that $\ker L \subseteq T_x(\partial C \cap U)$, and take $x' \in \partial C$ so that $L(x') = L(x)$ but $x' \neq x$. Since C is convex, we have $[x, x'] \subseteq \partial C$. Now, $\partial C \cap U$ is an analytic hypersurface and U is convex, so that the arbitrariness of x' implies that $\ell \cap U \subseteq \partial C$, where ℓ is the line passing through x and x'. Since each $x'' \in \ell \cap \partial C$ has a convex neighbourhood where ∂C is an analytic hypersurface, we see that the non-empty closed set $\ell \cap \partial C$ is open in ℓ. It follows that $\ell \subseteq \partial C$, which is absurd since then ℓ is contained in the compact set $L^{-1}(L(x)) \cap \partial C$. Therefore, $\ker L \not\subseteq T_x(\partial C \cap U)$, so that L' is open at x. The assertion follows. □

Theorem 7.28. *Assume that $d = 0$, that $\mathrm{Card}(H) = 1$, and that $W = \{ 0 \}$; take a positive integer $n_2' < n_2$, and assume that $\beta_{(\mathcal{L},(-iT_j)_{j=1,\dots,n_2'})}$ is a measure with base $\mathcal{H}^{n_2'+1}$. Then, the family $(\mathcal{L}, (-iT_j)_{j=1,\dots,n_2'})$ satisfies property (RL).*

Proof. **1.** Consider the mapping

$$L\colon E_{\mathcal{L}_A} \ni \lambda \mapsto (\lambda_1, (\lambda_{2,j})_{j=1}^{n_2'}) \in E_{(\mathcal{L},(-iT_j)_{j=1,\dots,n_2'})},$$

so that $L(\mathcal{L}_A) = (\mathcal{L}, (-iT_j)_{j=1,\dots,n_2'})$. Define, for every $\gamma \in \mathbb{N}^{n_1}$,

$$\beta_\gamma\colon \mathcal{D}(E_{\mathcal{L}_A}) \ni \varphi \mapsto \int_{\mathfrak{g}_2^*} \varphi(\widetilde{\mu}_\omega(\mathbf{1}_{n_1} + 2\gamma), \omega(\mathbf{T}))|\mathrm{Pf}(\omega)| \,\mathrm{d}\omega,$$

so that $\beta_{\mathcal{L}_A} = \frac{1}{(2\pi)^{n_1+n_2}} \sum_{\gamma\in\mathbb{N}^{n_1}} \beta_\gamma$. Define

$$\rho_0\colon \mathbb{R}^{n_2'} \ni \omega' \mapsto \min\{ \mu_\omega(\mathbf{n_1})\colon \omega(\mathbf{T}) = \omega' \},$$

and

$$C_0 := \Big\{ (\rho_0(\omega') + r, \omega')\colon \omega' \in \mathbb{R}^{n_2'}, r \geqslant 0 \Big\},$$

so that ρ_0 is a norm on $\mathbb{R}^{n_2'}$ and $C_0 = L(\mathrm{Supp}(\beta_0)) = L(\sigma(\mathcal{L}_A))$.

2. Now, Corollary 7.10 implies that $\mathrm{Supp}(\beta_0) \setminus \{ 0 \}$ is an analytic submanifold of $E_{\mathcal{L}_A}$. Therefore, Corollary 2.27 implies that $L_*(\beta_0)$ is equivalent to $\beta_{(\mathcal{L},(-iT_j)_{j=1,\dots,n_2'})}$ and that, if $(\beta_{0,\lambda})$ is a disintegration of β_0 relative to L, then $\mathrm{Supp}(\beta_{0,\lambda}) = L^{-1}(\lambda) \cap \mathrm{Supp}(\beta_0)$ for $L_*(\beta_0)$-almost every $\lambda \in C_0$. In addition, Lemma 7.27 implies that the mapping $L\colon \mathrm{Supp}(\beta_0) \to C_0$ is open. The assertion then follows from Proposition 2.21 and Lemma 7.19. □

Corollary 7.29. *Assume that $d = 0$, that $\mathrm{Card}(H) = 1$, and that $W = \{ 0 \}$; take $n_2' < n_2$. Then, the following hold:*

1. *for almost every $(T_1', \dots, T_{n_2'}) \in \mathfrak{g}_2^{n_2'}$, $(\mathcal{L}, (-iT_j')_{j=1,\dots,n_2'})$ satisfies property (RL);*
2. *if $\Omega = \mathfrak{g}_2^* \setminus \{ 0 \}$, then $(\mathcal{L}, (-iT_j)_{j=1,\dots,n_2'})$ satisfies property (RL).*

Proof. **1.** Let X be the polar in $\mathfrak{g}_2^*$ of the subspace of $\mathfrak{g}_2$ generated by some free family $(T_1', \dots, T_{n_2'}')$ of elements of $\mathfrak{g}_2$,[5] and let $(T_{n_2'+1}', \dots, T_{n_2}')$ be a basis of $\mathfrak{g}_2 \cap X^{\circ\perp}$. Consider the linear mapping $L\colon E_{(\mathcal{L},(-iT_j')_{j=1,\dots,n_2})} \to E_{(\mathcal{L},(-iT_j')_{j=1,\dots,n_2'})}$ such that $L(\mathcal{L}, (-iT_j')_{j=1,\dots,n_2}) = (\mathcal{L}, (-iT_j')_{j=1,\dots,n_2'})$. Then, there is a unique linear mapping $L'\colon \mathfrak{g}_2^* \to (X^\circ)^*$ such that

$$L(\lambda, (\omega(T_j'))_{j=1,\dots,n_2}) = (\lambda, (L'(\omega)(T_j'))_{j=1,\dots,n_2'})$$

for every $\omega \in \mathfrak{g}_2^*$ and for every $\lambda \in \mathbb{R}$. Then, ${}^tL'\colon X^\circ \to \mathfrak{g}_2$ is the canonical inclusion, so that $\ker L' = X$. Now, take $\gamma \in \mathbb{N}^{n_1}$ and define β_γ as in the proof of Theorem 7.28; let $\mathcal{C}$ be the set

[5] Observe that the set of non-free families in $\mathfrak{g}_2^{n_2'}$ is negligible, since it is the set of zeroes of the multilinear mapping $(T_1', \dots, T_{n_2}') \mapsto T_1' \wedge \dots \wedge T_{n_2'}'$.

of components of Ω. Choose $C \in \mathcal{C}$, and observe that $\widetilde{\mu}'_\omega(\mathbf{1}_{n_1} + 2\gamma)$ does not vanish identically on C, since the mapping $\omega \mapsto \widetilde{\mu}'_\omega(\mathbf{1}_{n_1} + 2\gamma)$ is proper on $\mathfrak{g}_2^*$ and C is unbounded. Then, take $\omega_{C,\gamma} \in C$ so that $\widetilde{\mu}'_{\omega_{C,\gamma}}(\mathbf{1}_{n_1} + 2\gamma) \neq 0$, and observe that L is generically a submersion on $\operatorname{Supp}(\beta_\gamma) \cap (\mathbb{R} \times C)$ if

$$X = \ker L' \not\subseteq \ker \widetilde{\mu}'_{\omega_{C,\gamma}}(\mathbf{1}_{n_1} + 2\gamma).$$

Therefore, if X is not contained in the negligible set

$$\bigcup_{\gamma \in \mathbb{N}^{n_1}, C \in \mathcal{C}} \ker \widetilde{\mu}'_{\omega_{C,\gamma}}(\mathbf{1}_{n_1} + 2\gamma),$$

then $\beta_{(\mathcal{L}, -iT'_1, \dots, -iT'_{n'_2})}$ is a measure with base $\mathcal{H}^{n'_2+1}$ thanks to Proposition 2.26. Hence, Proposition 7.28 shows that $(\mathcal{L}, -iT'_1, \dots, -iT'_{n'_2})$ satisfies property (RL); the assertion follows.

2. If $\Omega = \mathfrak{g}_2^* \setminus \{ 0 \}$, one only has to observe that the arguments we used to prove that $L_*(\beta_0)$ is a measure with base $\mathcal{H}^{n'_2+1}$ in Proposition 7.28 apply to prove that $L_*(\beta_\gamma)$ is a measure with base $\mathcal{H}^{n'_2+1}$. The assertion then follows from Proposition 7.28. ☐

Theorem 7.30. *Define $C_\gamma := \{ (\widetilde{\mu}_\omega(\mathbf{1}_{n_1} + 2\gamma), \omega(\mathbf{T})) : \omega \in \mathfrak{g}_2^* \}$ for every $\gamma \in \mathbb{N}^{n_1}$. In addition, take $n'_2 < n_2$ and define $L := I_\mathbb{R} \times \mathrm{pr}_{1,\dots,n'_2}$ on $E_{\mathcal{L}_A}$. Assume that the following hold:*

1. *$\operatorname{Card}(H) = 1$ and $d = 0$;*
2. *$\chi_{C_0} \cdot \beta_{\mathcal{L}_A}$ is L-connected;*
3. *for every $f \in L^1_{\mathcal{L}_A}(G)$ and for every $\gamma \in \mathbb{N}^{n_1}$, $\mathcal{M}_{\mathcal{L}_A}(f)$ equals $\beta_{\mathcal{L}_A}$-almost everywhere a continuous function on C_γ.*

Then, $L(\mathcal{L}_A)$ satisfies property (RL).

Observe that condition **2** holds if C_0 is the boundary of a convex polyhedron (cf. Corollary 2.24) and if $\Omega = \mathfrak{g}_2^* \setminus \{ 0 \}$ (cf. Lemma 7.27). With a little more effort, one may prove that condition **2** holds if $n'_2 = 1$.

We shall prepare the proof of Theorem 7.30 through several lemmas.

Lemma 7.31. *Let V be a topological vector space, C a convex subset of V with non-empty interior, and W an affine subspace of V such that $W \cap \mathring{C} \neq \emptyset$. Then, $W \cap \partial C$ is the boundary of $W \cap C$ in W.*

Proof. Indeed, take $x_0 \in W \cap \mathring{C}$, and take x in the interior of $W \cap C$ in W. Then, there is $y \in W \cap C$ such that $x \in [x_0, y[$, so that [18, Proposition 16 of Chapter II, § 2, No. 7] implies that $x \in \mathring{C}$. By the arbitrariness of x, $W \cap \mathring{C}$ is the interior of $W \cap C$ in W. Analogously, one proves that $W \cap \overline{C}$ is the closure of $W \cap C$ in W, so that the assertion follows. ☐

Lemma 7.32. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex function which is differentiable on an open subset U of $\mathbb{R}^n$. Let L be a linear mapping of $\mathbb{R}^n$* onto *$\mathbb{R}^k$ for some $k \leqslant n$, and assume that (f, L) has rank k on U. Then, for every $y \in (f, L)(U)$, the fibre $(f, L)^{-1}(y)$ is a closed convex set which contains $L^{-1}(y_2) \cap U$.*

Proof. Define $\pi := (f, L) : \mathbb{R}^n \to \mathbb{R} \times \mathbb{R}^k$, and observe that $\ker L \subseteq \ker f'(x)$ for every $x \in U$ since π has rank k on U. Therefore, if $y \in \pi(U)$, then f is locally constant on $L^{-1}(y_2) \cap U$. Now, take two components C_1 and C_2 of $L^{-1}(y_2) \cap U$, and observe that they are open in $L^{-1}(y_2)$. Take $x_1 \in C_1$ and $x_2 \in C_2$. Then, $[x_1, x_2] \subseteq L^{-1}(y_2)$, so that there are $x'_1, x'_2 \in]x_1, x_2[$ such that f is constant on $[x_1, x'_1]$ and on $[x'_2, x_2]$. By convexity, f must be constant on $[x_1, x_2]$, hence on $C_1 \cup C_2$. By the arbitrariness of C_1 and C_2, we infer that $\pi^{-1}(y) \supseteq L^{-1}(y_2) \cap U$.

Now, consider the closed convex set $C := \{ (\lambda, x) : x \in \mathbb{R}^n, \lambda \geqslant f(x) \}$, and observe that

$$\mathring{C} = \{ (\lambda, x) : x \in \mathbb{R}^n, \lambda > f(x) \}$$

since f is continuous, so that ∂C is the graph of f. Next, define $W := (I_\mathbb{R} \times L)^{-1}(y) = \{ y_1 \} \times L^{-1}(y_2)$, and observe that $W \cap \partial C = \{ y_1 \} \times \pi^{-1}(y)$. Assume by contradiction that

$W \cap \overset{\circ}{C} \neq \emptyset$. Then, Lemma 7.31 implies that $W \cap \partial C$ is the boundary of $W \cap C$ in W, so that $\pi^{-1}(y)$ has empty interior in $L^{-1}(y_2)$. However, $\pi^{-1}(y)$ contains $L^{-1}(y_2) \cap U$, which is open in $L^{-1}(y_2)$: contradiction. Therefore, $\{ y_1 \} \times \pi^{-1}(y) = W \cap C$ is a closed convex set, whence the result. □

Lemma 7.33. *Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be a convex function which is analytic on some open subset Ω of $\mathbb{R}^n$ whose complement is $\mathcal{H}^n$-negligible. Let L be a linear mapping of $\mathbb{R}^n$ onto $\mathbb{R}^k$ for some $k \leqslant n$, and let U be the union of the components of Ω where (f, L) has rank k. Then,*

$$(f, L)^{-1}(y) = \overline{L^{-1}(y_2) \cap U}$$

for $\mathcal{H}^k$-almost every $y \in (f, L)(U)$.

Proof. Define $\pi \coloneqq (f, L)$. Since the complement of Ω is $\mathcal{H}^n$-negligible, there is an $\mathcal{H}^k$-negligible subset N_1 of $\mathbb{R}^k$ such that $L^{-1}(y) \setminus \Omega$ is $\mathcal{H}^{n-k}$-negligible for every $y \in \mathbb{R}^k \setminus N_1$ (cf. [39, Theorem 3.2.11]). In addition, observe that the set R_k of $x \in \Omega \setminus U$ such that $\ker L \subseteq \ker f'(x)$, that is, such that $\pi'(x)$ has rank k, is $\mathcal{H}^n$-negligible by the analyticity of f. Then, there is an $\mathcal{H}^k$-negligible subset N_2 of $\mathbb{R}^k$ such that $L^{-1}(y) \cap R_k$ is $\mathcal{H}^{n-k}$-negligible for every $y \in \mathbb{R}^k \setminus N_1$ (*loc. cit.*). Now, define $N \coloneqq \mathbb{R} \times (N_1 \cup N_2)$, and observe that $\chi_{\pi(U)} \cdot \mathcal{H}^k$ is a pseudo-image measure of $\chi_U \cdot \mathcal{H}^n$ under π thanks to Corollary 2.27;[6] since $U \cap \pi^{-1}(N) = U \cap L^{-1}(N_1 \cup N_2)$ is $\mathcal{H}^n$-negligible, it follows that $\pi(U) \cap N$ is $\mathcal{H}^k$-negligible.

Now, take $y \in \pi(U) \setminus N$. Then, Lemma 7.32 implies that $\pi^{-1}(y)$ is a closed convex set which contains $L^{-1}(y_2) \cap U$, so that its interior in $L^{-1}(y_2)$ is not empty. Let U' be a component of Ω which is not contained in U, and assume that $\pi^{-1}(y) \cap U' \neq \emptyset$. Since f is analytic on U', and since $\pi^{-1}(y)$ is a convex set with non-empty interior in $L^{-1}(y_2)$, we see that a component C of $L^{-1}(y_2) \cap U'$ is contained in R_k. By the choice of N_2, this implies that C is $\mathcal{H}^{n-k}$-negligible; since C is non-empty and open in $L^{-1}(y_2)$, this leads to a contradiction. Therefore,

$$L^{-1}(y_2) \cap U \subseteq \pi^{-1}(y) \subseteq L^{-1}(y_2) \cap [U \cup (\mathbb{R}^n \setminus \Omega)].$$

By our choice of N_1, the set $L^{-1}(y_2) \setminus \Omega$ is $\mathcal{H}^{n-k}$-negligible; on the other hand, the support of $\chi_{\pi^{-1}(y)} \cdot \mathcal{H}^{n-k}$ is $\pi^{-1}(y)$ by convexity. Hence, $L^{-1}(y_2) \cap U$ is dense in $\pi^{-1}(y)$, whence the result. □

Lemma 7.34. *Keep the hypotheses and the notation of Lemma 7.33. Assume, in addition, that $\lim_{x \to \infty} f(x) = +\infty$ and that $\mathcal{H}^n$ is (f, L)-connected. Then, for every $m \in C(\mathbb{R}^n)$ such that $m = m' \circ (f, L)$ $\mathcal{H}^n$-almost everywhere for some $m'\colon \mathbb{R} \times \mathbb{R}^k \to \mathbb{C}$, there is $m'' \in C(\mathbb{R} \times \mathbb{R}^k)$ such that $m = m'' \circ (f, L)$ pointwise.*

Proof. Define $\pi \coloneqq (f, L)$. Let $(\beta_{1,y})_{y \in \mathbb{R} \times \mathbb{R}^k}$ be a disintegration of $\chi_U \cdot \mathcal{H}^n$ relative to π and let $(\beta_{2,y})_{y \in \mathbb{R} \times \mathbb{R}^k}$ be a disintegration of $\chi_{\Omega \setminus U} \cdot \mathcal{H}^n$ relative to π. Then, Corollary 2.27 implies that:

- $\pi_*(\chi_U \cdot \mathcal{H}^n)$ is equivalent to $\chi_{\pi(U)} \cdot \mathcal{H}^k$;
- $\pi_*(\chi_{\Omega \setminus U} \cdot \mathcal{H}^n)$ is equivalent to $\chi_{\pi(\Omega \setminus U)} \cdot \mathcal{H}^{k+1}$;
- $\operatorname{Supp}(\beta_{1,y}) = \overline{\pi^{-1}(y) \cap U}$ for $\mathcal{H}^k$-almost every $y \in \pi(U)$;
- $\operatorname{Supp}(\beta_{2,y}) = \overline{\pi^{-1}(y) \cap \Omega \setminus U}$ for $\mathcal{H}^{k+1}$-almost every $y \in \pi(\Omega \setminus U)$.

In addition, $\pi(U)$ has Hausdorff dimension k, so that $\mathcal{H}^{k+1}(\pi(U)) = 0$; in particular, $\pi_*(\chi_U \cdot \mathcal{H}^n)$ and $\pi_*(\chi_{\Omega \setminus U} \cdot \mathcal{H}^n)$ are alien measures. If we define $\beta_y \coloneqq \beta_{1,y}$ for every $y \in \pi(U)$ and $\beta_y \coloneqq \beta_{2,y}$ for every $y \in (\mathbb{R} \times \mathbb{R}^k) \setminus \pi(U)$, then (β_y) is a disintegration of $\mathcal{H}^n$ relative to π.

Now, Lemma 7.33 implies that $\overline{\pi^{-1}(y) \cap U} = \pi^{-1}(y)$ for $\mathcal{H}^k$-almost every $y \in \pi(U)$; let us prove that $\pi^{-1}(y) = \overline{\pi^{-1}(y) \cap \Omega \setminus U}$ for $\mathcal{H}^{k+1}$-almost every $y \notin \pi(U)$. Let us first prove that $\pi^{-1}(y)$ is the boundary of a compact convex set with non-empty interior in $L^{-1}(y_2)$ for $\mathcal{H}^{k+1}$-almost every $y \in \pi(\Omega \setminus U)$.

Indeed, by [75] there is an $\mathcal{H}^{k+1}$-negligible subset N of $\pi(\Omega \setminus U)$ such that $\pi'(x)$ has rank $k+1$ for every $x \in L^{-1}(y) \cap \Omega \setminus U$ and for every $y \in \pi(\Omega \setminus U) \setminus N$. Now, define $C \coloneqq$

[6] In other words, $\chi_{\pi(U)} \cdot \mathcal{H}^k$ is equivalent to $\pi_*(\chi_U \cdot \mathcal{H}^n)$.

$\{ (\lambda, x) \colon x \in \mathbb{R}^n, \lambda \geqslant f(x) \}$, and observe that $I_\mathbb{R} \times L$ is proper on C since $\lim_{x \to \infty} f(x) = +\infty$. Therefore, $\{ y_1 \} \times \pi^{-1}(y) = (I_\mathbb{R} \times L)^{-1}(y) \cap \partial C$ is compact for every $y \in \mathbb{R} \times \mathbb{R}^k$. In addition, if $y \in \pi(\Omega \setminus U) \setminus N$, then $(I_\mathbb{R} \times L)^{-1}(y) \cap \mathring{C} \neq \emptyset$, so that Lemma 7.31 implies that $\pi^{-1}(y)$ is the boundary of a compact convex set with non-empty interior in $L^{-1}(y_2)$.

Therefore, $\pi^{-1}(y)$ is bi-Lipschitz homeomorphic to $\mathbb{S}^{n-k-1}$, so that the support of $\chi_{\pi^{-1}(y)} \cdot \mathcal{H}^{n-k-1}$ is $\pi^{-1}(y)$ for such y. In addition, since $\mathbb{R}^n \setminus \Omega$ is $\mathcal{H}^n$-negligible, [39, 3.2.11] implies that $\pi^{-1}(y) \setminus \Omega$ is $\mathcal{H}^{n-k-1}$-negligible for $\mathcal{H}^{k+1}$-almost every $y \in \mathbb{R} \times \mathbb{R}^k$. Hence, $\overline{\pi^{-1}(y) \cap \Omega \setminus U} = \pi^{-1}(y)$ for $\mathcal{H}^{k+1}$-almost every $y \notin \pi(U)$.

Then, Proposition 2.21 implies that there is $m''' \colon \pi(\mathbb{R}^n) \to \mathbb{C}$ such that $m = m''' \circ \pi$; since π is proper, this implies that m''' is continuous on $\pi(\mathbb{R}^n)$. In addition, $\pi(\mathbb{R}^n)$ is closed, so that the assertion follows from [17, Corollary to Theorem 2 of Chapter IX, § 4, No. 2]. □

Proof of Theorem 7.30. Until the end of this proof, we shall identify $\mathbb{R}^{n_2}$ and $\mathfrak{g}_2^*$ by means of the bijection $\omega \mapsto \omega(\mathbf{T})$; L' will denote $\mathrm{pr}_{1,\dots,n_2'}$, so that $L = I_\mathbb{R} \times L'$. In addition, for every $\gamma \in \mathbb{N}^{n_1}$, we define $\pi_\gamma \colon \mathbb{R}^{n_2} \ni \omega \mapsto (\mu_\omega(\mathbf{1}_{n_1} + 2\gamma), \omega)$, so that π_γ is continuous and C_γ is its graph.

Take $f \in L^1_{L(\mathcal{L}_A)}(G)$, and let m be a representative of $\mathcal{M}_{L(\mathcal{L}_A)}(f)$. Take, for every $\gamma \in \mathbb{N}^{n_1}$, a continuous function m_γ on C_γ such that $m_\gamma = \mathcal{M}_{\mathcal{L}_A}(f)$ $\chi_{C_\gamma} \cdot \beta_{\mathcal{L}_A}$-almost everywhere. Then, Lemma 7.34 implies that there is a continuous function $m_0' \colon E_{L(\mathcal{L}_A)} \to \mathbb{C}$ such that $m_0 = m_0' \circ L$ *on* C_0. Since $\beta_{L(\mathcal{L}_A)}$ need *not* be equivalent to $L_*(\chi_{C_0} \cdot \beta_{\mathcal{L}_A})$, though, this is not sufficient to conclude.

For every $\gamma \in \mathbb{N}^{n_1}$, define $\beta_\gamma \coloneqq \chi_{C_\gamma} \cdot \beta_{\mathcal{L}_A}$, and observe that β_γ is equivalent to $(\pi_\gamma)_*(\mathcal{H}^{n_2})$. Let $U_{\gamma,1}$ be the union of the components C of Ω such that $\ker L' \not\subseteq \ker \widetilde{\mu}'_\omega(\mathbf{1}_{n_1} + 2\gamma)$ for some $\omega \in C$, and let $U_{\gamma,2}$ be the complement of $U_{\gamma,1}$ in Ω. Then, Corollary 2.27 implies that the following hold:

- $L_*(\chi_{\mathbb{R} \times U_{\gamma,1}} \cdot \beta_\gamma)$ is equivalent to $\chi_{L(\pi_\gamma(U_{\gamma,1}))} \cdot \mathcal{H}^{n_2'+1}$;
- $L_*(\chi_{\mathbb{R} \times U_{\gamma,2}} \cdot \beta_\gamma)$ is equivalent to $\chi_{L(\pi_\gamma(U_{\gamma,2}))} \cdot \mathcal{H}^{n_2'}$;
- $\chi_{\mathbb{R} \times U_{\gamma,2}} \cdot \beta_\gamma$ has a disintegration $(\beta_{\gamma,2,\lambda})_{\lambda \in E_{L(\mathcal{L}_A)}}$ relative to L such that $\beta_{\gamma,2,\lambda}$ is equivalent to $\chi_{L^{-1}(\lambda) \cap \pi_\gamma(U_{\gamma,2})} \cdot \mathcal{H}^{n_2 - n_2'}$ and $L^{-1}(\lambda) \cap \pi_\gamma(U_{\gamma,2}) \subseteq \mathrm{Supp}(\beta_{\gamma,2,\lambda})$ for $\mathcal{H}^{n_2'}$-almost every $\lambda \in L(\pi_\gamma(U_{\gamma,2}))$.

In particular, $\beta_{L(\mathcal{L}_A)}$ is equivalent to $\chi_{\sigma(L(\mathcal{L}_A))} \cdot \mathcal{H}^{n_2'+1} + \mu$, where μ is a measure with base $\mathcal{H}^{n_2'}$ alien to $\mathcal{H}^{n_2'+1}$. Now, observe that $L_*(\chi_{\mathbb{R} \times U_{\gamma,1}} \cdot \beta_\gamma)$ is a measure with base $L_*(\beta_0)$; since $(m - m_0') \circ L$ is β_0-negligible, we see that there is an $L_*(\beta_0)$-negligible subset N of $E_{L(\mathcal{L}_A)}$ such that $m = m_0'$ on $E_{L(\mathcal{L}_A)} \setminus N$. Since N is $L_*(\chi_{\mathbb{R} \times U_{\gamma,1}} \cdot \beta_\gamma)$-negligible, this implies that $(m - m_0') \circ L$ vanishes $\chi_{\mathbb{R} \times U_{\gamma,1}} \cdot \beta_\gamma$-almost everywhere. Since $m \circ L = m_\gamma$ β_γ-almost everywhere, it follows that $m_0' \circ L = m_\gamma$ $\chi_{\mathbb{R} \times U_{\gamma,1}} \cdot \beta_\gamma$-almost everywhere, hence *pointwise* on

$$\mathrm{Supp}\big(\chi_{\mathbb{R} \times U_{\gamma,1}} \cdot \beta_\gamma\big) = \mathrm{Supp}\big((\pi_\gamma)_*(\chi_{U_{\gamma,1}} \cdot \mathcal{H}^{n_2})\big) = \pi_\gamma(\overline{U_{\gamma,1}})$$

since $m_0' \circ L$ and m_γ are continuous, while π_γ is proper.

Next, consider $\chi_{\mathbb{R} \times U_{\gamma,2}} \cdot \beta_\gamma$. Observe that Tonelli's theorem implies that $L'^{-1}(\lambda_2) \setminus \Omega$ is $\mathcal{H}^{n_2 - n_2'}$-negligible for $\mathcal{H}^{n_2'}$-almost every $\lambda_2 \in \mathbb{R}^{n_2'}$. Now, if $\widetilde{N}$ is an $\mathcal{H}^{n_2'}$-negligible subset of $\mathbb{R} \times \mathbb{R}^{n_2'}$, then $\mathrm{pr}_2(\widetilde{N})$ is $\mathcal{H}^{n_2'}$-negligible since pr_2 is Lipschitz. Therefore, there is an $\mathcal{H}^{n_2'}$-negligible subset N' of $\mathbb{R}^{n_2'}$ such that, for every $\lambda \in L(\pi_\gamma(U_{\gamma,2})) \setminus (\mathbb{R} \times N')$,

- $m \circ L = m_\gamma$ $\beta_{\gamma,2,\lambda}$-almost everywhere;
- $L^{-1}(\lambda) \cap \pi_\gamma(U_{\gamma,2}) \subseteq \mathrm{Supp}(\beta_{\gamma,2,\lambda})$;
- $L'^{-1}(\lambda_2) \setminus \Omega$ is $\mathcal{H}^{n_2 - n_2'}$-negligible.

Hence, if $\lambda \in L(\pi_\gamma(U_{\gamma,2})) \setminus (\mathbb{R} \times N')$, then m_γ is constant on $L^{-1}(\lambda) \cap \pi_\gamma(U_{\gamma,2})$. In addition, fix $\lambda \in L(\pi_\gamma(U_{\gamma,2})) \setminus (\mathbb{R} \times N')$; then,

$$L'^{-1}(\lambda_2) = \overline{L'^{-1}(\lambda_2) \cap U_{\gamma,1}} \cup \overline{L'^{-1}(\lambda_2) \cap U_{\gamma,2}},$$

so that either $\overline{L'^{-1}(\lambda_2)\cap U_{\gamma,1}}\cap\overline{L'^{-1}(\lambda_2)\cap U_{\gamma,2}}\neq\emptyset$ or $L'^{-1}(\lambda_2)\cap U_{\gamma,1}=\emptyset$ by connectedness.

Now, let $\mathcal{C}$ be the set of components of $L'^{-1}(\lambda_2)\cap U_{\gamma,2}$; observe that $\mathcal{C}$ is finite since $L'^{-1}(\lambda_2)\cap\Omega$ is semi-algebraic (cf. [29, Proposition 4.13]) and since $L'^{-1}(\lambda_2)\cap U_{\gamma,2}$ is open and closed in $L'^{-1}(\lambda_2)\cap\Omega$. In addition, observe that $\mathrm{pr}_1\circ\pi_\gamma$ is constant on each $C\in\mathcal{C}$; let $\lambda_{1,C}$ be its constant value. In particular, since $\mathrm{pr}_1\circ\pi_\gamma$ is proper and since $\mathcal{C}$ is finite, this implies that $L'^{-1}(\lambda_2)\cap U_{\gamma,1}\neq\emptyset$. Further, m_γ is constant on $\pi_\gamma(C)\subseteq L^{-1}(\lambda_{1,C},\lambda_2)\cap\pi_\gamma(U_{\gamma,2})$ for every $C\in\mathcal{C}$. Now, there is $C_1\in\mathcal{C}$ such that $L'^{-1}(\lambda_2)\cap\overline{U_{\gamma,1}}\cap\overline{C_1}\neq\emptyset$; since $m_\gamma\circ\pi_\gamma=m'_0\circ L\circ\pi_\gamma$ on $\overline{U_{\gamma,1}}$, and since m_γ is continuous, it follows that $m_\gamma\circ\pi_\gamma=m'_0\circ L\circ\pi_\gamma$ on $\overline{C}_1$. Iterating this procedure, we eventually see that $m_\gamma\circ\pi_\gamma=m'_0\circ L\circ\pi_\gamma$ on $L'^{-1}(\lambda_2)$. Therefore, $m_\gamma=m'_0\circ L$ on $L^{-1}(\lambda)\cap C_\gamma$ for every $\lambda\in L(\pi_\gamma(U_{\gamma,2}))\setminus(\mathbb{R}\times N')$.

Now, observe that $L^{-1}(\mathbb{R}\times N')\cap\pi_\gamma(U_{\gamma,2})$ is $\mathcal{H}^{n'_2}$-negligible since $\mathrm{pr}_2\circ L\circ\pi_\gamma=L'$ and since $\mathcal{H}^{n''_2}$ is a pseudo-image measure of $\mathcal{H}^{n_2}$ under L'. Therefore, $m_\gamma=m'_0\circ L$ β_γ-almost everywhere, hence *pointwise* on C_γ by continuity. By the arbitrariness of γ, this implies that $m'_0\circ L$ is a representative of $\mathcal{M}_{\mathcal{L}_A}(f)$, so that m'_0 is a continuous representative of $\mathcal{M}_{L(\mathcal{L}_A)}(f)$. The assertion follows. □

7.4 Property (S)

Keep the notation of Section 7.2.

The results of this section are basically a generalization of the techniques employed in [4, 5]. The first two results concern the case in which $d=0$; the former one has very restrictive hypotheses, for the same reasons explained while discussing property (RL), but hold for the 'full family' $\mathcal{L}_A$ (cf. Theorem 7.36); on the contrary, the latter one holds under more general assumptions, but only for families of the form $(\mathcal{L},(-iT_1,\dots,-iT_{n'_2}))$ for $n'_2<n_2$ (cf. Theorem 7.39).

Notice that, even though Theorem 7.45 is the main application of Theorem 7.36, there are other families to which it applies as well. See Section 7.7 for some examples.

Our last result concerns the case in which $d>0$ (cf. Theorem 7.40). As for property (RL), the case in which $d>0$ is simpler than the case in which $d=0$, and the results so obtained are more general. Theorem 7.40 applies, for example, to the free 2-step nilpotent group on three generators, and to the product of a Métivier group and a non-trivial abelian group.

Notice that in all of our results we imposed the condition $W=\{0\}$; this is unavoidable (with our methods), since on W we cannot infer any kind of regularity from the 'inversion formulae' employed. Indeed, our auxiliary functions, such as $|x_\omega|^2$ or P_0, are not differentiable on W, in general. Nevertheless, this does not mean that property (S) cannot hold when $W\neq\{0\}$, as Theorems 7.41, 7.48, and 7.50 show.

We begin with a lemma which will allow us to get some 'Taylor expansions' of multipliers corresponding to Schwartz kernels under suitable hypotheses. We state it in a slightly more general context for later use.

Lemma 7.35. *Let $\mathcal{L}'_{A'}$ be a Rockland family on a homogeneous group G', and let $T'_1,\dots,T'_n$ be a free family of homogeneous elements of the centre of the Lie algebra $\mathfrak{g}'$ of G'. Let π_1 be the canonical projection of G' onto its quotient by the normal subgroup $\exp(\mathbb{R}T'_1)$, and assume that the following hold:*

- $(\mathcal{L}'_{A'},iT'_1,\dots,iT'_n)$ *satisfies property (RL);*
- $\mathrm{d}\pi_1(\mathcal{L}'_{A'},iT'_2,\dots,iT'_n)$ *satisfies property $(S)_0$.*

Take $\varphi\in\mathcal{S}_{(\mathcal{L}'_{A'},iT'_1,\dots,iT'_n)}(G')$. Then, there are two families $(\widetilde{\varphi}_\gamma)_{\gamma\in\mathbb{N}^n}$ and $(\varphi_\gamma)_{\gamma\in\mathbb{N}^n}$ of elements of $\mathcal{S}(G',\mathcal{L}'_{A'})$ and $\mathcal{S}_{(\mathcal{L}'_{A'},iT'_1,\dots,iT'_n)}(G')$, respectively, such that

$$\varphi=\sum_{|\gamma|<h}\mathbf{T}'^\gamma\widetilde{\varphi}_\gamma+\sum_{|\gamma|=h}\mathbf{T}'^\gamma\varphi_\gamma$$

for every $h\in\mathbb{N}$.

Proof. For every $k\in\{1,\dots,n\}$, let G'_k be the quotient of G' by the normal subgroup $\exp(\mathbb{R}T'_k)$. Endow $\mathfrak{g}'$ with a scalar product which turns $(T'_1,\dots,T'_n)$ into an orthonormal family. Then, Theorem 4.1 implies that $(\pi_1)_*(\varphi)\in\mathcal{S}_{\mathrm{d}\pi_1(\mathcal{L}'_{A'},iT'_2,\dots,iT'_n),0}(G'_1)$, so that there is $\widetilde{m}_1 \in \mathcal{S}(E_{\mathrm{d}\pi_1(\mathcal{L}'_{A'}, iT'_2,\dots,iT'_n)})$ such that $(\pi_1)_*(\varphi) = \mathcal{K}_{\mathrm{d}\pi_1(\mathcal{L}'_{A'}, iT'_2,\dots,iT'_n)}(\widetilde{m}_1)$. Define $\widetilde{\varphi}_{0,1} \coloneqq \mathcal{K}_{(\mathcal{L}'_{A'}, iT'_2,\dots,iT'_n)}(\widetilde{m}_1)$, so that Theorem 4.1 implies that $(\pi_1)_*(\varphi - \widetilde{\varphi}_{0,1}) = 0$. In other words,

$$\int_{\mathbb{R}} (\varphi - \widetilde{\varphi}_{0,1})(\exp(x + sT'_1))\, \mathrm{d}s = 0$$

for every $x \in (\mathbb{R}T'_1)^\perp$. Identifying $\mathcal{S}(G')$ with $\mathcal{S}(\mathbb{R}T'_1; \mathcal{S}((\mathbb{R}T'_1)^\perp))$, by means of Corollary 2.17 we see that there is $\varphi_1 \in \mathcal{S}(G')$ such that

$$\varphi = \widetilde{\varphi}_{0,1} + T'_1 \varphi_1.$$

Now, let us prove that $\varphi_1 \in \mathcal{S}_{(\mathcal{L}'_{A'}, iT'_1,\dots,iT'_n)}(G')$. Indeed,

$$T'_1 \mathcal{K}_{(\mathcal{L}'_{A'}, iT'_2,\dots,iT'_n)} \mathcal{M}_{(\mathcal{L}'_{A'}, iT'_2,\dots,iT'_n)}(\varphi_1) = \varphi - \widetilde{\varphi}_{0,1} = T'_1 \varphi_1.$$

Since clearly $\mathcal{K}_{(\mathcal{L}'_{A'}, iT'_2,\dots,iT'_n)} \mathcal{M}_{(\mathcal{L}'_{A'}, iT'_2,\dots,iT'_n)}(\varphi_1) \in L^2(G')$, and since T'_1 is one-to-one on $L^2(G')$, the assertion follows. If $n \geqslant 2$, then we can apply the same argument to $\widetilde{\varphi}_{0,1}$ considering the quotient G'_2, since we already know that $\widetilde{\varphi}_{0,1}$ has a Schwartz multiplier. Then, we obtain $\widetilde{\varphi}_{0,2} \in \mathcal{S}(G', (\mathcal{L}'_{A'}, iT'_3, \dots, iT'_n))$ and $\varphi_2 \in \mathcal{S}_{(\mathcal{L}'_{A'}, iT'_1,\dots,iT'_n)}(G')$ such that

$$\varphi = \widetilde{\varphi}_{0,2} + T'_1 \varphi_1 + T'_2 \varphi_2.$$

Iterating this procedure, we find $\widetilde{\varphi}_0 \in \mathcal{S}(G', \mathcal{L}'_{A'})$ and $\varphi_1, \dots, \varphi_n \in \mathcal{S}_{(\mathcal{L}'_{A'}, iT'_1,\dots,iT'_n)}(G')$ such that

$$\varphi = \widetilde{\varphi}_0 + \sum_{k=1}^n T'_k \varphi_k.$$

The assertion follows proceeding inductively. □

Theorem 7.36. *Assume that $d = 0$, that $\Omega = \mathfrak{g}_2^* \setminus \{ 0 \}$, that $\dim_{\mathbb{Q}} \mu_\omega(\mathbb{Q}^{\overline{h}}) = \dim_{\mathbb{R}} \mu_\omega(\mathbb{R}^{\overline{h}})$ for every $\omega \in \Omega$, and that μ is constant where $\mu_{\eta_0}(\mathbf{n_1})$ is constant. Then, $\mathcal{L}_A$ satisfies property (S).*

Proof. We proceed by induction on $n_2 \geqslant 1$.

1. Notice that the inductive hypothesis, Corollary 6.10, Theorem 7.25, and Lemma 7.35 imply that we may find a family $(\widetilde{\varphi}_\gamma)$ of elements of $\mathcal{S}(G, \mathcal{L}_H)$, and a family (φ_γ) of elements of $\mathcal{S}_{\mathcal{L}_A}(G)$ such that

$$\varphi = \sum_{|\gamma| < h} \mathbf{T}^\gamma \widetilde{\varphi}_\gamma + \sum_{|\gamma| = h} \mathbf{T}^\gamma \varphi_\gamma$$

for every $h \in \mathbb{N}$.

Define $\widetilde{m}_\gamma \coloneqq \mathcal{M}_{\mathcal{L}_H}(\widetilde{\varphi}_\gamma) \in \mathcal{S}(\sigma(\mathcal{L}_H))$ and $m_\gamma \coloneqq \mathcal{M}_{\mathcal{L}_A}(\varphi_\gamma) \in C_0(\sigma(\mathcal{L}_A))$ for every γ (cf. Theorem 7.25). Then,

$$m_0(\lambda, \omega) = \sum_{|\gamma| < h} \omega^\gamma \widetilde{m}_\gamma(\lambda) + \sum_{|\gamma| = h} \omega^\gamma m_\gamma(\lambda, \omega)$$

for every $h \in \mathbb{N}$ and for every $(\lambda, \omega) \in \sigma(\mathcal{L}_A)$.

2. Assume that $m_\gamma = 0$ for every $\gamma \in \mathbb{N}^{n_2}$, and define $N(\omega) \coloneqq \mu_\omega(\mathbf{n_1})$ for every $\omega \in \mathfrak{g}_2^*$, so that N is a norm on $\mathfrak{g}_2^*$ which is analytic on Ω. Define, in addition, $\Sigma \coloneqq \mu_\omega(\mathbf{n_1} + 2\mathbb{N}^{\overline{h}})$ for some (hence every) $\omega \in \mathfrak{g}_2^*$ such that $N(\omega) = 1$. Then, set $d \coloneqq \inf_{\sigma \in \Sigma} d(\sigma, \Sigma \setminus \{ \sigma \})$, and observe that $d > 0$ since $\dim_{\mathbb{Q}} \mu_\omega(\mathbb{Q}^{\overline{h}}) = \dim_{\mathbb{R}} \mu_\omega(\mathbb{R}^{\overline{h}})$. Finally, identify $\mathfrak{g}_2^*$ with $\mathbb{R}^{n_2}$ by means of the mapping $\omega \mapsto \omega(\mathbf{T})$, take $r \in \left]0, \frac{\min_{\sigma \in \Sigma} |\sigma|}{4d}\right[$, and choose $\varphi \in \mathcal{D}(\mathbb{R}^H)$ so that $\chi_{B(0,r)} \leqslant \varphi \leqslant \chi_{B(0,2r)}$. Define

$$\widetilde{m}(\lambda) \coloneqq \begin{cases} \sum_{\sigma \in \Sigma} \widetilde{m}_0(N(\lambda_2)\sigma, \lambda_2) \varphi\left(\frac{1}{d} \left(\frac{\lambda_1}{N(\lambda_2)} - \sigma \right) \right) & \text{if } \lambda_2 \neq 0 \\ 0 & \text{if } \lambda_2 = 0 \end{cases}$$

for every $\lambda \in E_{\mathcal{L}_A}$. Proceeding as in the proof of [4, Lemma 3.1], one sees that $\widetilde{m} \in \mathcal{S}(E_{\mathcal{L}_A})$, so that $\varphi \in \mathcal{S}(G, \mathcal{L}_A)$.

3. Now, consider the general case. By a vector-valued version of Borel's lemma (cf. [50, Theorem 1.2.6] for the scalar, one-dimensional case), we see that there is $\widehat{m} \in \mathcal{D}(\mathfrak{g}_2^*; \mathcal{S}(\mathbb{R}^H))$ such that $\widehat{m}^{(\gamma)}(0) = \widetilde{m}_\gamma$ for every $\gamma \in \mathbb{N}^{n_2}$. Interpret $\widehat{m}$ as an element of $\mathcal{S}(E_{\mathcal{L}_A})$. Then, **2** implies that $m - \widehat{m}$ equals a Schwartz function on $\sigma(\mathcal{L}_A)$. The assertion follows. □

For the case $d = 0$, $\mathrm{Card}(H) = 1$, and $n_2' < n_2$, we need a suitable version of the Morse lemma.

Lemma 7.37. *Let U be an open subset of $\mathbb{R}\times\mathbb{R}^n$, r an element of $\mathbb{N}\cup\{\infty\}$ such that $r \geqslant 3$, and φ a mapping of class C^r of U into $\mathbb{R}$. Assume that $\partial_1\varphi(x_0) = 0$ and $\partial_1^2\varphi(x_0) > 0$ for some $x_0 \in U$.*[7]

Then, there are an open neighbourhood V_1 of 0 in $\mathbb{R}$, an open neighbourhood V_2 of $x_{0,2}$ in $\mathbb{R}^n$, and a C^{r-2}-diffeomorphism ψ from $V_1 \times V_2$ onto an open subset of U such that $\psi(0, x_{0,2}) = x_0$, $\psi_2 = \mathrm{pr}_2$, and

$$\varphi(\psi(y)) = \varphi(\psi(0, y_2)) + y_1^2$$

for every $y \in V_1 \times V_2$. If, in addition, φ is analytic, then also ψ can be taken so as to be analytic.

Proof. Up to restricting U, we may assume that $U = U_1 \times U_2$ for some open subset U_1 of $\mathbb{R}$ and for some open subset U_2 of $\mathbb{R}^n$. Then, by means of the implicit function theorem we may assume that there is a function $f\colon U_2 \to U_1$ of class C^{r-1} such that, for every $x \in U$, $\partial_1\varphi(x) = 0$ if and only if $x_1 = f(x_2)$. In particular, $f(x_{0,2}) = x_{0,1}$. Now, up to restricting U_2, we may assume that there is an open interval U_1' of $\mathbb{R}$ such that $0 \in U_1'$ and $x_1 + f(x_2) \in U_1$ for every $x_1 \in U_1'$ and for every $x_2 \in U_2$. Then, define $U' := U_1' \times U_2$ and

$$\widetilde{\varphi}\colon U' \ni x \mapsto \varphi(x_1 + f(x_2), x_2) - \varphi(f(x_2), x_2) \in \mathbb{R},$$

so that $\widetilde{\varphi}$ is of class C^{r-1} on U', $\widetilde{\varphi}(0, x_2) = 0$ for every $x_2 \in U_2$ and, for every $x \in U'$, $\partial_1\widetilde{\varphi}(x) = 0$ if and only if $x_1 = 0$. Then, define

$$\psi_1\colon U' \ni x \mapsto \int_{[0,1]} (\partial_1^2\widetilde{\varphi})(tx_1, x_2)(1-t)\,\mathrm{d}t,$$

so that Taylor's theorem implies that

$$\widetilde{\varphi}(x) = x_1^2\psi_1(x)$$

for every $x \in U'$. In addition, it is easily seen that ψ_1 is of class C^{r-2} on U', and that $\psi_1(0, x_{0,2}) = \frac{1}{2}\partial_1^2\widetilde{\varphi}(0, x_{0,2}) > 0$. Hence, we may assume that $\psi_1(x) > 0$ for every $x \in U'$, so that the mapping

$$\psi_2\colon U' \ni x \mapsto x_1\sqrt{\psi_1(x)} \in \mathbb{R}$$

is of class C^{r-2} on U'. In addition, $\partial_1\psi_2(0, x_{0,2}) > 0$, so that we may assume that $\psi_3 = (\psi_2, \mathrm{pr}_2)\colon U' \to \mathbb{R}\times\mathbb{R}^n$ is a C^{r-2}-diffeomorphism of U' onto its image V, which is then an open subset of $\mathbb{R}\times\mathbb{R}^n$. Then, V is an open neighbourhood of $(0, x_{0,2})$ in $\mathbb{R}\times\mathbb{R}^n$, so that we may find an open neighbourhood V_1 of 0 in $\mathbb{R}$ and an open neighbourhood V_2 of $x_{0,2}$ in $\mathbb{R}^n$ such that $V_1 \times V_2 \subseteq V$. Let $\psi_4\colon V \to U'$ be the inverse of ψ_3, so that $\psi_4 = (\psi_{4,1}, \mathrm{pr}_2)$. Then, $\psi_2(\psi_4(0, y_2)) = 0$ for every $y_2 \in V_2$. However, the preceding arguments show that $\psi_2(x) = 0$ if and only if $x_1 = 0$, so that $\psi_{4,1}(0, y_2) = 0$ for every $y_2 \in V_2$. Then,

$$\psi\colon V_1 \times V_2 \ni y \mapsto (\psi_{4,1}(y) + f(y_2), y_2) \in U$$

satisfies the conditions of the statement. The assertions concerning analyticity are easily established. □

Corollary 7.38. *Keep the hypotheses and the notation of Lemma 7.37, and assume that $r = \infty$. Take a function $f \in C^\infty(\psi(V_1 \times V_2) \times \mathbb{R})$ and a function $g\colon V_2 \times \mathbb{R} \to \mathbb{C}$ so that*

$$f(x, \varphi(x)) = g(x_2, \varphi(x))$$

for every $x \in \psi(V_1 \times V_2)$. Then, g can be modified so as to be of class C^∞ in a neighbourhood of $(x_{0,2}, \varphi(x_0))$.

[7] Here, ∂_1 denotes the partial derivative in the $\mathbb{R}$-component of $\mathbb{R}\times\mathbb{R}^n$, *not* in the first component of $\mathbb{R}^n$.

Proof. Indeed, the assumption means that

$$f(y,\varphi(\psi(0,y_2))+y_1^2)=g(y_2,\varphi(\psi(0,y_2))+y_1^2)$$

for every $y\in V_1\times V_2$. Define, for every $y_2\in V_2$,

$$\widetilde{f}_{y_2}\colon V_1\ni t\mapsto f((t,y_2),\varphi(\psi(0,y_2))+t^2)$$

and

$$\widetilde{g}_{y_2}\colon \mathbb{R}\ni t\mapsto g(y_2,\varphi(\psi(0,y_2))+t).$$

Then, the mapping $V_2\ni y_2\mapsto \widetilde{f}_{y_2}$ belongs to $\mathcal{E}(V_2;\mathcal{E}(V_1))$, and

$$\widetilde{f}_{y_2}(y_1)=\widetilde{g}_{y_2}(y_1^2)$$

for every $y_1\in V_1$ and for every $y_2\in V_2$.

Now, [86] implies that the mapping

$$\Phi_1\colon \mathcal{E}_{\mathbb{R}}(\mathbb{R}_+)\ni h\mapsto h\circ(\,\cdot\,)^2\in\mathcal{E}(\mathbb{R})$$

is an isomorphism onto the set of even functions. Since there is a continuous linear extension operator $\mathcal{E}_{\mathbb{R}}(\mathbb{R}_+)\to\mathcal{E}(\mathbb{R})$, we find a continuous linear mapping $\Phi_2\colon \Phi_1(\mathcal{E}_{\mathbb{R}}(\mathbb{R}_+))\to\mathcal{E}(\mathbb{R})$ such that

$$\Phi_2(h)\circ(\,\cdot\,)^2=h$$

for every even function $h\in\mathcal{E}(\mathbb{R})$. Then, take $\tau\in\mathcal{D}(V_1)$ so that τ equals 1 on a neighbourhood V_1' of 0 in V_1, and define

$$\widetilde{G}_{y_2}\colon V_1\ni t\mapsto \Phi_2(\tau\widetilde{f}_{y_2})(t).$$

Then, $\widetilde{G}_{y_2}(t^2)=\widetilde{g}_{y_2}(t^2)$ for every $t\in V_1'$ and for every $y_2\in V_2$. In addition, the mapping $y_2\mapsto\widetilde{G}_{y_2}$ belongs to $\mathcal{E}(V_2;\mathcal{E}(\mathbb{R}))$, so that there is $\widetilde{G}\in\mathcal{E}(V_2\times\mathbb{R})$ such that $\widetilde{G}(y_2,t)=\widetilde{G}_{y_2}(t)$ for every $y_2\in V_2$ and for every $t\in\mathbb{R}$. Then,

$$g(y_2,\varphi(\psi(0,y_2))+t^2)=\widetilde{G}(y_2,t^2)$$

for every $y_2\in V_2$ and for every $t\in V_1'$. Define

$$G\colon V_2\times\mathbb{R}\ni(y_2,t)\mapsto \widetilde{G}(y_2,t-\varphi(\psi(0,y_2))),$$

so that $G\in\mathcal{E}(V_2\times\mathbb{R})$ and

$$f(x,\varphi(x))=G(x_2,\varphi(x))$$

for every $x\in\psi(V_1'\times V_2)$. The assertion follows. □

Theorem 7.39. *Assume that* $\mathrm{Card}(H)=1$, $d=0$, *and* $W=\{0\}$, *and let S be the analytic hypersurface* $\{\,\omega\in\mathfrak{g}_2^*\colon \mu_\omega(\mathbf{n}_{1,\omega})=1\,\}$. *Take* $n_2'\in\{0,\dots,n_2-1\}$ *and define* $\mathcal{L}_{A'}:=(\mathcal{L},(-iT_1,\dots,-iT_{n_2'}))$. *Assume that the following hold:*

- $\beta_{\mathcal{L}_{A'}}$ *is a measure with base* $\mathcal{H}^{n_2'+1}$*;*
- *if* $\omega\in S$ *and* $\langle T_1,\dots,T_{n_2'}\rangle^\circ\subseteq T_\omega(S)$, *then the Gaussian curvature of S at ω is non-zero.*

Then, $\mathcal{L}_{A'}$ *satisfies property* (S).

The hypotheses are satisfied if, for example, $\omega\mapsto\mu_\omega(\mathbf{n}_{1,\omega})$ is a hilbertian norm. Observe, in addition, that the Gaussian curvature of S vanishes on a negligible set in virtue of the strict convexity of the norm $\omega\mapsto\mu_\omega(\mathbf{n}_{1,\omega})$. Therefore, for almost every $(T_1,\dots,T_{n_2'})$ the family $(\mathcal{L},(-iT_1,\dots,-iT_{n_2'}))$ satisfies property (S).

Proof. **1.** Assume first that $n_2'=n_2-1$ and that $\mathrm{d}\pi_1(\mathcal{L}_{A'})$ satisfies property (S), where π_1 is the canonical projection of G onto its quotient by its normal subgroup $\exp_G(\mathbb{R}T_1)$.

Take $\varphi\in\mathcal{S}_{\mathcal{L}_{A'}}(G)$. Then, Proposition 7.28 and Lemma 7.35 imply that we may find a family $(\widetilde{\varphi}_\gamma)_{\gamma\in\mathbb{N}^{n_2'}}$ of elements of $\mathcal{S}(G,\mathcal{L})$, and a family $(\varphi_\gamma)_{\gamma\in\mathbb{N}^{n_2'}}$ of elements of $\mathcal{S}_{\mathcal{L}_{A'}}(G)$ such that

$$\varphi=\sum_{|\gamma|<h}\mathbf{T}^\gamma\widetilde{\varphi}_\gamma+\sum_{|\gamma|=h}\mathbf{T}^\gamma\varphi_\gamma$$

for every $h\in\mathbb{N}$.

Define $\widetilde{m}_\gamma \coloneqq \mathcal{M}_{\mathcal{L}}(\widetilde{\varphi}_\gamma)\in\mathcal{S}(\sigma(\mathcal{L}))$ and $m_\gamma\coloneqq \mathcal{M}_{\mathcal{L}_{A'}}(\varphi_\gamma)\in C_0(\sigma(\mathcal{L}_{A'}))$ for every γ. Then,

$$m_0(\lambda,\omega')=\sum_{\lvert\gamma\rvert<h}\omega'^\gamma\widetilde{m}_\gamma(\lambda)+\sum_{\lvert\gamma\rvert=h}\omega'^\gamma m_\gamma(\lambda,\omega')$$

for every $h\in\mathbb{N}$ and for every $(\lambda,\omega')\in\sigma(\mathcal{L}_{A'})$.

2. As in the proof of Theorem 7.36, we may reduce to the case in which $\widetilde{m}_\gamma=0$ for every γ. Let $L\colon E_{\mathcal{L}_A}\to E_{\mathcal{L}_{A'}}$ be the unique linear mapping such that $L(\mathcal{L}_A)=\mathcal{L}_{A'}$, and denote by L' the mapping $\mathfrak{g}_2^*\ni\omega\mapsto(\omega(T_1),\dots,\omega(T_{n_2'}))\in\mathbb{R}^{n_2'}$, so that

$$L(\{\,r\,\}\times rS(\mathbf{T}))=\{\,r\,\}\times rL'(S)=\left(\{\,r\,\}\times\mathbb{R}^{n_2'}\right)\cap\sigma(\mathcal{L}_{A'})$$

for every $r>0$. Now, define

$$\widetilde{M}(\omega)\coloneqq\int_G\varphi(x,t)e^{-\frac{1}{4}\lvert x_\omega\rvert^2+i\omega(t)}\,\mathrm{d}(x,t)$$

for every $\omega\in\mathfrak{g}_2^*$. Reasoning as in the proof of [4, Lemma 3.1] and taking into account Proposition 7.12, we see that $\widetilde{M}\in\mathcal{S}(\mathfrak{g}_2^*)$ and that $\widetilde{M}$ vanishes of order ∞ at 0. Now, observe that

$$m_0(\mu_\omega(\mathbf{n_{1,\omega}}),L'(\omega))=\widetilde{M}(\omega)$$

for every $\omega\in\mathfrak{g}_2^*$. In addition, $\Sigma\coloneqq\mathbb{R}_+(\{\,1\,\}\times S(\mathbf{T}))$ is a closed semianalytic subset of $E_{\mathcal{L}_A}$ since it is the closure of the graph of an analytic function (defined on $\mathfrak{g}_2^*\setminus\{\,0\,\}$); in addition, L is proper on Σ and $L(\Sigma)=\sigma(\mathcal{L}_{A'})$ is a subanalytic closed convex cone, hence Nash subanalytic. By Theorem 2.29, in order to prove that $m_0\in\mathcal{S}_{E_{\mathcal{L}_{A'}},0}(\sigma(\mathcal{L}_{A'}))$ it suffices to show that $\widetilde{M}$ is a formal composite of L'. Now, the assertion is clear at 0 since $\widetilde{M}$ vanishes of order ∞ at 0. Then, take $\omega\in S$. If $\ker L\not\subseteq T_{\omega(\mathbf{T})}(S(\mathbf{T}))$, then L' is a local diffeomorphism at ω, so that the assertion follows in this case. Otherwise, as in the proof of Lemma 7.27 we see that $L'^{-1}(L'(\omega))=\{\,\omega\,\}$, so that the assertion follows from Corollary 7.38. By homogeneity, the assertion follows for every $\omega\neq0$. Therefore, $m_0\in\mathcal{S}_{E_{\mathcal{L}_{A'}},0}(\sigma(\mathcal{L}_{A'}))$, whence the result in this case.

3. Now, let us prove the statement for $n_2'=n_2-1$ by induction on n_2. Observe that the assertion follows from Theorem 6.7 when $n_2=0$. Then, assume that $n_2>0$ and take $\mathcal{L}_{A'}$ as in the statement. Arguing as in the proof of Corollary 7.29, we see that, since $\beta_{\mathcal{L}_{A'}}$ is a measure with base $\mathcal{H}^{n_2'+1}$, $X^\circ\not\subseteq\ker\mu'_\omega(\mathbf{n_1}+2\gamma)$ for every $\gamma\in\mathbb{N}^{\overline{h}}$ and for almost every $\omega\in\mathfrak{g}_2^*$, where $X=\langle T_1,\dots,T_{n_2'}\rangle$. Now, taking the quotient of G by $\exp_G(Y)$, for some vector subspace Y of X, corresponds to restricting $\mathfrak{g}_2^*$ to $Y^\circ\supseteq X^\circ$. Therefore, for almost every line Y in X we have $Y^\circ\cap\Omega\neq\emptyset$ and $X^\circ\not\subseteq\ker\mu'_\omega(\mathbf{n_1}+2\gamma)$ for every $\gamma\in\mathbb{N}^{\overline{h}}$ and for almost every $\omega\in Y^\circ\cap\Omega$. Hence, we may find a basis $T'_1,\dots,T'_{n_2'}$ of X such that $\beta_{\mathrm{d}\pi_1(\mathcal{L}_{A'})}$ is a measure with base $\mathcal{H}^{n_2'}$. Then, **1** and **2** imply that $\beta_{\mathcal{L}_{A'}}$ satisfies property (S).

4. Finally, let us prove the assertion for $n_2'<n_2-1$. Keep the notation of **3** above, and observe that we may find a hyperplane X' of $\mathfrak{g}_2^*$ which contains X such that $X'^\circ\not\subseteq\ker\mu'_\omega(\mathbf{n_1}+2\gamma)$ for every $\gamma\in\mathbb{N}^{\overline{h}}$ and for almost every $\omega\in\mathfrak{g}_2^*$. Therefore, **3** above implies that the family $(\mathcal{L},(-iT_1,\dots,-iT_{n_2'},-iT'_{n_2'+1},\dots,-iT'_{n_2-1}))$ satisfies property (S), so that the assertion follows by means of Corollary 2.30. ☐

Theorem 7.40. *Assume that $d>0$ and that $W=\{\,0\,\}$. Take $n_2'\in\{\,0,\dots,n_2\,\}$. Then, $(\mathcal{L},(-iT_k)_{k=1}^{n_2'})$ satisfies property (S).*

Proof. Notice that Theorems 7.20 and 7.22 imply that $(\mathcal{L},(-iT_k)_{k=1}^{n_2'})$ satisfies property (RL). Therefore, by means of Corollary 2.30 we see that it will suffice to prove the assertion for $n_2'=n_2$. We proceed by induction on $n_2\geqslant1$.

1. Observe first that the inductive hypothesis, Theorems 6.7, and Lemma 7.35 imply that we may find a family $(\widetilde{\varphi}_\gamma)$ of elements of $\mathcal{S}(G,\mathcal{L})$, and a family (φ_γ) of elements of $\mathcal{S}_{\mathcal{L}_A}(G)$ such that

$$\varphi=\sum_{\lvert\gamma\rvert<h}\mathbf{T}^\gamma\widetilde{\varphi}_\gamma+\sum_{\lvert\gamma\rvert=h}\mathbf{T}^\gamma\varphi_\gamma$$

for every $h \in \mathbb{N}$.

Define $\widetilde{m}_\gamma \coloneqq \mathcal{M}_{\mathcal{L}}(\widetilde{\varphi}_\gamma) \in \mathcal{S}(\sigma(\mathcal{L}))$ and $m_\gamma \coloneqq \mathcal{M}_{\mathcal{L}_A}(\varphi_\gamma) \in C_0(\beta_{\mathcal{L}_A})$ for every γ. Then,

$$m_0(\lambda, \omega) = \sum_{|\gamma|<h} \omega^\gamma \widetilde{m}_\gamma(\lambda) + \sum_{|\gamma|=h} \omega^\gamma m_\gamma(\lambda, \omega)$$

for every $h \in \mathbb{N}$ and for every $(\lambda, \omega) \in \sigma(\mathcal{L}_A)$. As in the proof of Theorem 7.36, we may reduce to the case in which $\widetilde{m}_\gamma = 0$ for every γ; we shall simply write m instead of m_0.

2. Consider the norm $N \coloneqq \mu(\mathbf{n_1})$ on $\mathfrak{g}_2^*$ and let S be the associated unit sphere. Define $\sigma(\omega) \coloneqq \frac{\omega}{N(\omega)}$ for every $\omega \in \mathfrak{g}_2^* \setminus \{0\}$. Then, the mapping

$$S \ni \omega \mapsto (\pi_\omega)_*(\varphi) \in \mathcal{S}(\mathfrak{g}_1 \oplus \mathbb{R})$$

is of class C^∞. Next, fix $\omega_0 \in S$. It is not hard to see that we may find a dilation-invariant open neighbourhood U of ω_0 and an analytic mapping $\psi\colon U \times (\mathfrak{g}_1 \oplus \mathbb{R}) \to \mathbb{R}^{2n_1} \times \mathbb{R} \times \mathbb{R}^d$ such that, for every $\omega \in U$, $\psi_\omega \coloneqq \psi(\omega, \cdot)$ is an isometry of $\mathfrak{g}_1 \oplus \mathbb{R}$ onto $\mathbb{R}^{2n_1} \times \mathbb{R} \times \mathbb{R}^d$ such that $\psi_\omega(P_{0,\omega}(\mathfrak{g}_1)) = \{0\} \times \mathbb{R}^d$ and $\psi_\omega(\{0\} \times \mathbb{R}) = \{0\} \times \mathbb{R} \times \{0\}$. Take $\omega \in U$. By transport of structure, we may put on $\mathbb{R}^{2n_1} \times \mathbb{R}$ a group structure for which $\mathbb{R}^{2n_1} \times \mathbb{R}$ is isomorphic to $\mathbb{H}^{n_1}$ and which turns ψ_ω into an isomorphism of Lie groups.[8] Then, there is a sub-Laplacian $\mathcal{L}'_\omega$ on $\mathbb{R}^{2n_1} \times \mathbb{R}$ such that, if T denotes the derivative along $\{0\} \times \mathbb{R} \subseteq \mathbb{R}^{2n_1} \times \mathbb{R}$ and Δ is the standard (positive) Laplacian on $\mathbb{R}^d$, then

$$\mathrm{d}(\psi_\omega \circ \pi_\omega)(\mathcal{L}_A) = (\mathcal{L}'_\omega + \Delta, \omega(\mathbf{T})T).$$

Therefore, Theorem 4.1 and Corollary 4.10 imply that

$$(\psi_\omega \circ \pi_\omega)_*(\varphi_\gamma)((y,t), \cdot) \in \mathcal{S}_\Delta(\mathbb{R}^d)$$

for every $(y,t) \in \mathbb{R}^{2n_1} \times \mathbb{R}$ and for every $\gamma \in \mathbb{N}^{n_2}$. Define

$$\widehat{\varphi}_\gamma\colon (U \cap S) \times \mathbb{R}_+ \times (\mathbb{R}^{2n_1} \times \mathbb{R}) \ni (\omega, \xi, (y,t)) \mapsto \mathcal{M}_\Delta((\psi_\omega \circ \pi_\omega)_*(\varphi_\gamma)((y,t), \cdot))(\xi),$$

so that $\widehat{\varphi}_\gamma(\omega, \cdot, (y,t)) \in \mathcal{S}(\mathbb{R}_+)$ for every $\omega \in U \cap S$ and for every $(y,t) \in \mathbb{R}^{2n_1} \times \mathbb{R}$ thanks to Theorem 6.7. In addition, the mapping

$$\omega \mapsto [(y,t) \mapsto (\psi_\omega \circ \pi_\omega)_*(\varphi_\gamma)((y,t), \cdot)]$$

belongs to $\mathcal{E}(S \cap U; \mathcal{S}(\mathbb{R}^{2n_1} \times \mathbb{R}; \mathcal{S}_\Delta(\mathbb{R}^d)))$, so that the mapping

$$\omega \mapsto [(y,t) \mapsto \widehat{\varphi}_\gamma(\omega, \cdot, (y,t))]$$

belongs to $\mathcal{E}(S \cap U; \mathcal{S}(\mathbb{R}^{2n_1} \times \mathbb{R}; \mathcal{S}_\mathbb{R}(\mathbb{R}_+)))$. Now, observe that the mapping

$$U \ni \omega \mapsto \psi_\omega^{-1} \in \mathcal{L}(\mathbb{R}^{2n_1} \times \mathbb{R} \times \mathbb{R}^d; \mathfrak{g}_1 \oplus \mathbb{R})$$

is of class C^∞, so that also the mapping

$$f\colon U \times \mathbb{R}^{n_1} \ni (\omega, y) \mapsto |(\psi_{\sigma(\omega)}^{-1}(y,0,0))_\omega|^2$$

is of class C^∞, thanks to Proposition 7.12. In addition, by means of Proposition 7.16 we see that

$$m_\gamma(\xi + N(\omega), \omega(\mathbf{T})) = \int_{\mathbb{R}^{2n_1} \times \mathbb{R}} \widehat{\varphi}_\gamma(\sigma(\omega), \xi, (y,t)) e^{-\frac{1}{4} f(\omega,y) + iN(\omega)t} \,\mathrm{d}(y,t)$$

for every $\gamma \in \mathbb{N}^{n_2}$, for every $\omega \in U$ and for every $\xi \geqslant 0$. Therefore, the preceding arguments and some integrations by parts show that

$$\begin{aligned} m(\xi + N(\omega), \omega(\mathbf{T})) &= \sum_{|\gamma|=h} \sigma(\omega(\mathbf{T}))^\gamma \int_{\mathbb{H}^{n_1}} T^h \widehat{\varphi}_\gamma(\sigma(\omega), \xi, (y,t)) e^{-\frac{1}{4} f(\omega,y) + iN(\omega)t} \,\mathrm{d}(y,t) \\ &= \sum_{|\gamma|=h} (-i\omega(\mathbf{T}))^\gamma \int_{\mathbb{H}^{n_1}} \widehat{\varphi}_\gamma(\sigma(\omega), \xi, (y,t)) e^{-\frac{1}{4} f(\omega,y) + iN(\omega)t} \,\mathrm{d}(y,t) \end{aligned}$$

[8]Obviously, this structure depends on ω.

for every $h \in \mathbb{N}$, for every $\omega \in U$ and for every $\xi \geqslant 0$. Now, fix $p_1, p_2, p_3 \in \mathbb{N}$, and take $h \in \mathbb{N}$. Apply Faà di Bruno's formula and integrate by parts p_3 times in the t variable. Then, we see that there is a constant $C > 0$ such that

$$|(\partial_1^{p_1}\partial_2^{p_2} m)(\xi, \omega(\mathbf{T}))| \leqslant C N(\omega)^{h-p_2-p_3}(1+N(\omega))^{p_2} \int_{\mathbb{H}^{n_1}} (1+|(y,t)|)^{2p_2} \times \\ \times \max_{\substack{|\gamma|=h \\ q_2+q_3=0,\dots,p_2}} |\partial_1^{q_2}\partial_2^{p_1+q_3}\partial_{(3,2)}^{p_3}\widehat{\varphi}_\gamma(\sigma(\omega), \xi - N(\omega), (y,t))| \,\mathrm{d}(y,t)$$

for every $(\xi, \omega(\mathbf{T})) \in \sigma(\overset{\circ}{\mathcal{L}}_A) \cap (\mathbb{R} \times U)$. Here, $|(y,t)| = |y| + \sqrt{|t|}$ is a homogeneous norm on $\mathbb{R}^{2n_1} \times \mathbb{R}$.

Now, take a compact subset K of $U \cap S$. Then, the properties of the $\widehat{\varphi}_\gamma$ imply that for every $p_4 \in \mathbb{N}$ there is a constant C' such that

$$|\widehat{\varphi}_\gamma^{(q)}(\omega, \xi, (y,t))| \leqslant \frac{C'}{(1+\xi)^{p_4}(1+|(y,t)|)^{2p_2+2n_1+3}}$$

for every γ with length h, for every $q = 0, \dots, p_1+p_2+p_3$, for every $\omega \in K$, for every $\xi \geqslant 0$ and for every $(y,t) \in \mathbb{R}^{2n_1} \times \mathbb{R}$. Therefore, there is a constant $C'' > 0$ such that

$$|(\partial_1^{p_1}\partial_2^{p_2} m)(\xi, \omega(\mathbf{T}))| \leqslant C'' N(\omega)^{h-p_2-p_3} \frac{(1+N(\omega))^{p_2}}{(1+\xi-N(\omega))^{p_4}}$$

for every $(\xi, \omega(\mathbf{T})) \in \sigma(\overset{\circ}{\mathcal{L}}_A) \cap (\mathbb{R} \times U)$ such that $\sigma(\omega) \in K$. By the arbitrariness of U and K, and by the compactness of S, we see that we may take C'' so that the preceding estimate holds for every $(\xi, \omega(\mathbf{T})) \in \sigma(\overset{\circ}{\mathcal{L}}_A) \cap (\mathbb{R} \times \Omega)$.

Now, taking $h - p_3 > p_2$ we see that $\partial_1^{p_1}\partial_2^{p_2} m$ extends to a continuous function on $\sigma(\mathcal{L}_A)$ which vanishes on $\mathbb{R}_+ \times \{0\}$. If $N(\omega) \leqslant \frac{1}{3}$, then take $h - p_3 = p_2$ and observe that

$$\frac{1}{3} + \xi + N(\omega) \leqslant \frac{2}{3} + \xi \leqslant 1 + \xi - N(\omega)$$

for every $\xi \geqslant N(\omega)$. On the other hand, if $N(\omega) \geqslant \frac{1}{3}$, then take $p_3 = p_4 + h$ and observe that

$$1 + \xi + N(\omega) \leqslant (1+2N(\omega))(1+\xi-N(\omega)) \leqslant 5N(\omega)(1+\xi-N(\omega))$$

for every $\xi \geqslant N(\omega)$. Hence, for every $p_4 \in \mathbb{N}$ we may find a constant $C''' > 0$ such that

$$|(\partial_1^{p_1}\partial_2^{p_2} m)(\xi, \omega(\mathbf{T}))| \leqslant C''' \frac{1}{(1+\xi+N(\omega))^{p_4}}.$$

Then, Proposition 2.12 implies that $m \in \mathcal{S}_{E_{\mathcal{L}_A},0}(\sigma(\mathcal{L}_A))$. □

Theorem 7.41. *Assume that G is the product of a finite family $(G_\eta)_{\eta \in H}$ of 2-step stratified groups which do not satisfy the MW^+ condition; endow each G_η with a sub-Laplacian $\mathcal{L}_\eta$ and assume that $(\mathcal{L}_\eta, i\mathcal{T}_\eta)$ satisfies property (RL) (resp. (S)) for some finite family $\mathcal{T}_\eta$ of elements of the second layer of the Lie algebra of G_η. Define $\mathcal{L} := \sum_{\eta \in H} \mathcal{L}_\eta$ (on G), and let $\mathcal{T}$ be a finite family of elements of the vector space generated by the $\mathcal{T}_\eta$. Then, the family $(\mathcal{L}, -i\mathcal{T})$ satisfies property (RL) (resp. (S)).*

Proof. Observe first that, by means of Propositions 2.21, 2.22, and 2.26, and Corollary 2.30, we may reduce to the case in which $\mathcal{T}$ is the union of the $\mathcal{T}_\eta$. Then, Theorems 4.8 and 4.11 imply that the family $(\mathcal{L}_H, -i\mathcal{T})$ satisfies property (RL) (resp. (S)). Therefore, the assertion follows easily from Propositions 2.21, 2.22, and 2.26, and Corollary 2.30. □

7.5 Examples: H-Type Groups

In this section we deal with the following situation: G is an H-type group and there is a finite family $(\mathfrak{v}_\eta)_{\eta \in H}$ of vector subspaces of $\mathfrak{g}_1$ such that $\mathfrak{v}_\eta \oplus \mathfrak{g}_2$, with the induced structure, is an H-type Lie algebra for every $\eta \in H$, and such that $\mathfrak{v}_{\eta_1}$ and $\mathfrak{v}_{\eta_2}$ commute and are orthogonal for every $\eta_1, \eta_2 \in H$ such that $\eta_1 \neq \eta_2$. We define $\mathbf{n_1} := \left(\frac{1}{2}\dim \mathfrak{v}_\eta\right)_{\eta \in H}$.

We shall then consider, for every $\eta \in H$, the group of linear isometries $O(\mathfrak{v}_\eta)$ of $\mathfrak{v}_\eta$, and define a canonical action of $O := \prod_{\eta \in H} O(\mathfrak{v}_\eta)$ on the *vector space* subjacent to $\mathfrak{g}$ as follows: $(L_\eta)((v_\eta), t) := ((L_\eta \cdot v_\eta), t)$ for every $(L_\eta) \in O$ and for every $((v_\eta), t) \in \mathfrak{g}_1 \oplus \mathfrak{g}_2$.

A projector of $\mathcal{D}'(G)$ is then canonically defined as follows:

$$\pi_*(T) := \int_O (L\,\cdot)_*(T)\,\mathrm{d}\nu_O(L)$$

for every $T \in \mathcal{D}'(G)$; here, ν_O denotes the *normalized* Haar measure on O.

Proposition 7.42. *The following hold:*

1. *π_* induces a continuous projection on $\mathcal{D}'^r(G)$, $\mathcal{S}'(G)$, $\mathcal{E}'^r(G)$, $\mathcal{D}^r(G)$, $\mathcal{S}(G)$, $\mathcal{E}^r(G)$ and $L^p(G)$ for every $r \in \mathbb{N} \cup \{\infty\}$ and for every $p \in [1, \infty]$;*

2. *if $\varphi_1, \varphi_2 \in \mathcal{D}(G)$, then*

$$\langle \pi_*(\varphi_1), \varphi_2\rangle = \langle \varphi_1, \pi_*(\varphi_2)\rangle \qquad \text{and} \qquad \langle \pi_*(\varphi_1)|\varphi_2\rangle = \langle \varphi_1|\pi_*(\varphi_2)\rangle;$$

3. *if μ is a positive measure on G, then also $\pi_*(\mu)$ is a positive measure; in addition, $\pi_*(\nu_G) = \nu_G$;*

4. *if $T \in \mathcal{D}'(G)$ is O-invariant, then also $\check{T}$ is O-invariant;*

5. *if T is supported at e, then $\pi_*(T)$ is supported at e;*

6. *if $\varphi_1, \varphi_2 \in \mathcal{D}(G)$ are O-invariant, then also $\varphi_1 * \varphi_2$ is O-invariant and $\varphi_1 * \varphi_2 = \varphi_2 * \varphi_1$.*

The proof is based on [30] and is omitted.

Now, let $\mathcal{L}_\eta$ be the differential operator corresponding to the restriction of the scalar product to $\mathfrak{v}_\eta^*$; in other words, $\mathcal{L}_\eta$ is minus the sum of the squares of the elements of any orthonormal basis of $\mathfrak{v}_\eta$. Let $T_1, \dots, T_{n_2}$ be an orthonormal basis of $\mathfrak{g}_2$, and define $\mathcal{L}_A := ((\mathcal{L}_\eta)_{\eta \in H}, (-iT_1, \dots, -iT_{n_2}))$.

Proposition 7.43. *$\mathcal{L}_A$ is a Rockland family and generates the unital algebra of left-invariant differential operators which are π-radial.*

Notice that a left-invariant differential operator X is π-radial if and only if $\pi_*(X_e) = X_e$, that is, if and only if X_e is O-invariant. Nevertheless, this does *not* imply that X is O-invariant.

Proof. Since $\sum_{\eta \in H} \mathcal{L}_\eta$ is the operator associated with the scalar product of $\mathfrak{g}_1^*$, it is clear that $\mathcal{L}_A$ is a Rockland family.

Now, take an O-invariant distribution S on G which is supported at e. Let $p\colon G \to G/[G,G]$ be the canonical projection. Then, it is clear that $p(S)$ is O-invariant and supported at $p(e)$. By means of the Fourier transform, we see that there is a unique polynomial $P_0 \in \mathbb{R}[H]$ such that $p(S) = P_0(p(\mathcal{L}_{H,e}))$. Therefore, there are $S_1, \dots, S_{n_2} \in \mathcal{D}'(G)$ such that $\mathrm{Supp}(S_k) \subseteq \{e\}$ for every $k = 1, \dots, n_2$, and such that

$$S = P_0(\mathcal{L}_{H,e}) + \sum_{k=1}^{n_2} T_{k,e} S_k.$$

Reasoning by induction, it follows that S belongs to the unital algebra generated by $\mathcal{L}_{A,e}$. Conversely, it is clear that $T_1, \dots, T_{n_2}$ are π-radial. On the other hand, a direct computation shows that $\mathcal{L}_{\eta,e} = -\sum_{v \in B} (\partial_v^2)_e$, where v is any orthonormal basis of $\mathfrak{v}_\eta$. Hence, $\mathcal{L}_{\eta,e}$ is O-invariant. ☐

Now, we shall consider some image families of $\mathcal{L}_A$. More precisely, we shall fix $\mu \in (\mathbb{R}^H)^{H'}$ so that the induced mapping from $\mathbb{R}^H$ into $\mathbb{R}^{H'}$ is proper on $\mathbb{R}^H_+$. Then, we shall define $L\colon E_{\mathcal{L}_A} \ni (\lambda_1, \lambda_2) \mapsto (\mu(\lambda_1), \lambda_2) \in \mathbb{R}^{H'} \times \mathbb{R}^{n_2}$ and consider the family $L(\mathcal{L}_A)$. Then, $L(\mathcal{L}_A)$ is a Rockland family since L is proper on $\sigma(\mathcal{L}_A)$ by construction.

Proposition 7.44. *Set* $r \coloneqq \dim_{\mathbb{Q}} \mu(\mathbb{Q}^H)$. *Then, there are a* $\beta_{L(\mathcal{L}_A)}$*-measurable function* $m\colon E_{L(\mathcal{L}_A)} \to \mathbb{C}^r$ *and a linear mapping* $L'\colon \mathbb{R}^r \to \mathbb{C}^{H'}$ *such that the following hold:*

- *there is* $\mu' \in ((\mathbb{Q}_+^*)^H)^r$ *such that* $m(\mathcal{L}_A) = \mu'(\mathcal{L}_H)$;
- $(L'(m(\mathcal{L}_A)), (-iT_j)_{j=1}^{n_2}) = L(\mathcal{L}_A)$;
- *m equals* $\beta_{L(\mathcal{L}_A)}$*-almost everywhere a continuous function if and only if* $r = \dim_{\mathbb{R}} \mu(\mathbb{R}^H)$.

Proof. Indeed, we may find r linearly independent $\mathbb{Q}$-linear functionals $p_1, \dots, p_r$ on $\mu(\mathbb{Q}^H)$. Let $\mu'_1, \dots, \mu'_r$ be the elements of $\mathbb{Q}^H$ associated with $p_1 \circ \mu, \dots, p_r \circ \mu$. Then, $\mu'_1, \dots, \mu'_r$ are linearly independent over $\mathbb{Q}$, hence over $\mathbb{C}$ by tensorization. Now, define $\mathcal{L}''_h \coloneqq \sum_{\eta \in H} \mu'_{h,\eta} \mathcal{L}_\eta$, so that the family $(\mathcal{L}''_1, \dots, \mathcal{L}''_r)$ is linearly independent over $\mathbb{C}$. Next, take $h \in \{1, \dots, r\}$, and observe that, if $\lambda \in \mathbb{R}^{n_2} \setminus \{0\}$ and $\gamma_1, \gamma_2 \in \mathbb{N}^H$ are such that

$$(|\lambda| \mu(\mathbf{n_1} + 2\gamma_1), \lambda) = (|\lambda| \mu(\mathbf{n_1} + 2\gamma_2), \lambda),$$

then $\mu(\gamma_1 - \gamma_2) = 0$, so that

$$(|\lambda| \mu'(\mathbf{n_1} + 2\gamma_1), \lambda) = (|\lambda| \mu'(\mathbf{n_1} + 2\gamma_2), \lambda).$$

Hence, there is a $\beta_{\mathcal{L}_A}$-measurable function $m\colon E_{L(\mathcal{L}_A)} \to \mathbb{C}^r$ such that

$$m_h(L(\lambda')) = \mu'_h(\lambda')$$

for every $\lambda' \in \sigma(\mathcal{L}_A) \cap (\mathbb{R}^H \times (\mathbb{R}^{n_2} \setminus \{0\}))$; hence, $\mathcal{L}''_h \delta_e = \mathcal{K}_{\mathcal{L}_A}(m_h)$ for every $h = 1, \dots, r$. Next, observe that for every $\eta' \in H'$ there is $(L'_{\eta',1}, \dots, L'_{\eta',r}) \in \mathbb{Q}^r$ such that

$$\sum_{h=1}^{r} L'_{\eta',h}(p_h \circ \mu) = \mu_{\eta'}$$

on $\mathbb{Q}^H$. Therefore, $\sum_{h=1}^r L'_{\eta',h} \mu'_h = \mu_{\eta'}$, whence $(L'(m(\mathcal{L}_A)), (-iT_j)_{j=1}^{n_2}) = L(\mathcal{L}_A)$.

Now, if $r = \dim_{\mathbb{R}} \mu(\mathbb{R}^H)$, then $m \times I_{\mathbb{R}^{n_2}}$ is clearly a homeomorphism of $\sigma(L(\mathcal{L}_A))$ onto $\sigma(\mathcal{L}''_1, \dots, \mathcal{L}''_r, (-iT_h)_{h=1}^{n_2})$. Conversely, assume that m can be taken so as to be continuous. Then, clearly $m \times I_{\mathbb{R}^{n_2}}$ and $L' \times I_{\mathbb{R}^{n_2}}$ are inverse of one another between $\sigma(L(\mathcal{L}_A))$ and $\sigma(\mathcal{L}''_1, \dots, \mathcal{L}''_r, (-iT_h)_{h=1}^{n_2})$. In particular, L' induces a homeomorphism of $\mu'(\mathbb{R}_+^H)$ onto $\mu(\mathbb{R}_+^H)$, so that these two cones must have the same dimension. Hence, $r = \dim_{\mathbb{R}}(\mu(\mathbb{R}^H))$. ☐

Theorem 7.45. *The following conditions are equivalent:*

(i) $\chi_{L(\mathcal{L}_A)}$ *has a continuous representative;*

(ii) $L(\mathcal{L}_A)$ *satisfies property (RL);*

(iii) every element of $\mathcal{S}_{L(\mathcal{L}_A)}(G)$ *has a continuous multiplier;*

(iv) $L(\mathcal{L}_A)$ *satisfies property (S);*

(v) $L(\mathcal{L}_A)$ *is complete;*

(vi) $\dim_{\mathbb{Q}} \mu(\mathbb{Q}^H) = \dim_{\mathbb{R}} \mu(\mathbb{R}^H)$.

If, in addition, $L(\mathcal{L}_A)$ *is not complete, then there is* L'*, corresponding to some* $\mu' \in (\mathbb{R}^H)^{H''}$*, such that* $L'(\mathcal{L}_A)$ *is complete and functionally equivalent to* $L(\mathcal{L}_A)$.

Proof. **(i) $\Longrightarrow$ (ii).** Obvious.

(ii) $\Longrightarrow$ (iii). Obvious.

(iii) $\Longrightarrow$ (vi). Assume, on the contrary, that $\dim_{\mathbb{Q}} \mu(\mathbb{Q}^H) > \dim_{\mathbb{R}} \mu(\mathbb{R}^H)$, and keep the notation of Proposition 7.44. Then, m_q cannot be taken so as to be continuous for some $q \in \{1, \dots, r\}$. Take $\varphi \in \mathcal{S}(E_{L(\mathcal{L}_A)})$ so that $\varphi(\lambda) \neq 0$ for every $\lambda \in E_{\mathcal{L}_A}$. Then,

$$\mathcal{K}_{L(\mathcal{L}_A)}(m_q \varphi) = \mu'_q(\mathcal{L}_H) \mathcal{K}_{L(\mathcal{L}_A)}(\varphi) \in \mathcal{S}(G),$$

but $m_q \varphi$ is not equal $\beta_{L(\mathcal{L}_A)}$-almost everywhere to any continuous functions, whence the result.

(vi) $\Longrightarrow$ (iv). This follows from Theorem 7.36.

(iv) $\Longrightarrow$ (v). This follows from Proposition 4.21.

(v) $\Longrightarrow$ (vi). This follows from Proposition 7.44.

(vi) $\Longrightarrow$ (i). This follows from Theorem 7.24. ☐

7.6 Examples: Products of Heisenberg Groups

In this section, $(G_\alpha)_{\alpha\in A}$ denotes a family of Heisenberg groups each of which is endowed with a sub-Laplacian $\mathcal{L}_\alpha$. Define $\mathcal{L} \coloneqq \sum_{\alpha\in A}\mathcal{L}_\alpha$, and denote by $\mathcal{T}$ a finite family of elements of $\mathfrak{g}_2$, which is the centre of the Lie algebra of $G \coloneqq \prod_{\alpha\in A} G_\alpha$.

Before we proceed to the main results of these section, let us introduce some more notation. For every $\alpha \in A$, we denote by T_α a basis of the centre of the Lie algebra of G_α, so that we may identify canonically $\mathfrak{g}_2$ with $\bigoplus_{\alpha\in A}\mathbb{R}T_\alpha$. Then, there is a basis $(X_{\alpha,1},\dots,X_{\alpha,2n_{1,\alpha}},T_\alpha)$ of the Lie algebra of G_α such that $[X_{\alpha,k}, X_{\alpha,n_{1,\alpha}+k}] = T_\alpha$ for every $k = 1,\dots,n_{1,\alpha}$, while the other commutators vanish, and such that there is $\mu_\alpha \in (\mathbb{R}_+^*)^{n_{1,\alpha}}$ such that

$$\mathcal{L}_\alpha = -\sum_{k=1}^{n_{1,\alpha}} \mu_{\alpha,k}(X_{\alpha,k}^2 + X_{\alpha,n_{1,\alpha}+k}^2).$$

We shall denote by $\mathfrak{g}_{1,\alpha}$ the vector space generated by $X_{\alpha,1},\dots,X_{\alpha,2n_{1,\alpha}}$, and we shall set $\mathbf{n_1} \coloneqq (n_{1,\alpha})_{\alpha\in A}$.

Proposition 7.46. *Assume that* $\mathrm{Card}(A) \geqslant 2$. *If* $\mathcal{T}$ *generates* $\mathfrak{g}_2$, *then the families* $(\mathcal{L}, -i\mathcal{T})$ *and* $(\mathcal{L}_A, -i\mathcal{T})$ *are functionally equivalent. In addition,* $(\mathcal{L}, -i\mathcal{T})$ *does not satisfy properties* (RL) *and* (S).

Proof. The assertion follows from Proposition 7.26 and its proof. □

Lemma 7.47. *Let* μ *be a linear mapping of* $\mathbb{R}^n$ *onto* $\mathbb{R}^m$ *such that* $\ker\mu\cap\mathbb{R}_+^n = \{0\}$. *Define* $\Sigma_0 \coloneqq \mu(\mathbb{R}_+^n)\times\{0\}$, *and*

$$\Sigma \coloneqq \{ (\lambda\mu(\mathbf{1}_n + 2\gamma), \lambda) \colon \lambda > 0, \gamma \in \mathbb{N}^n \} \cup \Sigma_0.$$

If $\varphi \in C^\infty(\mathbb{R}^m\times\mathbb{R})$ *vanishes on* Σ, *then* φ *vanishes of order* ∞ *on* Σ_0.

Proof. Take $x = (\lambda\mu(\mathbf{1}_n + 2\gamma), 0)$ for some $\lambda > 0$ and some $\gamma \in \mathbb{N}^n$. Then, for every $k \in \mathbb{N}$,

$$\left(x_1, \frac{\lambda}{2k+1}\right) = \left(\frac{\lambda}{2k+1}\mu(\mathbf{1}_n + 2((2k+1)\gamma + k\mathbf{1}_n)), \frac{\lambda}{2k+1}\right) \in \Sigma.$$

Therefore, it is easily seen that $\partial_2^h\varphi(x) = 0$ for every $h \in \mathbb{N}$. Since the set

$$\{ (\lambda\mu(\mathbf{1}_n + 2\gamma), 0) \colon \lambda > 0, \gamma \in \mathbb{N}^n \}$$

is dense in Σ_0, it follows that $\partial_2^h\varphi$ vanishes on Σ_0 for every $h \in \mathbb{N}$. Then, observe that, since we assumed that $\mu(\mathbb{R}^n) = \mathbb{R}^m$, the closed convex cone Σ_0 generates $\mathbb{R}^m\times\{0\}$, so that Σ_0 is the closure of its interior in $\mathbb{R}^m\times\{0\}$. The assertion follows easily. □

Theorem 7.48. *Assume that* $\mathrm{Card}(A) \geqslant 2$. *If* $\mathcal{T}$ *does not generate* $\mathfrak{g}_2$, *then the family* $(\mathcal{L}, -i\mathcal{T})$ *satisfies properties* (RL) *and* (S).

Proof. **1.** Let us prove that $(\mathcal{L}, -i\mathcal{T})$ satisfies property (RL). Consider the Rockland family $(\mathcal{L}, -iT_A)$ and take $\alpha \in A$. Take $\omega \in \mathbb{R}^A$, and define

$$C_\gamma \coloneqq \left\{ \left(\sum_{\alpha\in A}|\omega_\alpha|\mu_\alpha(\mathbf{1}_{n_{1,\alpha}} + 2\gamma_\alpha), \omega\right) \colon \omega \in \mathbb{R}^A \right\}$$

for every $\gamma \in \mathbb{N}^{\mathbf{n_1}}$, so that C_0 is the boundary of a convex polyhedron. If $L\colon E_{(\mathcal{L},-iT_A)} \to E_{(\mathcal{L},-i\mathcal{T})}$ is the unique continuous linear mapping such that $L(\mathcal{L}, -iT_A) = (\mathcal{L}, -i\mathcal{T})$, then $\chi_{C_0}\cdot\beta_{(\mathcal{L},-iT_A)}$ is L-connected by Corollary 2.24. Now, define

$$\mathcal{L}'_{A'} \coloneqq ((-X_{\alpha,k}^2 - X_{\alpha,n_{1,\alpha}+k}^2)_{k=1,\dots,n_{1,\alpha}}, -iT_\alpha)_{\alpha\in A},$$

so that $\mathcal{L}'_{A'}$ satisfies properties (RL) and (S) by Theorems 4.8, 4.11, and 7.45. Take $f \in L^1_{(\mathcal{L},-i\mathcal{T})}(G)$, and let $\widetilde{m}$ be its continuous multiplier relative to $\mathcal{L}'_{A'}$ (cf. Theorem 7.45). Then,

$$m_\gamma\colon C_\gamma \ni \left(\sum_{\alpha\in A}|\omega_\alpha|\mu_\alpha(\mathbf{1}_{n_{1,\alpha}} + 2\gamma_\alpha), \omega\right) \mapsto \widetilde{m}((|\omega_\alpha|(\mathbf{1}_{n_{1,\alpha}} + 2\gamma_\alpha), \omega_\alpha)_{\alpha\in A})$$

is a continuous function on C_γ which equals $\mathcal{M}_{(\mathcal{L},-iT_A)}(f)\ \chi_{C_\gamma}\cdot\beta_{(\mathcal{L},-iT_A)}$-almost everywhere. Therefore, $(\mathcal{L},-i\mathcal{T})$ satisfies property (RL) thanks to Theorem 7.30.

2. Assume that $\mathcal{T}$ generates a hyperplane of $\mathfrak{g}_2$, and let us prove that $(\mathcal{L},-i\mathcal{T})$ satisfies property (S). Take $m\in C_0(E_{(\mathcal{L},-i\mathcal{T})})$ such that $\mathcal{K}_{(\mathcal{L},-i\mathcal{T})}(m)\in\mathcal{S}(G)$, and consider the (unique) linear mapping

$$L'\colon E_{\mathcal{L}'_{A'}}\to E_{(\mathcal{L},-i\mathcal{T})}$$

such that $L'(\mathcal{L}'_{A'})=(\mathcal{L},-i\mathcal{T})$. Then, there is $m_0\in\mathcal{S}(E_{\mathcal{L}'_{A'}})$ such that $m\circ L'=m_0$ on $\sigma(\mathcal{L}'_{A'})$. Next, define, for every $\varepsilon\in\{\,-,+\,\}^A$ and for every $\gamma\in\mathbb{N}^{\mathbf{n_1}}$,

$$S_{\varepsilon,\gamma}:=\left\{\,\big(|\omega_\alpha|(\mathbf{1}_{n_{1,\alpha}}+2\gamma_\alpha),\omega_\alpha\big)_{\alpha\in A}\colon \omega\in\prod_{\alpha\in A}\mathbb{R}_{\varepsilon_\alpha}\,\right\},$$

so that $S_{\varepsilon,\gamma}$ is a closed convex semi-algebraic set of dimension $\mathrm{Card}(A)$. Assume that L' is not one-to-one on $S_{\varepsilon,\gamma}$. Since L' is proper on $\sigma(\mathcal{L}'_{A'})$, for every $\lambda\in L'(S_{\varepsilon,\gamma})$ the fibre $L'^{-1}(\lambda)$ intersects $S_{\varepsilon,\gamma}$ on a closed segment whose end-points lie in the relative boundary of $S_{\varepsilon,\gamma}$. Therefore, $L'(S_{\varepsilon,\gamma})$ gives no contribution to $\bigcup_{\varepsilon'\in\{\,-,+\,\}^A}L'(S_{\varepsilon',\gamma})$; in particular, we may find a subset E_0 of $\{\,-,+\,\}^A$ such that $\bigcup_{\varepsilon\in E_0}L'(S_{\varepsilon,0})=L'(\sigma(\mathcal{L}'_{A'}))$ and such that L' is one-to-one on $S_{\varepsilon,0}$ for every $\varepsilon\in E_0$.

Now, Corollary 2.30 implies that for every $\varepsilon\in E_0$ there is $m'_\varepsilon\in\mathcal{S}(E_{(\mathcal{L},-i\mathcal{T})})$ such that $m'_\varepsilon\circ L'=m_0$ on $S_{\varepsilon,0}$. Nevertheless, we must prove that these functions m'_ε can be patched together to form a Schwartz multiplier of $\mathcal{K}_{(\mathcal{L},-i\mathcal{T})}(m)$. Then, take $\lambda\in\sigma(\mathcal{L},-i\mathcal{T})$. We shall distinguish some cases.

Assume first that there are $\varepsilon_1,\varepsilon_2\in E_0$ such that $L'(S_{\varepsilon_1,0})\cap L'(S_{\varepsilon_2,0})$ has non-empty interior and such that $\lambda\in L'(S_{\varepsilon_1,0})\cap L'(S_{\varepsilon_2,0})$. Then, $m'_{\varepsilon_1}=m'_{\varepsilon_2}$ on $L'(S_{\varepsilon_1,0})\cap L'(S_{\varepsilon_2,0})$, so that $m'_{\varepsilon_1}-m'_{\varepsilon_2}$ vanishes of order infinity on the closure of the interior of $L'(S_{\varepsilon_1,0})\cap L'(S_{\varepsilon_2,0})$, which is $L'(S_{\varepsilon_1,0})\cap L'(S_{\varepsilon_2,0})$ by convexity. In particular, $m'_{\varepsilon_1}-m'_{\varepsilon_2}$ vanishes of order infinity at λ.

Next, assume that there are $\varepsilon_1,\varepsilon_2\in E_0$ and $\lambda'\in S_{\varepsilon_1,0}\cap S_{\varepsilon_2,0}$ such that $L'(\lambda')=\lambda$. Then, $\lambda'\in S_{\varepsilon_k,\gamma}$ for every $\gamma\in\mathbb{N}^{\mathbf{n_1}}$ such that $\gamma_\alpha=0$ for every $\alpha\in A$ such that $\varepsilon_{1,\alpha}=\varepsilon_{2,\alpha}$, and for every $k=1,2$; let $\Gamma_{\varepsilon_1,\varepsilon_2}$ be the set of such γ. Now, clearly $m'_{\varepsilon_k}\circ L'=m_0$ on $S_{\varepsilon_k,\gamma}$ for every $\gamma\in\Gamma_{\varepsilon_1,\varepsilon_2}$. Taking into account Lemma 7.47, we see that the restriction of $(m'_{\varepsilon_1}-m'_{\varepsilon_2})\circ L'$ to $\prod_{\alpha\in A}V_\alpha$ vanishes of order ∞ at λ', where V_α is $\mathbb{R}(\mathbf{1}_{n_{1,\alpha}},\varepsilon_{1,\alpha})$ if $\varepsilon_{1,\alpha}=\varepsilon_{2,\alpha}$ while $V_\alpha=\mathbb{R}^{n_{1,\alpha}+1}$ otherwise. Since either $\varepsilon_1=\varepsilon_2$ or $L'\colon\prod_{\alpha\in A}V_\alpha\to E_{(\mathcal{L},-i\mathcal{T})}$ is onto, it follows that $m'_{\varepsilon_1}-m'_{\varepsilon_2}$ vanishes of order ∞ at λ.

Then, assume that there are $\varepsilon_1,\varepsilon_2\in E_0$ such that $\lambda\in L'(S_{\varepsilon_1,0})\cap L'(S_{\varepsilon_2,0})$, but that $L'(S_{\varepsilon_1,0})\cap L'(S_{\varepsilon_2,0})$ has empty interior and $\lambda\notin L'(S_{\varepsilon_1,0}\cap S_{\varepsilon_2,0})$. Let us prove that there is $\varepsilon_3\in E_0$ such that $\lambda\in L'(S_{\varepsilon_1,0}\cap S_{\varepsilon_3,0})$ and such that $L'(S_{\varepsilon_2,0})\cap L'(S_{\varepsilon_3,0})$ has non-empty interior. Indeed, observe that there is a unique liner mapping L'' such that $L''(\mathcal{L}'_{A'})=(\mathcal{L},-iT_A)$. In addition, if $S_0=\bigcup_{\varepsilon\in\mathbb{N}^A}S_{\varepsilon,0}$, then L'' induces a homeomorphism of S_0 onto $S'_0=L''(S_0)$. In addition, S'_0 is the boundary of the convex envelope C'_0 of $\sigma(\mathcal{L},-iT_A)$, which is a convex polyhedron; further, $\ker L''$ has dimension 1. Next, observe that $L(S'_0)=L(C'_0)=\sigma(\mathcal{L},-i\mathcal{T})$ and that L is proper on C'_0; put an orientation on $\ker L$, fix a linear section ℓ of L, and define

$$g_+\colon\sigma(\mathcal{L},-i\mathcal{T})\ni\lambda\mapsto\max\{\,t\in\ker L\colon \ell(\lambda)+t\in S'_0\,\}$$

and

$$g_-\colon\sigma(\mathcal{L},-i\mathcal{T})\ni\lambda\mapsto\min\{\,t\in\ker L\colon \ell(\lambda)+t\in S'_0\,\}.$$

Then, g_- and g_+ are convex and concave, respectively, hence continuous on the interior of $\sigma(\mathcal{L},-i\mathcal{T})$. Observe that the union of the graphs of g_- and g_+ is $\bigcup_{\varepsilon\in E_0}L''(S_{\varepsilon,0})$. Now, let $E_{0,\pm}$ be the set of $\varepsilon\in E_0$ such that $L''(S_{\varepsilon,0})$ is contained in the graph of $g_\pm$. Observe that E_0 is the disjoint union of $E_{0,-}$ and $E_{0,+}$, since $g_-(\lambda)\neq g_+(\lambda)$ for every λ in the interior of $\sigma(\mathcal{L},-i\mathcal{T})$ (cf. the proof of Corollary 2.24). Therefore, $\sigma(\mathcal{L},-i\mathcal{T})=\bigcup_{\varepsilon\in E_{0,\pm}}L'(S_{\varepsilon,0})$; since $L'(S_{\varepsilon,0})$ is closed for every $\varepsilon\in E_0$ and since E_0 is finite, this proves that the union of the $L'(S_{\varepsilon,0})$ such that $\varepsilon\in E_{0,\pm}$ and $\lambda\in L'(S_{\varepsilon,0})$ is a neighbourhood of λ in $\sigma(\mathcal{L},-i\mathcal{T})$. Next, since $\lambda\notin L'(S_{\varepsilon_1,0}\cap S_{\varepsilon_2,0})$, we may assume that $\varepsilon_1\in E_{0,+}$ and $\varepsilon_2\in E_{0,-}$. Then, there is $\varepsilon_3\in E_{0,+}$ such that $\lambda\in L'(S_{\varepsilon_3,0})$ and $L'(S_{\varepsilon_2,0})\cap L'(S_{\varepsilon_3,0})$ has non-empty interior, so that $\lambda\in L'(S_{\varepsilon_1,0}\cap S_{\varepsilon_2,0})$. Therefore, the preceding arguments show that $m'_{\varepsilon_2}-m'_{\varepsilon_1}$ vanishes of order ∞ at λ.

Hence, by means of Theorem 2.29 we see that there is $m' \in \mathcal{S}(E_{(\mathcal{L},-i\mathcal{T})})$ such that $m' \circ L = m_0$ on $\sigma(\mathcal{L}'_{A'})$, so that $m' = m$ on $\sigma(\mathcal{L}, -i\mathcal{T})$, whence the result in this case.

3. Now, consider the general case, and take $m \in C_0(E_{(\mathcal{L},-i\mathcal{T})})$ such that $\mathcal{K}_{(\mathcal{L},-i\mathcal{T})}(m) \in \mathcal{S}(G)$. Take a finite subset $\mathcal{T}'$ of $\mathfrak{g}_2$ which contains $\mathcal{T}$ and generates a hyperplane of $\mathfrak{g}_2$, so that **2** implies that $(\mathcal{L}, -i\mathcal{T}')$ satisfies property (S). Observe that $\sigma(\mathcal{L}, -i\mathcal{T}')$ is a convex semi-algebraic set. Therefore, the assertion follows easily from Corollary 2.30. □

Lemma 7.49. *Let G' and G'' be two non-trivial homogeneous groups, $\mathcal{L}'$ and $\mathcal{L}''$ two positive Rockland operators on G' and G'', respectively. Then, the operator $\mathcal{L}' + \mathcal{L}''$ on $G' \times G''$ satisfies property (S).*

Proof. The assertion is an easy consequence of Theorem 6.7. □

Theorem 7.50. *Let G' be a homogeneous group endowed with a positive Rockland operator $\mathcal{L}'$ which is homogeneous of degree 2. Then, the following hold:*

1. *$(\mathcal{L} + \mathcal{L}', -i\mathcal{T})$ (on $G \times G'$) satisfies property (RL);*

2. *if $\mathcal{L}'$ satisfies property (S), then also $(\mathcal{L} + \mathcal{L}', -i\mathcal{T})$ satisfies property (S).*

Notice that we do *not* require that G' is graded, so that the requirement that $\mathcal{L}'$ has homogeneous degree 2 can be always met up to rescaling the dilations of G'. In addition, if $\mathcal{L}'$ is not positive, then $(\mathcal{L} + \mathcal{L}', -i\mathcal{T})$ is *not* a Rockland family, since the mapping $\sigma(\mathcal{L}, -i\mathcal{T}, \mathcal{L}') \ni (\lambda_1, \lambda_2, \lambda_3) \mapsto (\lambda_1 + \lambda_3, \lambda_2)$ is not proper.

Finally, observe that the first assertion follows from Theorems 7.20 and 7.22 when G' is abelian.

Proof. **1.** Let us prove that $\mathcal{L}_A$ satisfies property (RL). Observe that, if the assertion holds when $\mathcal{T}$ generates $\mathfrak{g}_2$, then it holds in general thanks to Propositions 2.21 and 2.22. Therefore, we may assume that $\mathcal{T}$ is a basis of $\mathfrak{g}_2$.

Define $\mathcal{L}'_{A'} := (((-X_1^2 - X_{1+n_{1,\alpha}}^2, \dots, -X_{n_{1,\alpha}}^2 - X_{2n_{1,\alpha}}^2), -iT_\alpha)_{\alpha \in A}, \mathcal{L}')$, and observe that $\mathcal{L}'_{A'}$ satisfies property (RL) by Theorems 4.8 and 7.45. Define

$$S_0 := \left\{ \left(|\omega_\alpha| \mathbf{1}_{n_{1,\alpha}}, \omega_\alpha\right)_{\alpha \in A} : \omega \in \mathbb{R}^A \right\},$$

so that S_0 is a closed semi-algebraic set of dimension $\mathrm{Card}(A)$. Then, apply Proposition 2.21 with $\beta := \chi_{S_0 \times \mathbb{R}_+} \beta_{\mathcal{L}'_{A'}}$, observing that $L\colon S_0 \times \mathbb{R}_+ \to \sigma(\mathcal{L} + \mathcal{L}', -i\mathcal{T})$ is a proper bijective mapping, hence a homeomorphism. Since $L_*(\beta_{(\mathcal{L}+\mathcal{L}',-i\mathcal{T})})$ is equivalent to $L_*(\beta)$ thanks to Proposition 2.26, the assertion follows.

2. Next, assume that $\mathcal{L}'$ satisfies property (S), and let us prove that $(\mathcal{L} + \mathcal{L}', -i\mathcal{T})$ satisfies property (S). Observe that, if we prove that the assertion holds when $\mathcal{T}$ generates $\mathfrak{g}_2$, then the general case will follow by means of Corollary 2.30. Therefore, we shall assume that $\mathcal{T} = (T_\alpha)_{\alpha \in A}$.

Observe first that $\mathcal{L}'_{A'}$ satisfies property (S) by Theorems 7.45 and 4.11. Then, take $m \in C_0(\sigma(\mathcal{L}_A))$ such that $\mathcal{K}_{\mathcal{L}_A}(m) \in \mathcal{S}(G \times G')$, so that there is $m_1 \in \mathcal{S}(E_{\mathcal{L}'_{A'}})$ such that

$$m \circ L = m_1$$

on $\sigma(\mathcal{L}'_{A'})$. Since $S_0 \times \mathbb{R}_+$ is a closed semi-algebraic set, by Theorem 2.29 it will suffice to show that the class of m_1 in $\mathcal{S}(S_0 \times \mathbb{R}_+)$ is a formal composite of L. Now, this is clear at the points of the form $\left(\sum_{\alpha \in A} |\omega_\alpha| \mu_\alpha(\mathbf{n_{1,\alpha}}) + r, \omega\right)$, where $\omega \in (\mathbb{R}^*)^A$ and $r \geqslant 0$. Arguing by induction on $\mathrm{Card}(A)$ and taking Lemma 7.49 into account, the assertion follows by means of Lemma 7.35. □

Remark 7.51. Let (X, Y, T) be the standard basis of $\mathbb{H}^1$, and let U be the derivative on $\mathbb{R}$. Then, $(-X^2 - Y^2 - U^2, -iT)$ satisfies property (RL) but $\chi_{(-X^2-Y^2-U^2,-iT)}$ has no continuous representatives.

Indeed, the first assertion follows from Theorem 7.50, while the second one is a consequence of Proposition 7.16.

As a complement to Theorem 7.50, we present the following pathological case.

Proposition 7.52. *Let (X, Y, T) be a standard basis of $\mathbb{H}^1$, and let $\mathcal{L}'$ be a positive Rockland operator on a homogeneous group G. Assume that $(\mathcal{L}')$ satisfies property (S) and that $\mathcal{L}'^h$ is homogeneous of degree 2 for some $h \geqslant 2$. Then, the Rockland family $(-X^2 - Y^2 + \mathcal{L}'^h, -iT)$ is complete and satisfies property (RL), but does* not *satisfy property (S).*

Proof. **1.** Define $\mathcal{L} \coloneqq -X^2 - Y^2$. Then, Proposition 7.50 implies that $(\mathcal{L} + \mathcal{L}'^h, -iT)$ is a Rockland family which satisfies the property (RL). Next, take $\varphi \in \mathcal{D}(E_{(\mathcal{L}, -iT, \mathcal{L}')})$ such that φ is supported in $\{ (\lambda'_1, \lambda'_2, \lambda'_3) \colon \lambda'_1 < 3|\lambda'_2| - \lambda'^h_3 \}$ and equals pr_3 on a neighbourhood of $(1, 1, 0)$. Then, clearly

$$m \colon (\lambda_1, \lambda_2) \mapsto \varphi(|\lambda_2|, \lambda_2, \sqrt[h]{\lambda_1 - |\lambda_2|})$$

does not induce an element of $\mathcal{S}_{\sigma(\mathcal{L}_A)}(E_{\mathcal{L}_A})$. On the other hand, $\mathcal{K}_{\mathcal{L}_A}(m) = \mathcal{K}_{(\mathcal{L}, -iT, \mathcal{L}')}(\varphi) \in \mathcal{S}(\mathbb{H}^1 \times \mathbb{R})$, so that $\mathcal{L}_A$ does not satisfy property (S).

2. Now, let us prove that $\mathcal{L}_A$ is complete. Take $m \in C(E_{\mathcal{L}_A})$ such that $\mathcal{K}_{\mathcal{L}_A}(m)$ is supported in $\{ e \}$, and observe that we may assume that m is continuous since $\mathcal{L}_A$ satisfies property (RL). Projecting onto the quotient by $\{ e \} \times \mathbb{R}$, we see that there is a unique polynomial P on $E_{\mathcal{L}_A}$ which coincides with m on $\sigma(\mathcal{L}, -iT)$. On the other hand, the family $(\mathcal{L}, -iT, \mathcal{L}')$ is complete since it satisfies property (S) (cf. Theorem 4.11 and Proposition 4.21). Hence, there is a unique polynomial Q on $E_{(\mathcal{L}, -iT, \mathcal{L}')}$ such that

$$m(\lambda_1 + \lambda_3^h, \lambda_2) = Q(\lambda_1, \lambda_2, \lambda_3)$$

for every $(\lambda_1, \lambda_2, \lambda_3) \in \sigma(\mathcal{L}, -iT, \mathcal{L}')$. Hence,

$$P(\lambda_1 + \lambda_3^h, \lambda_2) = Q(\lambda_1, \lambda_2, \lambda_3)$$

for every $(\lambda_1, \lambda_2, \lambda_3) \in \left\{ \left(k_1|r|, r, \sqrt[h]{k_2|r|}\right) \colon r \in \mathbb{R}, k_1 \in 2\mathbb{N} + 1, k_2 \in 2\mathbb{N} \right\}$. Now, the closure of this latter set in the Zariski topology is $E_{(\mathcal{L}, -iT, \mathcal{L}')}$, so that $m = P$ on $\sigma(\mathcal{L}_A)$. The assertion follows. ☐

7.7 Miscellaneous Examples

7.7.1 The Complexified Heisenberg Group

Here, G denotes $\mathbb{C}^2 \times \mathbb{C}$ with a group law which is the complexification of that of the Heisenberg group $\mathbb{H}^1$. In other words,

$$(x_1, t_1)(x_2, t_2) = \left(x_1 + x_2, t_1 + t_2 + \frac{x_{1,1}x_{2,2} - x_{1,2}x_{2,1}}{2} \right)$$

for every $(x_1, t_1), (x_2, t_2) \in G$. We shall denote by $X_{1,1}, X_{1,2}, X_{2,1}, X_{2,2}, T_1, T_2$ the left-invariant vector fields on G whose exponentials are $(1, 0, 0)$, $(i, 0, 0)$, $(0, 1, 0)$, $(0, i, 0)$, $(0, 0, 1)$, $(0, 0, i)$, respectively. Then,

$$[X_{1,1}, X_{2,1}] = [X_{2,2}, X_{1,2}] = T_1$$
$$[X_{1,2}, X_{2,1}] = [X_{1,1}, X_{2,2}] = T_2,$$

while the other commutators vanish. In particular,

$$[X_{1,1} + iX_{1,2}, X_{2,1} + iX_{2,2}] = 2(T_1 + iT_2).$$

In addition, observe that ${}^t\mathrm{d}_{B_\omega} \mathrm{d}_{B_\omega} = |\omega| I$ for every $\omega \in \mathbb{R}^2$. In other words, G is an H-type group with respect to the scalar product which turns the chosen basis of the Lie algebra of G into an orthonormal one.

Now, if $\mathcal{L}$ is the homogeneous sub-Laplacian associated with a positive, non-degenerate, symmetric bilinear form Q, then the characteristic polynomial of $J_{Q,\omega}$ has the form $\lambda^4 + q(\omega)\lambda^2 + |\omega|^4$, where q is a positive non-degenerate quadratic form on $\mathbb{R}^2$. Hence, $\widetilde{\mu}_{\omega,1} + \widetilde{\mu}_{\omega,2} = \sqrt{q(\omega) + 2|\omega|^2}$ for every $\omega \in \mathbb{R}^2$.

If $T \in \mathfrak{g}_2$, then Theorems 6.7, 7.28, and 7.39 imply that $(\mathcal{L}, -iT)$ satisfies properties (RL) and (S).

Next, take $a, b, c, d > 0$ and assume that

$$\mathcal{L} = -(aX_{1,1}^2 + bX_{1,2}^2 + cX_{2,1}^2 + dX_{2,2}^2).$$

Then, $\widetilde{\mu}_{\omega,1}^2$ and $\widetilde{\mu}_{\omega,2}^2$ are the solutions of the equation

$$\lambda^2 - ((ac+bd)\omega_1^2 + (ad+bc)\omega_2^2)\lambda + abcd|\omega|^4.$$

We have the following cases:

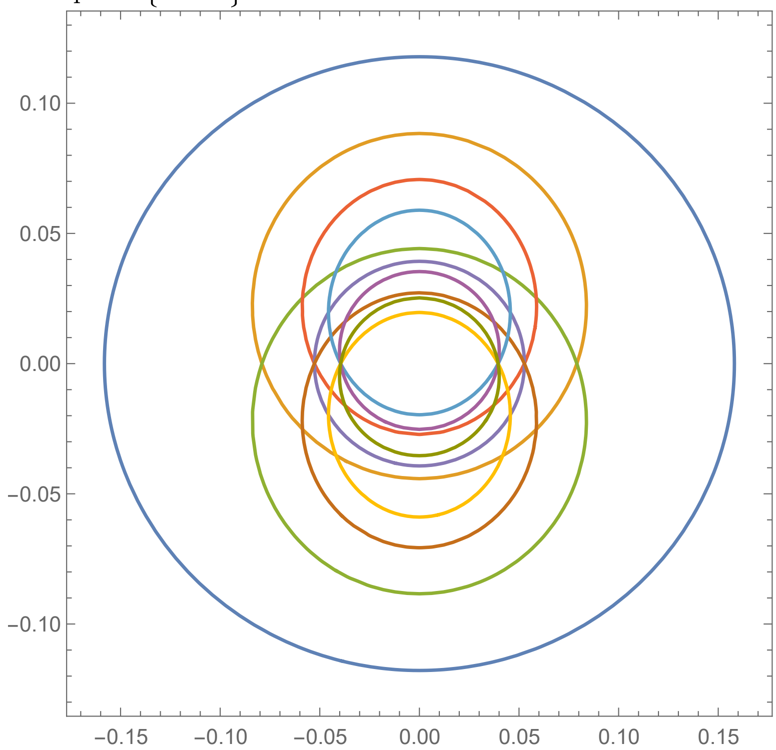
- $a = b$ and $c = d$: then $\widetilde{\mu}_{\omega,1} = \widetilde{\mu}_{\omega,2} = \sqrt{ac}|\omega|$. In addition, $\mathcal{L}$ is the standard sub-Laplacian on G with respect to the scalar product which turns $a^{-\frac{1}{2}}X_{1,1}, a^{-\frac{1}{2}}X_{1,2}, c^{-\frac{1}{2}}X_{2,1}, c^{-\frac{1}{2}}X_{2,2}$, $(ac)^{-\frac{1}{2}}T_1, (ac)^{-\frac{1}{2}}T_2$ into an orthonormal basis;
- $ac = bd$ and $ad \neq bc$ (resp. $ac \neq bd$ and $ad = bc$): then $\Omega = \mathbb{R} \times \mathbb{R}^*$ (resp. $\Omega = \mathbb{R}^* \times \mathbb{R}$);
- $ac \neq bd$ and $ad \neq bc$: then $\Omega = \mathbb{R}^2 \setminus \{ 0 \}$.

It is easily seen that the preceding cases cover all possibilities. In addition, the ratio $\frac{\widetilde{\mu}_{\omega,1}}{\widetilde{\mu}_{\omega,2}}$ is constant if and only if $ac+bd = ad+bc$, that is, $a = b$ or $c = d$. In this case, easy computations show that this ratio is rational if and only if $\sqrt{\frac{a}{b}}, \sqrt{\frac{c}{d}} \in \mathbb{Q}$.

Then, Theorems 7.24, 7.25, and 7.36 imply that the following hold:

- if $a = b$ and $c = d$, then $(\mathcal{L}, -iT_1, -iT_2)$ satisfies property and (S); further $\chi_{(\mathcal{L},-iT_1,-iT_2)}$ has a continuous representative;
- if $\sqrt{\frac{a}{b}}, \sqrt{\frac{c}{d}} \in \mathbb{Q}$ and one of them is 1, then $(\mathcal{L}, -iT_1, -iT_2)$ satisfies properties (RL) and (S).

We conclude this subsection portraying the outer part of the section of $\sigma(\mathcal{L}, -iT_1, -iT_2)$ with the plane $\{ 2^{-1/2} \} \times \mathbb{R}^2$ for $a = d = 1$ and $b = c = 5$.



As the reader may see, the spectrum presents some 'singularities' even in the 'smooth' region Ω, these singularities being generated by the intersections of multiple 'smooth' components. The reader should not think that this kind of 'singularities' do not arise when $\Omega = \mathbb{R}^2 \setminus \{ 0 \}$. Indeed, a moment's reflection shows that these 'multiple points' appear when some relation (over the integers) between the eigenvalues holds somewhere but not everywhere, and this is the case unless $a = b$ or $c = d$, in which case the eigenvalues are radial functions.

In the next example we shall see that more 'singularities' may arise, in the sense that the spectrum *cannot* be decomposed into the union of countably many analytic submanifolds, outside of Ω.

7.7.2 The Quaternionic Heisenberg Group

Here, G denotes $\mathbb{H} \times \mathrm{Im}\mathbb{H}$, where $\mathbb{H}$ is the division ring of quaternions, with the following group law:

$$(x_1, t_1)(x_2, t_2) = (x_1 + x_2, t_1 + t_2 + \mathrm{Im}(x_1 \overline{x_2}))$$

for every $(x_1, t_1), (x_2, t_2) \in G$. We shall denote by $X_1, X_2, X_3, X_4, T_1, T_2, T_3$ the left-invariant vector fields on G whose exponentials are $(1, 0)$, $(i, 0)$, $(j, 0)$, $(k, 0)$, $(0, i)$, $(0, j)$, $(0, k)$, respectively, where i, j, k is the canonical basis of $\mathrm{Im}\mathbb{H}$. Then, with respect to this basis,

$$\mathrm{d}_{B_\omega} = \begin{pmatrix} 0 & -\omega_1 & -\omega_2 & -\omega_3 \\ \omega_1 & 0 & -\omega_3 & \omega_2 \\ \omega_2 & \omega_3 & 0 & -\omega_1 \\ \omega_3 & -\omega_2 & \omega_1 & 0 \end{pmatrix}.$$

In particular, ${}^t\mathrm{d}_{B_\omega} \mathrm{d}_{B_\omega} = |\omega| I$ for every $\omega \in \mathbb{R}^3$. In other words, G is an H-type group with respect to the scalar product which turns the chosen basis of the Lie algebra of G into an orthonormal one.

Now, if $\mathcal{L}$ is the homogeneous sub-Laplacian associated with a positive, non-degenerate, symmetric bilinear form Q, then the characteristic polynomial of $J_{Q,\omega}$ has the form $\lambda^4 + q(\omega)\lambda^2 + |\omega|^4$, where q is a positive non-degenerate quadratic form on $\mathbb{R}^3$. Hence, $\widetilde{\mu}_{\omega,1} + \widetilde{\mu}_{\omega,2} = \sqrt{q(\omega) + 2|\omega|^2}$ for every $\omega \in \mathbb{R}^3$. Therefore, if $T_1', T_2' \in \mathfrak{g}_2$, then Theorems 6.7, 7.28, and 7.39 imply that $(\mathcal{L}, -iT_1', -iT_2')$ satisfies properties (RL) and (S).

Now, assume that

$$\mathcal{L} = -(aX_1^2 + bX_2^2 + cX_3^2 + dX_4^2)$$

for some $a, b, c, d > 0$. Then, $\widetilde{\mu}_{\omega,1}^2$ and $\widetilde{\mu}_{\omega,2}^2$ are the solutions of the equation

$$\lambda^2 + ((ab + cd)\omega_1^2 + (ac + bd)\omega_2^2 + (ad + bc)\omega_3^2)\lambda + abcd|\omega|^4.$$

We have the following cases:

- $a = b = c = d$: then $\widetilde{\mu}_{\omega,1} = \widetilde{\mu}_{\omega,2} = a|\omega|$ and $\mathcal{L}$ is the standard sub-Laplacian on G;
- $a = b \neq c = d$ (resp. $a = c \neq b = d$, $a = d \neq c = d$): then $\Omega = \mathbb{R}^* \times \mathbb{R}^2$ (resp. $\Omega = \mathbb{R} \times \mathbb{R}^* \times \mathbb{R}$, $\mathbb{R}^2 \times \mathbb{R}^*$);
- $ad = bc$ and $a \neq b, c$ (resp. $ac = bd$ and $a \neq b, d$, $ab = cd$ and $a \neq c, d$): then Ω is the complement of $\{0\}^2 \times \mathbb{R}$ (resp. $\{0\} \times \mathbb{R} \times \{0\}$, $\mathbb{R} \times \{0\}^2$) in $\mathbb{R}^3$;
- $ad \neq bc$, $ac \neq bd$, and $ab \neq cd$: then $\Omega = \mathbb{R}^3 \setminus \{0\}$.

It is easily seen that the preceding cases cover all possibilities. In addition, the ratio $\frac{\widetilde{\mu}_{\omega,1}}{\widetilde{\mu}_{\omega,2}}$ is constant if and only if $ab + cd = ac + bd = ad + bc$, that is, at least three among a, b, c, d are equal. In this case, easy computations show that this ratio is rational if and only if the square roots of the ratios of any two elements among a, b, c, d are rational.

Then, Theorems 7.24, 7.25, and 7.36 imply that the following hold:

- if $a = b = c = d$, then $(\mathcal{L}, -iT_1, -iT_2, -iT_3)$ satisfies property (S); in furthermore, $\chi_{(\mathcal{L}, -iT_1, -iT_2, -iT_3)}$ has a continuous representative;
- if, up to a re-ordering of the X_j, $a = b = c$ and $\sqrt{\frac{d}{a}} \in \mathbb{Q}$, then $(\mathcal{L}, -iT_1, -iT_2, -iT_3)$ satisfies properties (RL) and (S).

Observe, finally, that if $ad = bc$ and $a \neq b, c$ (up to a re-ordering of the X_j), then $\widetilde{\mu}$ *cannot* be taken so as to be differentiable at any (fixed) point of $\{0\}^2 \times \mathbb{R}$, as easy computations show.

7.7.3 A Métivier Group which is not of Heisenberg Type

We consider the group constructed in [63, Appendix]. In this case, $G = \mathbb{R}^8 \times \mathbb{R}^2$; if $X_1, \dots, X_8, T_1, T_2$ is the basis of its Lie algebra corresponding to the canonical basis of partial derivatives at the origin, then

$$[X_1, X_5] = [X_2, X_6] = [X_3, X_7] = [X_4, X_8] = T_1$$
$$[X_8, X_1] = [X_2, X_5] = [X_3, X_6] = [X_4, X_7] = T_2,$$

while the other commutators vanish. Now, assume that

$$\mathcal{L} = -(X_1^2 + \cdots + X_8^2).$$

Then, easy computations show that, for every $\omega \in \mathbb{R}^2$, we may take

$$\widetilde{\mu}_{\omega,1} = \sqrt{|\omega|^2 - \sqrt{2}\omega_1\omega_2} \qquad \text{and} \qquad \widetilde{\mu}_{\omega,1} = \sqrt{|\omega|^2 + \sqrt{2}\omega_1\omega_2}.$$

Notice, in particular, that $\widetilde{\mu}_\omega$ is analytic on $\mathbb{R}^2 \setminus \{ 0 \}$ even though $\Omega = \mathbb{R}^* \times \mathbb{R}^*$. In addition,

$$\widetilde{\mu}_{\omega,1} + \widetilde{\mu}_{\omega,2} = \sqrt{2}\sqrt{|\omega|^2 + \sqrt{\omega_1^4 + \omega_2^4}},$$

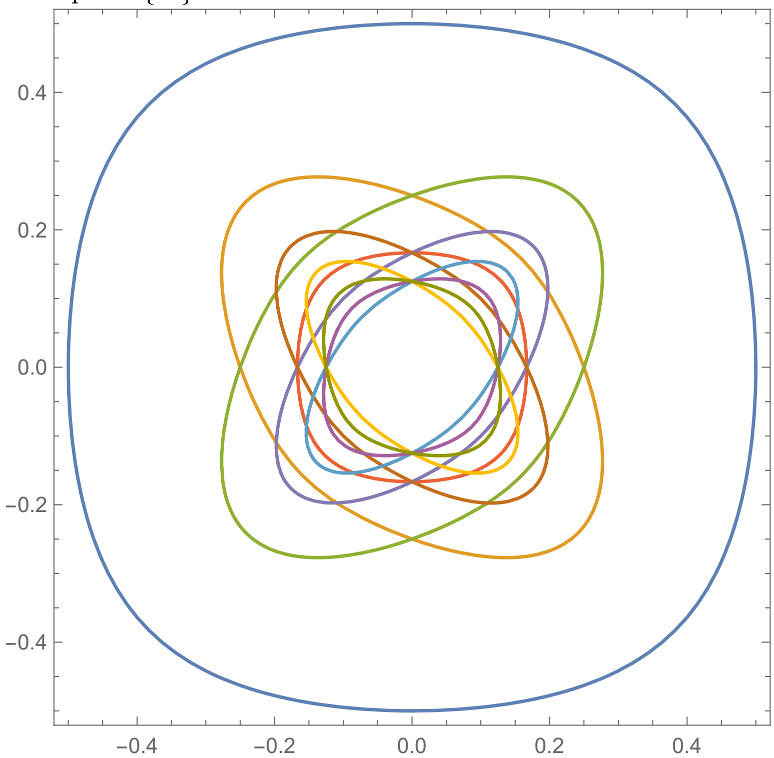
and the curve $\left\{ \omega \in \mathbb{R}^2 \colon |\omega|^2 + \sqrt{\omega_1^4 + \omega_2^4} = 1 \right\}$ has never vanishing curvature. Indeed, observe that

$$\left\{ \omega \in \mathbb{R}^2 \colon |\omega|^2 + \sqrt{\omega_1^4 + \omega_2^4} = 1 \right\} = \left\{ \omega \in \mathbb{R}^2 \colon |\omega| < 1, 2\omega_1^2\omega_2^2 - 2|\omega|^2 + 1 = 0 \right\}.$$

Now, the second derivative of the polynomial $2\omega_1^2\omega_2^2 - 2|\omega|^2 + 1$ is non-degenerate for $|\omega|^2 + 4\omega_1^2\omega_2^2 \neq 1$. It is easily seen that there is no $\omega \in \mathbb{R}^2$ such that $6\omega_1^2\omega_2^2 = 1$ and $3|\omega|^2 = 2$, so that the assertion follows.

Therefore, Theorems 7.28 and 7.39 imply that $(\mathcal{L}, -iT)$ satisfies properties (RL) and (S) for every $T \in \mathfrak{g}_2$.

We finish this subsection with a portrait of the outer part of the section of $\sigma(\mathcal{L}, -iT_1, -iT_2)$ with the plane $\{ 1 \} \times \mathbb{R}^2$:



7.7.4 An 'Irreducible' MW^+-Group which is not a Métivier Group

Assume that $G = \mathbb{R}^4 \times \mathbb{R}^3$; let $X_1, X_2, X_3, X_4, T_1, T_2, T_3$ be a basis of left-invariant vector fields on G which at the origin are the standard basis of partial derivatives of $\mathbb{R}^4 \times \mathbb{R}^3$. Assume that

$$[X_1, X_2] = T_1, \qquad [X_2, X_3] = T_2, \qquad [X_3, X_4] = T_3,$$

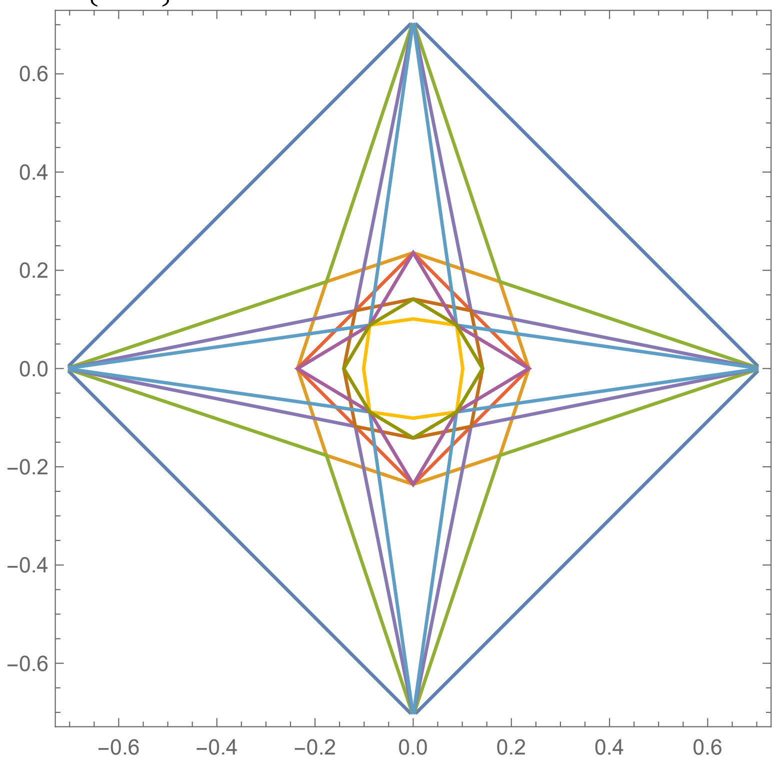
while the other commutators vanish. Then, $d = 0$ and $W = \{ \omega \in \mathbb{R}^3 \colon \omega_1 \omega_3 = 0 \}$, so that G is an MW^+-group but not a Métivier group. In addition, it is not hard to see that G is actually irreducible, in the sense that there are no non-trivial projectors of $\mathfrak{g}_1$ which are B_ω-self-adjoint for every $\omega \in \mathfrak{g}_2^*$.

Then, assume that $\mathcal{L} = -(X_1^2 + X_2^2 + X_3^2 + X_4^2)$. Then,

$$\widetilde{\mu}_{\omega,1} = 2^{-\frac{1}{2}} \sqrt{|\omega|^2 - \sqrt{|\omega|^4 - 4\omega_1^2\omega_3^2}} \qquad \widetilde{\mu}_{\omega,2} = 2^{-\frac{1}{2}} \sqrt{|\omega|^2 + \sqrt{|\omega|^4 - 4\omega_1^2\omega_3^2}}.$$

In particular, $\Omega = \{ \omega \in \mathbb{R}^3 \colon \omega_1\omega_3 \neq 0 \wedge (\omega_2 \neq 0 \vee \omega_1 \neq \omega_3) \}$.

We only portray the outer part of the section of the spectrum $\sigma(\mathcal{L}, -iT_1, -iT_2, -iT_3)$ with the plane $\left\{ 2^{-\frac{1}{2}} \right\} \times \mathbb{R} \times \{ 0 \} \times \mathbb{R}$.



7.8 Supplement: General Sub-Laplacians

In this section we show how the techniques developed so far can be effectively employed to address the study of non-homogeneous operators.

Observe that properties (RL) and (S) can be investigated for more general families of operators; for example, for weighted subcoercive systems of operators on general Lie groups and on Lie groups of polynomial growth, respectively; see [58] for more details on the 'kernel transform' associated with a weighted subcoercive system of operators.

Nonetheless, here we shall refrain from making use of the notion of a weighted subcoercive system of operators as much as possible, trying to rely on the results we proved for homogeneous operators instead.

Proposition 7.53. *Let G be a connected, simply-connected 2-step nilpotent Lie group, and let $\mathcal{L}$ be a (hypoelliptic) sub-Laplacian on G. Take T in the Lie algebra $\mathfrak{g}$ of G and c in $\mathbb{R}$. Then, we may define $\mathcal{K}_{\mathcal{L}+iT+c}$ in the fashion of Definition 3.4.*

Assume that $\mathcal{K}_{\mathcal{L}+iT+c}$ *maps some* continuous *multiplier which does not vanish at any point of* $\sigma(\mathcal{L}+iT+c)$ *into an element of* $L^1(G)$.[9] *Then,* $\mathcal{L}+iT+c$ *satisfies properties* (RL) *and* (S).

With different methods, A. Martini, D. Müller, F. Ricci and L. Tolomeo proved that a quasi-homogeneous sub-Laplacian on a stratified group satisfies properties (RL) and (S) (personal communication).

Proof. **1.** Let us first prove that we may endow G with a suitable stratification so that $\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2$, where $\mathcal{L}_1$ and $\mathcal{L}_2$ are homogeneous sums of squares of homogeneous degree 2 and 4, respectively. Indeed, $\mathcal{L}$ is the differential operator associated with a positive symmetric bilinear form Φ on $\mathfrak{g}^*$. Set $\mathfrak{g}_2 \coloneqq [\mathfrak{g},\mathfrak{g}]$, and let $\pi\colon G \to G/_{[G,G]}$ be the canonical projection. Observe that $\mathrm{d}\pi(\mathcal{L})$ is a homogeneous sum of squares; in addition, it is hypoelliptic thanks to Corollary 1.96. Hence, Φ is non-degenerate on $\mathfrak{g}_2^\circ$ thanks to Proposition 6.2, so that $\mathfrak{g}^*$ is the direct sum of $\mathfrak{g}_2^\circ$ and its Φ-orthogonal complement $(\mathfrak{g}_2^\circ)^\perp$ thanks to [15, Proposition 1 of Chapter 9, § 4, No. 1]; thus, $\mathfrak{g}_1 \coloneqq ((\mathfrak{g}_2^\circ)^\perp)^\circ$ is a direct complement of $\mathfrak{g}_2$ in $\mathfrak{g}$. Then, $\mathcal{L}_2$ is induced by the restriction of Φ to $\mathfrak{g}_1^\circ$, while $\mathcal{L}_1$ is induced by the restriction of Φ to $\mathfrak{g}_2^\circ$.

Now, let us prove that we may assume that $T \in \mathfrak{g}_2$ and $c = 0$. Since $\mathfrak{g}$ is the direct sum of $\mathfrak{g}_1$ and $\mathfrak{g}_2$, there are $T_1' \in \mathfrak{g}_1$ and $T_2' \in \mathfrak{g}_2$ such that $T = T_1' + T_2'$. Now, if $T_1' \neq 0$, that is, if $T \notin \mathfrak{g}_2$, then there is $d > 0$ so that we may complete dT_1' to a basis $dT_1', X_2, \dots, X_{n_1}$ of $\mathfrak{g}_1$ such that the corresponding dual basis of $\mathfrak{g}_2^\circ$ is Φ-orthonormal. Endow $G/_{[G,G]}$ with coordinates corresponding to the basis $\mathrm{d}\pi(T_1'), \mathrm{d}\pi(X_2), \dots, \mathrm{d}\pi(X_{n_1})$, and define $h \coloneqq \mathrm{pr}_1 \circ \pi$. Then, $T_1' h = (\mathrm{d}\pi(T_1')\,\mathrm{pr}_1) \circ \pi = \chi_G$, while $X_2 h = \cdots = X_{n_1} h = T_2' h = 0$. Therefore, if we consider the unitary operator $U\colon f \mapsto e^{\frac{i}{2d^2} h} f$ of $L^2(G)$, we have

$$U^{-1}(\mathcal{L} + iT + c)U = \mathcal{L} + iT_2' + c - \frac{1}{4d^2}.$$

Since $e^{\frac{i}{2d^2} h}$ is a character of G and since U preserves $\mathcal{D}(G)$, we have

$$U^{-1}(\mathcal{K}_{\mathcal{L}+iT+c}) = \mathcal{K}_{\mathcal{L}+iT_2'+c-\frac{1}{4d^2}},$$

in the sense that the left-hand side is defined if and only if the right-hand side is, and then they are equal. Since U^{-1} preserves both $L^1(G)$ and $\mathcal{S}(G)$, we may reduce to proving our assertions for $\mathcal{L} + iT_2' + c - \frac{1}{4d^2}$. In addition, it is also clear that

$$\mathcal{K}_{\mathcal{L}+iT_2'+c-\frac{1}{4d^2}}(m) = \mathcal{K}_{\mathcal{L}+iT_2'}\left(m\left(\,\cdot\, + c - \frac{1}{4d^2}\right)\right),$$

with the same meaning as before. Hence, we may reduce to proving our assertions for $\mathcal{L} + iT_2'$. In other words, from now on we may work under the additional assumptions that $T \in \mathfrak{g}_2$ and that $c = 0$.

Then, choose a basis $(T_1, \dots, T_{n_2})$ of $\mathfrak{g}_2$ with dual basis $(T_1^*, \dots, T_{n_2}^*)$ such that $(T_1^*, \dots, T_{n_2}^*)$ is Φ-orthogonal and $\Phi(T_k^*, T_k^*) = 1$ for every $k = 1, \dots, n_2'$ for some $n_2' \leqslant n_2$, while $\Phi(T_k^*, T_k^*) = 0$ for every $k = n_2'+1, \dots, n_2$. We shall then generally identify $\mathfrak{g}_2^*$ with $\mathbb{R}^{n_2}$ by means of the basis $T_1, \dots, T_{n_2}$ without further comments. Choose $s \in \mathbb{R}^{n_2}$ so that $T = \sum_{k=1}^{n_2} s_j T_j$, and define $P\colon E_{\mathcal{L}_A} \to \mathbb{R}$ so that

$$P(\lambda) \coloneqq \lambda_1 + \sum_{k=1}^{n_2} s_k \lambda_{2,k} + \sum_{k=1}^{n_2'} \lambda_{2,k}^2$$

for every $\lambda \in E_{\mathcal{L}_A}$; thus, $\mathcal{L} + iT = P(\mathcal{L}_A)$.

Now, by means of Theorem 3.19, applied to the right regular representation of G in $L^2(G)$, we see that $\mathcal{L} + iT$ is essentially self-adjoint on $W^{0,\infty,2}(G)$, hence on $\mathcal{D}(G)$. Therefore, $\mathcal{K}_{\mathcal{L}+iT}$ can be defined in the fashion of Definition 3.4, so that the first assertion of the statement holds.

From now on, assume that there is a nowhere vanishing $m \in C(\sigma(\mathcal{L} + iT + c))$ such that $\mathcal{K}_{\mathcal{L}+iT}(m) \in L^1(G)$.[10] Since $\mathcal{K}_{\mathcal{L}_A}(m \circ P) = \mathcal{K}_{\mathcal{L}+iT}(m) \in L^1(G)$, clearly $m \circ P \in C_0(\sigma(\mathcal{L}_A))$.

---

[9] This is the case, for example, if the one-element family $(\mathcal{L} + iT + c)$ is weighted subcoercive, since one may then take the multiplier to be a Gaussian (cf. [58, Proposition 4.2.1]).

[10] Notice that this condition is preserved under the preceding operations.

Arguing as in the proof of Proposition 4.13, we then see that P is proper on $\sigma(\mathcal{L}_A)$;[11] as a consequence, $P_*(\beta_{\mathcal{L}_A})$ is a well-defined Radon measure and plays the role of a Plancherel measure for $\mathcal{L}+iT$, so that we shall denote it by $\beta_{\mathcal{L}+iT}$.

2.a. Assume first that $d=0$; let us prove that $\mathcal{L}+iT$ satisfies property (RL). If $\mathcal{L}_2=0$, then $\mathcal{L}+iT$ is homogeneous, so that $\chi_{\mathcal{L}+iT}$ has a continuous representative and property (RL) follows. Therefore, we may assume that $\mathcal{L}_2\neq 0$.

Define $C\coloneqq \{\,(\mu_\omega\cdot \mathbf{n}_{\mathbf{1},\omega},\omega)\colon \omega\in\mathbb{R}^{n_2}\setminus W\,\}$, and observe that every element of $L^1_{\mathcal{L}_A}(G)$ has a multiplier which is continuous on $\overline{C}$ by Proposition 7.20; let β be the measure induced by $\beta_{\mathcal{L}_A}$ on $\overline{C}$. Define $S\coloneqq \mathrm{pr}_2((\{\,1\,\}\times\mathbb{R}^{n_2})\cap C)$, so that β is the image of a measure equivalent to $\mathcal{H}^{n_2}$ under the mapping $S\times\mathbb{R}\ni(\omega,r)\mapsto(|r|,r\omega)$. Observe that $\overline{S}=\mathrm{pr}_2((\{\,1\,\}\times\mathbb{R}^{n_2})\cap\overline{C})$ and define P' on $\overline{S}\times\mathbb{R}$ by

$$P'(\omega,r)\coloneqq P(|r|,r\omega)=|r|+ra(\omega)+r^2b(\omega)$$

for every $(\omega,r)\in\overline{S}\times\mathbb{R}$, where a and b are suitable continuous functions on $\overline{S}$ which are analytic on S; observe that $b\geqslant 0$. Now, P' is proper, so that the image of P' is a closed interval I of $\mathbb{R}$ containing $\mathbb{R}_+$; let γ be the greatest lower bound of I. Next, observe that $b\neq 0$ since $\mathcal{L}_2\neq 0$, and that, if $b(\omega)\neq 0$, then $P'(\omega,\,\cdot\,)$ is strictly convex, hence a submersion except at a single point. Since b is generically non-zero, P' is generically a submersion.

Let us prove that we may find $\omega\in\overline{S}$ so that the image of $P'(\omega,\,\cdot\,)$ is I. Assume first that $\gamma\neq-\infty$, and observe that there are $\omega\in\overline{S}$ and $r\in\mathbb{R}$ so that $P'(\omega,r)=\gamma$. If $b(\omega)>0$, then it is clear that $P'(\omega,\,\cdot\,)$ has image I. If, otherwise, $b(\omega)=0$, then the image of $P'(\omega,\,\cdot\,)$ is the union of two half lines with origin 0, so that it must be $\mathbb{R}_+$; in particular, $\gamma=0$. Next, assume that $\gamma=-\infty$. Observe that, since P' is proper, $a(\omega)\neq\pm1$ for every $\omega\in\overline{S}$ such that $b(\omega)=0$. If $|a(\omega)|<1$ for every $\omega\in\overline{S}$ such that $b(\omega)=0$, then by compactness there is $\delta\in[0,1[$ such that $|a(\omega)|\leqslant\delta$ for every ω in a neighbourhood U (in $\overline{S}$) of the set of $\omega\in\overline{S}$ such that $b(\omega)=0$. Since b has then a strictly positive minimum on $\overline{S}\setminus U$, it is clear that the image of P' is bounded from below: contradiction. Therefore, there is $\omega\in\overline{S}$ such that $|a(\omega)|>1$ and $b(\omega)=0$, so that the image of $P'(\omega,\,\cdot\,)$ is clearly $\mathbb{R}$.

Now, take $\widetilde{\omega}$ so that he image of $P'(\widetilde{\omega},\,\cdot\,)$ is I and observe that, since $P'(\widetilde{\omega},\,\cdot\,)$ is convex and takes γ at most one point, the mapping $P'(\widetilde{\omega},\,\cdot\,)\colon\mathbb{R}\to I$ is open. Therefore, β is P-connected.

Next, Proposition 2.26 implies that $P_*(\beta)$ is equivalent to $\chi_I\cdot\mathcal{H}^1$, and that β admits a disintegration $(\beta_\xi)_{\xi\geqslant\gamma}$ such that $\mathrm{Supp}(\beta_\xi)\supseteq P^{-1}(\xi)\cap C$ for $\mathcal{H}^1$-almost every $\xi\in I$. Take $\xi\in I$ such that $\xi\neq\gamma,0$, and $(\omega,r)\in\overline{S}\times\mathbb{R}$ such that $P'(\omega,r)=\xi$. Since $P'(\omega,\,\cdot\,)$ is convex, we may find a sequence (ω_k,r_k) of elements of $(S\cap b^{-1}(\mathbb{R}_+^*))\times\mathbb{R}^*$ such that (ω_k,r_k) converges to (ω,r) and $P'(\omega_k,r_k)<\xi$ for every $k\in\mathbb{N}$. Since clearly the image of $P'(\omega_k,\,\cdot\,)$ is an interval containing $\mathbb{R}_+$, we see that there is $r'_k\in\mathbb{R}^*$ such that $P'(\omega_k,r'_k)=\xi$; in addition, we may choose the r'_k so that they converge to r. In other words, the sequence $(|r'_k|,r'_k\omega_k)$ converges to $(|r|,r\omega)$, and its elements belong to $P^{-1}(\xi)\cap C$. By the arbitrariness of ξ and (ω,r), this implies that $\mathrm{Supp}(\beta_\xi)=P^{-1}(\xi)\cap\overline{C}$ for $\mathcal{H}^1$-almost every $\xi\in I$.

A similar argument which employs Ω instead of $\mathbb{R}^{n_2}\setminus W$ shows that $\beta_{\mathcal{L}+iT}=P_*(\beta_{\mathcal{L}_A})$ is equivalent to $\chi_I\cdot\mathcal{H}^1$, hence to $P_*(\beta)$. Therefore, Proposition 2.21 implies that for every $f\in L^1_{\mathcal{L}+iT}(G)$ there are $m_1\in\mathcal{E}^0(\overline{C})$ and $m_2\colon I\to\mathbb{C}$ such that $f=\mathcal{K}_{\mathcal{L}+iT}(m_2)$ and

$$m_2\circ P=m_1$$

on $\overline{C}$; since P is proper on $\overline{C}$, it follows that m_2 is continuous on I. Hence, $\mathcal{L}+iT$ satisfies property (RL).

2.b. Assume that $d>0$ and that $T\in\mathfrak{g}_2$; let us prove that $\mathcal{L}+iT$ satisfies property (RL). Arguing as in **2.a**, we may assume that $\mathcal{L}_2\neq 0$. Define, then,

$$C\coloneqq\{\,(\lambda,\omega)\in E_{\mathcal{L}_A}\colon \omega\notin W,\ \lambda\geqslant\mu_\omega\cdot\mathbf{n}_{\mathbf{1},\omega}\,\}\cup\{\,(\mu_\omega\cdot\mathbf{n}_{\mathbf{1},\omega},\omega)\colon\omega\in\mathbb{R}^{n_2}\,\},$$

so that every element of $L^1_{\mathcal{L}_A}(G)$ has a multiplier which is continuous on C by Proposition 7.20. Now, it is clear that P is a submersion, so that Proposition 2.26 implies that the restriction β of $\beta_{\mathcal{L}_A}$ to C admits a disintegration (β_ξ) relative to P such that $\mathrm{Supp}(\beta_\xi)=P^{-1}(\xi)\cap C$ for $\beta_{\mathcal{L}+iT}=P_*(\beta)$-almost every $\xi\in\mathbb{R}$ (argue as in **2.a**). In addition, arguing as in **2.a** we see that

[11]Conversely, it is not difficult to see that, if P is proper of $\sigma(\mathcal{L}_A)$, then $\mathcal{K}_{\mathcal{L}+iT+c}$ maps some continuous nowhere vanishing multiplier into an element of $\mathcal{S}(G)$.

β is P-connected. Therefore, Proposition 2.21 implies that for every $f \in L^1_{\mathcal{L}+iT}(G)$ there are $m_1 \in \mathcal{E}^0(C)$ and $m_2 \colon I \to \mathbb{C}$ such that $f = \mathcal{K}_{\mathcal{L}+iT}(m_2)$ and

$$m_2 \circ P = m_1$$

on C. Now, $P(C) = P(\partial\sigma(\mathcal{L}_A)) = P(\sigma(\mathcal{L}_A))$ and P is proper on $\partial\sigma(\mathcal{L}_A) \subseteq C$; therefore, m_2 is continuous. Hence, $\mathcal{L} + iT$ satisfies property (RL).

3. Let us prove that $\mathcal{L} + iT$ satisfies property (S). Take $\omega \in \mathbb{R}^{n_2}$ such that the image of $r \mapsto P(\mu_{r\omega} \cdot \mathbf{n}_{\mathbf{1},r\omega}, r\omega)$ is $\sigma(\mathcal{L} + iT)$ (cf. **2.a**), and let G_ω be the quotient of G by $\exp_G(\ker(\sum_{k=1}^{n_2} \omega_k T_k^*))$; let $\pi_\omega \colon G \to G_\omega$ be the canonical projection. Then, $\sigma(\mathcal{L} + iT) = \sigma(\mathrm{d}\pi_\omega(\mathcal{L} + iT))$, so that we may assume that $\dim \mathfrak{g}_2 = 1$, thanks to property (RL) and Theorem 3.19 (applied to $\mathcal{L}_A$).

The assertion follows by means of Propositions 4.16, 7.36, and 7.40 if $\mathcal{L}_2 = 0$. Therefore, assume that $\mathcal{L}_2 \neq 0$. In this case, observe that $\sigma(\mathcal{L}+iT)$ cannot be the whole of $\mathbb{R}$, since we assumed that $\dim \mathfrak{g}_2 = 1$. Now, if $\sigma(\mathcal{L}+iT) = \mathbb{R}_+$, then we may consider the canonical projection $\pi \colon G \to G/[G,G]$ and complete the proof, since $\mathrm{d}\pi(\mathcal{L}+iT)$ is a homogeneous Laplacian. Otherwise, $\sigma(\mathcal{L} + iT) = [\gamma, +\infty[$ for some $\gamma \in \mathbb{R}_-^*$. Then, P maps $L \coloneqq \{ (\lambda\mu_\omega \cdot \mathbf{n}_{\mathbf{1},\omega}, \varepsilon\lambda) \colon \lambda \geqslant 0 \}$ onto $\sigma(\mathcal{L}+iT)$ for some $\varepsilon \in \{ \pm 1 \}$. In addition, by means of the preceding techniques (cf., for example, Theorem 7.40) it is not difficult to see that every element of $L^1_{\mathcal{L}_A}(G)$ has a multiplier whose restriction to L can be extended to an element of $\mathcal{S}(E_{\mathcal{L}_A})$. Since P induces a polynomial of degree 2 on L, [86] leads to the conclusion. □

Chapter 8

The Heat Kernel on H-Type Groups

The results of this chapter are joint work with Tommaso Bruno, cf. [24].

Estimates at infinity for the heat kernel on the Heisenberg group or, more generally, H-type groups have attracted a lot of interest in the last decades (see, e.g., [44, 51, 8, 36, 55, 56]). In the context of H-type groups, in particular, some results were recently obtained by N. Eldredge [36] and H.-Q. Li [56] independently. In [36], Eldredge provides precise upper and lower bounds for the heat kernel p_s and its horizontal gradient $\nabla_{\mathcal{H}} p_s$. In [56], Li provides asymptotic estimates for the heat kernel p_s, as well as upper bounds for all its derivatives. In this chapter, we provide asymptotic expansions at infinity of the heat kernel and of all its derivatives.

We divide this chapter into three sections. In the next section we fix the notation and recall some preliminary facts on the method of stationary phase. In Section 8.2 we provide asymptotic estimates for p_{s,k_1,k_2} in the case $m = 1$, namely when G is a Heisenberg group; in Section 8.3 we extend the results of Section 8.2 to the more general class of H-type groups. This is done via a reduction to the case $m = 1$ when m is odd; a descent method is then applied in order to cover the case in which m is even. As the reader may see, our Theorem 8.24 and Corollary 8.37 cover the cases of [56, Theorems 1.4 and 1.5] and [36, Theorem 4.2] as particular instances, and imply [56, Theorems 1.1 and 1.2] and [36, Theorem 4.4] as easy corollaries.

We emphasize that our methods are strongly related to those employed by B. Gaveau [44] and then H. Hueber and D. Müller [51] in the case of the Heisenberg group $\mathbb{H}^1$; some ideas are also taken from the work of Eldredge [36]. In particular, we borrow from [44] and [51] the use of the method of *stationary phase*, though in a stronger form provided by L. Hörmander [50].

8.1 Preliminaries

8.1.1 The Heat Kernel

Let G be an H-type group, with $\dim \mathfrak{g}_1 = 2n$ and $\dim \mathfrak{g}_2 = m$, and let $\mathcal{L}$ be the sub-Laplacian on G associated with the scalar product on $\mathfrak{g}_1^*$ induced by that on $\mathfrak{g}$. Then, the corresponding heat kernel $(p_s)_{s>0}$ has the form

$$p_s(x,t) = \frac{1}{(4\pi)^n (2\pi)^m s^{n+m}} \int_{\mathbb{R}^m} e^{\frac{i}{s}\langle \lambda | t\rangle - \frac{|x|^2}{4s}|\lambda| \coth(|\lambda|)} \left(\frac{|\lambda|}{\sinh |\lambda|} \right)^n \mathrm{d}\lambda, \tag{1}$$

for every $s > 0$ and every $(x,t) \in G$ (cf. [71] or [88]). For the sake of clarity, we shall sometimes stress the dependence of p_s on m by writing $p_s^{(m)}$ instead of p_s.

We begin by writing the heat kernel (1) in a more convenient form. Let $\mathcal{R}$ be an isometry such that $\mathcal{R}t = |t| u_1$, where u_1 is the first element of the canonical basis[1] of the centre of G, namely $\mathbb{R}^m$. Making the change of variables $\lambda \mapsto \mathcal{R}^{-1}\lambda$ in (1), we get

$$p_s(x,t) = \frac{1}{(4\pi)^n (2\pi)^m s^{n+m}} \int_{\mathbb{R}^m} e^{\frac{i}{s}\langle \lambda | u_1\rangle |t| - \frac{|x|^2}{4s}|\lambda| \coth(|\lambda|)} \left(\frac{|\lambda|}{\sinh |\lambda|} \right)^n \mathrm{d}\lambda. \tag{2}$$

[1] The choice of u_1 is actually irrelevant.

It is now clear that p_s depends only on $|x|$ and $|t|$. This leads us to the following definition.

Definition 8.1. Let $R = \frac{|x|^2}{4}$. For all $s > 0$ and for all $k_1, k_2 \in \mathbb{N}$, define

$$p_{s,k_1,k_2}(x,t) \coloneqq \frac{\partial^{k_1}}{\partial R^{k_1}} \frac{\partial^{k_2}}{\partial |t|^{k_2}} p_s(x,t) = \frac{(-1)^{k_1} i^{k_2}}{(4\pi)^n (2\pi)^m s^{n+m+k_1+k_2}} \times \\ \times \int_{\mathbb{R}^m} e^{\frac{i}{s}|t|\langle\lambda|u_1\rangle - \frac{|x|^2}{4s}|\lambda|\coth|\lambda|} \frac{|\lambda|^{n+k_1} \cosh(|\lambda|)^{k_1}}{\sinh(|\lambda|)^{n+k_1}} \langle\lambda|u_1\rangle^{k_2} \,\mathrm{d}\lambda. \tag{3}$$

Notice that p_s is a smooth function of R and $|t|$ by formula (2), so that the definition of p_{s,k_1,k_2} is meaningful on the whole of G. In addition, consider a differential operator on G of the form

$$X = \frac{\partial^{|\gamma|}}{\partial x^{\gamma_1} \partial t^{\gamma_2}}$$

for some $\gamma = (\gamma_1, \gamma_2) \in \mathbb{N}^{2n} \times \mathbb{N}^m$. By means of Faà di Bruno's formula, the function Xp_s can be written on $\{t \neq 0\}$ as a finite linear combination with smooth coefficients of the functions p_{s,k_1,k_2}, for suitable k_1 and k_2. Since Xp_s is uniformly continuous, the value of $Xp_s(x, 0)$ can then be recovered by continuity uniformly in $x \in \mathbb{R}^{2n}$. Therefore, one can obtain asymptotic estimates for Xp_s by combining appropriately some given estimates of p_{s,k_1,k_2} (see also Remark 8.38).

Observe that it will be sufficient to study p_{1,k_1,k_2}, since

$$p_{s,k_1,k_2}(x,t) = \frac{1}{s^{n+m+k_1+k_2}} p_{1,k_1,k_2}\left(\frac{x}{\sqrt{s}}, \frac{t}{s}\right)$$

for every $s > 0$, k_1, $k_2 \in \mathbb{N}$ and $(x,t) \in G$. Hence, we shall focus only on p_{1,k_1,k_2}. Furthermore, from now on we shall fix the integers $k_1, k_2 \geq 0$. Of course, the choice $k_1 = k_2 = 0$ gives the heat kernel p_s.

Remark 8.2. It is well known (see [35] or [12, Remark 3.6.7]) that there exist n and m for which $\mathbb{R}^{2n} \times \mathbb{R}^m$ cannot represent any H-type group. Nevertheless, (1) and hence (3) make sense for every positive $n, m \in \mathbb{N}$, and for such n and m we shall then study p_{s,k_1,k_2}.

Definition 8.3. (cf. [51]) For every $(x,t) \in G$, define[2]

$$\omega \coloneqq \frac{|t|}{R}, \qquad \delta \coloneqq \sqrt{\frac{R}{\pi|t|}}, \qquad \kappa \coloneqq 2\sqrt{\pi|t|R}.$$

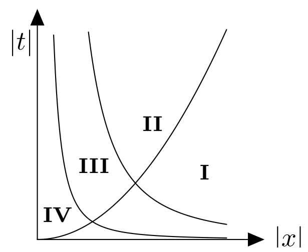
We shall split the asymptotic condition $(x,t) \to \infty$ into four cases, some of which depend on an arbitrary constant $C > 1$. In particular, the first one covers the case in which $|t|/|x|^2$ is bounded, while the other three are a suitable splitting of the case $|t|/|x|^2 \to \infty$.

I. $(x,t) \to \infty$ while $\omega = 4|t|/|x|^2 \leq C$;

II. $\delta \to 0^+$ and $\kappa \to +\infty$;

III. $\delta \to 0^+$ and $\kappa \in [1/C, C]$;

IV. $\kappa \to 0^+$ and $|t| \to +\infty$.



We shall describe the asymptotic behaviour of p_{1,k_1,k_2} in each of these four cases. The first two will both need the method of stationary phase (Theorem 8.7 below), while the other two can be treated through Taylor expansions.

In order to simplify the notation, we give some definitions.

Definition 8.4. Define the function $\theta\colon (-\pi,\pi) \to \mathbb{R}$ by

$$\theta(\lambda) \coloneqq \begin{cases} \frac{2\lambda - \sin(2\lambda)}{2\sin^2(\lambda)}, & \text{if } \lambda \neq 0, \\ 0, & \text{if } \lambda = 0. \end{cases}$$

[2] Actually, ω is defined for $x \neq 0$ and δ for $t \neq 0$, but we shall not recall it again in the following.

Lemma 8.5. [44, § 3, Lemma 3] *θ is an odd, strictly increasing analytic diffeomorphism between $(-\pi, \pi)$ and $\mathbb{R}$.*

Definition 8.6. For every $\omega \in \mathbb{R}$, set $y_\omega := \theta^{-1}(\omega)$. For every $(x,t) \in G$ define

$$d(x,t) := \begin{cases} |x| \frac{y_\omega}{\sin(y_\omega)} & \text{if } x \neq 0 \text{ and } t \neq 0, \\ |x| & \text{if } t = 0, \\ \sqrt{4\pi|t|} & \text{if } x = 0. \end{cases}$$

It is worth observing that $d(x,t)$ is the Carnot-Carathéodory distance between (x,t) and the origin with respect to the horizontal distribution generated by the vector fields $X_1, \dots, X_{2n}$. See [81] but also [72, 8, 36] for a proof and further details.

8.1.2 The Method of Stationary Phase

The main tool that we shall use is an easy corollary of Hörmander's theorem of stationary phase [50, Theorem 7.7.5]. We include a proof for the sake of clarity. If f is a twice differentiable function on an open neighbourhood of 0, we write $P_{2,0}f$ for the Taylor polynomial of order 2 about 0 of f.

Theorem 8.7. *Let V be an open neighbourhood of 0 in $\mathbb{R}^m$, and let $\mathcal{F}$, $\mathcal{G}$ be bounded subsets of $\mathcal{E}(V)$ such that*

1. $\mathrm{Im} f(\lambda) \geq 0$ *for every* $\lambda \in V$ *and every* $f \in \mathcal{F}$*. Moreover, there exist* $\eta > 0$ *and* $c_1 > 0$ *such that* $B(0, 2\eta) \subseteq V$ *and* $\mathrm{Im} f(\lambda) \geq c_1|\lambda|$ *whenever* $|\lambda| \geq \eta$ *and* $f \in \mathcal{F}$*;*
2. $\mathrm{Im} f(0) = f'(0) = 0$ *and* $\det f''(0) \neq 0$ *for all* $f \in \mathcal{F}$*;*
3. *there exists* $c_2 > 0$ *such that* $|f'(\lambda)| \geq c_2|\lambda|$ *for all* $|\lambda| \leq 2\eta$ *and for all* $f \in \mathcal{F}$*;*
4. *there exists* $c_3 > 0$ *such that* $|g(\lambda)| \leq c_3 e^{c_3|\lambda|}$ *whenever* $\lambda \in V$*, for every* $g \in \mathcal{G}$*.*

Then, for every $k \in \mathbb{N}$,

$$\int_V e^{iRf(\lambda)} g(\lambda)\,\mathrm{d}\lambda = e^{iRf(0)} \sqrt{\frac{(2\pi i)^m}{R^m \det f''(0)}} \sum_{j=0}^{k} \frac{L_{j,f}g}{R^j} + O\Big(\frac{1}{R^{\frac{m}{2}+k+1}}\Big) \tag{4}$$

as $R \to +\infty$, uniformly as $f \in \mathcal{F}$ and $g \in \mathcal{G}$, where

$$L_{j,f}g = i^{-j} \sum_{\mu=0}^{2j} \frac{\langle f''(0)^{-1}\partial|\partial\rangle^{\mu+j}[(f - P_{2,0}f)^\mu g](0)}{2^{\mu+j}\mu!(\mu+j)!}.$$

In particular, $L_{0,f}g = g(0)$.

Proof. Take some $\tau \in C_c^\infty(\mathbb{R}^m)$ such that $\chi_{B(0,\eta)} \leq \tau \leq \chi_{B(0,2\eta)}$. Then, split the integral as

$$\int_V e^{iRf(\lambda)} g(\lambda)\,\mathrm{d}\lambda = \int_V e^{iRf(\lambda)} g(\lambda)\tau(\lambda)\,\mathrm{d}\lambda + \int_V e^{iRf(\lambda)} g(\lambda)(1-\tau(\lambda))\,\mathrm{d}\lambda$$

and apply [50, Theorem 7.7.5] to the first term, thanks to the first assumption in 1 and the assumptions 2 and 3: this represents the main contribution to the integral, and gives the right hand side of (4). The second term is instead negligible, since by the second assumption in 1 and by 4 we get, if R is large enough,

$$\begin{aligned} \left|\int_V e^{iRf(\lambda)} g(\lambda)(1-\tau(\lambda))\,\mathrm{d}\lambda\right| &\leq c_3 \int_{|\lambda|\geq\eta} e^{-R\,\mathrm{Im} f(\lambda) + c_3|\lambda|}\,\mathrm{d}\lambda \\ &= c_3\,\omega_{m-1} \int_\eta^\infty e^{-Rc_1\rho + c_3\rho} \rho^{m-1}\,\mathrm{d}\rho \\ &= c_3\,\omega_{m-1} \int_\eta^\infty e^{-(c_1R\rho - (1+c_3)\rho) - \rho} \rho^{m-1}\,\mathrm{d}\rho \\ &\leq c_3\,\omega_{m-1}\, e^{-(c_1R-(1+c_3))\eta} \int_0^\infty e^{-\rho}\rho^{m-1}\,\mathrm{d}\rho, \end{aligned}$$

which is $O\big(e^{-Rc_1\eta}\big)$; here, ω_{m-1} is the measure of the unit sphere in $\mathbb{R}^m$. □

Remark 8.8. Theorem 8.7 covers more cases than only oscillatory integrals. Indeed, assume we have an integral of the form

$$\int_V e^{-Rf(\lambda)} g(\lambda)\,\mathrm{d}\lambda$$

where f is real. Under suitable assumptions, such integrals are usually treated via Laplace's method (see, e.g., [37] and [87]). In this case, one can use directly Theorem 8.7, by substituting $\mathrm{Im} f$ by f in the assumptions 1-4, thus getting

$$\int_V e^{-Rf(\lambda)} g(\lambda)\,\mathrm{d}\lambda = \sqrt{\frac{(2\pi)^m}{R^m \det f''(0)}} \sum_{j=0}^{k} \frac{L_{j,f} g}{R^j} + O\left(\frac{1}{R^{\frac{m}{2}+k+1}}\right), \tag{5}$$

with the obvious modifications on $L_{j,f} g$. In such cases, Theorem 8.7 will be referred to as *Laplace's method*.

8.2 Heisenberg Groups

In this section we deal with the case $m = 1$, namely when $G = \mathbb{H}^n$ is a Heisenberg group. The function p_{1,k_1,k_2} of Definition 8.1 here reads

$$p_{1,k_1,k_2}(x,t) = \frac{2(-1)^{k_1} i^{k_2}}{(4\pi)^{n+1}} \int_{\mathbb{R}} e^{i\lambda|t| - \frac{|x|^2}{4}\lambda\coth(\lambda)} \frac{\lambda^{n+k_1+k_2}\cosh(\lambda)^{k_1}}{\sinh(\lambda)^{n+k_1}}\,\mathrm{d}\lambda.$$

Indeed, the absolute values of λ in the integral (3) can be removed by parity reasons. We begin by introducing some functions which greatly simplify the notation.

Definition 8.9. Define

$$\begin{aligned} h_{k_1,k_2}(R,t) &:= (-1)^{k_1} i^{k_2} \int_{\mathbb{R}} e^{i\lambda|t| - R\lambda\coth(\lambda)} \frac{\lambda^{n+k_1+k_2}\cosh(\lambda)^{k_1}}{\sinh(\lambda)^{n+k_1}}\,\mathrm{d}\lambda \\ &= \int_{\mathbb{R}} e^{iR\varphi_\omega(\lambda)} a_{k_1,k_2}(\lambda)\,\mathrm{d}\lambda, \end{aligned}$$

where

$$\begin{aligned} a_{k_1,k_2}(\lambda) &= \begin{cases} (-1)^{k_1} i^{k_2} \frac{\lambda^{n+k_1+k_2}\cosh(\lambda)^{k_1}}{\sinh(\lambda)^{n+k_1}} & \text{if } \lambda \notin \pi i\mathbb{Z}, \\ (-1)^{k_1} i^{k_2} \delta_{k_2,0} & \text{if } \lambda = 0, \end{cases} \\ \varphi_\omega(\lambda) &= \begin{cases} \omega\lambda + i\lambda\coth(\lambda) & \text{if } \lambda \notin \pi i\mathbb{Z}, \\ i & \text{if } \lambda = 0. \end{cases} \end{aligned} \tag{6}$$

Notice that

$$p_{1,k_1,k_2}(x,t) = \frac{2}{(4\pi)^{n+1}} h_{k_1,k_2}(R,t)$$

for all $(x,t) \in \mathbb{H}^n$; hence we can reduce matters to studying $h_{k_1,k_2}(R,t)$. Observe, in addition, that $y_\omega = \theta^{-1}(\omega) \in [0,\pi)$, since $\omega \geq 0$.

It will be convenient to reverse the dependence relation between (R,ω) and (x,t): hence, we shall no longer consider R and ω as functions of (x,t), but rather as 'independent variables.' In this order of ideas, the formula $|t| = R\omega$ should sound as a definition.

Our intent will be to apply Theorem 8.7 to a function closely related to h_{k_1,k_2}; hence we shall find some stationary points of the phase of h_{k_1,k_2}, namely φ_ω. The lemma below is of fundamental importance.

Lemma 8.10. [44, § 3, Lemma 6] *$\varphi'_\omega(\lambda) = \omega + \widetilde{\theta}(i\lambda)$ for all $\lambda \notin \pi i\mathbb{Z}^*$, where $\widetilde{\theta}$ is the analytic continuation of θ to* $\mathrm{Dom}(\varphi_\omega)$. *In particular, iy_ω is a stationary point of φ_ω.*

8.2.1 Estimates for $(x,t) \to \infty$ while $4|t|/|x|^2 \le C$

Theorem 8.11. *Fix $C > 0$. If $(x,t) \to \infty$ while $0 \le \omega \le C$, then*

$$p_{1,k_1,k_2}(x,t) = \frac{1}{|x|} e^{-\frac{1}{4}d(x,t)^2} \Psi(\omega) \left[(-1)^{k_1+k_2} \frac{y_\omega^{n+k_1+k_2} \cos(y_\omega)^{k_1}}{\sin(y_\omega)^{n+k_1}} + O\left(\frac{1}{|x|^2}\right)\right] \tag{7}$$

where

$$\Psi(\omega) = \begin{cases} \frac{1}{4^n \pi^{n+1}} \sqrt{\frac{\pi \sin(y_\omega)^3}{\sin(y_\omega) - y_\omega \cos(y_\omega)}}, & \text{if } \omega \neq 0, \\ \frac{(3\pi)^{1/2}}{4^n \pi^{n+1}}, & \text{if } \omega = 0. \end{cases}$$

It is worthwhile to stress that the above estimates may *not* be sharp when $\omega \to 0$ and $k_2 > 0$, as well as when $\omega \to \frac{\pi}{2}$ and $k_1 > 0$. In these cases indeed $y_\omega \to 0$ and $y_\omega \to \frac{\pi}{2}$, respectively, and the first term of the asymptotic expansion (7) may be smaller than the remainder. Since the *sharp* asymptotic behaviour of p_{1,k_1,k_2} when ω remains bounded is rather involved, we avoid to outline the complete picture for the moment. The statement above is just a simplified version of Theorem 8.24, where the general case of H-type groups is completely described.

In this section we then limit ourselves to consider Theorem 8.11 in the stated form. Its proof mostly consists in a straightforward generalization of [44, Theorem 2 of § 3], but it can also be seen as Proposition 8.26 in the current setting of Heisenberg groups. Nevertheless, for the sake of completeness we give a brief sketch of the proof.

The main idea is to change the contour of integration in the integral defining h_{k_1,k_2} in order to meet a stationary point of φ_ω. Since $\operatorname{Im} \varphi_\omega(\lambda) = \omega \operatorname{Im} \lambda + \operatorname{Re}[\lambda \coth(\lambda)]$ for every $\lambda \notin \pi i \mathbb{Z}$, to make this change we need to deepen our knowledge of $\operatorname{Re}[\lambda \coth(\lambda)]$ and $|a_{k_1,k_2}|$; this is done in the following lemma, which we state without proof.

Lemma 8.12. *For all $\lambda, y \in \mathbb{R}$ such that $|\lambda| > |y|$,*

$$\operatorname{Re}[(\lambda + iy)\coth(\lambda + iy)] = \frac{\lambda \sinh(2\lambda) + y \sin(2y)}{2(\sinh(\lambda)^2 + \sin(y)^2)} > 0.$$

Moreover, for all $\lambda, y \in \mathbb{R}$ such that either $y \notin \pi\mathbb{Z}$ or $\lambda \neq 0$,

$$|a_{k_1,k_2}(\lambda + iy)| = \frac{|\lambda + iy|^{n+k_1+k_2} (\sinh(\lambda)^2 + \cos(y)^2)^{\frac{k_1}{2}}}{(\sinh(\lambda)^2 + \sin(y)^2)^{\frac{n+k_1}{2}}}.$$

In the following lemma we perform the change of the contour of integration in the definition of h_{k_1,k_2}. Its proof is a simple adaptation of that of [51, Lemma 1.4].

Lemma 8.13. *For all $y \in [0, +\infty) \setminus \pi\mathbb{N}^*$*

$$h_{k_1,k_2}(R,t) = \int_{\mathbb{R}} e^{iR\varphi_\omega(\lambda+iy)} a_{k_1,k_2}(\lambda + iy)\, \mathrm{d}\lambda + 2\pi i \sum_{\substack{k \in \mathbb{N}^* \\ k\pi \in [0,y]}} \operatorname{Res}\big(e^{iR\varphi_\omega} a_{k_1,k_2}, k\pi i\big).$$

Proof of Theorem 8.11. Define

$$\psi_\omega = \varphi_\omega(\cdot + iy_\omega) - \varphi_\omega(iy_\omega)$$

and observe that

$$\varphi_\omega(iy_\omega) = i\omega\, y_\omega + iy_\omega \cot(y_\omega) = i \frac{y_\omega^2}{\sin(y_\omega)^2},$$

since $\omega = \theta(y_\omega)$. Therefore, by Lemma 8.13 (recall that $0 \le y_\omega < \pi$, so that there are no residues)

$$h_{k_1,k_2}(R,t) = e^{-\frac{1}{4}d(x,t)^2} \int_{\mathbb{R}} e^{iR\psi_\omega(\lambda)} a_{k_1,k_2}(\lambda + iy_\omega)\, \mathrm{d}\lambda.$$

Our intent is to apply Theorem 8.7 to the bounded subsets $\mathcal{F} = \{\psi_\omega \colon \omega \in [0,C]\}$ and $\mathcal{G} = \{a_{k_1,k_2}(\cdot + iy_\omega) \colon \omega \in [0,C]\}$ of $\mathcal{E}(\mathbb{R})$. Therefore we first verify that the four conditions of its statement hold.

2. Lemmas 8.10 and 8.5 imply that $i\varphi''_\omega(iy_\omega) = -\theta'(-y_\omega) < 0$ for all $\omega \in \mathbb{R}_+$. From the definition of ψ_ω we then get

$$\psi_\omega(0) = \psi'_\omega(0) = 0, \qquad i\psi''_\omega(0) < 0. \tag{8}$$

3. Consider the mapping $\psi\colon \mathbb{R} \times (-\pi, \pi) \ni (\lambda, y) \mapsto \psi_{\theta(y)}(\lambda)$. By (8), $\partial_1\psi(0, y) = 0$ and $i\partial_1^2\psi(0, y) < 0$ for all $y \in [0, \pi)$; moreover, ψ is analytic thanks to Lemma 8.5. Therefore, by Taylor's formula we may find two constants $\eta > 0$ and $C' > 0$ such that $|\partial_1\psi(\lambda, y)| \geq C'|\lambda|$ for all $\lambda \in [-2\eta, 2\eta]$ and for all $y \in [0, \theta^{-1}(C)]$.

1. Lemma 8.12 implies that

$$\operatorname{Im}\psi(\lambda, y) = \frac{\lambda\cosh(\lambda)\sinh(\lambda) - y\cot(y)\sinh(\lambda)^2}{\sinh(\lambda)^2 + \sin(y)^2}$$

for all $\lambda \in \mathbb{R}$ and for all $y \in (-\pi, \pi)$, $y \neq 0$; moreover, the mapping $(0, \pi) \ni y \mapsto y\cot(y)$ is strictly decreasing and tends to 1 as $y \to 0^+$. Therefore, if $\lambda \neq 0$ and $y \in [0, \pi)$, then

$$\operatorname{Im}\psi(\lambda, y) \geq \frac{\lambda\coth(\lambda) - 1}{1 + \frac{1}{\sinh(\lambda)^2}} > 0$$

since $\lambda\coth(\lambda) - 1 > 0$. Observe finally that, since $\frac{\lambda\coth(\lambda)-1}{1+\frac{1}{\sinh(\lambda)^2}} \sim |\lambda|$ for $\lambda \to \infty$, the second condition is also satisfied.

4. Just observe that $\mathcal{G}$ is bounded in $L^\infty(\mathbb{R})$.

By Theorem 8.7,

$$\int_{\mathbb{R}} e^{iR\psi_\omega(\lambda)} a_{k_1,k_2}(\lambda + iy_\omega)\,\mathrm{d}\lambda = \frac{(2\pi)(4\pi)^n}{|x|}\Psi(\omega)a_{k_1,k_2}(iy_\omega) + O\left(\frac{1}{|x|^3}\right)$$

for $R \to +\infty$, uniformly as ω runs through $[0, C]$. □

From now on, we shall consider the case $\omega \to +\infty$. The method of stationary phase cannot be applied directly in this case, since $y_\omega \to \pi$, and $i\pi$ is a pole of the phase (as well as of the amplitude). Although it seems possible to adapt the techniques developed by Li [56] to this situation, our proof follows the idea presented by Hueber and Müller [51, Theorem 1.3 (i)] for the Heisenberg group $\mathbb{H}^1$. We shall take advantage of this singularity to get the correct behaviour of h_{k_1,k_2}, by means of the residues obtained by Lemma 8.13.

8.2.2 Estimates for $\delta \to 0^+$ and $\kappa \to +\infty$

We state below the main result of this section.

Theorem 8.14. *For $\delta \to 0^+$ and $\kappa \to +\infty$*

$$p_{1,k_1,k_2}(x, t) = \frac{(-1)^{k_2}\pi^{k_1+k_2}}{4^n(\pi\delta)^{n+k_1-1}\sqrt{2\pi\kappa}} e^{-\frac{1}{4}d(x,t)^2}\left[1 + O\left(\frac{1}{\kappa} + \delta\right)\right].$$

The proof of Theorem 8.14 will be prepared by several lemmas. The first step will be to invoke Lemma 8.13, of which we keep the notation, to move the contour of integration beyond the singularity at πi; since at $2\pi i$ there is another one, it seems convenient to stop at $\frac{3\pi i}{2}$. We first notice that the integral on $\mathbb{R} + \frac{3\pi i}{2}$ may be neglected in some circumstances, as the following lemma shows. It is essentially [51, Lemma 1.4], so we omit the proof.

Lemma 8.15. *There exists a constant $C' > 0$ such that*

$$\left|\int_{\mathbb{R}} e^{iR\varphi_\omega\left(\lambda + \frac{3\pi i}{2}\right)} a_{k_1,k_2}\left(\lambda + \frac{3\pi i}{2}\right)\mathrm{d}\lambda\right| \leq C' e^{-\frac{3\pi|t|}{2}}.$$

Hence, matters are reduced to the computation of the residue. First of all, define

$$r(\lambda) = \begin{cases} 1 + \frac{1}{\lambda} - \pi(1+\lambda)\cot(\pi\lambda), & \text{if } \lambda \notin \mathbb{Z}, \\ 0, & \text{if } \lambda = 0, \end{cases}$$

and observe that r is holomorphic on its domain. It will be useful to define also

$$\widetilde{\varphi}_{k_1,k_2}(R,\xi) := \begin{cases} e^{R\,r(-\xi)} \frac{(\pi\xi)^{n+k_1}\cos(\pi\xi)^{k_1}(1-\xi)^{n+k_1+k_2}}{\sin(\pi\xi)^{n+k_1}}, & \text{if } \xi \notin \mathbb{Z}, \\ 1, & \text{if } \xi = 0, \end{cases}$$

and

$$\varphi_{\delta,k_1,k_2}(s) := e^{-i(n+k_1-1)s}\widetilde{\varphi}_{k_1,k_2}(0,\delta e^{is}) \tag{9}$$

whenever $\delta e^{is} \notin \mathbb{Z}^*$. The following lemma may be proved again on the lines of [51, Lemma 1.4].

Lemma 8.16. *For every $\delta < 1$*

$$2\pi i \operatorname{Res}\big(e^{iR\varphi_\omega} a_{k_1,k_2}, \pi i\big) = \frac{(-1)^{k_2}\pi^{k_2+1}}{\delta^{n+k_1-1}} e^{-R-\pi|t|} \int_{-\pi}^{\pi} e^{\kappa\cos(s)+Rr(-\delta e^{is})}\varphi_{\delta,k_1,k_2}(s)\,\mathrm{d}s. \tag{10}$$

Therefore, it remains only to estimate the integral in (10), namely

$$H_{k_1,k_2}(R,t) := \int_{-\pi}^{\pi} e^{\kappa\cos(s)+Rr(-\delta e^{is})}\varphi_{\delta,k_1,k_2}(s)\,\mathrm{d}s = \int_{-\pi}^{\pi} e^{\kappa q_\delta(-is)}\varphi_{\delta,k_1,k_2}(s)\,\mathrm{d}s, \tag{11}$$

where

$$q(\delta,\zeta) = q_\delta(\zeta) := \cosh(\zeta) + \frac{\delta}{2} r(-\delta e^{-\zeta}). \tag{12}$$

Notice that we may apply Theorem 8.7 only when $\kappa \to +\infty$, and this is why we confined ourselves to the case where $\delta \to 0^+$ (and we shall assume $0 < \delta < 1$) and $\kappa \to +\infty$.

Again for technical convenience, we shall reverse the dependence relation between (δ,κ) and $(R,|t|)$, thus assuming that δ and κ are 'independent variables.' Indeed, δ and κ completely describe our problem, since

$$|t| = \frac{\kappa}{2\pi\delta}, \qquad R = \frac{\kappa\delta}{2},$$

and $|t| + R \to +\infty$ if $\delta \to 0^+$ and $\kappa \to +\infty$. We shall sometimes let δ take complex values. The following lemma is essentially [51, Lemma 1.2]. We present a slightly shorter proof.

Lemma 8.17. *The function q is holomorphic on the set $\{(\delta,\zeta) \in \mathbb{C}\times\mathbb{C} | \, \delta e^{-\zeta} \notin \mathbb{Z}^*\}$. Further, there exist two constants $\delta_1 \in (0,1)$ and $\eta_1 > 0$ such that for all $\delta \in \mathrm{B}_{\mathbb{C}}(0,\delta_1)$ there is a unique $\sigma_\delta \in \mathrm{B}_{\mathbb{C}}(0,\eta_1)$ such that $q'_\delta(\sigma_\delta) = 0$. Then, the mapping $\mathrm{B}_{\mathbb{C}}(0,\delta_1) \ni \delta \mapsto \sigma_\delta$ is holomorphic and real on $(-\delta_1,\delta_1)$. Finally, $\sigma_\delta = O(\delta^2)$ and $q_\delta(\sigma_\delta) = 1 + O(\delta^2)$ for $\delta \to 0$.*

Proof. q is holomorphic since r is. Furthermore, $\partial_2 q(0,0) = 0$ and $\partial_2^2 q(0,0) = 1$. Therefore, the implicit function theorem (cf. [26, Proposition 6.1 of IV.5.6]) implies the existence of some δ_1 and η_1 as in the statement, the holomorphy of the mapping $\delta \mapsto \sigma_\delta$, and that $\frac{d}{d\delta}\sigma_\delta|_{\delta=0} = 0$. Notice also that $\sigma_0 = 0$, so that $\sigma_\delta = O(\delta^2)$ for $\delta \to 0$ by Taylor's formula.

Since q_δ is real on real numbers, $q'_\delta(\overline{\sigma_\delta}) = \overline{q'_\delta(\sigma_\delta)} = 0$; thus $\sigma_\delta = \overline{\sigma_\delta}$ for the uniqueness of σ_δ, and hence $\sigma_\delta \in \mathbb{R}$ for all $\delta \in (-\delta_1,\delta_1)$.

The last assertion follows from Taylor's formula, since $q_0(\sigma_0) = q_0(0) = 1$ and $\frac{d}{d\delta}q_\delta(\sigma_\delta)|_{\delta=0} = \partial_1 q(0,0) + \partial_2 q(0,0)\frac{d}{d\delta}\sigma_\delta|_{\delta=0} = 0$. ☐

The contour of integration can now be changed in order to apply the method of stationary phase. For the remainder of this section, we keep δ_1 and η_1 of Lemma 8.17 fixed.

Lemma 8.18. *Let $\tau \in C_c^\infty(\mathbb{R})$ such that $\chi_{[-\frac{\pi}{2},\frac{\pi}{2}]} \leq \tau \leq \chi_{[\pi,\pi]}$. Define, for all $\delta \in (-\delta_1,\delta_1)$, the path $\gamma_\delta(s) := s + i\sigma_\delta\,\tau(s)$, and*

$$F_\delta(s) := -iq_\delta(-i\gamma_\delta(s)) + iq_\delta(\sigma_\delta) \qquad \text{and} \qquad \psi_{\delta,k_1,k_2} := (\varphi_{\delta,k_1,k_2}\circ\gamma_\delta)\,\gamma'_\delta.$$

Then

$$H_{k_1,k_2}(R,t) = e^{\kappa\,q_\delta(\sigma_\delta)} \int_{-\pi}^{\pi} e^{i\kappa F_\delta(s)}\psi_{\delta,k_1,k_2}(s)\,\mathrm{d}s.$$

Proof of Theorem 8.14. We shall apply Theorem 8.7 to the bounded subsets $\mathcal{F} = \{F_\delta \colon \delta \in (0, \delta_2)\}$ and $\mathcal{G} = \{\psi_{\delta,k_1,k_2} \colon \delta \in (0, \delta_2)\}$ of $\mathcal{E}((-\pi, \pi))$, depending on some δ_2 to be fixed later. Hence we check that the four conditions of the statement are satisfied.

1. The mapping $F \colon (-\delta_1, \delta_1) \times \mathbb{R} \ni (\delta, s) \mapsto F_\delta(s)$ is of class C^∞, and $\partial_2^2 F(0, 0) = i$; thus we may find $\delta_2 \in (0, \delta_1)$, $\eta_2 \in \left(0, \frac{\pi}{2}\right)$ and $C'' > 0$ such that $\operatorname{Im} \partial_2^2 F(\delta, s) \geq 2C''$ for all $\delta \in [-\delta_2, \delta_2]$ and for all $s \in [-2\eta_2, 2\eta_2]$. From Taylor's formula then
$$\operatorname{Im} F(\delta, s) = \int_0^s \partial_2^2 \operatorname{Im} F(\delta, \tau)(s - \tau) \,\mathrm{d}\tau \geq C'' s^2$$
for all $s \in [-2\eta_2, 2\eta_2]$ and for all $\delta \in [-\delta_2, \delta_2]$. Since $\operatorname{Im} F(0, s) = 1 - \cos(s)$ for all $s \in [-\pi, \pi]$, by reducing δ_2 and C'' if necessary one may assume that $\operatorname{Im} F(\delta, s) \geq C''\pi^2 \geq C'' s^2$ for all $s \in \mathbb{R}$ such that $2\eta_2 \leq |s| \leq \pi$ and for all $\delta \in [-\delta_2, \delta_2]$.

2. It is immediately seen that $F_\delta(0) = F_\delta'(0) = 0$ by definition.

3. For every $\delta \in [-\delta_2, \delta_2]$ and $s \in [-2\eta_2, 2\eta_2]$
$$|\partial_2 F(\delta, s)| \geq |\partial_2 \operatorname{Im} F(\delta, s)| = \left| \int_0^s \partial_2^2 \operatorname{Im} F(\delta, \tau) \,\mathrm{d}\tau \right| \geq 2C''|s|.$$

4. Just observe that $\mathcal{G}$ is bounded in $L^\infty((-\pi, \pi))$.

By Theorem 8.7, then,
$$\int_{-\pi}^{\pi} e^{i\kappa F_\delta(s)} \tau_2(s) \psi_{\delta,k_1,k_2}(s) \,\mathrm{d}s = \sqrt{\frac{2\pi i}{\kappa F_\delta''(0)}} \psi_{\delta,k_1,k_2}(0) + O\left(\frac{1}{\kappa^{3/2}}\right).$$

It is then easily seen that $F_\delta''(0) = i q_\delta''(\sigma_\delta) = i(1 + O(\delta))$ and $\psi_{\delta,k_1,k_2}(0) = \varphi_{\delta,k_1,k_2}(i\sigma_\delta) = 1 + O(\delta)$ for $\delta \to 0^+$.

Now, by construction,
$$-R - \pi|t| + \kappa q_\delta(s) = iR\, \varphi_\omega(\pi i(1 - \delta e^{-s}))$$
for s in a neighbourhood of σ_δ. Take $\delta_3 \in (0, \delta_2]$ so that $(1 - \delta e^{-\sigma_\delta}) \in (-1, 1)$ for all $\delta \in [0, \delta_3]$, and fix $\delta \in (0, \delta_3)$ and $t \neq 0$. We shall prove that
$$y_\omega = \pi(1 - \delta e^{-\sigma_\delta}).$$
Indeed, y_ω is the unique element of $(-\pi, \pi)$ such that $\varphi_\omega'(iy_\omega) = 0$; furthermore, $\pi(1 - \delta e^{-\sigma_\delta}) \in (-\pi, \pi)$ for the choice of δ_3, and $-R\pi\,\delta\, e^{-\sigma_\delta} \varphi_\omega'(\pi i(1 - \delta e^{-\sigma_\delta})) = \kappa\, q_\delta'(\sigma_\delta) = 0$. Therefore, $y_\omega = \pi(1 - \delta e^{-\sigma_\delta})$. Finally, equality holds by analyticity whenever both sides are defined. It then follows that
$$-R - \pi|t| + \kappa q_\delta(\sigma_\delta) = iR\varphi_\omega(iy_\omega) = -\frac{1}{4} d(x, t)^2. \tag{13}$$
Finally observe that, by definition of κ and δ, and by Lemma 8.17,
$$-\frac{3\pi|t|}{2} + R + \pi|t| - \kappa q_\delta(\sigma_\delta) + \log\kappa \leq -\frac{\kappa}{2\pi\delta}\left[\frac{\pi}{2} - \pi\delta^2 + 2\pi\delta\big(1 + O(\delta^2)\big) - 2\pi\delta\frac{\log\kappa}{\kappa}\right],$$
which tends to $-\infty$ as $\delta \to 0^+$ and $\kappa \to +\infty$. This means that
$$e^{-\frac{3\pi|t|}{2}} = o\left(\frac{e^{-R - \pi|t| + \kappa q_\delta(\sigma_\delta)}}{\kappa}\right)$$
for $\kappa \to +\infty$, uniformly as δ runs through $(0, \delta_2]$. Our assertion is then a consequence of Lemmas 8.13 and 8.15. □

8.2.3 Estimates for $\delta \to 0^+$ and κ bounded

Strictly speaking, cases **III** and **IV** have already been considered together by Hueber and Müller [51, Theorem 1.3 (ii)] on the Heisenberg group $\mathbb{H}^1$, i.e. when $n = 1$. Since their method does not apply when $n > 1$, we shall follow a different approach similar to that of Li [55].

We first recall that, for all $\nu \in \mathbb{Z}$ and $\zeta \in \mathbb{C}$, the modified Bessel function I_ν of order ν is defined as

$$I_\nu(\zeta) = \sum_{k \in \mathbb{N}} \frac{\zeta^{2k+\nu}}{2^{2k+\nu} k!\, \Gamma(k+\nu+1)}.$$

If $s > 0$, then also

$$I_\nu(s) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{s\cos(\xi) - i\nu\xi}\, \mathrm{d}\xi,$$

as one can verify from [38, 7.3.1 (2)] by applying the change of variables $\psi = \frac{\pi}{2} - \varphi$ and by taking into account the relationship [38, 7.2.2 (12)] between $I_\nu = I_{-\nu}$ and J_ν, and also the periodicity of the integrand. Notice that for $s > 0$ and $\nu \in \mathbb{Z}$, $I_\nu(s)$ is strictly positive unless $s = 0$ and $\nu \neq 0$. The main result of this section is the following.

Theorem 8.19. *Fix $C > 1$. If $\delta \to 0^+$ while $1/C \leq \kappa \leq C$, then*

$$p_{1,k_1,k_2}(x,t) = \frac{(-1)^{k_2}\pi^{k_1+k_2}}{4^n(\pi\delta)^{n+k_1-1}} e^{-\frac{1}{4}d(x,t)^2} e^{-\kappa} I_{n+k_1-1}(\kappa)[1 + O(\delta)]. \tag{14}$$

When $\kappa \to 0^+$ and $|t| \to +\infty$

$$p_{1,k_1,k_2}(x,t) = \frac{(-1)^{k_2}\pi^{k_1+k_2}}{4^n(n+k_1-1)!} |t|^{n+k_1-1} e^{-\frac{1}{4}d(x,t)^2} \left[1 + O\left(\frac{1}{|t|} + \kappa\right)\right]. \tag{15}$$

Lemma 8.20. *For every $N \in \mathbb{N}$*

$$H_{k_1,k_2}(R,t) = 2\pi \sum_{|\alpha| \leq N} I_{n+k_1-1-\alpha_2}(\kappa) \frac{\partial^\alpha \widetilde{\varphi}_{k_1,k_2}(0,0)\kappa^{\alpha_1}}{2^{\alpha_1}\alpha!} \delta^{|\alpha|} + O\big(\delta^{N+1}\big)$$

for $\delta \to 0^+$, uniformly as κ runs through $[0, C]$.

Proof. By substituting (9) in (10) and by Taylor's formula applied to $\widetilde{\varphi}_{k_1,k_2}$,

$$\begin{aligned} H_{k_1,k_2}(R,t) &= \int_{-\pi}^{\pi} e^{\kappa\cos(s)} e^{-i(n+k_1-1)s} \widetilde{\varphi}_{k_1,k_2}(R, \delta e^{is})\, \mathrm{d}s \\ &= \sum_{|\alpha|\leq N} \frac{\partial^\alpha \widetilde{\varphi}_{k_1,k_2}(0,0)}{\alpha!} R^{\alpha_1}\delta^{\alpha_2} \int_{-\pi}^{\pi} e^{\kappa\cos(s)} e^{-i(n+k_1-1-\alpha_2)s}\, \mathrm{d}s + \mathcal{R}_{N+1}(\delta,\kappa) \\ &= 2\pi \sum_{|\alpha|\leq N} I_{n+k_1-1-\alpha_2}(\kappa) \frac{\partial^\alpha \widetilde{\varphi}_{k_1,k_2}(0,0)\kappa^{\alpha_1}}{2^{\alpha_1}\alpha!} \delta^{|\alpha|} + \mathcal{R}_{N+1}(\delta,\kappa), \end{aligned}$$

where the last equality holds since $R = \frac{\delta\kappa}{2}$. Moreover, $\mathcal{R}_{N+1}(\delta,\kappa)$ is easily seen to be $O\big(\delta^{N+1}\big)$ for $\delta \to 0^+$ uniformly as κ runs through $[0, C]$. This completes the proof. □

Proof of Theorem 8.19. Lemmas 8.15 and 8.16 imply that

$$p_{1,k_1,k_2}(x,t) = \frac{(-1)^{k_2} 2\pi^{k_2-n}}{4^{n+1}\delta^{n+k_1-1}} e^{-R-\pi|t|} H_{k_1,k_2}(R,t) + O\Big(e^{-\frac{3\pi|t|}{2}}\Big).$$

Moreover, recall that $\delta|t| = \frac{\kappa}{2\pi}$ and $R = \frac{\kappa\delta}{2}$; therefore, for every $N \in \mathbb{N}$,

$$e^{-\frac{3\pi|t|}{2}} = o\Big(\delta^{N+2-n-k_1} e^{-R-\pi|t|}\Big) \tag{16}$$

as $\delta \to 0^+$, uniformly as κ runs through $[1/C, C]$. By (13) and Lemma 8.17, the first assertion follows from Lemma 8.20 for $N = 0$.

As for (15), observe first that $\kappa \to 0^+$ and $|t| \to +\infty$ is equivalent to saying $\delta, \kappa \to 0^+$ and $\delta = o(\kappa)$. Then, Lemma 8.20 with $N = n + k_1 - 1$ and an easy development of the Bessel function in a neighbourhood of 0 imply that

$$p_{1,k_1,k_2}(x,t) = \frac{\pi^{k_1+k_2}(-1)^{k_2}}{4^n(\pi\delta)^{n+k_1-1}} e^{-\pi|t|-R}\Bigg[\kappa^{n+k_1-1}\frac{I_{n+k_1-1}^{(n+k_1-1)}(0)}{(n+k_1-1)!} + O(\kappa^{n+k_1}) + \sum_{1\le|\alpha|\le n+k_1-1} O\Big(I_{n+k_1-1-\alpha_2}(\kappa)\kappa^{\alpha_1}\delta^{|\alpha|}\Big) + O(\delta^{n+k_1})\Bigg] + O\Big(e^{-\frac{3\pi|t|}{2}}\Big).$$

Since $\delta = o(\kappa)$, one has $\delta^{\alpha_2+\alpha_1-1} = O(\kappa^{\alpha_2+\alpha_1-1})$ for every $\alpha \neq 0$. Therefore,

$$\sum_{1\le|\alpha|\le n+k_1-1} O\Big(I_{n+k_1-1-\alpha_2}(\kappa)\kappa^{\alpha_1}\delta^{|\alpha|}\Big) = \sum_{1\le|\alpha|\le n+k_1-1} O\big(\kappa^{n+k_1-2+2\alpha_1}\delta\big) = O\big(\kappa^{n+k_1-2}\delta\big).$$

Since $\frac{\kappa}{2\pi\delta} = |t|$ and $I_{n+k_1-1}^{(n+k_1-1)}(0) = \frac{1}{2^{n+k_1-1}}$, we get

$$p_{1,k_1,k_2}(x,t) = \frac{\pi^{k_1+k_2}(-1)^{k_2}}{4^n(n+k_1-1)!} e^{-\pi|t|-R}|t|^{n+k_1-1}\left[1 + O\left(\frac{1}{|t|} + \kappa + \delta\right)\right] + O\Big(e^{-\frac{3\pi|t|}{2}}\Big).$$

Finally, $\delta = o\left(\frac{1}{|t|}\right)$ since $\delta|t| = \frac{\kappa}{2\pi}$; moreover

$$e^{-\frac{3\pi|t|}{2}} = o\Big(e^{-\pi|t|-R}|t|^{n+k_1-2}\Big)$$

since $R \to 0^+$ and $|t| \to +\infty$. The assertion follows. □

The estimates in cases **II**, **III**, and **IV** can be put together. This is done in the following corollary, which will turn out to be fundamental later on. Define first, for $\zeta \in \mathbb{C}$ and $\nu \in \mathbb{Z}$,

$$\widetilde{I}_\nu(\zeta) := \sum_{k\ge 0} \frac{\zeta^{2k}}{2^{2k+\nu}k!\Gamma(k+\nu+1)}.$$

From now on we shall use the following abbreviation. We keep the notation of Lemma 8.17.

Definition 8.21. For $\delta \in B_{\mathbb{C}}(0, \delta_1)$, define $\rho(\delta) := q_\delta(\sigma_\delta)$.

By Lemma 8.17, ρ is a holomorphic function such that $\rho(0) = 1$ and $\rho'(0) = 0$, so that $\rho(\delta) = 1 + O(\delta^2)$ as $\delta \to 0$.

Corollary 8.22. *When $(x,t) \to \infty$ and $\delta \to 0^+$*

$$p_{1,k_1,k_2}(x,t) = \frac{(-1)^{k_2}\pi^{k_1+k_2}}{2^{n-k_1+1}}|t|^{n+k_1-1}e^{-\frac{1}{4}d(x,t)^2}e^{-\kappa\rho(\delta)}\widetilde{I}_{n+k_1-1}(\kappa\rho(\delta))[1+g(|x|,|t|)],$$

where

$$g(|x|,|t|) = \begin{cases} O\big(\delta + \frac{1}{\kappa}\big) & \text{if } \delta \to 0^+ \text{ and } \kappa \to +\infty, \\ O(\delta) & \text{if } \delta \to 0^+ \text{ and } \kappa \in [1/C, C], \\ O\Big(\frac{1}{|t|} + \kappa\Big) & \text{if } \delta \to 0^+ \text{ and } \kappa \to 0^+ \end{cases} \qquad (17)$$

for every $C > 1$.

Proof. **1.** Assume first that $\kappa \to +\infty$. Since $I_\nu(s) = \frac{e^s}{\sqrt{2\pi s}}\big[1 + O\big(\frac{1}{s}\big)\big]$ for $s \to +\infty$, $\nu \in \mathbb{Z}$ (cf. [38, 7.13.1 (5)]),

$$\widetilde{I}_\nu(s) = \frac{e^s}{s^\nu\sqrt{2\pi s}}\left[1 + O\left(\frac{1}{s}\right)\right] \quad \text{for } s \to +\infty. \qquad (18)$$

Therefore, Theorem 8.14 implies that

$$\begin{aligned}
p_{1,k_1,k_2}(x,t) &= \frac{(-1)^{k_2}\pi^{k_1+k_2}}{4^n(\pi\delta)^{n+k_1-1}\sqrt{2\pi\kappa}} e^{-\frac{1}{4}d(x,t)^2}\left[1+O\left(\frac{1}{\kappa}+\delta\right)\right] \\
&= \frac{(-1)^{k_2}\pi^{k_1+k_2}\widetilde{I}_{n+k_1-1}(\kappa\rho(\delta))}{2^{n-k_1+1}}|t|^{n+k_1-1}e^{-\frac{1}{4}d(x,t)^2}e^{-\kappa\rho(\delta)} \\
&\qquad \times\left[1+O\left(\frac{1}{\kappa\rho(\delta)}\right)\right]\left[1+O\left(\frac{1}{\kappa}+\delta\right)\right] \\
&= \frac{(-1)^{k_2}\pi^{k_1+k_2}\widetilde{I}_{n+k_1-1}(\kappa\rho(\delta))}{2^{n-k_1+1}}|t|^{n+k_1-1}e^{-\frac{1}{4}d(x,t)^2}e^{-\kappa\rho(\delta)}\left[1+O\left(\frac{1}{\kappa}+\delta\right)\right],
\end{aligned}$$

since $\rho(\delta)=1+O(\delta^2)$ and $\frac{2|t|}{\kappa}=\frac{1}{\pi\delta}$.

2. Assume now that $\kappa\in[1/C,C]$ for some $C>1$. Then, by Theorem 8.19,

$$\begin{aligned}
p_{1,k_1,k_2}(x,t) &= \frac{(-1)^{k_2}\pi^{k_1+k_2}}{4^n(\pi\delta)^{n+k_1-1}} e^{-\frac{1}{4}d(x,t)^2}e^{-\kappa}I_{n+k_1-1}(\kappa)[1+O(\delta)] \\
&= \frac{(-1)^{k_2}\pi^{k_1+k_2}}{2^{n-k_1+1}}|t|^{n+k_1-1}e^{-\frac{1}{4}d(x,t)^2}e^{-\kappa\rho(\delta)}\widetilde{I}_{n+k_1-1}(\kappa\rho(\delta))\left[1+O(\delta^2)\right][1+O(\delta)] \\
&= \frac{(-1)^{k_2}\pi^{k_1+k_2}}{2^{n-k_1+1}}|t|^{n+k_1-1}e^{-\frac{1}{4}d(x,t)^2}e^{-\kappa\rho(\delta)}\widetilde{I}_{n+k_1-1}(\kappa\rho(\delta))[1+O(\delta)],
\end{aligned}$$

where the second equality holds since $I_{n+k_1-1}(\kappa\rho(\delta))-I_{n+k_1-1}(\kappa)=O(\kappa(\rho(\delta)-1))=O(\delta^2)$ uniformly as κ runs through $[1/C,C]$ by Taylor's formula.

3. Finally, if $\kappa\to 0^+$ then

$$\widetilde{I}_{n+k_1-1}(\kappa)=\widetilde{I}_{n+k_1-1}(0)+O(\kappa)=\frac{1}{2^{n+k_1-1}(n+k_1-1)!}+O(\kappa)$$

by the definition of $\widetilde{I}_{n+k_1-1}$. The assertion follows by means of Theorem 8.19. □

8.3 *H*-type Groups

In this section we deal with the general case $m\geq 1$. In particular, we prove a refined version of Theorem 8.11, and extend Theorems 8.14 and 8.19: this is done through Theorems 8.24, 8.35 and 8.36 respectively. Theorem 8.24 treats case **I** and is still inspired by [44, Theorem 2 of § 3]. The asymptotic estimates in the other three cases are first obtained in the case m odd, 'reducing' to the case $m=1$; the case m even is then achieved through a descent method.

The first step in order to apply the method of stationary phase is to extend the integrand to a meromorphic function on $\mathbb{C}^m$. If $m>1$, such extension is no longer automatic as when $m=1$. A natural way consists in taking advantage of the parity of the functions that appear, as in [36]. Indeed, any continuous branch of $\lambda\mapsto\sqrt{\lambda^2}$ is a holomorphic function which coincides with $\lambda\mapsto\pm|\lambda|$ on $\mathbb{R}^m$; therefore, whenever g is an even holomorphic function defined on a symmetric open subset of $\mathbb{C}$, the function $\lambda\mapsto g(\sqrt{\lambda^2})$ is well-defined, holomorphic, and coincides with $\lambda\mapsto g(|\lambda|)$ on $\mathbb{R}^m$. Hence, we are led to the following definition, which is the analogue of Definition 8.9. We shall use the same notation as before, without stressing the (new) dependence on m.

Definition 8.23. Define

$$h_{k_1,k_2}(R,t)=\int_{\mathbb{R}^m}e^{iR\varphi_\omega(\lambda)}a_{k_1,k_2}(\lambda)\,\mathrm{d}\lambda$$

where

$$\begin{aligned}
a_{k_1,k_2}(\lambda)&=\begin{cases}(-1)^{k_1}i^{k_2}\frac{\sqrt{\lambda^2}^{n+k_1}\cosh(\sqrt{\lambda^2})^{k_1}}{\sinh(\sqrt{\lambda^2})^{n+k_1}}\langle\lambda|u_1\rangle^{k_2} & \text{if } \sqrt{\lambda^2}\notin i\pi\mathbb{Z}^*,\\ (-1)^{k_1}i^{k_2}\delta_{k_2,0} & \text{if } \lambda=0,\end{cases}\\
\varphi_\omega(\lambda)&=\begin{cases}\omega\,\langle\lambda|u_1\rangle+i\sqrt{\lambda^2}\coth(\sqrt{\lambda^2}) & \text{if } \sqrt{\lambda^2}\notin i\pi\mathbb{Z}^*,\\ i & \text{if } \lambda=0.\end{cases}
\end{aligned}\tag{19}$$

Define also

$$a_{k_1,k_2,\omega}(\lambda) \coloneqq a_{k_1,k_2}(\lambda + i y_\omega u_1). \tag{20}$$

Observe again that

$$p_{1,k_1,k_2}(x,t) = \frac{1}{(4\pi)^n (2\pi)^m} h_{k_1,k_2}(R,t)$$

for all $(x,t) \in \mathbb{R}^{2n} \times \mathbb{R}^m$, and that $y_\omega = \theta^{-1}(\omega) \in [0,\pi)$, since $\omega \geq 0$.

8.3.1 Estimates for $(x,t) \to \infty$ while $4|t|/|x|^2 \leq C$

The main result of this section is Theorem 8.24 below. As already said, the main ingredient of its proof is the method of stationary phase (cf. Proposition 8.26), which is already employed in [44, Theorem 2 of § 3] to treat the case $n = m = 1$ and $k_1 = k_2 = 0$.

The novelty of considering all the derivatives of the heat kernel p_1 (in other words, all the cases $k_1 \geq 0$ and $k_2 \geq 0$) introduces additional complexity to the developments, since the choice $k = 0$ in (4) may not give the sharp asymptotic behaviour of p_{1,k_1,k_2} at infinity, while ω remains bounded. In particular, this happens in the cases $\omega \to 0$ and $k_2 > 0$, or $\omega \to \frac{\pi}{2}$ and $k_1 > 0$. If ω remains bounded and away from 0 and $\frac{\pi}{2}$, the first term is instead enough.

Theorem 8.24. *Fix $\varepsilon, C > 0$. If $(x,t) \to \infty$ while $0 \leq \omega \leq C$, then*

$$p_{1,k_1,k_2}(x,t) = \frac{1}{|x|^m} e^{-\frac{1}{4} d(x,t)^2} \Psi(\omega) \Upsilon(x,t)$$

where

$$\Psi(\omega) = \begin{cases} \frac{1}{4^n \pi^{n+m}} \sqrt{\frac{(2\pi)^m y_\omega^{m-1} \sin(y_\omega)^3}{2\omega^{m-1}(\sin(y_\omega) - y_\omega \cos(y_\omega))}}, & \text{if } \omega \neq 0, \\ \frac{(3\pi)^{m/2}}{4^n \pi^{n+m}}, & \text{if } \omega = 0, \end{cases} \tag{21}$$

and

1. *if $\varepsilon \leq \omega \leq \frac{\pi}{2} - \varepsilon$ or $\frac{\pi}{2} + \varepsilon \leq \omega \leq C$,*

$$\Upsilon(x,t) = (-1)^{k_1+k_2} \frac{y_\omega^{n+k_1+k_2} \cos(y_\omega)^{k_1}}{\sin(y_\omega)^{n+k_1}} + O\left(\frac{1}{|x|^2}\right); \tag{22}$$

2. *if $\omega \to 0$ and k_2 is even,*

$$\Upsilon(x,t) = \sum_{j=0}^{k_2/2} c_{k_1,k_2,j} \frac{\omega^{k_2-2j}}{|x|^{2j}} + O\left(\sum_{j=0}^{k_2/2} \frac{\omega^{k_2-2j+1}}{|x|^{2j}} + \frac{1}{|x|^{k_2+2}}\right); \tag{23}$$

3. *if $\omega \to 0$, k_2 is odd and $|t| \to \infty$,*

$$\Upsilon(x,t) = \sum_{j=0}^{(k_2-1)/2} c_{k_1,k_2,j} \frac{\omega^{k_2-2j}}{|x|^{2j}} + O\left(\sum_{j=0}^{(k_2+1)/2} \frac{\omega^{k_2-2j+1}}{|x|^{2j}}\right); \tag{24}$$

4. *if $\omega \to 0$, k_2 is odd and $0 \leq |t| \leq C$*

$$\Upsilon(x,t) = c_{k_1,k_2+1,(k_2+1)/2} \frac{|t|}{|x|^{k_2+1}} + O\left(\frac{|t|}{|x|^{k_2+3}}\right); \tag{25}$$

5. *if $\omega \to \frac{\pi}{2}$ and k_1 is even,*

$$\Upsilon(x,t) = \sum_{j=0}^{k_1/2} b_{k_1,k_2,j} \frac{\left(\omega - \frac{\pi}{2}\right)^{k_1-2j}}{|x|^{2j}} + O\left(\sum_{j=0}^{k_1/2} \frac{\left(\omega - \frac{\pi}{2}\right)^{k_1-2j+1}}{|x|^{2j}} + \frac{1}{|x|^{k_1+2}}\right); \tag{26}$$

6. *if $\omega \to \frac{\pi}{2}$ and k_1 is odd,*

$$\Upsilon(x,t) = \sum_{j=0}^{(k_1-1)/2} b_{k_1,k_2,j} \frac{\left(\omega - \frac{\pi}{2}\right)^{k_1-2j}}{|x|^{2j}} + \frac{b_{k_1,k_2,(k_1+1)/2}}{|x|^{k_1+1}} + O\left(\sum_{j=0}^{(k_1-1)/2} \frac{\left(\omega-\frac{\pi}{2}\right)^{k_1-2j+1}}{|x|^{2j}} + \frac{\omega-\frac{\pi}{2}}{|x|^{k_1+1}} + \frac{1}{|x|^{k_1+3}}\right). \quad (27)$$

The coefficients $c_{k_1,k_2,j}$ and $b_{k_1,k_2,j}$ are explicitly given by (33)*,* (35) *and* (36)*.*

The remainder of this section is devoted to the proof of Theorem 8.24. Since it is quite involved, we split this section into two parts: in the first one we apply the method of stationary phase, while in the second one we find the asymptotics of the development given by Theorem 8.7 which are required to get the sharp developments (23)–(27). These proofs go through several lemmas.

Remark 8.25. Notice that any pair of terms in the sums appearing in the developments (23), (24), (26), and (27) are not comparable with each other under the stated asymptotic condition. Therefore, these developments cannot be simplified. Observe, in addition, that for k_1 and k_2 fixed the coefficients $b_{k_1,k_2,j}$ (resp. $c_{k_1,k_2,j}$) have the same sign; thus, no cancellation can occur, and our developments are indeed *sharp*. A more detailed description will be given in Section 8.3.1.

Finally, notice that it is possible to obtain even more precise expansions if one does not develop the terms $L_{j,\psi_\omega} a_{k_1,k_2,\omega}$ which appear in Proposition 8.26 below. In particular, in the cases when $\omega \to 0^+$ and $k_2 = 0$, or $\omega \to \frac{\pi}{2}$ and $k_1 = 0$, the explicit computation of $L_{0,\psi_\omega} a_{k_1,k_2,\omega} = a_{k_1,k_2}(iy_\omega u_1)$ leads to better remainders than those in (23) and (26) respectively.

Application of the Method of Stationary Phase

As already said, Proposition 8.26 below is an easy generalization of Theorem 8.11.

Proposition 8.26. *Fix $C > 0$ and let $k \in \mathbb{N}$. Then, if $(x,t) \to \infty$ while $0 \le \omega \le C$,*

$$p_{1,k_1,k_2}(x,t) = \frac{1}{|x|^m} e^{-\frac{1}{4}d(x,t)^2} \Psi(\omega) \left[\sum_{j=0}^{k} \frac{4^j L_{j,\psi_\omega} a_{k_1,k_2,\omega}}{|x|^{2j}} + O\left(\frac{1}{|x|^{2k+2}}\right)\right] \quad (28)$$

where Ψ is defined by (21)*.*

In the same way as in Section 8.2.1, we begin by finding some stationary points of the phase of h_{k_1,k_2}, namely φ_ω.

Lemma 8.27. [36, Formula (5.7)] *For all λ such that $\sqrt{\lambda^2} \notin i\pi\mathbb{Z}^*$,*

$$\varphi'_\omega(\lambda) = \omega u_1 + \lambda \frac{\widetilde{\theta}(i\sqrt{\lambda^2})}{\sqrt{\lambda^2}}$$

where $\widetilde{\theta}$ is the analytic continuation of θ to $\mathrm{Dom}(\varphi_\omega)$*. In particular, $iy_\omega u_1$ is a stationary point of φ_ω.*

We then change the contour of integration in the integral defining h_{k_1,k_2} in order to meet a stationary point of φ_ω. This is done in the following lemma, which is the analogue of Lemma 8.13.

Lemma 8.28. *For every $y \in [0,\pi)$*

$$h_{k_1,k_2}(R,t) = \int_{\mathbb{R}^m} e^{iR\varphi_\omega(\lambda + iyu_1)} a_{k_1,k_2}(\lambda + iyu_1)\,\mathrm{d}\lambda.$$

Proof. The theorem is proved in a similar fashion to [36, Lemma 5.4]. It may be useful to observe that for every $\lambda \in \mathbb{C}^m$ such that either $\mathrm{Im}\sqrt{\lambda^2} \notin \pi\mathbb{Z}$ or $\mathrm{Re}\sqrt{\lambda^2} \neq 0$, we have

$$|a_{k_1,k_2}(\lambda)| = \frac{|\lambda|^{n+k_1}\left(\sinh\left(\mathrm{Re}\sqrt{\lambda^2}\right)^2 + \cos\left(\mathrm{Im}\sqrt{\lambda^2}\right)^2\right)^{k_1/2}}{\left(\sinh\left(\mathrm{Re}\sqrt{\lambda^2}\right)^2 + \sin\left(\mathrm{Im}\sqrt{\lambda^2}\right)^2\right)^{(n+k_1)/2}} |\langle\lambda|u_1\rangle|^{k_2},$$

by Lemma 8.12, since $|\sqrt{\lambda^2}| = |\lambda|$. Moreover, a_{k_1,k_2} is bounded on the set $\{\lambda + iyu_1 \colon \lambda \in \mathbb{R}^m, y \in [0, C']\}$ for every $C' \in (0,\pi)$. □

Proof of Proposition 8.26. Define

$$\psi_\omega = \varphi_\omega(\,\cdot\, + iy_\omega u_1) - \varphi_\omega(iy_\omega u_1)$$

and observe that, since $\sqrt{(iy_\omega u_1)^2} = \pm i y_\omega$ and $\omega = \theta(y_\omega)$, $\varphi_\omega(iy_\omega u_1) = i\frac{y_\omega^2}{\sin(y_\omega)^2}$. Therefore, by Lemma 8.28

$$h_{k_1,k_2}(R,t) = e^{-\frac{1}{4}d(x,t)^2}\int_{\mathbb{R}} e^{iR\psi_\omega(\lambda)} a_{k_1,k_2}(\lambda + iy_\omega u_1)\,\mathrm{d}\lambda.$$

We apply Theorem 8.7 to the bounded subsets $\mathcal{F} = \{\psi_\omega \colon \omega \in [0,C]\}$ and $\mathcal{G} = \{a_{k_1,k_2,\omega} \colon \omega \in [0,C]\}$ of $\mathcal{E}(\mathbb{R}^m)$.

2. Elementary computations show that

$$-i\psi_\omega''(0) = \theta'(y_\omega) u_1 \otimes u_1 + \frac{\omega}{y_\omega}\sum_{j=2}^{m} u_j \otimes u_j, \tag{29}$$

so that $\det(-i\psi_\omega''(0)) = \theta'(y_\omega)\left(\frac{\theta(y_\omega)}{y_w}\right)^{m-1} > 0$. The conditions $\psi_\omega(0) = \psi_\omega'(0) = 0$ hold by construction.

3. Consider the mapping $\psi \colon \mathbb{R}^m \times (-\pi,\pi) \ni (\lambda, y) \mapsto \psi_{\theta(y)}(\lambda)$. Then, by the preceding arguments, there is $c > 0$ such that $\partial_1\psi(0,y) = 0$ and $-i\partial_1^2\psi(0,y) \geq c\langle\cdot|\cdot\rangle$ for all $y \in [0,\pi)$; moreover, ψ is analytic by Lemma 8.5. Therefore, by Taylor's formula we may find two constants $\eta > 0$ and $C' > 0$ such that $|\partial_1\psi(\lambda,y)| \geq C'|\lambda|$ for all $\lambda \in B_{\mathbb{R}^m}(0,2\eta)$ and for all $y \in [0, \theta^{-1}(C)]$.

1. Combining [36, Lemmas 5.3 and 5.7], we infer that there is a constant $C'' > 0$ such that

$$\mathrm{Im}\,\psi(\lambda,y) = y\theta(y) + \mathrm{Re}\left[\sqrt{(\lambda + iyu_1)^2}\coth\sqrt{(\lambda+iyu_1)^2}\right] - \frac{y^2}{\sin^2 y} \geq C''|\lambda|$$

whenever $|\lambda| \geq \eta$ and $0 \leq y \leq \theta^{-1}(C)$.

4. Just observe that $\mathcal{G}$ is bounded in $L^\infty(\mathbb{R}^m)$.

By Theorem 8.7, then,

$$\int_{\mathbb{R}^m} e^{iR\psi_\omega(\lambda)} a_{k_1,k_2}(\lambda + iy_\omega u_1)\,\mathrm{d}\lambda = \frac{(2\pi)^m(4\pi)^n}{|x|^m}\Psi(\omega)\sum_{j=0}^{k}\frac{4^j L_{j,\psi_\omega}a_{k_1,k_2,\omega}}{|x|^{2j}} + O\left(\frac{1}{|x|^{m+2k+2}}\right)$$

for $R \to +\infty$, uniformly as ω runs through $[0,C]$. □

Further Developments and Completion of the Proof of Theorem 8.24

We begin by recalling that, for every $j \in \mathbb{N}$,

$$L_{j,\psi_\omega} a_{k_1,k_2,\omega} = i^{-j} \sum_{\mu=0}^{2j} \frac{\langle \psi''_\omega(0)^{-1}\partial|\partial\rangle^{\mu+j}[(\psi_\omega - P_{2,0}\psi_\omega)^\mu a_{k_1,k_2,\omega}](0)}{2^{\mu+j}\mu!(\mu+j)!}. \tag{30}$$

Thus, **1** of Theorem 8.24 follows immediately taking $k = 0$ in Proposition 8.26, since

$$L_{0,\psi_\omega} a_{k_1,k_2,\omega} = a_{k_1,k_2,\omega}(0) = a_{k_1,k_2}(iy_\omega u_1).$$

As for the other developments, observe that, by (29),

$$\begin{aligned}\langle \psi''_\omega(0)^{-1}\partial|\partial\rangle^{\mu+j}&[(\psi_\omega - P_{2,0}\psi_\omega)^\mu a_{k_1,k_2,\omega}](0)\\ &= \sum_{|\alpha|=\mu+j} \frac{(\mu+j)!}{\alpha!} \frac{1}{(i\theta'(y_\omega))^{\alpha_1}} \left(\frac{y_\omega}{i\omega}\right)^{|\alpha|-\alpha_1} \partial^{2\alpha}[(\psi_\omega - P_{2,0}\psi_\omega)^\mu a_{k_1,k_2,\omega}](0), \end{aligned}\tag{31}$$

where

$$\begin{aligned}\partial^{2\alpha}&[(\psi_\omega - P_{2,0}\psi_\omega)^\mu a_{k_1,k_2,\omega}](0)\\ &= \sum_{\substack{\beta\le 2\alpha,\\ |\beta|\ge 3\mu}} \frac{(2\alpha)!}{\beta!\,(2\alpha-\beta)!} \partial^\beta[(\psi_\omega - P_{2,0}\psi_\omega)^\mu](0)\,\partial^{2\alpha-\beta} a_{k_1,k_2}(iy_\omega u_1).\end{aligned}\tag{32}$$

The sum above is restricted to $|\beta| \ge 3\mu$ since $\psi_\omega(\lambda) - P_{2,0}\psi_\omega(\lambda)$ is infinitesimal of order at least 3 for $\lambda \to 0$. Observe moreover that, since $|2\alpha-\beta| = 2|\alpha| - |\beta| \le 2j - \mu$, we have $|2\alpha - \beta| \le 2j$ and $|2\alpha - \beta| = 2j$ if and only if $\mu = 0$ and $\beta = 0$. We first consider the case $\omega \to 0$.

Lemma 8.29. *For every $j \in \mathbb{N}$ such that $2j \le k_2$, define*

$$c_{k_1,k_2,j} \coloneqq (-1)^{k_1+k_2} \frac{3^{k_2-j} k_2!}{2^{k_2-2j}(k_2-2j)!j!}. \tag{33}$$

Then

$$4^j L_{j,\psi_\omega} a_{k_1,k_2,\omega} = c_{k_1,k_2,j}\omega^{k_2-2j} + O\big(\omega^{k_2-2j+1}\big)$$

for $\omega \to 0$.

Proof. Recall that a_{k_1,k_2} is an analytic function on its domain, and observe that[3]

$$a_{k_1,k_2}(\lambda) = (-1)^{k_1} i^{k_2} \lambda_1^{k_2} + O\Big(|\lambda|^{k_2+2}\Big)$$

for $\lambda \to 0$. Therefore, for every $h = 0, \dots, k_2$ we have

$$a^{(h)}_{k_1,k_2}(\lambda) = (-1)^{k_1} i^{k_2} \frac{k_2!}{(k_2-h)!} \lambda_1^{k_2-h} u_1^{\otimes h} + O\Big(|\lambda|^{k_2-h+2}\Big) \tag{34}$$

as $\lambda \to 0$.

We now consider (32). If $|2\alpha - \beta| < 2j$, then by (34)

$$\partial^\beta[(\psi_\omega - P_{2,0}\psi_\omega)^\mu](0)\partial^{2\alpha-\beta} a_{k_1,k_2}(iy_\omega u_1) = O\Big(y_\omega^{k_2-|2\alpha-\beta|}\Big) = O\big(y_\omega^{k_2-2j+1}\big)$$

for $\omega \to 0$. Otherwise, let $|2\alpha - \beta| = 2j$, so that $\mu = 0$ and $\beta = 0$. If $\alpha \ne ju_1$, then (34) implies that

$$\partial^{2\alpha} a_{k_1,k_2}(iy_\omega u_1) = O\big(y_\omega^{k_2-2j+2}\big) = O\big(y_\omega^{k_2-2j+1}\big),$$

while, if $\alpha = ju_1$,

$$\partial_1^{2j} a_{k_1,k_2}(iy_\omega u_1) = (-1)^{k_1+k_2} i^{-2j} \frac{k_2!}{(k_2-2j)!} y_\omega^{k_2-2j}.$$

From this and the fact that

$$\theta'(0) = \lim_{\omega\to 0} \frac{\omega}{y_\omega} = \frac{2}{3}$$

we get the asserted estimate. □

[3]Here and in the following, λ_1 stands for $\langle \lambda | u_1 \rangle$.

Lemma 8.29 above gives the expansions 2 and 3 of Theorem 8.24. Indeed, it allows us to choose k in Proposition 8.26 as

2. $k = k_2/2$ if k_2 is even, since in this case the last term of the sum in (28) is

$$\frac{c_{k_1,k_2,k_2/2}}{|x|^{k_2}} + O\left(\frac{\omega}{|x|^{k_2}}\right)$$

which is bigger than the remainder.

3. $k = (k_2-1)/2$ if k_2 is odd and $|t| \to \infty$, since in this case the last term of the sum in (28) is

$$c_{k_1,k_2,(k_2-1)/2}\frac{\omega}{|x|^{k_2-1}} + O\left(\frac{\omega^2}{|x|^{k_2-1}}\right) = c_{k_1,k_2,(k_2-1)/2}\frac{|t|}{|x|^{k_2+1}} + O\left(\frac{|t|^2}{|x|^{k_2+3}}\right)$$

which is bigger than the remainder, since $|t| \to \infty$.

The case 4 of Theorem 8.24, that is the case when k_2 is odd, $\omega \to 0$ and $|t|$ is bounded, has to be treated in a different way, since $\omega/|x|^{k_2-1}$ may be comparable with the remainder $1/|x|^{k_2+1}$ or even smaller. Thus, the development given above may not be sharp in this case. To overcome this difficulty, we make use of the following lemma. For the reader's convenience, we also consider k_2 even and a stronger statement than the one we need (see Remark 8.38).

Lemma 8.30. *Let $N \in \mathbb{N}$. Then, when $\omega \to 0$,*

$$p_{1,k_1,k_2}(x,t) = \sum_{h=0}^{N} \frac{1}{(2h+1)!}|t|^{2h+1}p_{1,k_1,k_2+2h+1}(x,0) + O\Big(|t|^{2N+3}p_{1,k_1,k_2+2N+3}(x,0)\Big)$$

if k_2 odd; if k_2 is even, then

$$p_{1,k_1,k_2}(x,t) = \sum_{h=0}^{N} \frac{1}{(2h)!}|t|^{2h}p_{1,k_1,k_2+2h}(x,0) + O\Big(|t|^{2N+2}p_{1,k_1,k_2+2N+2}(x,0)\Big).$$

Proof. Assume that k_2 is odd. Then

$$\begin{aligned}
&(4\pi)^n(2\pi)^m\left|p_{1,k_1,k_2}(x,t) - \sum_{h=0}^{N}\frac{1}{(2h+1)!}|t|^{2h+1}p_{1,k_1,k_2+2h+1}(x,0)\right| \\
&= \left|\int_{\mathbb{R}^m} e^{-\frac{|x|^2}{4}|\lambda|\coth|\lambda|}\frac{|\lambda|^{n+k_1}\cosh(|\lambda|)^{k_1}}{\sinh(|\lambda|)^{n+k_1}}\langle\lambda|u_1\rangle^{k_2}\left\{e^{i|t|\langle\lambda|u_1\rangle} - \sum_{h=0}^{N}\frac{[i|t|\langle\lambda|u_1\rangle]^{2h+1}}{(2h+1)!}\right\}\mathrm{d}\lambda\right| \\
&\le \frac{|t|^{2N+3}}{(2N+3)!}\int_{\mathbb{R}^m} e^{-\frac{|x|^2}{4}|\lambda|\coth|\lambda|}\frac{|\lambda|^{n+k_1}\cosh(|\lambda|)^{k_1}}{\sinh(|\lambda|)^{n+k_1}}\langle\lambda|u_1\rangle^{k_2+2N+3}\,\mathrm{d}\lambda \\
&= \frac{(4\pi)^n(2\pi)^m}{(2N+3)!}|t|^{2N+3}|p_{1,k_1,k_2+2N+3}(x,0)|.
\end{aligned}$$

The first assertion is then proved. The proof in when k_2 is even is analogous. □

Thus, the case $\omega \to 0$ while $|t|$ remains bounded when k_2 is odd can be related to the same case when k_2 is even, which is completely described by Lemma 8.29. Observe that the expansion appearing in **4** of Theorem 8.24 is obtained with the choice $N = 0$ in Lemma 8.30.

We finally consider the case $\omega \to \frac{\pi}{2}$, which as above provides the expansions **5** and **6** of Theorem 8.24.

Lemma 8.31. *Define, for $j \in \mathbb{N}$ such that $2j \le k_1$,*

$$b_{k_1,k_2,j} := (-1)^{k_2}\frac{k_1!}{2^{k_1-2j}(k_1-2j)!j!}\left(\frac{\pi}{2}\right)^{n+k_1+k_2}, \tag{35}$$

and, when k_1 is odd,

$$\begin{aligned}
b_{k_1,k_2,(k_1+1)/2} := (-1)^{k_2}\frac{(k_1+1)!}{[(k_1+1)/2]!}\left(\frac{\pi}{2}\right)^{n+k_1+k_2-1}\times \\
\times\left(n+k_1+k_2+\frac{\pi^2}{24}(k_1+2)+\frac{3}{2}(m-1)\right).
\end{aligned} \tag{36}$$

Then, for $\omega \to \frac{\pi}{2}$, if $2j \le k_1$

$$4^j L_{j,\psi_\omega} a_{k_1,k_2,\omega} = b_{k_1,k_2,j}\Big(\omega - \frac{\pi}{2}\Big)^{k_1-2j} + O\Big(\Big(\omega - \frac{\pi}{2}\Big)^{k_1-2j+1}\Big)$$

while if k_1 is odd, then

$$2^{k_1+1} L_{(k_1+1)/2,\psi_\omega} a_{k_1,k_2,\omega} = b_{k_1,k_2,(k_1+1)/2} + O\Big(\omega - \frac{\pi}{2}\Big).$$

Proof. By elementary computations,

$$a_{k_1,k_2,\pi/2}(\lambda) = (-1)^{k_1} i^{k_2-n}\Big(i\frac{\pi}{2}\Big)^{n+k_1+k_2}\lambda_1^{k_1} + (-1)^{k_1} i^{k_2-n}\Big(i\frac{\pi}{2}\Big)^{n+k_1+k_2-1}\times$$
$$\times\Big((n+k_1+k_2)\lambda_1^{k_1+1} + \frac{k_1}{2}\lambda_1^{k_1-1}(\lambda^2-\lambda_1^2)\Big) + O\Big(|\lambda|^{k_1+2}\Big). \qquad (37)$$

Since $a_{k_1,k_2,\pi/2}$ is analytic on its domain, we infer that, for every $h = 0,\dots,k_1$,

$$a^{(h)}_{k_1,k_2,\pi/2}(\lambda) = (-1)^{k_1} i^{k_2-n}\Big(i\frac{\pi}{2}\Big)^{n+k_1+k_2}\frac{k_1!}{(k_1-h)!}\lambda_1^{k_1-h}u_1^{\otimes h} + O\Big(|\lambda|^{k_1-h+1}\Big), \qquad (38)$$

as $\lambda \to 0$.

Consider first j such that $2j \le k_1$. Then, arguing as in the proof of Lemma 8.29 and taking into account (38) and the fact that

$$y_\omega - \frac{\pi}{2} = \frac{1}{2}\Big(\omega - \frac{\pi}{2}\Big) + O\Big[\Big(\omega - \frac{\pi}{2}\Big)^2\Big]$$

when $\omega \to \pi/2$, the first assertion follows.

Let now k_1 be odd, so that $(k_1+1)/2$ is an integer. We shall prove that

$$2^{k_1+1} L_{(k_1+1)/2,\psi_{\pi/2}} a_{k_1,k_2,\pi/2} = b_{k_1,k_2,(k_1+1)/2}.$$

The estimate in the statement will be a consequence of this equality by Taylor expansion.

Since $(\psi''_{\pi/2}(0)^{-1}\partial,\partial)^{\mu+(k_1+1)/2}$ is a differential operator of degree $2\mu+k_1+1$ while $[(\psi_\omega - P_{2,0}\psi_\omega)^\mu a_{k_1,k_2,\omega}]$ is infinitesimal of degree $3\mu+k_1$ at 0, the only terms in the sum (30) (with $j=(k_1+1)/2$) which are not zero are clearly those for which

$$2\mu + k_1 + 1 \ge 3\mu + k_1,$$

namely $\mu \le 1$. Consider first $\mu = 0$. Then, since $\theta'(y_{\pi/2}) = 2$, by (31)

$$\Big\langle \psi''_{\pi/2}(0)^{-1}\partial|\partial\Big\rangle^{(k_1+1)/2} a_{k_1,k_2,\pi/2}(0) = i^{-(k_1+1)/2}\sum_{|\alpha|=(k_1+1)/2}\frac{[(k_1+1)/2]!}{2^{\alpha_1}\alpha!}\partial^{2\alpha}a_{k_1,k_2,\pi/2}(0).$$

Observe that, by (37), $\partial^{2\alpha}a_{k_1,k_2,\pi/2}(0) \ne 0$ only if $\alpha = ((k_1-1)/2)u_1 + u_h$ for some $h = 1,\dots,m$. For the choice $h=1$,

$$\partial_1^{k_1+1}a_{k_1,k_2,\pi/2}(0) = (-1)^{k_1} i^{k_2-n}\Big(i\frac{\pi}{2}\Big)^{n+k_1+k_2-1}(k_1+1)!(n+k_1+k_2)$$

while, for $h = 2,\dots,m$,

$$\partial_1^{k_1-1}\partial_h^2 a_{k_1,k_2,\pi/2}(0) = (-1)^{k_1} i^{k_2-n}\Big(i\frac{\pi}{2}\Big)^{n+k_1+k_2-1}k_1!$$

so that

$$\Big\langle \psi''_{\pi/2}(0)^{-1}\partial|\partial\Big\rangle^{(k_1+1)/2} a_{k_1,k_2,\pi/2}(0) = (-1)^{k_1}\frac{i^{k_2-n-\frac{k_1+1}{2}}}{2^{\frac{k_1+1}{2}}}\times$$
$$\times\Big(i\frac{\pi}{2}\Big)^{n+k_1+k_2-1}(k_1+1)!(n+k_1+k_2+m-1).$$

Consider now $\mu = 1$. Then, by (31)

$$\Big\langle \psi''_{\pi/2}(0)^{-1}\partial|\partial\Big\rangle^{(k_1+3)/2}\big[(\psi_{\pi/2} - P_{2,0}\psi_{\pi/2})a_{k_1,k_2,\pi/2}\big](0)$$
$$= i^{-(k_1+3)/2}\sum_{|\alpha|=(k_1+3)/2}\frac{[(k_1+3)/2]!}{2^{\alpha_1}\alpha!}\partial^{2\alpha}\big[(\psi_{\pi/2} - P_{2,0}\psi_{\pi/2})a_{k_1,k_2,\pi/2}\big](0).$$

Since

$$\psi'''_{\pi/2}(0) = \pi u_1\otimes u_1\otimes u_1 + \frac{2}{\pi}\sum_{h=2}^{m}(u_1\otimes u_h\otimes u_h + u_h\otimes u_1\otimes u_h + u_h\otimes u_h\otimes u_1),$$

we deduce that the only α for which we get a non-zero term in the above sum are $u_1(k_1+1)/2+u_h$ for $h = 1,\dots,m$. Now,

$$\partial_1^{k_1+3}\big[(\psi_{\pi/2} - P_{2,0}\psi_{\pi/2})a_{k_1,k_2,\pi/2}\big](0) = \frac{(k_1+3)!}{3!}(-1)^{k_1}i^{k_2-n}\pi\Big(i\frac{\pi}{2}\Big)^{n+k_1+k_2},$$

while, for $h = 2,\dots,m$,

$$\partial_1^{k_1+1}\partial_h^2\big[(\psi_{\pi/2} - P_{2,0}\psi_{\pi/2})a_{k_1,k_2,\pi/2}\big](0) = \frac{2}{\pi}(-1)^{k_1}i^{k_2-n}\Big(i\frac{\pi}{2}\Big)^{n+k_1+k_2}(k_1+1)!.$$

Therefore,

$$\Big\langle \psi''_{\pi/2}(0)^{-1}\partial|\partial\Big\rangle^{(k_1+3)/2}\big[(\psi_{\pi/2} - P_{2,0}\psi_{\pi/2})a_{k_1,k_2,\pi/2}\big](0)$$
$$= (-1)^{k_1}i^{k_2-\frac{k_1+1}{2}}\frac{(k_1+1)!}{2^{(k_1+3)/2}}i^{-n}\Big(i\frac{\pi}{2}\Big)^{n+k_1+k_2-1}(k_1+3)\Big[\frac{\pi^2}{12}(k_1+2)+m-1\Big]$$

from which one gets the asserted estimate. □

Theorem 8.24 is now completely proved.

The Other Cases

We now consider the case $\omega\to+\infty$. We begin by showing that, when m is odd, matters can be reduced to the case $m = 1$.

Lemma 8.32. *When m is odd, $m\geq 3$,*

$$p^{(m)}_{1,k_1,k_2}(x,t) = \sum_{k=1}^{\frac{m-1}{2}}\frac{c_{m,k}(-1)^k}{(2\pi)^{\frac{m-1}{2}}}\sum_{r=0}^{k_2}\binom{k_2}{r}\frac{(-1)^r(m-1-k)_r}{|t|^{m-1-k+r}}p^{(1)}_{1,k_1,k_2+k-r}(x,|t|), \tag{39}$$

where

$$c_{m,k} = \frac{(m-k-2)!}{2^{\frac{m-1}{2}-k}\big(\frac{m-1}{2}-k\big)!(k-1)!}$$

and $(m-1-k)_r = (m-1-k)\cdots(m-1-k+r-1)$ is the Pochhammer symbol[4]*.*

Proof. Let m be odd, $m\geq 3$. We first pass to polar coordinates in (3) for $k_2 = 0$, and get

$$p^{(m)}_{1,k_1,0}(x,t) = \frac{(-1)^{\frac{m-1}{2}}}{(2\pi)^m(4\pi)^n}\int_0^\infty\int_{S^{m-1}}e^{i\rho|t|\langle\sigma|u_1\rangle}\,\mathrm{d}\mathcal{H}^{m-1}(\sigma)\,e^{-R\rho\coth(\rho)}a_{k_1,m-1}(\rho)\,\mathrm{d}\rho,$$

where $a_{k_1,m-1}$ is the function defined in (6). Since the Bessel function is an elementary function when m is odd, one can prove that (see e.g. [36, equation (6.5)] and references therein)[5]

$$\int_{S^{m-1}}e^{i\rho|t|\langle\sigma|u_1\rangle}\,\mathrm{d}\sigma = 2(2\pi)^{\frac{m-1}{2}}\,\mathrm{Re}\Bigg[\frac{e^{i\rho|t|}}{(\rho|t|)^{m-1}}\sum_{k=1}^{\frac{m-1}{2}}c_{m,k}(-i|t|\rho)^k\Bigg].$$

[4]See, e.g., [38].

[5]This is why we had to restrict to the case $k_2 = 0$; otherwise, we would get the additional term $(\sigma,u_1)^{k_2}$ in the integral on the sphere.

This yields

$$p^{(m)}_{1,k_1,0}(x,t) = \sum_{k=1}^{\frac{m-1}{2}} \frac{c_{m,k}(-1)^k}{(2\pi)^{\frac{m-1}{2}}} \frac{1}{|t|^{m-1-k}} p^{(1)}_{1,k_1,k}(x,|t|)$$

which gives (39), since $p^{(m)}_{1,k_1,k_2}(x,t) = \frac{\partial^{k_2}}{\partial |t|^{k_2}} p^{(m)}_{1,k_1,0}(x,t)$ by definition. □

Corollary 8.33. *Assume that m is odd. Then, when $(x,t)\to\infty$ and $\delta \to 0^+$,*

$$p^{(m)}_{1,k_1,k_2}(x,t) = \frac{(-1)^{k_2}\pi^{k_1+k_2}}{2^{n-k_1+1+\frac{m-1}{2}}} |t|^{n+k_1-1-\frac{m-1}{2}} e^{-\frac{1}{4}d(x,t)^2} \frac{\widetilde{I}_{n+k_1-1}(\kappa\rho(\delta))}{e^{\kappa\rho(\delta)}} [1+g(|x|,|t|)], \tag{40}$$

where g satisfies the estimates (17).

Proof. If $m=1$, the statement reduces to Corollary 8.22. Suppose, then, that $m\geq 3$. Since $p^{(1)}_{1,k_1,r} \asymp p^{(1)}_{1,k_1,k_2}$ for every $0\leq r\leq k_2$ by Corollary 8.22, the principal term in (39) corresponds to $r=0$ and $k=\frac{m-1}{2}$. Hence,

$$p^{(m)}_{1,k_1,k_2}(x,t) = \frac{(-1)^{\frac{m-1}{2}}}{(2\pi)^{\frac{m-1}{2}}} |t|^{-\frac{m-1}{2}} p^{(1)}_{1,k_1,k_2+\frac{m-1}{2}}(x,t) \left[1+O\left(\frac{1}{|t|}\right)\right]. \tag{41}$$

Now substitute the estimate given by Corollary 8.22 into (41). The remainder g in (40) still satisfies (17), since (17) is satisfied by $1/|t|$. □

Assume, now, that m is even and greater than 2. We start by a descent method, in the same spirit of [36]: indeed, observe that the Fourier inversion formula yields

$$p^{(m)}_{1,k_1,0}(x,t) = \int_{\mathbb{R}} p^{(m+1)}_{1,k_1,0}(x,(t,t_{m+1}))\,\mathrm{d}t_{m+1},$$

so that, by differentiating under the integral sign,

$$p^{(m)}_{1,k_1,k_2}(x,t) = \int_{\mathbb{R}} \frac{\partial^{k_2}}{\partial |t|^{k_2}} p^{(m+1)}_{1,k_1,0}(x,(t,t_{m+1}))\,\mathrm{d}t_{m+1}.$$

Observe now that $|(t,t_{m+1})| = |t|\sqrt{1+\frac{t_{m+1}^2}{|t|^2}}$. Therefore, Faà di Bruno's formula applied twice leads to

$$p^{(m)}_{1,k_1,k_2}(x,t) = \sum_{\sum_{j=1}^{k_2} jh_j = k_2} \frac{k_2!}{h!} \int_{\mathbb{R}} p^{(m+1)}_{1,k_1,|h|}(x,(t,t_{m+1}))\, F_h(t,t_{m+1})\,\mathrm{d}t_{m+1},$$

where

$$F_h(t,t_{m+1}) = \prod_{j=1}^{k_2} \left(\sum_{\ell_1+2\ell_2=j} \frac{2^{\ell_1}}{\ell!} (-1)^{|\ell|} \left(-\frac{1}{2}\right)_{|\ell|} |t|^{1-j} \left(1+\frac{t_{m+1}^2}{|t|^2}\right)^{\frac{1}{2}-|\ell|} \right)^{h_j}.$$

Since $F_{(k_2,0,\dots,0)} = \left(1+\frac{t_{m+1}^2}{|t|^2}\right)^{-k_2/2}$, while $F_h = O\left(\frac{1}{|t|}\left(1+\frac{t_{m+1}^2}{|t|^2}\right)^{-1/2}\right)$ otherwise, we have proved the following lemma.

Lemma 8.34. *When m is even, $m\geq 2$,*

$$\begin{aligned} p^{(m)}_{1,k_1,k_2}(x,t) = &\int_{\mathbb{R}} \left(1+\frac{t_{m+1}^2}{|t|^2}\right)^{-\frac{k_2}{2}} p^{(m+1)}_{1,k_1,k_2}(x,(t,t_{m+1}))\,\mathrm{d}t_{m+1} \\ &+ O\left[\frac{1}{|t|} \max_{0\leq r<k_2} \int_{\mathbb{R}} \left(1+\frac{t_{m+1}^2}{|t|^2}\right)^{-\frac{1}{2}} p^{(m+1)}_{1,k_1,r}(x,(t,t_{m+1}))\,\mathrm{d}t_{m+1}\right]. \end{aligned}$$

As a consequence of Lemma 8.34, matters can be reduced to finding the asymptotic expansions of the integrals

$$\int_{\mathbb{R}} \left(1 + \frac{t_{m+1}^2}{|t|^2}\right)^{\alpha} p_{1,k_1,r}^{(m+1)}(x,(t,t_{m+1}))\,dt_{m+1} \tag{42}$$

when $\alpha \in \mathbb{R}$ and $0 \le r \le k_2$. From these, it will also be proved that the remainder in Lemma 8.34 is indeed smaller than the principal part, which *a priori* is not obvious.

With this aim, we define the function $\sigma\colon \mathbb{R} \ni s \mapsto \sqrt{1+s^2}$, and write $t' = (t, t_{m+1}) \in \mathbb{R}^{m+1}$. It is straightforward to check that $|t'| = |t|\sigma\Big(\frac{t_{m+1}}{|t|}\Big)$. Thus, define

$$\delta(s) \coloneqq \frac{\delta}{\sqrt{\sigma(s)}}, \qquad \kappa(s) \coloneqq \kappa\sqrt{\sigma(s)} = 2\pi|t|\delta\sqrt{\sigma(s)}.$$

Obviously, $\delta(0) = \delta$ and $\kappa(0) = \kappa$. If we put a prime on the quantities introduced in Definition 8.3 relative to t', then

$$\delta' = \delta\left(\frac{t_{m+1}}{|t|}\right), \qquad \kappa' = \kappa\left(\frac{t_{m+1}}{|t|}\right).$$

In cases **II**, **III** and **IV**, $|t| \to \infty$ and $\delta \to 0^+$. By substituting (40) into (42) and by the change of variable $\frac{t_{m+1}}{|t|} \mapsto s$ in the integral, we get

$$(42) = \frac{(-1)^r \pi^{r+k_1}}{2^{n-k_1+1+\frac{m}{2}}} |t|^{n+k_1-1-\frac{m}{2}+1} e^{-\frac{1}{4}d(x,t)^2} e^{-\kappa\rho(\delta)} \mathcal{I}_{2\alpha+n+k_1-1-\frac{m}{2}},$$

where

$$\mathcal{I}_\beta = \int_{\mathbb{R}} \sigma(s)^\beta e^{-|t|\pi(\sigma(s)-1)} \widetilde{I}_{n+k_1-1}(\kappa(s)\rho(\delta(s)))[1 + g(|x|, |t|\sigma(s))]\,\mathrm{d}s, \tag{43}$$

and g satisfies the estimates (17). Therefore, matters can be reduced to finding some asymptotic estimates of the integrals $\mathcal{I}_\beta$.

8.3.2 Estimates for $\delta \to 0^+$ and $\kappa \to +\infty$

Theorem 8.35. *For $\delta \to 0^+$ and $\kappa \to +\infty$*

$$p_{1,k_1,k_2}(x,t) = \frac{(-1)^{k_2}\pi^{k_1+k_2}}{4^n(\pi\delta)^{n+k_1-\frac{m+1}{2}}\sqrt{2\pi\kappa^m}} e^{-\frac{1}{4}d(x,t)^2}\left[1 + O\left(\delta + \frac{1}{\kappa}\right)\right].$$

Proof. When m is odd, the theorem follows from Theorem 8.14 and (41). Therefore, we only consider m even. By the preceding arguments, it will be sufficient to study $\mathcal{I}_\beta$ in (43).

Since the argument of the modified Bessel function tends to $+\infty$, we use the development (18), which gives

$$\mathcal{I}_\beta = \frac{(2\pi)^{-n-k_1} e^{\kappa\rho(\delta)}}{\delta^{n+k_1-\frac{1}{2}}|t|^{n+k_1-\frac{1}{2}}} \int_{\mathbb{R}} e^{-|t|\varphi_\delta(s)} \frac{\sigma(s)^{\beta-\frac{1}{4}-\frac{n+k_1-1}{2}}}{\rho(\delta(s))^{n+k_1-\frac{1}{2}}} \times \left[1 + O\left(\frac{1}{\delta|t|\sqrt{\sigma(s)}}\right)\right][1 + g(|x|, |t|\sigma(s))]\,\mathrm{d}s,$$

where

$$\varphi_\delta(s) = \pi[\sigma(s) - 1] + 2\pi\delta\Big[\rho(\delta) - \sqrt{\sigma(s)}\rho(\delta(s))\Big].$$

We first study the principal part of the integral, to which we apply Laplace's method (see Remark 8.8) with

$$\mathcal{F} = \{\varphi_\delta \colon \delta \in [0,\delta_2]\}, \qquad \mathcal{G} = \left\{ \frac{\sigma(\cdot)^{\beta-\frac{1}{4}-\frac{n+k_1-1}{2}}}{\rho(\delta(\cdot))^{n+k_1-\frac{1}{2}}} : \delta \in [0,\delta_2] \right\}$$

for some δ_2, smaller than the δ_1 of Lemma 8.17, to be determined.

2. It is easily seen that $\varphi_\delta(0)=0$. In addition,

$$\varphi'_\delta(s)=\pi\frac{s}{\sigma(s)}\left[1-\delta(s)\rho(\delta(s))+\frac{\delta^2}{\sigma(s)^{\frac{3}{2}}}\rho'(\delta(s))\right], \tag{44}$$

so that $\varphi'_\delta(0)=0$ and $\varphi''_\delta(0)=\pi(1-\delta\rho(\delta)+\delta^2\rho'(\delta))$. Observe that there is $\delta_2>0$, which we may choose smaller than δ_1, such that

$$1-\delta(s)\rho(\delta(s))+\frac{\delta^2}{\sigma(s)^{\frac{3}{2}}}\rho'(\delta(s))\geq\frac{1}{2} \tag{45}$$

for every s and every $\delta\in[0,\delta_2]$. Therefore, $\varphi''_\delta(0)\geq\frac{\pi}{2}$ for every $\delta\in[0,\delta_2]$.

3. By (44) and (45), for $s\in\mathbb{R}$ and $\delta\in(0,\delta_2)$,

$$|\varphi'_\delta(s)|\geq\frac{\pi}{2\sigma(s)}|s|. \tag{46}$$

In particular, $|\varphi'_\delta(s)|\geq\frac{\pi}{2\sigma(2)}|s|$ for every $s\in[-2,2]$.

1. Observe that $\varphi'_\delta(s)=\operatorname{sgn}(s)|\varphi'_\delta(s)|$ by (44); then, by (46),

$$\varphi_\delta(s)=\int_0^s\operatorname{sgn}(s)|\varphi'_\delta(u)|\,\mathrm{d}u=\left|\int_0^s|\varphi'_\delta(u)|\,\mathrm{d}u\right|\geq\frac{\pi}{2\sigma(s)}\left|\int_0^s|u|\,\mathrm{d}u\right|\geq\frac{\pi s^2}{4\sigma(s)}$$

for every $s\in\mathbb{R}$, since σ is even and increasing on $[0,\infty)$.

4. Taking into account the definition of σ and the continuity of ρ at zero, we get $g(s)=O\left(|s|^{\beta-\frac{1}{4}-\frac{n+k_1-1}{2}}\right)$ for $s\to\infty$, uniformly in $g\in\mathcal{G}$.

By Theorem 8.7, then,

$$\begin{aligned}\int_{\mathbb{R}}e^{-|t|\varphi_\delta(s)}\frac{\sigma(s)^{\beta-\frac{1}{4}-\frac{n+k_1-1}{2}}}{\rho(\delta(s))^{n+k_1-\frac{1}{2}}}\,\mathrm{d}s&=\sqrt{\frac{2}{|t|(1-\delta\rho(\delta)+\delta^2\rho'(\delta))}}\left[1+O\left(\frac{1}{|t|}\right)\right]\\&=\sqrt{\frac{2}{|t|}}\left[1+O\left(\delta+\frac{1}{|t|}\right)\right].\end{aligned}$$

The remainder can be treated similarly, and with the same arguments as above one gets

$$\begin{aligned}\int_{\mathbb{R}}e^{-|t|\varphi_\delta(s)}\frac{\sigma(s)^{\beta-\frac{1}{4}-\frac{n+k_1-1}{2}}}{\rho(\delta(s))^{n+k_1-\frac{1}{2}}}&\left[O\left(\frac{1}{\delta|t|\sqrt{\sigma(s)}}\right)+O\left(\frac{1}{\kappa}+\delta\right)\right]\mathrm{d}s\\&=\sqrt{\frac{2}{|t|}}\left[1+O\left(\delta+\frac{1}{|t|}\right)\right]O\left(\frac{1}{\delta|t|}+\frac{1}{\kappa}+\delta\right)=\sqrt{\frac{2}{|t|}}O\left(\frac{1}{\kappa}+\delta\right)\end{aligned}$$

since $\frac{1}{\delta|t|}=\frac{2\pi}{\kappa}=O\left(\frac{1}{\kappa}\right)$ and $1/\sqrt{\sigma(s)}\leq1$ for every $s\in\mathbb{R}$. The proof is complete. □

8.3.3 Estimates for $\delta\to0^+$ and κ bounded

These two cases can be treated together and the principal part of $p^{(m)}_{1,k_1,k_2}$ is easy to get. The remainders are more tricky, since when passing from the m-dimensional variable t to the $(m+1)$-dimensional variable t' the asymptotic conditions in **II**, **III** and **IV** do not correspond to those in **II'**, **III'**, **IV'** (these symbols standing for the cases relative to $m+1$); on the contrary, they mix together according to the values of the additional variable t_{m+1}.

Theorem 8.36. *Fix $C>1$. If $\delta\to0^+$ while $1/C\leq\kappa\leq C$, then*

$$p_{1,k_1,k_2}(x,t)=\frac{(-1)^{k_2}\pi^{k_1+k_2}}{4^n(\pi\delta)^{n+k_1-\frac{m+1}{2}}\kappa^{\frac{m-1}{2}}}e^{-\frac{1}{4}d(x,t)^2}e^{-\kappa}I_{n+k_1-1}(\kappa)[1+O(\delta)].$$

When $\kappa\to0^+$ and $|t|\to+\infty$

$$p_{1,k_1,k_2}(x,t)=\frac{(-1)^{k_2}\pi^{k_1+k_2}}{2^{2n+\frac{m-1}{2}}(n+k_1-1)!}|t|^{n+k_1-1-\frac{m-1}{2}}e^{-\frac{1}{4}d(x,t)^2}\left[1+O\left(\kappa+\frac{1}{|t|}\right)\right].$$

Proof. The theorem holds when m is odd by Theorem 8.19 and (41). When m is even, we apply Laplace's method to $\mathcal{I}_\beta$. We first deal with the principal part. Define

$$\varphi(s) = \pi\sigma(s) - \pi,$$

so that Theorem 8.7 will be applied to

$$\mathcal{F} = \{\varphi\}, \qquad \mathcal{G} = \{\sigma(\cdot)^\beta \widetilde{I}_{n+k_1-1}(\kappa(\cdot)\rho(\delta(\cdot)))\colon \delta \in [0,\delta_1),\ \kappa \in [0,C]\}$$

where δ_1 is that of Lemma 8.17.

2. Notice that $\varphi(0) = 0$, that $\varphi'(s) = \pi\frac{s}{\sigma(s)}$, and that $\varphi''(0) = \pi$.

1. Observe that $\varphi(s) = \pi\frac{s^2}{1+\sqrt{1+s^2}} \geq \pi\frac{s^2}{2+|s|}$, for every $s \in \mathbb{R}$.

3. It is easily seen that $|\varphi'(s)| \geq \frac{\pi}{\sigma(1)}|s|$ for every $s \in [-1,1]$.

4. Recall that by (18)

$$\widetilde{I}_{n+k_1-1}(\kappa(s)\rho(\delta(s))) = O\Big(e^{\kappa(s)\rho(\delta(s))}\Big) = O\Big(e^{\kappa\sqrt{\sigma(s)}}\Big)$$

as $s \to \infty$, uniformly as $\kappa \in [0,C]$ and $\delta \in [0,\delta_1)$. Hence, there is a constant $c_1 > 0$ such that $|\sigma(s)^\beta \widetilde{I}_{n+k_1-1}(\kappa(s)\rho(\delta(s)))| \leq c_1 e^{c_1|s|}$.

Therefore, by Theorem 8.7

$$\int_{\mathbb{R}} e^{-|t|\varphi(s)}\sigma(s)^\beta \widetilde{I}_{n+k_1-1}(\kappa(s)\rho(\delta(s)))\,\mathrm{d}s = \sqrt{\frac{2}{|t|}}\widetilde{I}_{n+k_1-1}(\kappa\rho(\delta))\left[1 + O\left(\frac{1}{|t|}\right)\right]$$

uniformly in κ and δ. Since $\widetilde{I}_{n+k_1-1}(\kappa\rho(\delta)) - \widetilde{I}_{n+k_1-1}(\kappa) = O(\kappa\rho(\delta) - \kappa) = O(\delta^2)$ uniformly as $\kappa \in [0,C]$ by Taylor's formula, we are done with the principal part. We now deal with the remainders, namely

$$\mathcal{I}'_\beta = \int_{\mathbb{R}} e^{-|t|\varphi(s)}\sigma(s)^\beta \widetilde{I}_{n+k_1-1}(\kappa(s)\rho(\delta(s)))g(|x|,|t|\sigma(s))\,\mathrm{d}s,$$

where

$$g(|x|,|t|\sigma(s)) = \begin{cases} O\Big(\delta(s) + \frac{1}{\kappa(s)}\Big) & \text{if } \delta(s) \to 0^+ \text{ and } \kappa(s) \to +\infty, \\ O(\delta(s)) & \text{if } \delta(s) \to 0^+ \text{ and } \kappa(s) \in [1/C', C'], \\ O\Big(\frac{1}{|t|\sigma(s)} + \kappa(s)\Big) & \text{if } \delta(s) \to 0^+ \text{ and } \kappa(s) \to 0^+ \end{cases}$$

for every $C' > 1$. Since $\delta(s) \leq \delta$ for every $s \in \mathbb{R}$, we may find some positive constants C'', $\delta_2 \leq \delta_1$, where δ_1 is that of Lemma 8.17, and $\kappa_2 \leq \kappa_1$ such that

$$|g(|x|,|t|\sigma(s))| \leq \begin{cases} C''\Big(\delta(s) + \frac{1}{\kappa(s)}\Big) & \text{when } \delta \leq \delta_2,\ \kappa(s) \geq \kappa_1, \\ C''\delta(s) & \text{when } \delta \leq \delta_2,\ \kappa_2 \leq \kappa(s) \leq \kappa_1, \\ C''\Big(\frac{1}{|t|\sigma(s)} + \kappa(s)\Big) & \text{when } \delta \leq \delta_2,\ \kappa(s) \leq \kappa_2. \end{cases}$$

We shall split the integrals accordingly. Notice first that we may assume also that $\kappa_2 \leq 1/(2C) \leq 2C \leq \kappa_1$, and, up to taking a smaller δ_2, that

$$\varphi(s) - 2\pi\delta\sqrt{\sigma(s)}\rho(\delta(s)) \geq \frac{1}{2}|s|$$

whenever $|s| \geq 2$ and $\delta \in [0,\delta_2)$.

Consider case **III**, that is, $\kappa \in [1/C, C]$. We split

$$\mathcal{I}'_\beta = \int_{\kappa(s)\leq\kappa_1} + \int_{\kappa(s)\geq\kappa_1} = \mathcal{I}'_{\beta,1} + \mathcal{I}'_{\beta,2}.$$

Observe that $\kappa(s)\geq\kappa_1$ if and only if $|s|\geq\sqrt{\frac{\kappa_1^4}{\kappa^4}-1}=:s_{1,\kappa}\geq 2$. Since

$$\widetilde{I}_{n+k_1-1}(\kappa(s)\rho(\delta(s))e^{-|t|\varphi(s)}=O\Big(e^{|t|[2\pi\delta\sqrt{\sigma(s)}\rho(\delta(s))-\varphi(s)]}\Big)=O\Big(e^{-\frac{1}{2}|t||s|}\Big)$$

as $s\to\infty$, and since $\delta=O\big(\frac{1}{\kappa}\big)=O(1)$ in case **III**, we get

$$|\mathcal{I}'_{\beta,2}|\leq C'\left(\delta+\frac{1}{\kappa}\right)\int_{|s|\geq s_{1,\kappa}}\sigma(s)^{\beta-\frac{1}{2}}\widetilde{I}_{n+k_1-1}(\kappa(s)\rho(\delta(s)))e^{-|t|\varphi(s)}\,\mathrm{d}s=O\Big(e^{-\frac{s_{1,\kappa}}{4}|t|}\Big),$$

which is negligible relative to $\frac{1}{|t|^{3/2}}$. By Laplace's method, in addition,

$$|\mathcal{I}'_{\beta,1}|\leq C'\delta\int_{|s|\leq s_{1,\kappa}}\sigma(s)^{\beta-\frac{1}{2}}\widetilde{I}_{n+k_1-1}(\kappa(s)\rho(\delta(s)))e^{-|t|\varphi(s)}\,\mathrm{d}s=O\left(\delta\frac{1}{\sqrt{|t|}}\right)$$

with the same arguments as above. This concludes the study of case **III**.

Consider, now, case **IV**, that is, $\kappa\to 0^+$. We split

$$\mathcal{I}'_\beta=\int_{\kappa(s)\leq\kappa_2}+\int_{\kappa_2\leq\kappa(s)\leq\kappa_1}+\int_{\kappa(s)\geq\kappa_1}=\mathcal{I}'_{\beta,1}+\mathcal{I}'_{\beta,2}+\mathcal{I}'_{\beta,3}.$$

Observe that $\kappa(s)\geq\kappa_2$ if and only if $s\geq\sqrt{\frac{\kappa_2^4}{\kappa^4}-1}=:s_{2,\kappa}$, and $s_{1,\kappa}\geq s_{2,\kappa}\geq 2$ if κ is sufficiently small. Exactly as above, we get

$$|\mathcal{I}'_{\beta,3}|\leq C'\left(\delta+\frac{1}{\kappa}\right)\int_{|s|\geq s_{1,\kappa}}\sigma(s)^{\beta-\frac{1}{2}}\widetilde{I}_{n+k_1-1}(\kappa(s)\rho(\delta(s)))e^{-|t|\varphi(s)}\,\mathrm{d}s=O\left(\frac{1}{\kappa}e^{-\frac{s_{1,\kappa}}{4}|t|}\right)$$

which is negligible relative to $\frac{1}{|t|^{3/2}}$. Then,

$$|\mathcal{I}'_{\beta,2}|\leq C'\delta\int_{s_{2,\kappa}\leq|s|\leq s_{1,\kappa}}\sigma(s)^{\beta-\frac{1}{2}}\widetilde{I}_{n+k_1-1}(\kappa(s)\rho(\delta(s)))e^{-|t|\varphi(s)}\,\mathrm{d}s=O\Big(\delta\,e^{-\frac{s_{2,\kappa}}{4}|t|}\Big),$$

which is negligible relative to $\frac{1}{|t|^{3/2}}$ in case **IV**. Finally,

$$\begin{aligned}|\mathcal{I}'_{\beta,1}|&\leq C'\int_{|s|\leq s_{2,\kappa}}\sigma(s)^{\beta}\widetilde{I}_{n+k_1-1}(\kappa(s)\rho(\delta(s)))e^{-|t|\varphi(s)}\left(\sqrt{\sigma(s)}\kappa+\frac{1}{\sigma(s)|t|}\right)\mathrm{d}s\\&=O\left[\frac{1}{\sqrt{|t|}}\left(\frac{1}{|t|}+\kappa\right)\right],\end{aligned}$$

by Laplace's method as above. The proof is complete. □

We can finally state the following corollary, which is the natural extension of Corollary 8.22.

Corollary 8.37. *For $(x,t)\to\infty$ and $\delta\to 0^+$*

$$p_{1,k_1,k_2}(x,t)=\frac{(-1)^{k_2}\pi^{k_1+k_2}}{2^{n-k_1+1+\frac{m-1}{2}}}|t|^{n+k_1-\frac{m+1}{2}}e^{-\frac{1}{4}d(x,t)^2}\frac{\widetilde{I}_{n+k_1-1}(\kappa\rho(\delta))}{e^{\kappa\rho(\delta)}}[1+g(|x|,|t|)],$$

where

$$g(|x|,|t|)=\begin{cases}O\big(\delta+\frac{1}{\kappa}\big) & \text{if } \delta\to 0^+ \text{ and } \kappa\to+\infty,\\ O(\delta) & \text{if } \delta\to 0^+ \text{ and } \kappa\in[1/C,C],\\ O\Big(\frac{1}{|t|}+\kappa\Big) & \text{if } \delta\to 0^+ \text{ and } \kappa\to 0^+\end{cases}$$

for every $C>0$.

We have not been able to find a single function which displays the asymptotic behaviour of $p_{1,k_1,k_2}(x,t)$ as $(x,t)\to\infty$, though we showed that the exponential decrease is the same in the four cases. This is also the same decrease found by Eldredge [36, Theorems 4.2 and 4.4], when $k_1=k_2=0$ and for the horizontal gradient, and Li [56, Theorems 1.4 and 1.5], when $k_1=k_2=0$. Notice that in [56, Theorem 1.5 and the following Remark (1)] the remainders for $k_1=k_2=0$ seem to be better than the one we put in Corollary 8.37, but they reduce to ours when developing the estimates in a more convenient form in cases **II** and **IV**, as we did in Theorems 8.35 and 8.36.

Remark 8.38. Our sharp estimates for p_{1,k_1,k_2} can be used to obtain asymptotic estimates of all the derivatives of the heat kernel p_1. Indeed, Faà di Bruno's formula leads to

$$\begin{aligned}\frac{\partial^{|\gamma|}}{\partial x^{\gamma_1}\partial t^{\gamma_2}}p_1(x,t) = \gamma_1!\gamma_2! \sum_{\eta,\mu,\beta} \frac{|\mu|!2^{|\mu_1|-|\gamma_1|}}{\eta!\mu!\beta!}\left[\prod_{h=1}^{|\mu|}\left(\frac{\left(\frac{1}{2}\right)_h}{h!}\right)^{\beta_h}\right] x^{\eta_1}\operatorname{sgn}(t)^{\mu_1}\times \\ \times |t|^{|\beta|-|\gamma_2|} p_{1,|\eta|,|\beta|}(x,t),\end{aligned} \tag{47}$$

where the sum is extended to all $\eta = (\eta_1, \eta_2) \in \mathbb{N}^{2n} \times \mathbb{N}^{2n}$, $\mu = (\mu_1, \mu_2) \in \mathbb{N}^m \times \mathbb{N}^m$ and $\beta \in \mathbb{N}^{|\mu|}$ such that

$$\gamma_1 = \eta_1 + 2\eta_2, \qquad \gamma_2 = \mu_1 + 2\mu_2, \qquad \sum_{h=1}^{|\mu|} h\beta_h = |\mu|.$$

Anyway, the sharp asymptotic expansions we explicitly provided in Theorems 8.24, 8.35 and 8.36 may not be enough to get directly *sharp* asymptotic estimates of any desired derivative of p_1: some cancellations among the principal terms may indeed occur in (47). Nevertheless, by inspecting case by case, the interested reader could consider as many terms of the expansions given by Theorem 8.7 or Lemma 8.20 as necessary. In the case when $t \to 0$, one may also make use of Lemma 8.30 before expanding each term: a suitable choice for N gets rid of the negative powers of $|t|$ appearing in (47).

Appendix A

Appendix

A.1 Banach Algebras

Here we recall some basic notation concerning Banach algebras. See [67, 68] for more details.

Definition A.1. A Banach algebra A is an associative algebra over $\mathbb{C}$ which is endowed with a complete norm such that $\|xy\| \leqslant \|x\|\|y\|$ for every $x, y \in A$, and such that $\|e\| = 1$ if A has a unit e.

Definition A.2. Let A be a Banach algebra. Then, we denote by $\Delta'(A)$ the set of characters of A, that is, the set of (continuous) homomorphisms of $\mathbb{C}$-algebras from A into $\mathbb{C}$. We shall endow $\Delta'(A)$ with the topology of pointwise convergence, that is, the topology induced by the weak topology $\sigma(A', A)$.

We shall denote by $\Delta(A)$ the Gelfand spectrum of A, that is, $\Delta'(A) \setminus \{ 0 \}$.

Recall that $\Delta'(A)$ is a compact space which is also metrizable if A is separable. Therefore, $\Delta(A)$ is a locally compact space which has a countable base if A is separable; $\Delta(A)$ is actually compact if A has a unit.

Definition A.3. Let A be a Banach algebra and take $x \in A$. If A is a unital algebra, then the spectrum $\sigma(x)$ of x in A is the set of $\lambda \in \mathbb{C}$ such that $x - \lambda e$ is not invertible. If A is not unital, then the spectrum $\sigma(x)$ of x is the spectrum of $(x, 0)$ in the unitization of A, that is, the space $A \oplus \mathbb{C}$ with product $(x, \lambda)(y, \mu) \coloneqq (xy + \mu x + \lambda y, \lambda\mu)$ and norm $\|(x, \lambda)\| \coloneqq \|x\| + |\lambda|$ ($x, y \in A$ and $\lambda, \mu \in \mathbb{C}$).

Definition A.4. Let A be a Banach algebra. Then, we shall denote by $\mathcal{G}$ the Gelfand transform, that is, the homomorphism of A into $C_0(\Delta(A))$ defined by $\mathcal{G}(x)(\chi) \coloneqq \chi(x)$ for every $x \in A$ and for every $\chi \in \Delta(A)$.

Definition A.5. A Banach $*$-algebra A is a Banach algebra endowed with an isometry $*\colon A \to A$ such that $(\lambda v + \mu w)^* = \overline{\lambda} v + \overline{\mu} w$, $(vw)^* = w^* v^*$ and $(v^*)^* = v$ for every $v, w \in A$ and for every $\lambda, \mu \in \mathbb{C}$.

Definition A.6. Let A be a Banach $*$-algebra. Then, A is symmetric if $\sigma(x^* x) \subseteq \mathbb{R}_+$ for every $x \in A$. Equivalently, if $\sigma(x) \subseteq \mathbb{R}$ for every $x \in A$ such that $x = x^*$.

A.2 The Spectral Theorem

In this section we recall some basic facts and notation concerning the spectral theorem for commuting self-adjoint operators on a hilbertian space. H will denote a complex hilbertian space.

Definition A.7. An operator T on H is a linear mapping from a vector subspace $\operatorname{dom}(T)$ of H into H. If $\operatorname{dom}(T)$ is dense in H, then T is said to be densely defined. We shall generally write $T \cdot v$ instead of $T(v)$ to denote the evaluation of T at $v \in \operatorname{dom}(T)$.

If T is a densely defined operator on H, then we shall denote by T^* the operator on H defined as follows: $v \in \mathrm{dom}(T^*)$ if and only if there is $w \in H$ such that, for every $v' \in \mathrm{dom}(T)$,

$$\langle T \cdot v' | v \rangle = \langle v' | w \rangle.$$

In this case, w is uniquely determined and $T^* \cdot v \coloneqq w$.

Definition A.8. An operator T on H is closed if its graph is closed in $H \times H$; T is closable if the closure of its graph is the graph of an operator, which is denoted by $\overline{T}$.

If T is closed, its spectrum $\sigma(T)$ is the set of $\lambda \in \mathbb{C}$ such that $\lambda I_H - T$ has no continuous inverse.

Recall that, if T is densely defined, then T^* is closed; in addition, T is closable if and only if T^* is densely defined (cf. [74, Theorem 13.9 and 13.12]).

Definition A.9. Let T_1 and T_2 be two operators on H. Then, we define

$$T_1 + T_2 \colon \mathrm{dom}(T_1) \cap \mathrm{dom}(T_2) \ni v \mapsto T_1 \cdot v + T_2 \cdot v,$$

and

$$T_1 \cdot T_2 \colon T_2^{-1}(\mathrm{dom}(T_1)) \ni v \mapsto T_1 \cdot (T_2 \cdot v_2).$$

Definition A.10. Let T be a densely defined operator on H. Then:

- T is self-adjoint if $T = T^*$;
- T is essentially self-adjoint if it is closable and $\overline{T}$ is self-adjoint;
- T is symmetric if $T \subseteq T^*$; equivalently, if $\langle T \cdot v | v \rangle \in \mathbb{R}$ for every $v \in \mathrm{dom}(T)$;
- T is positive if $\langle T \cdot v | v \rangle \geqslant 0$ for every $v \in \mathrm{dom}(T)$.

Now we define the measures which will be used to define the functional calculus associated with a family of commuting self-adjoint operators on H. We call them spectral measures to simplify the notation, even though they need not be associated with any family of self-adjoint operators in the sense of Theorem A.14.

Definition A.11. Let μ be a Radon measure on a locally compact space X with values in $\mathcal{L}_s(H)$. We say that μ is a spectral measure if the following hold:

- $\mu(\varphi) \cdot \mu(\psi) = \mu(\varphi\psi)$ for every $\varphi, \psi \in C_c(X)$;
- $\mu(\varphi)^* = \mu(\overline{\varphi})$ for every $\varphi \in C_c(X)$;
- $\chi_X \in \mathcal{L}(\mu)$ and $\int_X \mathrm{d}\mu = I_H$.

If $f \colon X \to \mathbb{C}$ is μ-measurable, we shall denote by $\mathcal{D}_f$ the set of $v \in H$ such that $f \in \mathcal{L}(\mu \cdot v)$, where $\mu \cdot v$ is the measure $\varphi \mapsto \mu(\varphi) \cdot v$. We shall then define an operator $\int_X f \, \mathrm{d}\mu$ on H with domain $\mathcal{D}_f$ so that $\left(\int_X f \, \mathrm{d}\mu\right) \cdot v \coloneqq \int_X f \, \mathrm{d}(\mu \cdot v)$ for every $v \in \mathcal{D}_f$.

Proposition A.12. *Let μ be a spectral measure on a locally compact space X with values in $\mathcal{L}_s(H)$. Then, the following hold:*

- *for every μ-measurable function $f \colon X \to \mathbb{C}$, $\mathcal{D}_f$ is the set of $v \in H$ such that $f \in \mathcal{L}^2(\langle \mu \cdot v | v \rangle)$. In addition, $\mathcal{D}_f$ is dense in H and*

$$\left\| \int_X f \, \mathrm{d}\mu \cdot v \right\|^2 = \int_X |f|^2 \, \mathrm{d}\langle \mu \cdot v | v \rangle$$

  for every $v \in \mathcal{D}_f$;
- *$\left(\int_X f \, \mathrm{d}\mu\right)^* = \int_X \overline{f} \, \mathrm{d}\mu$ for every μ-measurable function $f \colon X \to \mathbb{C}$. In particular, $\int_X f \, \mathrm{d}\mu$ is a closed operator;*

- *if f and g are μ-measurable functions from X into $\mathbb{C}$, then*

$$\int_X f\,\mathrm{d}\mu + \int_X g\,\mathrm{d}\mu \subseteq \int_X (f+g)\,\mathrm{d}\mu \subseteq \overline{\int_X f\,\mathrm{d}\mu + \int_X g\,\mathrm{d}\mu}$$

and

$$\int_X f\,\mathrm{d}\mu \cdot \int_X g\,\mathrm{d}\mu \subseteq \int_X fg\,\mathrm{d}\mu \subseteq \overline{\int_X f\,\mathrm{d}\mu \cdot \int_X g\,\mathrm{d}\mu};$$

in addition, the domain of $\int_X f\,\mathrm{d}\mu \cdot \int_X g\,\mathrm{d}\mu$ is $\mathcal{D}_g \cap \mathcal{D}_{fg}$;

- *if $f\colon X \to \mathbb{C}$ is μ-measurable, then $\int_X f\,\mathrm{d}\mu$ is self-adjoint (resp. positive) if and only if f is real-valued (resp. positive).*

Definition A.13. Let T_1 and T_2 be two self-adjoint operators on H. Then, T_1 and T_2 commute if the bounded operators $(\lambda I_H - T_1)^{-1}$ and $(\lambda I_H - T_2)^{-1}$ commute in the ordinary sense for some (hence every) $\lambda \notin \sigma(T_1) \cup \sigma(T_2) \subseteq \mathbb{R}$.

Theorem A.14. *Let $(T_\alpha)_{\alpha\in A}$ be a finite family of commuting self-adjoint operators on H. Then, there is a unique spectral measure μ on $\mathbb{R}^A$ such that*

$$T_\alpha = \int_{\mathbb{R}^A} \lambda_\alpha \,\mathrm{d}\mu(\lambda)$$

for every $\alpha \in A$.

In addition, if $T \in \mathcal{L}(H)$ and $T \cdot T_\alpha \subseteq T_\alpha \cdot T$ for every $\alpha \in A$, then $T \cdot \int_X f\,\mathrm{d}\mu \subseteq \int_X f\,\mathrm{d}\mu \cdot T$ for every μ-measurable function $f\colon \mathbb{R}^A \to \mathbb{C}$.

Extending the notion of spectral measures to the case of Hausdorff spaces which are not necessarily locally compact, one may extend the preceding result to arbitrary families of commuting self-adjoint operators on H.

Definition A.15. Let $T_A = (T_\alpha)_{\alpha\in A}$ be a finite family of commuting self-adjoint operators on H. Then, we shall denote by $\sigma(T_A)$ the support of the unique spectral measure μ on $\mathbb{R}^A$ such that $T_\alpha = \int_{\mathbb{R}^A} \lambda_\alpha \,\mathrm{d}\mu(\lambda)$ for every $\alpha \in A$. In addition, we shall simply write $f(T_A)$ instead of $\int_{\mathbb{C}^A} f\,\mathrm{d}\mu$ for every μ-measurable function $f\colon \mathbb{R}^A \to \mathbb{C}$.

A.3 Transference

Here we recall some basic results on transference. We begin with the definition of amenable groups, which we state in terms of Leptin's condition (cf. [69, Corollary 4.14]). See [69] or [70] for other characterizations of amenable groups.

Definition A.16. A locally compact group G is amenable if for every compact subset K of G and for every $\varepsilon > 0$ there is a compact subset H of G such that, if β is a left Haar measure on G, then $\beta(KH) \leqslant (1+\varepsilon)\beta(H)$.

Notice that, since G is amenable if and only if its opposite G° is amenable (cf. [70, Corollary 4.20]), an analogous characterization holds for right Haar measures.

Definition A.17. Let G be a locally compact group, endowed with a right Haar measure β, and take $\mu \in \mathcal{M}^1(G)$, a Banach space F, and $p \in [1,\infty]$. Then, the mapping $f \mapsto f *^\beta \mu$ induces an endomorphism of $L^p(\beta;F)$ (cf. Proposition 1.69 for the scalar case); we shall denote by $N_{p,F}(\mu)$ its norm.

Notice that $N_{p,F}$ does *not* depend on the choice of β. The following result is a consequence of [89, Lemma 2.10 of Chapter XV].

Proposition A.18. *Let G be a locally compact group, endowed with a right Haar measure β, and take $\mu \in \mathcal{M}^1(G)$, a Hilbertian space H, and $p \in [1,\infty[$. Then, $N_{p,H}(\mu) = N_{p,\mathbb{R}}(\mu)$.*

The following result is a straightforward generalization of [46, Lemma 13.3].

Proposition A.19. *Let G be a locally compact group, endowed with a right Haar measure β, let ν be a positive measure on a σ-algebra of subsets of a set X, and take $\mu \in \mathcal{M}^1(G)$. Let F_1 be a Banach space and let F_2 be a closed subspace of $L^p(\nu; F_1)$ for some $p \in [1, \infty[$. Then, $N_{p,F_1}(\mu) = N_{p,F_2}(\mu)$.*

Theorem A.20. *Let G be a locally compact group, and take $\mu \in \mathcal{M}^1(G)$, a Banach space F, and $p \in [1, \infty[$. Let π be an equicontinuous representation of G in F, and define $C := \sup_{g \in G} \|\pi(g)\|$. Then,*

$$\|\pi_*(\mu)\| \leqslant C^2 N_{p,F}(\mu).$$

For the proof, follow that of [46, Theorem 13.1] with minor modifications. By approximation, one may then prove a similar result when G is a Lie group and $\mu \in W^{0,-\infty,1}(G)$.

A.4 Miscellaneous Results

Lemma A.21. *Let M be a locally compact manifold of class C^r ($r \in \mathbb{N} \cup \{\infty\}$) which admits a countable base, and let X be a Hausdorff space endowed with a Radon measure μ. Let $f : X \times M \to \mathbb{C}$ be a mapping such that $f(x, \cdot)$ is of class C^r for every $x \in X$, while $f(\cdot, y)$ is μ-measurable for every $y \in M$. Then, the mapping $X \ni x \mapsto f(x, \cdot) \in \mathcal{E}^r(M)$ is μ-measurable.*

Proof. Notice first that, since the notion of measurability is local, we may assume that X is compact. Now, since M admits a countable base, there is a sequence of local charts $(U_j, \varphi_j)_{j \in \mathbb{N}}$ of M which covers M. Hence, the mapping $\psi \mapsto (\psi \circ \varphi_j^{-1})_{j \in \mathbb{N}}$ identifies $\mathcal{E}(M)$ with a subspace of the countable product $\prod_{j \in \mathbb{N}} \mathcal{E}^r(\varphi_j(U_j))$. Since M is also locally compact, by means of [19, Theorem 1 of Chapter IV, § 5, No. 3] we may further assume that M is $\mathbb{R}^n$ for some n.

Then, it is clear that $\mathcal{E}^r(M)$ is a separable Fréchet space; denote by ν the Lebesgue measure on $\mathbb{R}^n$. Now, take $\varphi \in \mathcal{D}^r(M)$, and define

$$T_j := \sum_{k \in \mathbb{N}^n} \varphi(2^{-j} k) 2^{-nj} \delta_{2^{-j} k}$$

for every $j \in \mathbb{N}$. Then, clearly T_j is a (Radon) measure with finite support, and T_j converges to $\varphi \cdot \nu$ in $\mathcal{E}_c'^0(M)$ as $j \to \infty$. By assumption, the mappings

$$X \ni x \mapsto \langle f(x, \cdot), T_j \rangle$$

are μ-measurable for every $j \in \mathbb{N}$; hence, so is their pointwise limit

$$X \ni x \mapsto \langle f(x, \cdot), \varphi \cdot \nu \rangle.$$

Since $\mathcal{D}^r(M) \cdot \nu$ is sequentially dense in $\mathcal{E}_c'^r(M)$, as above we see that the mapping $X \ni x \mapsto \langle f(x, \cdot), T \rangle$ is μ-measurable for every $T \in \mathcal{E}'^r(M)$. Then, [19, Corollary 2 to Proposition 10 of Chapter IV, § 5, No. 5] completes the proof. □

The following result is a straightforward generalization of [28, Theorem 1.1.13]. The proof is omitted.

Lemma A.22. *Let $\mathfrak{g}$ be a homogeneous Lie algebra, and let $(\mathfrak{g}_j)_{j=1}^k$ be an increasing sequence of* homogeneous *ideals of $\mathfrak{g}$. Define $m_j := \dim \mathfrak{g}_j$ for every $j = 1, \dots, k$, and assume that $\mathfrak{g}_k = \mathfrak{g}$. Then, there is a basis $(X_h)_{h=1}^{m_k}$ of $\mathfrak{g}$ consisting of homogeneous elements such that, for every $h = 1, \dots, m_k$, the vector subspace of $\mathfrak{g}$ generated by $X_1, \dots, X_h$ is a (homogeneous) ideal of $\mathfrak{g}$; in addition, if $h = m_j$ for some $j = 1, \dots, k$, then $X_1, \dots, X_h$ generates $\mathfrak{g}_j$.*

Next we prove two results about Faà di Bruno's formula.

Lemma A.23. *Let A be a commutative ring in which $n \cdot 1_A$ is invertible for every $n \in \mathbb{Z}_+^*$. Let E_1, E_2, E_3 be three A-modules, with E_1 and E_2 free. Let $P : E_1 \to E_2$ and $Q : E_2 \to E_3$ be two polynomial mappings such that $P(0) = 0$. Denote by P_k and Q_k the homogeneous components of degree k of P and Q respectively, and let $\widetilde{Q}_k$ be the symmetric k-multilinear mapping associated with Q_k, for every $k \in \mathbb{N}$. Then,*

$$Q \circ P = \sum_{k \in \mathbb{N}} \sum_{\sum_{h \in \mathbb{N}} h \alpha_h = k} \frac{|\alpha|!}{\alpha!} \widetilde{Q}_{|\alpha|} \circ \mathbf{P}^\alpha,$$

where $\mathbf{P}^\alpha$ *denotes the polynomial mapping* $E_1 \to E_2^{|\alpha|}$ *which has* α_h *components equal to* P_h *for every* $h \in \mathbb{N}$*, arranged in any (fixed) order.*

Proof. Indeed,

$$Q \circ P = \sum_{k\in\mathbb{N}} \widetilde{Q}_k \left(\sum_{h\in\mathbb{N}} P_h, \ldots, \sum_{h\in\mathbb{N}} P_h \right) = \sum_{k\in\mathbb{N}} \sum_{|\alpha|=k} \frac{k!}{\alpha!} \widetilde{Q}_k \circ \mathbf{P}^\alpha,$$

and the assertion follows easily. □

Corollary A.24 (**Faà di Bruno's formula**)**.** *Let* E_1, E_2 *be two normed spaces, and let* E_3 *be a locally convex space. Let* V_j *be an open subset of* E_j *(*$j = 1, 2$*), and take* $f \in C^k(V_1; V_2)$ *and* $g \in C^k(V_2; E_3)$ *for some* $k \in \mathbb{N}$*. Take* $x_0 \in V_1$*. Then,*[1]

$$(g \circ f)^{(k)}(x_0) = \sum_{\sum_{h=1}^k h\alpha_h = k} \frac{k!}{\alpha!} g^{(|\alpha|)}(f(x_0)) \cdot \prod_{h=1}^k \left(\frac{f^{(h)}(x_0)}{h!} \right)^{\alpha_h}.$$

Proof. Indeed, if P denotes the Taylor polynomial oforder k of $f - f(x_0)$ about x_0 and Q denotes the Taylor polynomial of order k of Q about $f(x_0)$, clearly

$$\begin{aligned} g(f(x)) &= g\Big(f(x_0) + P(x) + o_1\Big(|x - x_0|^k\Big)\Big) \\ &= Q\Big(P(x) + o_1\Big(|x - x_0|^k\Big)\Big) + o_2\Big(\Big(P(x) + o_1\Big(|x - x_0|^k\Big)\Big)^k\Big) \\ &= Q(P(x)) + o_3\Big(|x - x_0|^k\Big) \end{aligned}$$

for $x \to x_0$, since $P(x) = O(|x - x_0|)$. The assertion then follows easily from Lemma A.23 and Taylor's formula. □

Lemma A.25. *Let* V *be an open subset of* $\mathbb{R}^n$*, let* F_1, F_2, F_3 *be three Hausdorff locally convex spaces and let* $\langle \cdot , \cdot \rangle \colon F_1 \times F_2 \to F_3$ *be a bilinear mapping which is hypocontinuous for the countable compact subsets of* F_1*.*

Take $f_1 \in C^r(V; F_1)$ *and* $f_2 \in C^r(V, F_2)$ *for some* $r \in \mathbb{N} \cup \{ \infty \}$*. Then,* $\langle f_1, f_2 \rangle \in C^r(V; F_3)$*. In addition, if* $r \geqslant 1$*, then*

$$\langle f_1, f_2 \rangle' = \langle f_1', f_2 \rangle + \langle f_1, f_2' \rangle$$

Proof. **1.** Assume first that $r = 0$. Take $x \in V$, and let (x_j) be a sequence which converges to x in V. Then [18, Proposition 4 of Chapter III, § 5, No. 3] implies that

$$\lim_{j\to\infty} \langle f_1(x_j), f_2(x_j) \rangle = \langle f_1(x), f_2(x) \rangle,$$

whence the result by the arbitrariness of x and (x_j).

2. Now, assume that $n = r = 1$. Take x and (x_j) as before, but now assume that $x_j \neq x$ for every $j \in \mathbb{N}$. Then

$$\frac{\langle f_1(x_j), f_2(x_j) \rangle - \langle f_1(x), f_2(x) \rangle}{x_j - x} = \left\langle f_1(x_j), \frac{f_2(x_j) - f_2(x)}{x_j - x} \right\rangle + \left\langle \frac{f_1(x_j) - f_1(x)}{x_j - x}, f_2(x) \right\rangle.$$

Therefore, [18, Proposition 4 of Chapter III, § 5, No. 3] again and the arbitrariness of x and (x_j) imply that $\langle f_1, f_2 \rangle$ is differentiable on V, with

$$\langle f_1, f_2 \rangle' = \langle f_1', f_2 \rangle + \langle f_1, f_2' \rangle.$$

3. Now, assume that $r = 1$ and that n is arbitrary. Then, **2** shows that $\langle f_1, f_2 \rangle$ is differentiable along every direction v of $\mathbb{R}^n$, and that

$$\partial_v \langle f_1, f_2 \rangle = \langle f_1' \cdot v, f_2 \rangle + \langle f_1, f_2' \cdot v \rangle.$$

[1]Here $\prod_{h=1}^k \left(\frac{f^{(h)}(x_0)}{h!} \right)^{\alpha_h}$ denotes the symmetrization of the mapping $E_1 \to E_2^{|\alpha|}$ which has the first α_1 components equal to $f'(x_0)$, the subsequent α_2 components equal to $\frac{f''(x_0)}{2}$, and so on. Notice that the product so defined is *not* commutative if $\dim E_2 > 1$; nevertheless, the actual order of the factors is not relevant, since $g^{(|\alpha|)}(f(x_0))$ is symmetric.

Since $\langle f_1', f_2\rangle + \langle f_1, f_2'\rangle$ is continuous on V by **1**, by means of standard techniques we see that $\langle f_1, f_2\rangle \in C^1(V; F_3)$.

4. Now, assume that $r \in \mathbb{Z}_+^*$. Then, **3** implies that

$$\langle f_1, f_2\rangle' = \langle f_1', f_2\rangle + \langle f_1, f_2'\rangle$$

Arguing by induction on r, we see that $\langle f_1', f_2\rangle + \langle f_1, f_2'\rangle \in C^{r-1}(V; F_3)$, so that $\langle f_1, f_2\rangle \in C^r(V; F_3)$. The assertion then follows also for the case $r = \infty$. □

Lemma A.26. *Let F_1, F_2, F_1', F_2' be four locally convex spaces, and let $T_1 \colon F_1 \to F_1'$ and $T_2 \colon F_2 \to F_2'$ be two strict morphisms; assume that either F_1' or F_2' is nuclear. Then, the linear mapping $T_1 \widehat{\otimes} T_2 \colon F_1 \widehat{\otimes} F_2 \to F_1' \widehat{\otimes} F_2'$ is a strict morphism.*

Proof. Let F_1'', F_2'' be the images of T_1, T_2, respectively. Denote by T_1', T_2' the co-restriction of T_1, T_2 to their images, respectively. Then [84, Proposition 43.9] implies that $T_1' \widehat{\otimes} T_2' \colon F_1 \widehat{\otimes} F_2 \to F_1'' \widehat{\otimes} F_2''$ is a strict morphism. In addition, since F_1' or F_2' is a nuclear space, also F_1'' or F_2'' is a nuclear space by [84, Proposition 50.1], so that $F_1'' \widehat{\otimes} F_2''$ is canonically isomorphic to $F_1'' \widehat{\otimes}_\varepsilon F_2''$ by [84, Theorem 50.1]. Analogously, $F_1' \widehat{\otimes} F_2'$ is canonically isomorphic to $F_1' \widehat{\otimes}_\varepsilon F_2'$. Therefore, by means of [84, Proposition 43.7] we see that the canonical mapping $F_1'' \widehat{\otimes} F_2'' \to F_1' \widehat{\otimes} F_2'$ is an isomorphism onto its image. The assertion follows. □

Bibliography

[1] Abraham, R., Marsden, J. E., *Foundations of Mechanics*, Addison-Wesley, 1987.

[2] Ambrosio, L., Fusco, N., Pallara, D., *Functions of Bounded Variation and Free Discontinuity Problems*, Oxford University Press, 2000.

[3] Astengo, F., Cowling, M., Di Blasio, B., Sundari, M., Hardy's Uncertainty Principle on Certain Lie Groups, *J. London Math. Soc.,* **62** (2000), pp. 461–472.

[4] Astengo, F., Di Blasio, B., Ricci, F., Gelfand transforms of polyradial Schwartz functions on the Heisenberg group, *J. Funct. Anal.,* **251** (2007), pp. 772–791.

[5] Astengo, F., Di Blasio, B., Ricci, F., Gelfand Pairs on the Heisenberg Group and Schwartz Functions, *J. Funct. Anal.,* **256** (2009), pp. 1565–1587.

[6] Aubin, T., *Some Nonlinear Problems in Riemannian Geometry*, Springer-Verlag, 1998.

[7] Beals, R., Opérateurs invariants hypoelliptiques sur un groupe de Lie nilpotent, *Séminaire Équations aux dérivées partielles (Polytechnique),* (1976-1977), pp. 1–8.

[8] Beals, R., Gaveau, B., Greiner, C., P., Hamilton-Jacobi theory and the heat kernel on Heisenberg groups, *J. Math. Pures Appl.* **79** (2000), p. 633–689.

[9] Bierstone, E., Milman, P., Composite Differentiable Functions, *Ann. Math.,* **116** (1982), pp. 541–558.

[10] Bierstone, E., Milman, P., Semianalytic and Subanalytic sets, *Inst. Hautes Études Sci. Publ. Math.,* **67** (1988), pp. 5–42.

[11] Bierstone, E., Schwarz, G. W., Continuous Linear Division and Extension of C^∞ Functions, *Duke Math. J.,* **50** (1983), pp. 233–271.

[12] Bonfiglioli, A., Lanconelli, E., Uguzzoni, F., *Stratified Lie groups and Potential Theory for their Sub-Laplacians*, Springer-Verlag, 2009.

[13] Bourbaki, N., *Algebra I*, chap. 1–3 (Elements of Mathematics), Springer-Verlag, 1989.

[14] Bourbaki, N., *Algebra II*, chap. 4–7 (Elements of Mathematics), Springer-Verlag, 1990.

[15] Bourbaki, N., *Algèbre*, chap. 9 (Éléments de Mathématique), Springer-Verlag, 2007.

[16] Bourbaki, N., *General Topology*, chap. 1–4 (Elements of Mathematics), Springer-Verlag, 1995.

[17] Bourbaki, N., *Topologie Générale*, chap. 5–10 (Éléments de Mathématique), Springer-Verlag, 2006.

[18] Bourbaki, N., *Topological Vector Spaces*, chap. 1–5 (Elements of Mathematics), Springer-Verlag, 2003.

[19] Bourbaki, N., *Integration I*, chap. 1–6 (Elements of Mathematics), Springer-Verlag, 2004.

[20] Bourbaki, N., *Integration II*, chap. 7–9 (Elements of Mathematics), Springer-Verlag, 2004.

[21] Bourbaki, N., *Groupes et algèbres del Lie*, chap. 1 (Éléments de Mathématique), Springer-Verlag, 2007.

[22] Bourbaki, N., *Groupes et algèbres del Lie*, chap. 2–3 (Éléments de Mathématique), Springer-Verlag, 2006.

[23] Bourbaki, N., *Théories spectrales*, chap. 1–2 (Éléments de Mathématique), Springer-Verlag, 2007.

[24] Bruno, T., Calzi, M., Asymptotics for the heat kernel on H-type Groups, *Ann. Mat. Pur. Appl.* **197** (2018), pp. 1017–1049.

[25] Calabi, E., Linear Systems of Real Quadratic Forms, *P. Am. Math. Soc.,* **15** (1964), pp. 844–846.

[26] Cartan, H., *Théorie élémentaire des fonctions analytiques d'une ou plusieurs variables complexes*, Hermann, 1985.

[27] Christ, M., L^p Bounds for Spectral Multipliers on Nilpotent Groups, *Trans. Am. Math. Soc.,* **328** (1991), pp. 73–81.

[28] Corwin, L. J., Greenleaf, F. P., *Representations of Nilpotent Lie Groups and their Applications, Part I: Basic Theory and Examples*, Cambridge University Press, 1990.

[29] Coste, M., An Introduction to Semialgebraic Geometry, URL `https://perso.univ-rennes1.fr/michel.coste/polyens/SAG.pdf`.

[30] Damek, E., Ricci, F., Harmonic Analysis on Solvable Extensions of H-type Groups, *J. Geom. Anal.,* **2** (1992), pp. 213–248.

[31] de Rham, G., *Differentiable Manifolds*, Springer-Verlag, 1984.

[32] Dierolf, P., Voigt, J., Convolution and $\mathcal{S}'$-Convolution of Distributions, *Collect. Math.* **29** (1978), pp. 185–196.

[33] Dixmier, J., *Les C^*-algèbres et leurs représentations*, Gauthier-Villars, 1969.

[34] Dixmier, J., Malliavin, P., Factorisations de fonctions et de vecteurs indéfiniment différentiables, *Bull. Sci. Math.* **102** (1978), pp. 305–330.

[35] Eckmann, B., Gruppentheoretischer Beweis des Satzes von Hurwitz-Radon über die Komposition quadratischer Formen, *Comment. Math. Helv.* **15** (1943), pp. 358–366.

[36] Eldredge, N., Precise estimates for the subelliptic heat kernel on H-type groups, *J. Math. Pures Appl.* **92** (2009), pp. 52–85.

[37] Erdélyi, A., *Asymptotic Expansions*, Dover Publications, Inc., 1956.

[38] Erdélyi, A., Magnus, W., Oberhettinger, F., Tricomi, F. G., *Higher Transcendental Functions, vol. II*, McGraw–Hill, 1953.

[39] Federer, H., *Geometric Measure Theory*, Springer, 1969.

[40] Fischer, V., Ricci, F., Gelfand transforms of $SO(3)$-invariant Schwartz functions on the free group $N_{3,2}$, *Ann. Inst. Fourier (Grenoble)* **59** (2009), no. 6, pp. 2143–2168.

[41] Fischer, V., Ricci, F., Yakimova, O., Nilpotent Gelfand pairs and Schwartz extensions of spherical transforms via quotient pairs, *J. Funct. Anal.* **274** (2018), no. 4, pp. 1076–1128.

[42] Folland, G. B., *A Course in Abstract Harmonic Analysis*, CRC Press, 1995.

[43] Folland, G. B., Stein, E. M., *Hardy Spaces on Homogeneous Group*, Princeton University Press, 1982.

[44] Gaveau, B., Principe de moindre action, propagation de la chaleur et estimées sous elliptiques sur certains groupes nilpotents, *Acta Math.* **139** (1977), pp. 95–153.

[45] Grothendieck, A., Produits tensoriels topologiques et espaces nucléaires, *Mem. Am. Math. Soc.,* **16** (1966).

[46] Haase, M., Lectures on Functional Calculus, 2018, URL `https://www.math.uni-kiel.de/isem21/en/course/phase1/isem21-lectures-on-functional-calculus`.

[47] Helffer, B., Nourrigat, J., Caracterisation des opérateurs hypoelliptiques homogènes invariants a gauche sur un groupe de Lie nilpotent gradué, *Commun. Part. Diff. Eq.,* **4** (1979), pp. 899–958.

[48] Helgason, S., *Groups and Geometric Analysis*, AMS, 2002.

[49] Hörmander, L., Hypoelliptic Second Order Differential Equations, *Acta Math.,* **119** (1967), pp. 147–171.

[50] Hörmander, L., *The Analysis of Linear Partial Differential Operators, I*, Springer-Verlag, 1990.

[51] Hueber, H., Müller, D., Asymptotics for some Green Kernels on the Heisenberg Group and the Martin Boundary, *Math. Ann.* **283** (1989), pp. 97–119.

[52] Hulanicki, A., A functional calculus for Rockland operators on nilpotent Lie groups, *Stud. Math.,* **78** (1984), pp. 253–266.

[53] Hulanicki, A., Ricci, F., A Tauberian Theorem and Tangential Convergence for Bounded Harmonic Functions on Balls in $\mathbb{C}^n$, *Invent. Math.,* **62** (1980), pp. 325–331.

[54] Kato, T., *Perturbation Theory for Linear Operators*, Springer-Verlag, 1980.

[55] Li, H.-Q., Estimations asymptotiques du noyau de la chaleur sur les groupes de Heisenberg, *C. R. Math. Acad. Sci. Paris* **344** (2007), pp. 497–502.

[56] Li, H.-Q., Estimations optimales du noyau de la chaleur sur les groupes de type Heisenberg, *J. Reine Angew. Math.* **646** (2010), pp. 195–233.

[57] Ludwig, J., Müller, D., Sub-Laplacians of Holomorphic L^p-type on Rank One AN-Groups and Related Solvable Groups, *J. Funct. Anal.* **170** (2000), pp. 366–427.

[58] Martini, A., *Algebras of Differential Operators on Lie Groups and Spectral Multipliers*, Ph.D. thesis, Scuola Normale Superiore, 2010, arXiv:1007.1119v1 [math.FA].

[59] Martini, A., Ricci, F., Tolomeo, L., Riemann-Lebesgue and Invariance of the Schwartz Class for Sub-Laplacians on Solvable Lie Groups of Polynomial Growth, preprint.

[60] Miller, K. G., Parametrices for Hypoelliptic Operators on Step Two Nilpotent Lie Groups, *Commun. Part. Diff. Eq.,* **5** (1980), pp. 1153–1184.

[61] Moore, C. C., Wolf, J. A., Square Integrable Representations of Nilpotent Groups, *T. Am. Math. Soc.* **185** (1973), pp. 445–462.

[62] Müller, D., Ricci, F., Solvability for a Class of Doubly Characteristic Differential Operators on 2-Step Nilpotent Groups, *Ann. Math.,* **143** (1996), pp. 1–49.

[63] Müller, D., Seeger, A., Singular Spherical Maximal Operators on a Class of Two Step Nilpotent Lie Groups, *Israel J. Math.,* **141** (2004), pp. 315–340.

[64] Nagel, A., Ricci, F., Stein, E. M., Singular Integrals with Flag Kernels and Analysis on Quadratic CR Manifolds, *J. Funct. Anal.,* **181** (2001), pp. 29–118.

[65] Nelson, E., *Operants: A Functional Calculus for Non-Commuting Operators*, Springer Berlin Heidelberg, 1970, pp. 172–187.

[66] Nelson, E., Stinespring, W. F., Representation of Elliptic Operators in an Enveloping Algebra, *Am. J. Math.,* **81** (1959), pp. 547–560.

[67] Palmer, T. W., *Banach Algebras and the General Theory of $*$-Algebras, I*, Cambridge University Press, 1994.

[68] Palmer, T. W., *Banach Algebras and the General Theory of $*$-Algebras, II*, Cambridge University Press, 2001.

[69] Paterson, A. L. T., *Amenability*, AMS, 1988.

[70] Pier, J.-P., *Ameanble Locally Compact Groups*, John Wiley & Sons, 1984.

[71] Randall, J., The heat kernel for generalized Heisenberg groups, *J. Geom. Anal.* **6** (1996), pp. 287–316.

[72] Rigot, S., Mass transportation in groups of type H, *Comm. in Cont. Math.* **7** (2005), no. 4, p. 509–537.

[73] Robbin, J. W., Salamon, D. A., The exponential Vandermonde matrix, *Linear Algebra Appl.* **317** (2000), pp. 225–226, URL `https://people.math.ethz.ch/ salamon/PREPRINTS/vandermonde.pdf`.

[74] Rudin, W., *Functional Analysis*, McGraw-Hill, 1973.

[75] Sard, A., Hausdorff Measure of Critical Images on Banach Manifolds, *Am. J. Math.,* **87** (1965), pp. 158–174.

[76] Schwartz, L., Définition intégrale de la convolution de deux distributions, *Seminaire Schwartz,* **1** (1953–1954), no. 22, pp. 1–7.

[77] Schwartz, L., Espaces de fonctions différentiables a valeurs vectorielles, *Jour. d'Analyse Math.,* **4** (1954–1955), pp. 88–148.

[78] Schwartz, L., *Théorie des distributions*, Hermann, 1978.

[79] Stein, E. M., *Singular Integrals and Differentiability Properties of Functions*, Princeton University Press, 1970.

[80] Stein, E. M., Weiss, G., *Introduction to Fourier Analysis on Euclidean Spaces*, Princeton University Press, 1971.

[81] Tan, K.-H., X.-P., Y., Characterisation of the sub-Riemannian isometry groups of H-type groups, *Bull. Austr. Math. Soc.* **70** (2004), no. 1, pp. 87–100.

[82] ter Elst, A. F. M., Robinson, D. W., Weighted Subcoercive Operators on Lie Groups, *J. Funct. Anal.* **157** (1998), pp. 88–163.

[83] Tolomeo, L., *Misure di Plancherel associate a sub-laplaciani su Gruppi di Lie*, Master's thesis, Scuola Normale Superiore, 2015, URL `https://core.ac.uk/download/pdf/79618830.pdf`.

[84] Treves, F., *Topological Vector Spaces, Distributions and Kernels*, Academic Press, 1967.

[85] Veneruso, A., Schwartz kernels on the Heisenberg group, *Bollettino dell'Unione Matematica Italiana* **6-B** (2003), no. 3, pp. 657–666.

[86] Whitney, H., Differentiable Even Functions, *Duke Math. J.,* **10** (1943), pp. 159–160.

[87] Wong, R., *Asymptotic approximations of integrals*, Academic Press, Inc., 1989.

[88] Yang, Q., Zhu, F., The heat kernel on H-type groups, *Proc. Amer. Math. Soc.* **136** (2008), pp. 1457–1464.

[89] Zygmund, A., *Trigonometric Series*, Cambridge University Press, 3rd ed., 2003.

Index of Notation

Notation	Description	Page
$\mathbb{N}$	natural numbers, starting from 0	
$\mathbb{N}^*$	natural numbers, starting from 1	
$\mathbb{Z}$	rational integers	
$\mathbb{Q}$	rational numbers	
$\mathbb{R}$	real numbers	
$\mathbb{C}$	complex numbers	
A_+	positive ($\geqslant 0$) elements of A	
x_+, x_-	positive and negative parts of x	
$\mathbb{R}_+^*$	strictly positive real numbers	
$T_x(M)$	tangent space of M at x	
$\varphi_*(X)$	push-forward of X under φ	
$\mathbf{X}^\gamma$	$X_1^{\gamma_1} \dots X_n^{\gamma_n}$	
$X^\dagger$	formal transpose of X	
XT	$\langle XT, \varphi\rangle = \langle T, X^\dagger\varphi\rangle$	
$X^\dagger T$	$\langle X^\dagger T, \varphi\rangle = \langle T, X\varphi\rangle$	
$A \otimes B$	tensor product, endowed with the π-topology	
$A \widehat{\otimes} B$	completed π-tensor product	
$A \otimes_\varepsilon B$	tensor product, endowed with the ε-topology	
$A \widehat{\otimes}_\varepsilon B$	completed ε-tensor product	
$\mathcal{L}(E;F)$	the space of linear mappings from E into F with the topology of bounded convergence	
$\mathcal{L}_c(E;F)$	the space of linear mappings from E into F with the topology of compact convergence	
$\mathcal{L}_s(E;F)$	the space of linear mappings from E into F with the topology of pointwise convergence	
${}^tT$	transpose of T	
$\mathrm{Pol}(A;B)$	the space of polynomial mappings from A into B	
$\mathcal{D}^r$	the space of C^r compactly supported functions	
$\mathcal{E}^r$	the space of C^r functions	
$\mathcal{S}$	the space of Schwartz functions	
$\mathcal{D}'^r$	the space of distributions of order r	
$\mathcal{E}'^r$	the space of distributions with compact support or order r	
$\mathcal{S}'$	the space of tempered distributions	
$C_0(X)$	the space of continuous functions vanishing at infinity on X	
χ_A	characteristic function of the set A	
I_X	identity mapping of X	
$\mathcal{M}$	the space of Radon measures	
$\mathcal{M}^1$	the space of bounded Radon measures	
$\mathcal{L}^p, L^p$	classical Lebesgue spaces	
$\mathcal{L}^p_{\mathrm{loc}}, L^p_{\mathrm{loc}}$	classical local Lebesgue spaces	
δ_x	Dirac delta at x	
$\mathcal{H}^n$	suitably normalized Hausdorff measure	
$f \cdot \mu$	measure with density f with respect to μ	
$\lvert\mu\rvert$	absolute value of the measure μ	
μ^*	outer measure associated with μ	
$\mu^\bullet$	essential outer measure associated with μ	

Symbol	Description	Page
$x_{h,\omega}, x_\omega$		100
$\pi_\omega \colon G \to G_\omega$		101
$\lvert\mathrm{Pf}(\omega)\rvert$	the Pfaffian	102
Λ_m^γ	the m-th Laguerre polynomial of order γ	102
R	$\frac{\lvert x\rvert^2}{4}$	138
p_{s,k_1,k_2} or $p_{s,k_1,k_2}^{(m)}$		138
ω	$\lvert t\rvert/R$	138
δ	$\left(R/\pi\lvert t\rvert\right)^{1/2}$	138
κ	$2\sqrt{\pi\lvert t\rvert R}$	138
θ	$\lambda \mapsto \frac{2\lambda-\sin(2\lambda)}{2\sin^2(\lambda)}$	138
$d(x,t)$	Carnot-Carathéodory distance	139
I_ν	modified Bessel function of the first kind of order ν	145
ρ		146
$\Delta'(A)$	the set of characters of A	161
$\Delta(A)$	the Gelfand spectrum of A	161
$\sigma(x)$	the spectrum of x in A	161
$\mathcal{G}$	the Gelfand transform	161
$\mathrm{dom}(T)$	the domain of T	161
T^*	the adjoint of T	162
$\overline{T}$	the closure of T	162
$\sigma(T)$	the spectrum of T	162
$T_1 + T_2, T_1 \cdot T_2$		162
$\sigma(T_A)$	joint spectrum of T_A	163
$f(T_A)$		163

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