Scuola Normale Superiore

Classe di Scienze Matematiche, Fisiche e Naturali


# Structure of non-smooth spaces with Ricci curvature bounded below

Tesi di Perfezionamento in Matematica


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Elia Bruè

*Relatore*:

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Anno Accademico 2019 – 2020

Pisa, 31 Gennaio 2020

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Abstract. This thesis is devoted to the study of structural properties of non-smooth spaces with Ricci curvature bounded from below.
The first part concerns with the structure theory of $\mathrm{RCD}(K,N)$ spaces: we prove the existence of the so-called *essential dimension*, along with rectifiability properties of the regular set. This theory is a result of many contributions [**43**, **72**, **91**, **95**, **109**, **121**], in our presentation we closely follow the recent works [**41**, **43**].
The second part of this thesis deals with codimension-1 structures on $\mathrm{RCD}(K,N)$ spaces. More precisely we study structural properties of boundaries of sets with finite perimeter, generalising the celebrated De Giorgi theory [**65**, **66**] to this framework. This is based on the works [**7**, **40**].

# Contents

# Introduction

This thesis is devoted to the study of structural properties of metric measure spaces with a synthetic notion of Ricci curvature bounded below. The results therein presented are part of the work done by the author during his PhD studies. Other papers written during the PhD studies and not completely related to this topic are summarized in the second part of the introduction.

### Non-smooth spaces with Ricci bounded below

In the last ten years the topic of *synthetic Ricci curvature bounds* has known a deep and fast development. Aim of this introduction is to present motivations, a brief historical account on this theory, and give an outline of the results contained in this work.

In the study of smooth Riemannian manifolds the Ricci tensor plays a fundamental role. Lower bounds on the Ricci curvature have many important analytical and geometrical implications such as volume and Laplacian comparisons [**125**, Chapter 6], parabolic Harnack inequality [**115**] and Guassian bounds on the heat kernel, local Poincarè inequalities [**44**], isoperimetric inequalities [**96**, Appendix C] [**47**], eigenvalue estimates [**123**], and many others. Moreover, as a consequence of Gromov pre-compactness theorem [**96**, Theorem 5.3] and a result [**81**] of K. Fukaya, one has that any sequence of $n$-dimensional Riemannian manifolds with Ricci bounded below by $K$ and diameter bounded by $D$ converges, up to subsequence, to a *metric measure spaces* $(X, \mathsf{d}, \mathfrak{m})$ in the *measured Gromov-Hausdorff* topology. The resulting spaces, called *Ricci limits*, are not Riemannian manifolds in general. Nevertheless, the deep theory devoleped by J. Cheeger and T. Colding [**50–53**] in the 90's ensures that Ricci limits enjoy analytical and structural properties typical of smooth spaces with Ricci tensor bounded below. With this results at hand it is natural to try to understand whether a "synthetic notions of having Ricci curvature bounded below" does exist. The following question was raised by J. Cheeger and T. Colding in [**51**, Appendix 2].

*Calling synthetic a set of conditions defining a class of metric spaces without refering to any notion of smoothness, can one provide a synthetic notion of having Ricci curvature bounded below?*

Despite its own theoretical interest, the study of Ricci limits and synthetic notions of Ricci bounded below is motivated by potential applications in Riemannian geometry. For instance, in the study of Ricci flow and other geometric flows, non-smooth limit spaces come up naturally when the flow becomes singular. Another important example of application is given by "almost rigidity results" for smooth manifolds via compactness arguments. These statements are usually obtained building upon two ingredients: the pre-compactness of a suitable family of manifolds in some topology (for instance $n$ dimensional manifolds with Ricci bounded below in the measured Gromov-Hausdorff topology) and the validity of the rigidity result for limit objects. Examples are the "almost splitting theorem" and the "almost volume cone implies almost metric cone" which follow from the splitting theorem [**50**, **84**] and the volume cone implies metric cone theorem [**50**, **70**] for non-smooth spaces. Let us point out that both theorems can be proven also via direct estimates (Cf. [**50**]).

The compactness approach to "almost rigidity results" is an instance of a general way of thinking which has important similarities with ideas coming from the field of functional analysis. In order to get new information on Riemannian manifolds with Ricci bounded below we identify them with points of an abstract metric space and we investigate its structure. This is in perfect analogy with the study of Banach and Hilbert spaces of functions, that in the last century has lead to striking applications in very classical fields like calculus of variations and partial differential equations. Adopting this point of view several notions of "generalized smooth functions" (e.g. Sobolev functions) come up naturally, and their understanding often gives new insights on problems involving smooth objects. The same phenomenon happens in the study of Riemannian manifold with Ricci bounded below and motivates the study of Ricci limits and the search for a "synthetic notion of having Ricci bounded below" in the non-smooth setting.

$\mathrm{RCD}(K,N)$ **spaces.** Given $K \in \mathbb{R}$, and $N \in [1,\infty)$, $\mathrm{RCD}(K,N)$ metric measure spaces are "Riemannian like" spaces with a synthetic notion of having Ricci curvature bounded below by $K$ and dimension bounded above by $N$. They provide a satisfactory answer to the above presented question, originally raised by J. Cheeger and T. Colding.

Unlike Alexandrov spaces [**45**], which are metric spaces $(X,\mathsf{d})$ with a synthetic notion of *sectional curvature* bounded below, the RCD conditions should be seen as a property coupling the measure and the distance.

The first step towards the RCD notion was given in the seminal and independent works by Lott-Villani [**116**] and Sturm [**135**, **136**] introducing the curvature-dimension condition $\mathrm{CD}(K,N)$ for metric measure spaces. This notion is formulated in terms of convexity-type properties of suitable entropies along optimal transport paths. The relation between Ricci curvature and convexity-type properties of entropies comes from work of Otto-Villani [**124**] and Cordero-Erausquin-McCann-Schmuckenschläger [**60**]. They showed that, on Riemannian manifolds, the Ricci curvature affects the convexity of certain entropy functionals along an optimal transport path.

The $\mathrm{CD}(K,N)$ theory is consistent with the Alexandrov theory (Cf. Petrunin [**127**]) and with the notion of having Ricci bounded below by $K$ and dimension bounded above by $N$ proposed by Bakry-Émery [**32**] in the smooth Riemannian setting (as proved by Ambrosio, Gigli, and Savaré in [**18**]). Moreover Ricci limit spaces are $\mathrm{CD}(K,N)$, being the latter stable under measured Gromov-Hausdorff convergence.

The so-called reduced curvature dimension condition $\mathrm{CD}^*(K,N)$, which is weaker than the $\mathrm{CD}(K,N)$ one, was introduced by Bacher and Sturm [**31**] to get around the lack of the local-to-global property for $\mathrm{CD}(K,N)$ spaces. It has been a long standing question whether the two notion does coincide. A positive answer for essentially non-branching spaces with finite measure was given recently in [**46**].

Despite the consistency results stated above, the $\mathrm{CD}(K,N)$ class of metric measure spaces is still too large to some extent. For instance, it includes smooth Finsler manifolds (see the last theorem in [**138**]) which are known to not appear as Ricci limit spaces after [**51**]. To single out spaces with a Riemannian-like behaviour from this broader class, Ambrosio-Gigli-Savaré introduced in [**17**] the $\mathrm{RCD}(K,\infty)$ notion, adding the request of linearity of the heat flow to the $\mathrm{CD}(K,\infty)$ condition. The definition of $\mathrm{RCD}(K,N)$ metric measure spaces, was proposed in [**86**] as a finite-dimensional counterpart to the $\mathrm{RCD}(K,\infty)$ condition, and provides especially a splitting theorem [**84**] similar to Cheeger-Gromoll's original result [**54**].

In [**17**] it was shown that one of the equivalent formulations of $\mathrm{RCD}(K,\infty)$ spaces is that any probability measure with finite second moment is the starting point of an $\mathrm{EVI}_K$-gradient flow of the entropy. This condition is known to imply $K$-convexity of the entropy along every geodesic [**64**]. This along with the result of T. Rajala and T. Sturm in [**131**] shows that $\mathrm{RCD}(K,N)$ spaces are essentially non-branching and therefore [**46**] applies guaranteeing the equivalence between $\mathrm{RCD}(K,N)$ and $\mathrm{RCD}^*(K,N)$. Where the latter is defined adding to the $\mathrm{CD}^*(K,N)$ condition the requirement of linearity of the heat flow.

Another crucial property of RCD spaces is the characterization via a weak version of the Bochner inequality. In the infinite-dimensional case this equivalence was studied in [**18**], then [**77**] established equivalence with the dimensional Bochner inequality for the class $\mathrm{RCD}^*(K,N)$ (see also [**25**]).

We refer to Chapter 1 for a more detailed presentation of RCD spaces.

**Structure theory.** Given an arbitrary metric measure space, there is a well defined notion of measured tangent space at a fixed point, as pointed measured Gromov-Hausdorff limits of a sequence of rescalings of the starting space. In particular, in the case of an $\mathrm{RCD}(K,N)$ metric measure space $(X,\mathsf{d},\mathfrak{m})$, we can define, for any $1\le k\le N$, the $k$-dimensional regular set $\mathcal{R}_k$ to be the set of those $x\in X$ such that $x$ belongs to the support of $\mathfrak{m}$ and the tangent space of $(X,\mathsf{d},\mathfrak{m})$ at $x$ is the $k$-dimensional Euclidean space. Better said, $x\in\mathcal{R}_k$ if

$$(X,r^{-1}\mathsf{d},\mathfrak{m}_r^x,x)\to(\mathbb{R}^k,\mathsf{d}_{\mathrm{Eucl}},c_k\mathscr{L}^k,0)\quad\text{as}\quad r\downarrow 0,$$

where

$$\mathfrak{m}_r^x:=\left(\int_{B_r(x)}1-\frac{\mathsf{d}(x,y)}{r}\,\mathrm{d}\mathfrak{m}(y)\right)^{-1}\mathfrak{m},\quad c_k:=\left(\int_{B_1(0)}(1-|y|)\,\mathrm{d}\mathscr{L}^k(y)\right)^{-1}$$

and the convergence is understood with respect to the pointed measured Gromov-Hasdorff topology.

In Chapter 2 we outline the rectifiability result for $\mathrm{RCD}(K,N)$ spaces [**72**, **91**, **95**, **109**, **121**], ensuring that regular sets $\mathcal{R}_k$ are $(\mathfrak{m},k)$-rectifiable,

$$\mathfrak{m}\left(X\setminus\bigcup_{k=1}^{[N]}\mathcal{R}_k\right)=0$$

and the restriction of the reference measure $\mathfrak{m}$ to $\mathcal{R}_k$ is absolutely continuous with respect to the $k$-dimensional Hausdorff measure $\mathcal{H}^k$.

Our presentation closely follows the recent paper [**40**] written by the author in collaboration with E. Pasqualetto and D. Semola, where a simpler approach compared to [**121**], to the rectifiability of RCD spaces, via $\delta$-splitting maps, is pursued.

In Chapter 3 we present the so-called constancy of the dimension theorem for $\mathrm{RCD}(K,N)$ spaces, proved by the author in collaboration with D. Semola in [**43**]. This theorem says that, given an $\mathrm{RCD}(K,N)$ metric measure space $(X,\mathsf{d},\mathfrak{m})$, it holds $\mathfrak{m}(\mathcal{R}_k)=0$ for any $k\neq n\in[1,N]$, where $n$ is called *essential dimension* of $X$.

The analogous statement in the setting of Ricci limits was proved by T. Colding and A. Naber [**58**] in 2012 closing a conjecture of J. Cheeger and T. Colding that has been open for twelve years.

The most important ingredient in the proof of the constancy of the dimension theorem for $\mathrm{RCD}(K,N)$ spaces is a new quantitative estimate for flows associated to Sobolev vector fields. It is known since the works [**27, 93**] that flows associated to Sobolev vector fields exist and act "almost transitively", in a suitable measure theoretic sense, on $X$. In particular, assuming by contradiction the existence of $m<n$ such that $\mathfrak{m}(\mathcal{R}_n)>0$ and $\mathfrak{m}(\mathcal{R}_m)>0$, we can bring a portion of positive $\mathfrak{m}$-measure of $\mathcal{R}_n$ to $\mathcal{R}_m$ by means of a flow map associated to a Sobolev vector field. Therefore, a strong enough regularity result for these flow maps (for instance an approximate bi-Lipschitz regularity would suffice) would give the sought conclusion.

The approximate bi-Lipschitz regularity for flow maps associated to Sobolev vector fields is a reasonable property, and in the Euclidean setting holds true as a consequence of Crippa-DeLellis' estimates [**63**].

In the joint work with D. Semola [**42**], we obtain a first result in this regard proving the analogue of Crippa-DeLellis' estimates in the setting of *Ahlfors regular* $\mathrm{RCD}(K,N)$ spaces. Even though this class of spaces is quite wide (it includes non-collapsed Ricci limits and Alexandrov spaces), it is still too restrictive for applications to the constancy of the dimension. In order to cover *collapsed* spaces it is of fundamental importance to get rid of the Ahlfors regularity assumption in the regularity statement. To do so we change completely point of view: we measure the approximate Lipschitz regularity of flows by means of a (quasi)-distance defined in terms of the *Green function* of the Laplacian. We refer to Section 1 for further details on this regularity result. Eventually in Section 2 we conclude the proof of the constancy of the dimension theorem.

**Theory of sets with finite perimeter.** At this stage of development of the theory we have reached a good understanding of the structure of $\mathrm{RCD}(K,N)$ spaces up to measure zero sets. It is therefore natural to try to push the theory by further studying codimension-1 structures like boundaries of *sets of finite perimeter*.

This line of research, which the author has pursued in [**7**] with L. Ambrosio and D. Semola, and in [**40**] with E. Pasqualetto and D. Semola, has lead to the complete generalization of De Giorgi's theory to the setting of RCD spaces, along with a new rigidity result Theorem 4.4 and a new integration by parts formula.

Let us recall that a set $E\subset\mathbb{R}^d$ is said to be of finite perimeter if $D\chi_E$ (the distributional derivative of the characteristic function of $E$) is a finite Borel measure. De Giorgi introduced the notions of *reduced boundary* $\mathcal{F}E$ and *generalized exterior normal* $\nu_E$ of $E$ as measure theoretic generalizations of the topological boundary and the exterior normal. He then proved that, for $E\subset\mathbb{R}^d$ of finite perimeter, $\mathcal{F}E$ is $(d-1)$-rectifiable and $|D\chi_E|=\mathscr{H}^{d-1}\llcorner\mathcal{F}E$.

A crucial step in the proof of De Giorgi's theorem was the *blow-up analysis* establishing that the sequence of rescaled spaces $\left(\frac{E-x}{r}\right)_{r\in(0,1)}$ converges in $L^1_{\mathrm{loc}}$ to the *Euclidean half space* orthogonal to $\nu_E(x)$ for any $x\in\mathcal{F}E$.

Trying to study the blow up of sets with finite perimeter on $\mathrm{RCD}(K,N)$ spaces one has several difficulties to overcome. The first one is that, neither the definition of reduced boundary nor the one of generalized exterior normal can be easily extended to this setting, since the tangent bundle of the ambient space may be defined just up to negligible sets. This is a manifestation of the non-smoothness of the ambient space.

In Chapter 4, we present the analysis of blow-ups first obtained in [**7**]. The approach therein introduced is based on *rigidities* in functional inequalities. We establish a new principle: at $|D\chi_E|$-a.e. point any blow-up of $E$ satisfies the equality in the *Bakry-Émery inequality* with exponent $p = 1$.

This, along with the characterization of equality cases in the Bakry-Émery inequality (Cf. Theorem 4.4), and an *iterated tangent* theorem in the spirit of Preiss, leads to the proof of the existence of tangent half-spaces at $|D\chi_E|$-a.e. point.

Aiming at concluding De Giorgi's analysis, in Chapter 5 we show uniqueness of tangents as well as rectifiability of reduced boundary of sets with finite perimeter on RCD spaces.

The strategy of proof builds upon several ingredients. The first one is a Gauss-Green formula tailored for $\mathrm{RCD}(K, N)$ spaces, where the exterior normal to the set of finite perimeter $E$ is obtained as an element of the newly defined *tangent module* $L^2_{|D\chi_E|}(TX)$. This functional space is a variant of the tangent module $L^2(TX)$ introduced by Gigli in [**87**] and its axiomatization comes from the theory developed in [**67**]. The Hilbertianity of $L^2_{|D\chi_E|}(TX)$ allows us to prove strong approximation results for the exterior normal of $E$, that turn out to be crucial in the proof of boundary rectifiability.

The second crucial ingredient is the notion of *$\delta$-splitting map, $\delta$-orthogonal to the unit normal of $E$*, which allows to control both the geometry of the ambient space and that of the reduced boundary. An important technical point is that the combination of $\delta$-orthogonality and $\delta$-splitting is suitable for propagation at many locations and any scale with maximal function arguments.

### Lusin's approximation result on infinite dimensional spaces

The class of $\mathrm{RCD}(K, \infty)$ metric measure structures has been thoroughly studied in the last ten years, and nowadays it is well understood from the analytical point of view. Nevertheless, nothing is known regarding the fine structure of this spaces. It is an interesting and completely open question whether infinite dimensional spaces with Ricci curvature bounded below admit a rectifiable structure, in some generalised sense. The first difficulty one meets approaching this problem is the lack of compactness of $\mathrm{RCD}(K, \infty)$ spaces, with respect to the pmGH topology. In particular it is unclear whether tangent cones to a given point exist; this seems to be a hard obstacle to get around when trying to adapt the techniques developed in the previous chapters to study structural properties of $\mathrm{RCD}(K, N)$ spaces.

In the last chapter of this thesis we present an analytical result regarding approximation of Sobolev maps by means of Lipschitz ones in the setting of $\mathrm{RCD}(K, \infty)$ spaces. Although it does not give any structural result we believe that it is a very first step toward a structure theory for this class of spaces. Indeed, as the analysis performed in the previous chapters shows, Sobolev functions play an important role in building good parametrizations of the space, and the fact that they are Lipschitz on big sets helps in showing rectifiability results.

## Other papers

In this second part of the introduction we give a short summary of the additional research made during the PhD studies. We briefly report the results obtained and we refer to the original papers for a more complete treatment of the problem and the relevant literature.

**Singular set and boundary of** RCD **spaces.** Since the first works concerning Ricci limits [**51**–**53**] in the 90's, the study of the singularities of these spaces has been of great interest.

Let us denote by $\mathcal{S} := X \setminus \cup_k \mathcal{R}_k$ the collection of singular points of an $\mathrm{RCD}(K,N)$ space $X$, i.e. the points having a non-Euclidean space in the tangent cone. When restricting the attention to *non-collapsed* spaces [**71**] ($(X,\mathsf{d},\mathfrak{m})$ is said to be a *non-collapsed* $\mathrm{RCD}(K,N)$ space if $\mathfrak{m} = \mathscr{H}^N$), one can stratify $\mathcal{S}$ according to the number of "symmetries" of tangent cones. Precisely, for $k \in [0,N]$ we say that $x \in \mathcal{S}^k$ if no tangent space of $(X,\mathsf{d},\mathfrak{m})$ at $x$ splits off isometrically a factor $\mathbb{R}^{k+1}$.

By means of topological arguments, Cheeger and Colding in [**51**] proved that, for any non-collapsed Ricci limit space $(X,\mathsf{d},\mathscr{H}^N)$, one has $S^{N-1} = \emptyset$. This corresponds to the rough statement "non-collapsed limits of manifolds with Ricci uniformly bounded below have no boundary points". While, concerning $\mathcal{S}^k$ for $k \le N-2$, Cheeger and Colding have shown that the Hausdorff dimension is smaller than or equal to $k$. Their proof, based on the reduction of the dimension argument, has been made quantitative by Cheeger and Naber in [**56**], exploiting deep ideas coming from *quantitative differentiation*.

In the work [**29**] in collaboration with G. Antonelli and D. Semola, we have adapted the argument by Cheeger and Naber to the setting of RCD spaces, developing all the needed tools such as the "almost volume cone implies almost metric cone" and "almost cone splitting" theorem.

Let us begin by introducing the quantitative stratification, that is built separating points according to the number of symmetries of balls of a definite size at any point.

**Definition 0.1.** For any $\eta > 0$ and any $0 < r < 1$, define the $k^{th}$-effective stratum $\mathcal{S}^k_{\eta,r}$ by

$$\mathcal{S}^k_{\eta,r} := \{y \mid \mathsf{d}_{GH}(B_s(y), B_s((0,z^*))) \ge \eta s \quad \text{for all } \mathbb{R}^{k+1}\times C(Z) \quad \text{and all } r \le s \le 1\},$$

where $B_s((0,z^*))$ denotes the ball in $\mathbb{R}^{k+1}\times C(Z)$ centered at $(0,z^*)$ with radius $s$.

Notice that on a smooth Riemannian manifold the strata $\mathcal{S}^k$ are all empty, instead the effective strata $\mathcal{S}^k_{\eta,r}$ are non trivial in general. Moreover the singular strata can be recovered from the effective ones by means of

$$\mathcal{S}^k = \bigcup_\eta \bigcap_r \mathcal{S}^k_{\eta,r}.$$

Let us now state the main result of [**29**].

**Theorem 0.2.** *Given $K \in \mathbb{R}$, $N \in [2,\infty)$, an integer $k \in [0,N)$ and $v,\eta > 0$, there exists a constant $c(K,N,v,\eta) > 0$ such that if $(X,\mathsf{d},\mathcal{H}^N)$ is a non-collapsed* $\mathrm{RCD}(K,N)$ *m.m.s. satisfying*

$$\frac{\mathcal{H}^N(B_1(x))}{V_{K,N}(1)} \ge v \qquad \forall x \in X, \tag{1}$$

*where $V_{K,N}$ is defined in* (1.14)*, then, for all $x \in X$ and $0 < r < 1/2$, it holds*

$$\mathcal{H}^N(\mathcal{S}^k_{\eta,r} \cap B_{1/2}(x)) \le c(K,N,v,\eta) r^{N-k-\eta}.$$

It is not difficult to see that Theorem 0.2 implies the Hausdorff dimension estimate $\dim_H(\mathcal{S}^k) \le k$, proven in [**71**]. Another interesting outcome is a bound of the *$r$-enlargement of the boundary* of non-collapsed $\mathrm{RCD}(K,N)$ spaces. In [**71**] the authors have proposed a definition of boundary $\partial X$ of a non-collapsed $\mathrm{RCD}(K,N)$ m.m.s. $X$ as

$$\partial X := \text{closure of } \mathcal{S}^{N-1} \setminus \mathcal{S}^{N-2}.$$

**Corollary 0.3.** *Given $K \in \mathbb{R}$, $N \in [2,\infty)$ and $v, \eta > 0$, there exist $c(K,N,v,\eta) > 0$ and $r(K,N) > 0$ such that, if $(X, \mathsf{d}, \mathcal{H}^N)$ is a* $\mathrm{RCD}(K,N)$ *m.m.s. satisfying* (1)*, then, for all $x \in X$ and $0 < r < r(K,N)$, it holds*

$$\mathcal{H}^N(T_r(\partial X) \cap B_{1/2}(x)) \le c(K,N,v,\eta) r^{1-\eta}.$$

*Where $T_r(E) := \cup_{x \in E} B_r(x)$ denotes the $r$-enlargement of $E \subset X$.*

**Lipschitz and Whitney type extension theorem.** In this section we briefly outline the results of the author [**34**] in collaboration with S. Di Marino and F. Strà. It concerns the study of the Lipschitz extensions in metric measure spaces. More precisely we consider the extension problem $\mathrm{Lip}(X;Z) \to \mathrm{Lip}(Y;Z)$ where $X \subset Y$ is a closed subset of a complete metric space $(Y, \mathsf{d})$ and $Z$ is a Banach space, under hypotheses just on the space $X$ alone and not on the ambient space $Y$.

We give two new and very simple proofs of the celebrated result of Lee and Naor.

**Theorem 0.4** ([**112**])**.** *Let $X \subset (Y, \mathsf{d})$ be a doubling metric space with doubling constant $\lambda_X$, and let $Z$ be a Banach space. Then there is an extension $T : \mathrm{Lip}(X;Z) \to \mathrm{Lip}(Y;Z)$ such that*

$$\mathrm{Lip}(Tf) \le C \log(\lambda_X) \,\mathrm{Lip}(f) \qquad \forall f \in \mathrm{Lip}(X;Z),$$

*where $C$ is a universal constant.*

Our approach is based on the notion of *random projection*, given in terms of *optimal transportation distances*, introduced by Ambrosio and Puglisi. We then give a $C^1$-extension result for infinite dimensional Banach spaces in the spirit of Whitney's extension theorem.

**Theorem 0.5.** *Let $Y$ be a Banach space whose norm belongs to $C^1(Y \setminus \{0\})$ and let $X \subset Y$ be a closed subset with doubling constant $\lambda$. Given two continuous functions $f : X \to \mathbb{R}$ and $L : X \to Y^*$, define the remainder*

$$R(x,y) = f(y) - f(x) - L_x(y-x) \qquad \text{for } x, y \in X,\ x \ne y$$

*and assume that the function*

$$(x,y) \mapsto \frac{R(x,y)}{|y-x|}$$

*can be extended to a continuous function on $X \times X$ that takes the value $0$ where $y = x$. Then there exists an extension $\tilde f \in C^1(Y)$ such that $d\tilde f_x = L_x$ for all $x \in X$.*

*Moreover, the extension operator $(f, L) \mapsto \tilde f$ is linear.*

**DiPerna-Lions-Ambrosio theory.** In the last thirty years, a big interest has grown on the study of the *transport equation*

$$\partial_t u(t,x) + b(t,x)\cdot\nabla u(t,x) = 0 \quad \text{with} \quad t\in[0,T],\ x\in\mathbb{R}^d \tag{Tr}$$

and the *ODE problem*

$$\frac{\mathrm{d}}{\mathrm{d}t}X(t,x) = b(t,X(t,x)), \quad X(0,x) = x \tag{ODE}$$

under mild regularity assumptions on the velocity field $b$.

Apart from the theoretical importance of such an investigation, the main motivation comes from the study of many nonlinear PDEs of the mathematical physics such as systems of conservation laws, the Euler equation, the Navier-Stokes equation, the Vlasov-Poisson equation, and many others.

Two pioneering contributions to this topic are [**5**, **76**], where it has been proved the well-posedness in $L^\infty([0,T]\times\mathbb{R}^d)$ for (Tr) and the existence and uniqueness of *regular Lagrangian flows* assuming the Sobolev and $BV$ regularity of the velocity field and the $L^\infty$-bound of the divergence. The regular Lagrangian flow is a suitable selection of trajectories of the related ODE satisfying additional compressibility properties:

**Definition 0.6.** The map $X:[0,T]\times\mathbb{R}^d\to\mathbb{R}^d$ is said to be a regular Lagrangian flow associated to a velocity field $b:[0,T]\times\mathbb{R}^d\to\mathbb{R}^d$ if it satisfies:

(i) for $\mathscr{L}^d$-a.e. $x\in\mathbb{R}^d$ the curve $t\to X(t,x)$ solves (ODE);
(ii) $X(t,\cdot)_*\mathscr{L}^d \le L\mathscr{L}^d$ for any $t\in[0,T]$.

A long-standing open question is whether the uniqueness of the regular Lagrangian flow is a corollary of the uniqueness of the trajectory of the ODE for $\mathscr{L}^d$-a.e. initial datum. A recent result of Crippa-Caravenna [**62**] gives a positive answer in the case of $W^{1,p}$ velocity fields with $p>d$ and bounded divergence. In [**33**] the author in collaboration with M. Colombo and C. De Lellis has given a complete answer to the question above by showing that $\mathscr{L}^d$-a.e. uniqueness of trajectories does not hold for $W^{1,p}$ velocity fields with $p<d$.

**Theorem 0.7.** *For every $d\ge 2$, $p<d$, $s<\infty$ and every $T>0$ there is a divergence-free vector field $b\in C([0,T],W^{1,p}(\mathbb{T}^d,\mathbb{R}^d)\cap L^s)$ such that the following holds for every Borel map $v$ with $b=v$ $\mathscr{L}^{d+1}$-a.e.: there is a measurable $A\subseteq\mathbb{T}^d$ with positive Lebesgue measure such that for every $x\in A$ there are at least two integral curves of $v$ starting at $x$.*

The proof builds upon Ambrosio's superposition principle and a new ill-posedness result for positive solutions of the transport equation (Tr).

**Theorem 0.8.** *Let $d\ge 2$, $q\in(1,\infty), p\in[1,\infty]$ be such that*

$$\frac{1}{p}+\frac{1}{q} > 1+\frac{1}{d}$$

*and denote by $q'$ the dual exponent of $q$, i.e. $\frac{1}{q}+\frac{1}{q'}=1$. Then for every $T>0$ there exists a divergence-free vector field $b\in C([0,T],W^{1,p}(\mathbb{T}^d,\mathbb{R}^d)\cap L^{q'})$ and a* nonconstant $u\in C([0,T],L^q(\mathbb{T}))$ *such that* (Tr) *holds with initial data $u(0,\cdot)=1$ and for which $u\ge c_0$ for some positive constant $c_0$.*

Theorem 0.8 is based on a convex integration type scheme introduced by C. De Lellis and L. Székelyhidi [**68**], [**69**] in the study of Euler equation and recently adapted to the setting of linear transport equations by Modena and Székelyhidi [**120**].

In the second part of [**33**] we introduce a new class of asymmetric Lusin-Lipschitz inequalities and use them to prove the uniqueness of positive solutions of the transport equation in an integrability range which goes beyond the DiPerna-Lions theory [**76**].

At the *quantitative* level there are important differences between smooth and non-smooth framework. On the one hand it is well-known that solutions to the transport equation and ODE flows inherit the Lipschitz regularity from the velocity field. But, on the other hand, recent examples [**1**, **104**] show *instantaneous loss of Sobolev regularity* when the vector field is Sobolev regular. More precisely, one can build a Sobolev vector field and a solution to the transport equation that is smooth at time $t = 0$ but does not belong to any Sobolev space, even of fractional order, at any time $t > 0$.

These new phenomena give rise to two natural questions that the author has investigated in collaboration with Q.-H. Nguyen:

(i) Does some propagation of regularity survive?
(ii) What is the minimal regularity property on the velocity field that guarantees a propagation of Sobolev regularity?

In the work [**35**] we give a satisfactory answer to the first question, proving sharp regularity results in the scale of *log-Sobolev spaces.*

**Theorem 0.9.** *Let* $b \in L^1([0,T], W^{1,p}(\mathbb{R}^d, \mathbb{R}^d))$ *with* $p > 1$ *and* $\|\operatorname{div} b\|_{L^\infty} \le L$. *Then*

$$\int_{B_{1/3}} \int_{\mathbb{R}^d} \frac{|u_t(x+h) - u_t(x)|^2}{|h|^d \log(1/|h|)^{1-p}} \,\mathrm{d}x\,\mathrm{d}h \lesssim_{p,d,L} \left( \int_0^t \|\nabla b_s\|_{L^p} \,\mathrm{d}s \right)^p + C(u_0, p),$$

*where* $u \in L^\infty_{t,x}$ *is a solution to* (Tr) *with* $\|u\|_{L^\infty} = 1$.

*For any* $p \ge 1$, *there exist* $b \in L^1([0,T], W^{1,p}(\mathbb{R}^d, \mathbb{R}^d))$ *and* $u_0 \in W^{1,d}(\mathbb{R}^d) \cap L^\infty$, *such that*

$$\int_{B_{1/3}} \int_{\mathbb{R}^d} \frac{|u_t(x+h) - u_t(x)|^2}{|h|^d \log(1/|h|)^{1-q}} \,\mathrm{d}x\,\mathrm{d}h = \infty, \qquad \forall t > 0,\ \forall q > p,$$

*where* $u \in L^\infty$ *solves* (Tr).

The positive part of this theorem generalizes the result in [**113**] using completely different techniques. Our approach is *Lagrangian* in spirit and exploits a variant of the strategy introduced by Crippa and De Lellis in [**63**], as well as an abstract lemma relating two different notions of having a "logarithm of the derivative" in $L^p$. In [**37**] we further investigate the topic of *log-Sobolev* spaces proving Sobolev embeddings, approximation results in the sense of Lusin, and interpolation inequalities. The counterexample in Theorem 0.9 is obtained following the strategy in [**1**].

In the work [**36**] we answer the question (ii), proving that a sought condition is the exponential integrability of the gradient of the velocity field $b$.

Under this assumption, we prove a propagation of regularity result in Sobolev spaces whose regularity index coarsens with time. It is worth remarking that this kind of propagation has been observed several time by different authors in the study of transport equations with "almost" Lipschitz drift and nonlinear equations of the fluid mechanics. We then show, by means of three different examples, that the exponential integrability of $\nabla b$ cannot be relaxed significantly in order to hope for some propagation of Sobolev regularity. We eventually give an application to the study of Yudovich solutions of the 2-D Euler equation with initial vorticity in fractional Sobolev spaces.

**Advection-diffusion equation and enhanced dissipation.** In the work [**38**], written in collaboration with Q.-H. Nguyen, we have studied advection diffusion equations associated to incompressible Sobolev velocity fields. More precisely, given a divergence free vector field $b \in L^1([0,T], W^{1,p}(\mathbb{T}^d, \mathbb{R}^d))$, for $p > 1$, and an initial datum $u_0 \in L^\infty(\mathbb{T}^d)$ we consider the Cauchy problem associated to the advection-diffusion equation

$$(E_\nu) \qquad \begin{cases} \partial_t u^\nu + b \cdot \nabla u^\nu - \nu \Delta u^\nu = 0 & \text{on } \mathbb{T}^d \times (0,T] \\ u^\nu(0,x) = u_0(x), \end{cases}$$

above, $\nu > 0$ is a constant molecular diffusivity.

Recent developments in the topic of *dissipation enhancement by mixing* [**59, 61**] have shown that, very often, solutions to $(E_\nu)$ dissipate the energy $\|u_t^\nu\|_{L^2}$ faster than $e^{-\nu t}$, the rate at which the heat equation dissipates energy. More rigorously, we give the following definition.

**Definition 0.10.** Let $r : (0, \nu_0) \to (0,1)$ be an increasing function satisfying

$$\lim_{\nu \to 0} \frac{\nu}{r(\nu)} = 0.$$

We say that a divergence free vector field $b$ is diffusion enhancing of rate $r(\nu)$, if for any $\nu \in (0, \nu_0)$ there exists $t_\nu > 0$ such that

$$\|u_t^\nu\|_{L^2}^2 \le C e^{-r(\nu)t} \|u_0\|_{L^2}^2 \quad \text{for every } t \ge t_\nu, \text{ and } u_0 \in W^{1,2}.$$

The constant $C > 0$ above depends only on $b$.

It is nowadays well known that, at least in the setting of smooth velocity fields, mixing in the diffusion free case is a responsible of diffusion enhancing [**59, 61**]. The picture is less clear in the non-smooth setting, where very few and highly non-optimal results are available. Other important open questions in the field regard the study of upper bounds on the diffusion enhancing rate and lower bounds on $L^2$ norms of solutions to $(E_\nu)$.

Aiming at better understanding these questions, the key quantity to study is the energy dissipation rate

$$2\nu \int_0^t \|\nabla u_s^\nu\|_{L^2}^2 \, \mathrm{d}s = \|u_0\|_{L^2}^2 - \|u_t^\nu\|_{L^2}^2 .$$

In the Sobolev setting we have proven the following "almost" sharp estimate.

**Theorem 0.11.** *Let $b \in L^\infty([0,T], W^{1,p}(\mathbb{T}^d, \mathbb{R}^d))$ be a divergence free vector field for some $p > 2$. Any solution $u^\nu$ to $(E_\nu)$ with $u_0 \in W^{1,2}(\mathbb{T}^d) \cap L^\infty$ satisfies*

$$\nu \int_0^t \|\nabla u_s^\nu\|_{L^2}^2 \, \mathrm{d}s \le C(\|u_0\|_{W^{1,2}}^2 + \|u_0\|_{L^\infty}^2) \left[ \nu t + \frac{t^p \|\nabla b\|_{L_t^\infty L_x^p}^p + 1}{\log\left(\frac{1}{\nu t} + 2\right)^{p-1}} \right] \qquad \forall\, t \ge 0,$$

*where $C = C(p,d)$. In particular, for any $t > 0$, we have*

$$\nu \int_0^t \|\nabla u_s^\nu\|_{L^2}^2 \, \mathrm{d}s \le C \log(1/\nu)^{-p+1} \quad \textit{for every } \nu \in (0, 1/5).$$

*Here $C = C(p, d, t, \|u_0\|_{W^{1,2}}^2 + \|u_0\|_{L^\infty}^2, \|\nabla b\|_{L_t^\infty L_x^p}) > 0$.*

*Moreover, for any $p > 2$ there exist a divergence free $b \in L^\infty([0,1], W^{1,p}(\mathbb{T}^d, \mathbb{R}^d))$ and $u_0 \in W^{1,2}(\mathbb{T}^d) \cap L^\infty$ such that*

$$\limsup_{\nu\to 0} \log(1/\nu)^r \, \nu \int_0^1 \|\nabla u_s^\nu\|_{L^2}^2 \, \mathrm{d}s = \infty$$

*for any $r > p\frac{(p-1)}{p-2}$.*

Key tools employed in the proof of Theorem 0.11 are a propagation of regularity result related to Theorem 0.9, and a new theorem connecting the energy dissipation rate to regularity estimates for transport equations.

Theorem 0.11 has important applications to the study of upper bounds on the enhancing dissipation rate, lower bounds on the $L^2$ norm of the density and quantitative vanishing viscosity estimates. Let us just present one of them referring to the original paper [**38**] for precise statements and more explanations.

**Corollary 0.12.** *Let $b \in L^\infty([0,\infty), W^{1,p}(\mathbb{T}^d, \mathbb{R}^d))$ be a divergence free vector field for some $p > 2$. Given $u_0 \in W^{1,2}(\mathbb{T}^d) \cap L^\infty$, if there exists $r : (0,\nu_0) \to (0,\infty)$ for some $0 < \nu_0 < 1$, which satisfies*

$$\|u_t^\nu\|_{L^2}^2 \le e^{-r(\nu)t} \, \|u_0\|_{L^2}^2 \quad \text{for any } t > 1/\nu_0 \text{ and } \nu \in (0,\nu_0),$$

*then*

$$\limsup_{\nu\downarrow 0} \frac{r(\nu)}{\log(1/\nu)^{-\frac{p-1}{p}}} < \infty.$$

In other words the upper bound on the enhancing dissipation rate $r(\nu) \le O(\log(1/\nu)^{-\frac{p-1}{p}})$ holds in the Sobolev setting.

**Maximal characterization of Sobolev functions among BV ones.** In the short note [**39**], written in collaboration with Q.-H. Nguyen and G. Stefani, we give a charaterization of $W^{1,1}$ functions among $BV$ ones by means of the *weighted sharp maximal function.*

**Theorem 0.13.** *Let $f \in BV(\mathbb{R}^d)$. Set*

$$\mathsf{A}f(x) := \sup_{r>0} \frac{1}{r} \fint_{B(x,r)} |f - (f)_{x,r}| \, dy \in [0,\infty].$$

*Then*

$$\liminf_{\lambda\to\infty} \lambda \, \mathscr{L}^d\Big(\{x \in \mathbb{R}^d : \mathsf{A}f(x) > \lambda\}\Big) \gtrsim_d |D^s f|(\mathbb{R}^d),$$

*where $|D^s f|(\mathbb{R}^d)$ denotes the total variation of the singular part of $Df$.*

*In particular, $f \in W^{1,1}(\mathbb{R}^d)$ if and only if*

$$\liminf_{\lambda\to\infty} \lambda \, \mathscr{L}^d\Big(\{x \in \mathbb{R}^d : \mathsf{A}f(x) > \lambda\}\Big) = 0.$$

We then show that a quantitative approach for the study of well-posedness for ODEs with $W^{1,1}$ coefficients, developed by Crippa-DeLellis and Jabin, cannot be further extended to the case of $BV$ velocity fields.

CHAPTER 1

# Preliminaries

In this thesis we are concerned with *metric measure spaces* $(X, \mathsf{d}, \mathfrak{m})$ where $(X, \mathsf{d})$ is a complete and separable metric space, while $\mathfrak{m} \geq 0$ is a Borel measure on $X$ that is finite on balls. Given $x \in X$ we denote by $B_r(x) = \{y \in X : \mathsf{d}(y,x) < r\}$ and $\overline{B}_r(x) = \{y \in X : \mathsf{d}(y,x) \leq r\}$ the open and closed balls respectively, by $\mathrm{C_{bs}}(X)$ the space of bounded continuous real valued functions with bounded support, by $\mathrm{Lip_{bs}}(X) \subset \mathrm{C_{bs}}(X)$ the subspace of Lipschitz functions. We shall adopt the notation $\mathrm{C_b}(X)$ and $\mathrm{Lip_b}(X)$ for bounded continuous and bounded Lipschitz functions respectively. For any $f \in \mathrm{Lip}(X)$ we shall denote by $\mathrm{Lip}\, f$ its global Lipschitz constant. The space of locally Lipschitz functions is denoted by $\mathrm{Lip_{loc}}(X)$.

The Borel $\sigma$-algebra is denoted by $\mathscr{B}(X)$. We shall denote by $\mathscr{M}(X)$ the space of signed Borel measures with finite total variation on $X$ and by $\mathscr{M}^+(X), \mathscr{M}^+_{\mathrm{loc}}(X), \mathscr{P}(X)$ the spaces of nonnegative finite Borel measures, nonnegative measures finite on bounded subsets of $X$ and probability measures, respectively. We will denote by $\mathrm{supp}\, \mathfrak{m}$ the support of any $\mathfrak{m} \in \mathscr{M}^+_{\mathrm{loc}}(X)$.

We will use the standard notation $L^p(X, \mathfrak{m})$, $L^p_{\mathrm{loc}}(X, \mathfrak{m})$ for the $L^p$ spaces, whenever $\mathfrak{m}$ is nonnegative, and $\mathscr{L}^n, \mathscr{H}^n$ for the $n$-dimensional Lebesgue measure on $\mathbb{R}^n$ and the $n$-dimensional Hausdorff measure on a metric space, respectively. We shall denote by $\omega_n$ the Lebesgue measure of the unit ball in $\mathbb{R}^n$. Given a measure $\mu$ and a Borel set $E$ we use the notation $\mu \llcorner E(A) := \mu(A \cap E)$ to indicate the restriction measure.

Below we list two useful lemmas. The proof of the first one, based on Cavalieri's formula, can be found for instance in [**18**, Lemma 3.3] (notice that since we are assuming that $\mu$ and all $\mu_n$ are probability measures, weak convergence in duality w.r.t. $\mathrm{C_{bs}}(Z)$ and w.r.t. $\mathrm{C_b}(Z)$ are equivalent).

**Lemma 1.1.** *Let $(Z, \mathsf{d}_Z)$ be a complete and separable metric space. Let $(\mu_n) \subset \mathscr{P}(Z)$ be weakly converging in duality with $\mathrm{C_{bs}}(Z)$ to $\mu \in \mathscr{P}(Z)$ and let $f_n$ be uniformly bounded Borel functions such that*

$$\limsup_{n\to\infty} f_n(z_n) \leq f(z) \quad \textit{whenever} \quad \mathrm{supp}\, \mu_n \ni z_n \to z \in \mathrm{supp}\, \mu, \tag{1.1}$$

*for some Borel function $f$. Then*

$$\limsup_{n\to\infty} \int_Z f_n \, \mathrm{d}\mu_n \leq \int_Z f \, \mathrm{d}\mu.$$

**Remark 1.2.** If $(Z, \mathsf{d}_Z)$ is proper, $f_n$ and $f$ are continuous, and $\mu_n$ have uniformly bounded supports, then the uniform bound from above for $f_n$ is a direct consequence of (1.1).

The proof of Lemma 1.1 can be easily adapted to the case when we need to estimate the liminf of $\int_Z f_n \, \mathrm{d}\mu_n$.

**Lemma 1.3.** *Let $(Z, \mathsf{d}_Z)$ be a complete and separable metric space. Let $(\mu_n)$ be a sequence of nonnegative Borel measures on $Z$ finite on bounded sets and assume that $\mu_n$ weakly converge to $\mu$ in duality w.r.t. $\mathrm{C_{bs}}(Z)$. Let $(f_n)$ and $f$ be nonnegative Borel functions on $Z$ such that*

$$f(z) \leq \liminf_{n\to\infty} f_n(z_n) \quad \textit{whenever } \operatorname{supp}\mu_n \ni z_n \to z \in \operatorname{supp}\mu. \tag{1.2}$$

*Then*

$$\int f \,\mathrm{d}\mu \leq \liminf_{n\to\infty} \int f_n \,\mathrm{d}\mu_n.$$

## 1. Calculus tools on metric measure spaces

Given a Lipschitz function $f\colon X \to \mathbb{R}$, we will denote by $\operatorname{lip}(f)\colon X \to [0,\infty)$ its *slope*, which is the function defined as

$$\operatorname{lip}(f)(x) \coloneqq \limsup_{y\to x} \frac{\big|f(x) - f(y)\big|}{\mathsf{d}(x,y)} \quad \text{for every accumulation point } x \in X$$

and $\operatorname{lip}(f)(x) \coloneqq 0$ elsewhere. Let us now present a brief discussion about Sobolev functions and Sobolev spaces over an arbitrary metric measure space, referring to [**15**, **16**, **87**] and [**9**] (see also the beautiful [**6**] paper) for a more detailed discussion about this topic.

### 1.1. Sobolev space $H^{1,p}$.

Following [**48**], we define the *Sobolev space* $H^{1,p}(X, \mathsf{d}, \mathfrak{m})$.

**Definition 1.4.** For $1 < p < \infty$ we set

$$H^{1,p}(X, \mathsf{d}, \mathfrak{m}) := \Big\{ f \in L^p(X, \mathfrak{m}) \,\Big|\, \mathsf{Ch}_p(f) < \infty \Big\},$$

where the *Cheeger energy* $\mathsf{Ch}_p : L^p(X, \mathfrak{m}) \to [0,\infty]$ is the convex, lower semicontinuous functional

$$\mathsf{Ch}_p(f) := \inf \left\{ \liminf_{n\to\infty} \int \operatorname{lip}(f_n)^p \,\mathrm{d}\mathfrak{m} \,\middle|\, (f_n)_n \subseteq L^p \cap \mathrm{Lip}_b(X) \ \lim_{n\to\infty} \|f_n - f\|_{L^p(\mathfrak{m})} = 0 \right\}. \tag{1.3}$$

It holds that $H^{1,p}(X, \mathsf{d}, \mathfrak{m})$ is a Banach space if endowed with the norm $\|\cdot\|_{H^{1,p}(X,\mathsf{d},\mathfrak{m})}$, given by

$$\|f\|^p_{H^{1,p}(X,\mathsf{d},\mathfrak{m})} := \|f\|^p_{L^p(\mathfrak{m})} + \mathsf{Ch}_p(f).$$

It is worth noticing that $H^{1,p}(X, \mathsf{d}, \mathfrak{m})$ is dense in $L^p(X, \mathfrak{m})$.

Given any $f \in H^{1,p}(X, \mathsf{d}, \mathfrak{m})$, by looking at the optimal approximating sequence in (1.3), one can select a canonical object $|\nabla f|_p \in L^p(\mathfrak{m})$ – called the *minimal relaxed slope* of $f$ – for which $\mathsf{Ch}_p(f)$ admits the integral representation

$$\mathsf{Ch}_p(f) = \int |\nabla f|_p^p \,\mathrm{d}\mathfrak{m}.$$

Moreover, the minimal relaxed slope is strongly local, meaning that

$$|\nabla f|_p = |\nabla g|_p \quad \mathfrak{m}\text{-a.e. on } \{f = g\} \quad \text{for any } f, g \in H^{1,p}(X, \mathsf{d}, \mathfrak{m}). \tag{1.4}$$

This, combined with the integral representation property, ensures that $|\nabla f|_p$ is unique as class of $L^p$-equivalent functions.

We further introduce, following [**10**], the space of functions of bounded variation.

**Definition 1.5.** For any $f \in L^1(X, \mathfrak{m})$ and for any open set $A \subset X$ we introduce the relaxed total variation of $f$ over $X$ by

$$|Df|(A) := \inf \left\{ \liminf_{n\to\infty} \int_A \mathrm{lip}(f_n)\, \mathrm{d}\mathfrak{m} \;\Big|\; f_n \in \mathrm{Lip}_{\mathrm{loc}}(A),\ f_n \to f \text{ in } L^1(A, \mathfrak{m}) \right\}.$$

Moreover we let $\mathrm{BV}(X, \mathsf{d}, \mathfrak{m}) := \{f \in L^1(X, \mathfrak{m}) \ : |Df|(X) < \infty\}$ be the space of functions with bounded variation on $(X, \mathsf{d}, \mathfrak{m})$.

It is proved in [**119**] that, for any $f \in \mathrm{BV}(X, \mathsf{d}, \mathfrak{m})$, the map $A \mapsto |Df|(A)$ is the restriction to open sets of a finite Borel measure, for which we keep the same notation.

**Definition 1.6.** We let $H^{1,1}(X, \mathsf{d}, \mathfrak{m})$ be the space of functions $f \in \mathrm{BV}(X, \mathsf{d}, \mathfrak{m})$ with the following property: there exists $|\nabla f|_1 \in L^1(X, \mathfrak{m})$ such that $|Df| = |\nabla f|_1 \mathfrak{m}$.

The space $H^{1,1}(X, \mathsf{d}, \mathfrak{m})$ endowed with the norm $\|f\|_{H^{1,1}} := \|f\|_{L^1} + \||\nabla f|_1\|_{L^1}$ is a Banach space.

1.1.1. *Locally Sobolev functions.* It is useful to talk about locally Sobolev functions in the abstract framework of metric measure spaces. It can be done building upon the strong locality property (1.4).

For every $1 < p < \infty$, we define $H^{1,p}_{\mathrm{loc}}(X, \mathsf{d}, \mathfrak{m})$ as the space of those $f \in L^p_{\mathrm{loc}}(X, \mathfrak{m})$ such that $f\eta \in H^{1,p}(X, \mathsf{d}, \mathfrak{m})$ for every $\eta \in \mathrm{Lip}_{\mathrm{bs}}(X)$. Using the strong locality property of the minimal relaxed gradient it is possible to define $|\nabla f|_p \in L^p_{\mathrm{loc}}(X, \mathfrak{m})$, which retains the same strong locality property, for every $f \in H^{1,p}_{\mathrm{loc}}(X, \mathsf{d}, \mathfrak{m})$. In an analogous way one can define the space $H^{1,1}_{\mathrm{loc}}(X, \mathsf{d}, \mathfrak{m})$, exploiting the strong locality of the relaxed gradient $|\nabla f|_1$ for $f \in H^{1,1}(X, \mathsf{d}, \mathfrak{m})$.

1.1.2. *Sobolev spaces on open sets.* Given any open set $\Omega \subseteq X$, we denote by $\mathrm{Lip}_{\mathrm{bs}}(\Omega)$ the family of all Lipschitz functions $f \colon \Omega \to \mathbb{R}$ whose support is bounded and satisfies $\mathrm{dist}\Big(\mathrm{spt}(f), X \setminus \Omega\Big) > 0$.

**Definition 1.7.** We define $H^{1,p}_{\mathrm{loc}}(\Omega, \mathsf{d}, \mathfrak{m})$ as the space of all those $f \in L^p_{\mathrm{loc}}(X, \mathfrak{m})$ such that $\eta f \in H^{1,p}(X, \mathsf{d}, \mathfrak{m})$ holds for every $\eta \in \mathrm{Lip}_{\mathrm{bs}}(\Omega)$.

We define $H^{1,p}_0(\Omega, \mathsf{d}, \mathfrak{m})$ considering the closure of $\mathrm{Lip}_{\mathrm{bs}}(\Omega)$ in $H^{1,p}(X, \mathsf{d}, \mathfrak{m})$.

Thanks to the locality property of the minimal relaxed slope, it makes sense to define $|\nabla f|_p \in L^p_{\mathrm{loc}}(X, \mathfrak{m})$ as

$$|\nabla f|_p := \Big|\nabla(\eta f)\Big|_p \ \mathfrak{m}\text{-a.e. on } \{\eta = 1\}, \qquad \text{for any } \eta \in \mathrm{Lip}_{\mathrm{bs}}(\Omega).$$

Finally, we define $H^{1,p}(\Omega, \mathsf{d}, \mathfrak{m})$ as the space of all $f \in H^{1,p}_{\mathrm{loc}}(\Omega, \mathsf{d}, \mathfrak{m})$ such that $f, |\nabla f|_p \in L^p(X, \mathfrak{m})$.

**1.2. The Sobolev class $S^2(X, \mathsf{d}, \mathfrak{m})$.** Another way to characterize Sobolev functions on the Euclidean space $\mathbb{R}^n$ is to require that for almost any line $\gamma$ in $\mathbb{R}^n$ the curve $f \circ \gamma$ is absolutely continuous. In this subsection we present a similar characterization in the setting of metric measure spaces.

The notion of "almost every line" can be given by introducing the concept of *test plan* (Cf. [**16**, Section 5]).

Let us start by introducing the class of absolutely continuous curves over $(X, \mathsf{d})$:

$$\mathrm{AC}([0,1], X) := \left\{ \gamma \in C([0,1], X) |\ \exists g \in L^1(0,1) \text{ s.t. } \mathsf{d}(\gamma(t), \gamma(s)) \le \int_s^t g(r)\, \mathrm{d}r,\ s < t \right\}.$$

Notice that in the Euclidean setting AC coincides with the classical class of absolutely continuous curves.

The next result shows that absolutely continuous curves on metric spaces have a weak notion of velocity.

**Theorem 1.8** (Metric derivative)**.** *For any $\gamma \in AC([0,1],X)$ there exists $|\gamma'| \in L^1(0,1)$, called metric derivative, such that*

$$\lim_{h\to 0} \frac{\mathsf{d}(\gamma(t+h),\gamma(t))}{|h|} = |\gamma'|(t) \quad \text{for } \mathscr{L}\text{-a.e. } t\in(0,1)$$

*and*

$$\mathsf{d}(\gamma(t),\gamma(s)) \le \int_s^t |\gamma'|(r)\,\mathrm{d}r \quad \text{for any } s<t.$$

**Definition 1.9** (Test plan)**.** A probability measure $\pi \in \mathscr{P}(C([0,1],X))$ is said to be a test plan if it is concentrated on $\mathrm{AC}([0,1],X)$ and satisfies

$$\int_0^1 \int |\gamma'|^2(s)\,\mathrm{d}\pi(\gamma)\,\mathrm{d}s < \infty,$$

$$(e_t)_*\pi \le C\mathfrak{m} \quad \text{for any } t\in[0,1],$$

for some $C>0$, where $e_t(\gamma) := \gamma(t)$ is the evaluation map.

**Definition 1.10.** The Sobolev class $S^2(X,\mathsf{d},\mathfrak{m})$ is by definition the set of $u \in L^2(X,\mathfrak{m})$ such that

$$\int |u(\gamma(1)) - u(\gamma(0))|\,\mathrm{d}\pi \le \int\int_0^1 G(\gamma(s))|\gamma'|(s)\,\mathrm{d}s\,\mathrm{d}\pi(\gamma) \quad \text{for any test plan } \pi,$$

for some function $G \in L^2(X,\mathfrak{m})$. Any such $G$ is called *weak upper gradient.*

For any $u \in S^2(X,\mathsf{d},\mathfrak{m})$ there exists a unique (up to $\mathfrak{m}$-negligible sets) minimal in the $\mathfrak{m}$-a.e. sense weak upper gradient, denoted by $|\nabla f|_*$ (see [**16**, Definition 5.1]).

The next result, taken from [**16**], ensures that the Sobolev classes $S^2$, $H^{1,2}$ do coincide.

**Theorem 1.11** (Relaxed and weak upper gradients coincide)**.** *Let $(X,\mathsf{d},\mathfrak{m})$ be a metric measure space with $\sigma$-finite measure. Then $S^2(X,\mathsf{d},\mathfrak{m}) = H^{1,2}(X,\mathsf{d},\mathfrak{m})$ and*

$$|\nabla f|_2 = |\nabla f|_* \quad \mathfrak{m}\text{-a.e., for any } f\in S^2(X,\mathsf{d},\mathfrak{m}).$$

**1.3. Infinitesmal Hilbertian spaces.** In the Euclidean space $\mathbb{R}^n$ equipped with the Hilbertian norm $|\cdot|_2$ and the Lebesgue measure $\mathscr{L}^n$ the 2-Cheeger energy $\mathsf{Ch}_2(f)$ coincides with the Dirichlet energy $\int |\nabla f|^2\,\mathrm{d}x$ for any $f \in H^{1,2}(\mathbb{R}^n)$. In particular $\mathsf{Ch}_2$ is a quadratic form and $H^{1,2}(\mathbb{R}^n)$ is a Hilbert space. Simple examples shows that this is not the case in general. Consider for instance $(\mathbb{R}^n, |\cdot|_q, \mathscr{L}^n)$ where $|x|_q^q := |x_1|^q + \ldots + |x_n|^q$ for $x = (x_1,\ldots,x_n)$. It is easily seen that $\mathsf{Ch}_2(f) = \int |\nabla f|_{q'}^2\,\mathrm{d}x$ where $q'$ is the conjugate exponent of $q$, i.e. $\frac{1}{q}+\frac{1}{q'}=1$. In particular for $q\neq 2$ $\mathsf{Ch}_2(f)$ is not quadratic.

The next definition is due to Nicola Gigli.

**Definition 1.12.** A metric measure space $(X,\mathsf{d},\mathfrak{m})$ is said to be *infinetisimally Hilbertian* if $H^{1,2}(X,\mathsf{d},\mathfrak{m})$ is a Hilbert space.

On infinetisimally Hilbertian spaces we can adopt the point of view of the well-established theory of Dirichlet form [**82, 117**]. It has been indeed proven in [**17, 86**] that, under the infinetisimally Hilbertian assumption, the function

$$\nabla f_1 \cdot \nabla f_2 := \lim_{\varepsilon \to 0} \frac{|\nabla(f_1 + \varepsilon f_2)|_2^2 - |\nabla f_1|_2^2}{2\varepsilon}$$

defines a symmetric bilinear form on $H^{1,2}(X, \mathsf{d}, \mathfrak{m}) \times H^{1,2}(X, \mathsf{d}, \mathfrak{m})$ with values into $L^1(X, \mathfrak{m})$. Moreover

$$\mathcal{E}(f_1, f_2) := \frac{1}{2} \int \nabla f_1 \cdot \nabla f_2 \, \mathrm{d}\mathfrak{m}$$

defines a strongly local *Dirichlet form* with domain $H^{1,2}(X, \mathsf{d}, \mathfrak{m}) \subset L^2(X, \mathfrak{m})$ and *carré du champ* $\Gamma(f_1, f_2) := \nabla f_1 \cdot \nabla f_2$.

It is possible to define the Laplacian operator $\Delta : \mathcal{D}(\Delta) \subset L^2(X, \mathfrak{m}) \to L^2(X, \mathfrak{m})$ in the following way.

**Definition 1.13.** We let $\mathcal{D}(\Delta)$ be the set of those $f \in H^{1,2}(X, \mathsf{d}, \mathfrak{m})$ such that, for some $h \in L^2(X, \mathfrak{m})$, one has

$$\int \nabla f \cdot \nabla g \, \mathrm{d}\mathfrak{m} = -\int hg \, \mathrm{d}\mathfrak{m} \qquad \forall g \in H^{1,2}(X, \mathsf{d}, \mathfrak{m}) \tag{1.5}$$

and in that case we put $\Delta f = h$ since $h$ is uniquely determined by (1.5).

It is easy to check that the definition is well-posed and that the Laplacian is linear (because $\mathsf{Ch}_2$ is a quadratic form).

More generally, we say that $f \in H^{1,2}_{\mathrm{loc}}(X, \mathsf{d}, \mathfrak{m})$ is in the domain of the measure valued Laplacian, and we write $f \in D(\boldsymbol{\Delta})$, if there exists a Radon measure $\mu$ on $X$ such that, for every $g \in \mathrm{Lip_{bs}}(X)$ with compact support, it holds

$$\int g \, \mathrm{d}\mu = -\int \nabla f \cdot \nabla g \, \mathrm{d}\mathfrak{m}.$$

In this case we write $\boldsymbol{\Delta} f := \mu$. If moreover $\boldsymbol{\Delta} f \ll \mathfrak{m}$ with $L^2_{\mathrm{loc}}$ density we denote by $\Delta f$ the unique function in $L^2_{\mathrm{loc}}(X, \mathfrak{m})$ such that $\boldsymbol{\Delta} f = \Delta f \mathfrak{m}$ and we write $f \in D_{\mathrm{loc}}(\Delta)$.

Let us now define the Laplacian operator on open domains $\Omega \subset X$.

**Definition 1.14.** A function $f \in H^{1,2}(\Omega, \mathsf{d}, \mathfrak{m})$ belongs to $D(\Delta, \Omega)$ if there exists $g \in L^2(\Omega, \mathfrak{m})$ satisfying

$$\int_\Omega \nabla f \cdot \nabla h \ \mathrm{d}\mathfrak{m} = -\int_\Omega f \, g \ \mathrm{d}\mathfrak{m} \quad \text{for any } h \in H^{1,2}_0(\Omega, \mathsf{d}, \mathfrak{m}).$$

With a slight abuse of notation we write $\Delta f = g$ in $\Omega$.

It is easily seen that, if $f \in D(\Delta, \Omega)$ and $\eta \in \mathrm{Lip_{bs}}(\Omega, \mathsf{d}) \cap D(\Delta)$, $\Delta \eta \in L^\infty(X, \mathfrak{m})$ then $\eta f \in D(\Delta)$.

**1.4. Heat flow.** The heat flow $P_t$ is defined as the $L^2(X, \mathfrak{m})$-gradient flow of $\frac{1}{2}\mathsf{Ch}_2$. Its existence and uniqueness follow from the Komura-Brezis theory which is briefly outlined below.

Let $(H, \langle\ ,\ \rangle)$ be a Hilbert space and $F : H \to [0, \infty]$ be a convex and lower semicontinuous function. We denote by $D(F) := \{x \in H \mid F(x) < \infty\}$ its finiteness domain.

In the classical theory of gradient flows, the Komura-Brezis theorem ensures that, for any starting point $x \in \overline{D(F)}$, there exists a unique absolutely continuous curve $(x(t))_{t>0} \subset H$ such that $\lim_{t \to 0} |x(t) - x| = 0$ and

$$x'(t) \in -\partial F(x(t)) \quad \text{for a.e. } t > 0 \tag{1.6}$$

where $\partial F$ denotes the sub-differential of $F$

$$\partial F(x) := \{p \in H \mid f(y) \geq f(x) + \langle p, y - x\rangle \quad \forall y \in H\}. \tag{1.7}$$

The curve $(x(t))_{t>0}$ is called gradient flow of $F$ starting from $x$.

Being $\frac{1}{2}\mathsf{Ch}_2$ lower semicontinuous and convex, and $\overline{D(\frac{1}{2}\mathsf{Ch}_2)} = \overline{H^{1,2}(X, \mathsf{d}, \mathfrak{m})} = L^2(X, \mathfrak{m})$ we can apply the Komura-Brezis theory to get a family of maps

$$P_t : L^2(X, \mathfrak{m}) \to L^2(X, \mathfrak{m}) \quad \text{for any } t \geq 0$$

such that $(P_t u)_{t>0} := (u_t)_{t>0}$ is the gradient flow of $\frac{1}{2}\mathsf{Ch}_2$ starting from $u \in L^2(X, \mathfrak{m})$.

When $(X, \mathsf{d}, \mathfrak{m})$ is infinetisimally Hilbertian (Definition 1.12) the heat flow provides a linear, continuous and self-adjoint contraction semigroup in $L^2(X, \mathfrak{m})$. It can equivalently be characterized by saying that for any $u \in L^2(X, \mathfrak{m})$ the curve $t \mapsto P_t u \in L^2(X, \mathfrak{m})$ is locally absolutely continuous in $(0, \infty)$ and satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t} P_t u = \Delta P_t u \quad \text{for } \mathscr{L}^1\text{-a.e. } t \in (0, \infty), \qquad \lim_{t \downarrow 0} P_t u = u \quad \text{in } L^2(X, \mathfrak{m}). \tag{1.8}$$

This is a consequence of the identity

$$\partial \frac{1}{2}\mathsf{Ch}_2(f) := \begin{cases} \{\Delta f\} & \text{whenever } f \in D(\Delta) \\ \emptyset & \text{otherwise.} \end{cases}$$

We recall the following regularization properties of $P_t$, ensured by the theory of gradient flows and maximal monotone operators:

$$\|P_t f\|_{L^2} \leq \|f\|_{L^2}, \quad \mathsf{Ch}_2(P_t f) \leq \frac{\|f\|_{L^2}^2}{2t} \quad \text{and} \quad \|\Delta P_t f\|_{L^2} \leq \frac{\|f\|_{L^2}}{t}, \tag{1.9}$$

for any $t > 0$ and for any $f \in L^2(X, \mathfrak{m})$.

Let us finally mention that $P_t$ extends to a linear, continuous and mass preserving operator, still denoted by $P_t$ , in all the $L^p$ spaces for $1 \leq p < \infty$.

## 2. Curvature-dimension condition via optimal transport

Let us start by recalling some notions of optimal transport theory. We refer to [**137**, Chapter 7], [**138**, Chapter 6] for more details.

Given a complete and separable metric space $(X, \mathsf{d})$ we denote by $\mathscr{P}(X)$ the class of Borel probability measures and by $\mathscr{P}_2(X)$ the subclass of those with finite second moment.

**Definition 1.15.** For $\mu_0, \mu_1 \in \mathscr{P}_2(X)$ the quadratic Wasserstein distance $W_2(\mu_0, \mu_1)$ is defined by

$$W_2^2(\mu_0, \mu_1) := \inf_\pi \int_{X \times X} \mathsf{d}^2(x, y) \, \mathrm{d}\pi(x, y),$$

where the infimum runs over all $\pi \in \mathscr{P}(X \times X)$ with $\mu_0$ and $\mu_1$ as the first and the second marginal, i.e. $\pi(A \times X) = \mu_0(A)$ and $\pi(X \times A) = \mu_1(A)$ for any Borel subset $A \subset X$.

A very important tool in optimal transport and metric geometry is the duality formula

$$\frac{1}{2} W_2^2(\mu_0, \mu_1) = \sup \left\{ \int \mathcal{Q}_1 f \, \mathrm{d}\mu_1 - \int f \, \mathrm{d}\mu_0 \;\middle|\; f \in \mathrm{Lip_b}(X, \mathsf{d}) \right\}$$

where

$$\mathcal{Q}_t f(x) := \inf_{y \in X} \left\{ f(y) + \frac{\mathsf{d}^2(x, y)}{2t} \right\} \quad \text{for any } (x, t) \in X \times (0, \infty), \tag{1.10}$$

denotes the so-called *Hopf-Lax* semigroup (see [**9**, **15**] for a detailed discussion about its properties).

The Wassertein space $(\mathscr{P}_2(X), W_2)$ is a metric space that inherits many geometrical properties of $(X, \mathsf{d})$. Before discussing them let us give a definition.

**Definition 1.16.** Let $(X, \mathsf{d})$ a metric spaces. A curve $\gamma : [0,1] \to X$ is said to be a (constant speed minimizing) $\mathsf{d}$-geodesic if

$$\mathsf{d}(\gamma(t), \gamma(s)) = |t - s| \mathsf{d}(\gamma(1), \gamma(0)) \quad \text{for any } s, t \in [0,1].$$

We denote by $\mathrm{Geo}(X)$ the space of $\mathsf{d}$-geodesics on $(X, \mathsf{d})$, parametrized on $[0,1]$, endowed with the sup distance.

A metric space $(X, \mathsf{d})$ is said to be a geodesic space if for any $x, y \in X$ there exists $\gamma \in \mathrm{Geo}(X)$ such that $\gamma(0) = x$ and $\gamma(1) = y$. If $(X, \mathsf{d})$ is a geodesic space $(\mathscr{P}_2(X), W_2)$ is a geodesic space as well. Moreover, it turns out that any $W_2$-geodesic $\mu_t \in \mathrm{Geo}(\mathscr{P}_2(X))$ can be lifted to a measure $\Pi \in \mathscr{P}(\mathrm{Geo}(X))$, that we shall call optimal geodesic plan, in such a way that $(e_t)_*\Pi = \mu_t$ for any $t \in [0,1]$. Here we denoted by $e_t : \mathrm{Geo}(X) \to X$ the evaluation map defined by $e_t(\gamma) := \gamma(t)$. We refer to [**12**, Theorem 2.10] for more details.

**2.1. CD condition.** The curvature-dimension condition $\mathrm{CD}(K, N)$ is a notion of having Ricci curvature bounded below by $K$ and dimension bounded above by $N$ for metric measure spaces. Its introduction dates back to the seminal and independent works [**116**] and [**135**, **136**].

Before presenting the CD notion we give a motivating result due to D. Cordero-Erausquin, R. McCann and M. Schmuckenschläger [**60**, Theorem 6.2] and K.-T. Sturm and, M.-K. Von Renesse [**132**, Theorem 0.1].

**Theorem 1.17.** *Let $(M, g)$ be a smooth Riemannian manifold endowed with the canonical Riemannian distance $\mathsf{d}_g$ and volume measure $\mathfrak{m}_g$. Then the following properties are equivalent:*

(i) $\mathrm{Ric}_g \geq K$,
(ii) *let $\mu_0, \mu_1 \in \mathscr{P}_2(M)$ be absolutely continuous with respect to $\mathfrak{m}_g$. For any $W_2$-geodesic $(\mu_t)_{t\in[0,1]}$ connecting $\mu_0$ and $\mu_1$ it holds*

$$Ent_{\mathfrak{m}_g}(\mu_{(1-\lambda)s+\lambda t}) \leq (1-\lambda) Ent_{\mathfrak{m}_g}(\mu_s) + \lambda Ent_{\mathfrak{m}_g}(\mu_t) - K \frac{\lambda(1-\lambda)}{2} W_2^2(\mu_s, \mu_t)$$

*for any $0 \leq s \leq t \leq 1$ and $\lambda \in [0,1]$. Here*

$$Ent_{\mathfrak{m}_g}(\mu) := \begin{cases} \int \rho \log \rho \, \mathrm{d}\mathfrak{m}_g & \text{if } \mu = \rho \mathfrak{m}_g \in \mathscr{P}_2(M) \\ \infty & \text{otherwise.} \end{cases} \tag{1.11}$$

*denotes the Shannon entropy relative to $\mathfrak{m}_g$.*

In other words, on smooth manifolds, the lower bound on the Ricci curvature is equivalent to the $K$-convexity of the Shannon Entropy. Notice that the condition (ii) does not require any smooth structure to be introduced. This motivates the next definition due to K.-J. Sturm.

**Definition 1.18** (Curvature bound)**.** Let $(X, \mathsf{d}, \mathfrak{m})$ be a complete and separable metric measure space satisfying the volume growth condition: for some $x \in X$ there exist $c_0$ and $c_1$

positive such that

$$\mathfrak{m}(B_r(x)) \le c_0 e^{c_1 r^2} \quad \text{for every } r > 0. \tag{1.12}$$

For any $K \in \mathbb{R}$, we say that $(X, \mathsf{d}, \mathfrak{m})$ is a $\mathrm{CD}(K,\infty)$ space if for any $\mu_0, \mu_1 \in \mathscr{P}_2(X)$, absolutely continuous with respect to $\mathfrak{m}$, there exists a $W_2$-geodesic $(\mu_t)_{t\in[0,1]}$ such that

$$\mathrm{Ent}_{\mathfrak{m}}(\mu_t) \le (1-t)\mathrm{Ent}_{\mathfrak{m}}(\mu_0) + t\mathrm{Ent}_{\mathfrak{m}}(\mu_1) - K\frac{t(1-t)}{2}W_2^2(\mu_0,\mu_1)$$

for any $0 \le t \le 1$.

The volume growth condition (1.12) ensures that $\mathrm{Ent}_{\mathfrak{m}}(\mu)$ is well defined for any $\mu \in \mathscr{P}_2(X)$. The $\mathrm{CD}(K,\infty)$ encodes the lower bound on the Ricci curvature, in order to introduce an upper bound on the dimension we need to replace the Shannon Entropy with a so-called Rényi entropy

$$\mathrm{S}^N_{\mathfrak{m}}(\mu) = \begin{cases} -\int \rho^{1-\frac{1}{N}}\,\mathrm{d}\mathfrak{m} & \text{if } \mu = \rho\mathfrak{m}_g \in \mathscr{P}(X) \\ \infty & \text{otherwise.} \end{cases}$$

In this presentation we closely follow [**31**].

**Definition 1.19** (Curvature dimension bounds)**.** Let $K \in \mathbb{R}$ and $1 \le N < \infty$. We say that a m.m.s. $(X, \mathsf{d}, \mathfrak{m})$ is a $\mathrm{CD}(K,N)$ space if, for any $\mu_0, \mu_1 \in \mathscr{P}(X)$ absolutely continuous w.r.t. $\mathfrak{m}$ with bounded support, there exists an optimal geodesic plan $\Pi \in \mathscr{P}(\mathrm{Geo}(X))$ such that for any $t \in [0,1]$ and for any $N' \ge N$ we have

$$\begin{aligned} &\mathrm{S}^{N'}_{\mathfrak{m}}(\mu_t) \\ &\le -\int \left\{ \tau^{(1-t)}_{K,N'}(\mathsf{d}(\gamma(0),\gamma(1)))\rho_0^{-\frac{1}{N'}}(\gamma(0)) + \tau^{(t)}_{K,N'}(\mathsf{d}(\gamma(0),\gamma(1)))\rho_1^{-\frac{1}{N'}}(\gamma(1)) \right\} \mathrm{d}\Pi(\gamma), \end{aligned}$$

where $(e_t)_*\Pi = \rho_t\mathfrak{m}$, $\mu_0 = \rho_0\mathfrak{m}$, $\mu_1 = \rho_1\mathfrak{m}$ and the distortion coefficients $\tau^t_{K,N}(\cdot)$ are defined as follows. First we define the coefficients $[0,1]\times[0,\infty) \ni (t,\theta) \mapsto \sigma^{(t)}_{K,N}(\theta)$ by

$$\sigma^{(t)}_{K,N}(\theta) := \begin{cases} \infty & \text{if } K\theta^2 \ge N\pi^2, \\ \frac{\sin(t\theta\sqrt{K/N})}{\sin(\theta\sqrt{K/N})} & \text{if } 0 < \theta < N\pi^2, \\ t & \text{if } K\theta^2 = 0, \\ \frac{\sinh(t\theta\sqrt{K/N})}{\sinh(\theta\sqrt{K/N})} & \text{if } K\theta^2 < 0, \end{cases}$$

then we set $\tau^{(t)}_{K,N}(\theta) := t^{1/N}\sigma^{(t)}_{K,N-1}(\theta)^{1-1/N}$.

Notice that the $\mathrm{CD}(0,N)$ condition is equivalent to convexity of Rényi's entropy along a $W_2$-geodesic for every choice of the end points among probabilities measures absolutely continuous w.r.t. $\mathfrak{m}$.

**2.2. Bishop-Gromov inequality and doubling property.** A very important geometric property of $\mathrm{CD}(K,N)$ metric measure spaces is the Bishop-Gromov inequality (see [**136**, **138**] for a more detailed discussion). That is to say

$$\frac{\mathfrak{m}(B(x,R))}{\mathfrak{m}(B(x,r))} \le \frac{V_{K,N}(R)}{V_{K,N}(r)}, \tag{1.13}$$

for any $0 < r < R$ and for any $x \in X$, where

$$V_{K,N}(r) := \begin{cases} \int_0^r \sin(t\sqrt{K/(N-1)})^{N-1}\,\mathrm{d}t & \text{if } K > 0, \\ r^N & \text{if } K = 0, \\ \int_0^r \sinh(t\sqrt{-K/(N-1)})^{N-1}\,\mathrm{d}t & \text{if } K < 0, \end{cases} \tag{1.14}$$

stands for the volume of the ball of radius $r$ in the model space for the curvature-dimension condition $\mathrm{CD}(K,N)$.

When $K \geq 0$, (1.13) implies that $(X, \mathsf{d}, \mathfrak{m})$ is doubling with doubling constant $2^N$, i.e.

$$\mathfrak{m}(B(x,2r)) \leq 2^N \mathfrak{m}(B(x,r)) \quad \text{for any } x \in X \text{ and for any } r > 0. \tag{1.15}$$

In the case of a possibly negative lower Ricci curvature bound we can achieve the weaker conclusion that $(X, \mathsf{d}, \mathfrak{m})$ is locally uniformly doubling, that is to say, for any $R > 0$ there exists $C_R > 0$ such that

$$\mathfrak{m}(B(x,2r)) \leq C_R \mathfrak{m}(B(x,r)) \quad \text{for any } x \in X \text{ and for any } 0 < r < R. \tag{1.16}$$

As a consequence of (1.16), any $\mathrm{CD}(K,N)$ m.m.s. is proper.

2.2.1. *Maximal function.* We recall that, when the space $(X, \mathsf{d}, \mathfrak{m})$ is doubling, the maximal operator

$$Mf(x) := \sup_{r>0} \fint_{B(x,r)} |f(z)|\,\mathrm{d}\mathfrak{m}(z), \qquad \forall x \in \operatorname{supp}\mathfrak{m}, \tag{1.17}$$

where

$$\fint_{B(x,r)} f(z)\,\mathrm{d}\mathfrak{m}(z) := \frac{1}{\mathfrak{m}(B(x,r))} \int_{B(x,r)} f(z)\,\mathrm{d}\mathfrak{m}(z),$$

is bounded from $L^p(X, \mathfrak{m})$ into itself for any $1 < p \leq \infty$. However if $X$ satisfies only the local doubling condition (1.16) the following local version of this fact holds. Let us fix $1 < p \leq \infty$ and a compact set $P \subset X$. Then, there exists a constant $C > 0$, depending only on the diameter of $P$ and the local doubling constant of the space, such that for every $f \in L^p(X, \mathfrak{m})$ with $\operatorname{supp} f \subset P$, it holds

$$\|Mf\|_{L^p(P)} \leq C \|f\|_{L^p(X)}. \tag{1.18}$$

**2.3. Local Poincaré inequality.** The local Poincaré inequality is a very important analytic tool in the study of metric measure spaces (see for instance [**102**], [**48**], [**98**]). In the setting of CD spaces its validity has been proven by T. Rajala [**129**].

**Theorem 1.20.** *Any* $\mathrm{CD}(K,N)$ *space, with* $N < \infty$*, supports the weak local* $(1,1)$*-Poincaré inequality*

$$\fint_B |f - f_B|\,\mathrm{d}\mathfrak{m} \leq C(N) r e^{\sqrt{2(N-1)K^-}r} \fint_{2B} \operatorname{lip}(f)\,\mathrm{d}\mathfrak{m} \quad \textit{for any } f \in \mathrm{Lip_{bs}}(X) \tag{1.19}$$

*where* $B \subset X$ *is any ball of radius* $r$*,* $f_B := \fint_B f\,\mathrm{d}\mathfrak{m}$*, and* $K^-$ *is the negative part of* $K$*.*

**Remark 1.21.** Let $1 < p < \infty$. From (1.19) and Definition 1.7 we get

$$\fint_B |f - f_B|\,\mathrm{d}\mathfrak{m} \leq C(N) r e^{\sqrt{2(N-1)K^-}r} \left( \fint_{2B} |\nabla f|_p^p\,\mathrm{d}\mathfrak{m} \right)^{1/p} \quad \text{for any } f \in H^{1,p}(2B, \mathsf{d}, \mathfrak{m}).$$

While for $p = 1$, (1.19) and Definition 1.5 imply

$$\fint_B |f - f_B| \, d\mathfrak{m} \le C(N) r e^{\sqrt{2(N-1)K^- r}} \fint_{2B} |Df| \, d\mathfrak{m} \quad \text{for any } f \in \mathrm{BV}(X, \mathsf{d}, \mathfrak{m}).$$

In particular $\mathrm{CD}(K, N)$ spaces support the $(1, p)$-Poincaré inequality according to Definition 1.67.

## 3. Riemannian curvature-dimension condition

The just introduced CD notion does not rule out purely *Finslerian* structures, as the following result due to D. Cordero-Erausquin, K.T.-Sturm and C. Villani, shows. We refer to [**138**, p. 912 ] for its proof.

**Example 1.22.** Let $N \in \mathbb{N}$ be positive. The space $(\mathbb{R}^N, |\cdot|_\infty, \mathscr{L}^N)$ is a $\mathrm{CD}(0, N)$ space. Here $|x|_\infty := \max\{|x_1|, \dots, |x_N|\}$ denotes the $\infty$-norm.

In order to single out Riemannian structures among possible Finslerian ones the notion of $\mathrm{RCD}(K, N)$ m.m.s. was proposed in [**86**] (see also [**25**, **77**]), as a finite dimensional counterpart to $\mathrm{RCD}(K, \infty)$ m.m. spaces which were introduced and firstly studied in [**17**] (see also [**13**], dealing with the case of $\sigma$-finite reference measures).

**Definition 1.23.** We say that a metric measure space $(X, \mathsf{d}, \mathfrak{m})$ satisfies the *Riemannian* $\mathrm{CD}(K, N)$ condition (it is an $\mathrm{RCD}(K, N)$ m.m.s. for short) for some $K \in \mathbb{R}$ and $1 \le N < \infty$ if it is a $\mathrm{CD}(K, N)$ m.m.s. and the Banach space $H^{1,2}(X, \mathsf{d}, \mathfrak{m})$ is Hilbert.

Notice that, if $(X, \mathsf{d}, \mathfrak{m})$ is an $\mathrm{RCD}(K, N)$ m.m.s., then so is $(\operatorname{supp} \mathfrak{m}, \mathsf{d}, \mathfrak{m})$, hence in the following we will always tacitly assume that $\operatorname{supp} \mathfrak{m} = X$.

Let us point out that, in the last few years, many results have been proven for spaces satisfying the so-called *reduced curvature dimension condition* $\mathrm{CD}^*(K, N)$ or *reduced Riemannian curvature-dimension condition* $\mathrm{RCD}^*(K, N)$, which were known to have better localization and tensorization properties since the work [**31**]. However, one of the main consequences of the recent work [**46**] is that the classes of $\mathrm{RCD}^*(K, N)$ and $\mathrm{RCD}(K, N)$ spaces do actually coincide provided $\mathfrak{m}(X) < \infty$.

In Proposition 1.24 below we collect some results concerning the improved regularity of $W_2$-geodesics on $\mathrm{RCD}(K, N)$ metric measure spaces. The results are mainly taken from [**83**–**85**, **130**].

**Proposition 1.24.** *Let $(X, \mathsf{d}, \mathfrak{m})$ be an $\mathrm{RCD}(K, N)$ m.m.s. for some $K \in \mathbb{R}$ and $1 < N < \infty$. Let $\mu_0, \mu_1 \in \mathscr{P}_2(X)$ be absolutely continuous w.r.t. $\mathfrak{m}$, with bounded densities and bounded supports. Then:*

- (i) *there exists a unique $W_2$-geodesic $(\mu_t)_{t \in [0,1]}$ joining $\mu_0$ and $\mu_1$. Moreover, it holds $\mu_t \le C\mathfrak{m}$ for any $t \in [0, 1]$ for some $C > 0$;*
- (ii) *letting $\rho_t$ be the density of $\mu_t$ w.r.t. $\mathfrak{m}$, it holds that, for any $t \in [0, 1]$ and for any sequence $(t_k)_{k \in \mathbb{N}}$ converging to $t$, there exists a subsequence $(t_{n_k})_{k \in \mathbb{N}}$ such that*

$$\rho_{t_{n_k}} \to \rho_t \quad \mathfrak{m}\text{-a.e. as } k \to \infty.$$

**3.1. Consequences of the RCD condition on the first order Sobolev calculus.** On $\mathrm{RCD}(K, \infty)$ spaces the minimal relaxed slope $|\nabla f|_p$ is independent of $p$, for any $1 < p < \infty$. More precisely in [**90**] the authors proved the following.

**Theorem 1.25.** *Let $(X, \mathsf{d}, \mathfrak{m})$ be an $\mathrm{RCD}(K,\infty)$ spaces with $K \in \mathbb{R}$. For any $p, q \in (1,\infty)$, if $f \in H^{1,p}_{\mathrm{loc}}(X, \mathsf{d}, \mathfrak{m})$ satisfies $|\nabla f|_p \in L^q(X, \mathfrak{m})$, then*

$$f \in H^{1,q}_{\mathrm{loc}}(X, \mathsf{d}, \mathfrak{m}) \quad \text{and} \quad |\nabla f|_p = |\nabla f|_q \quad \mathfrak{m}\text{-a.e.}.$$

*Moreover when $(X, \mathsf{d})$ is proper the same conclusion holds for $p = 1$ and/or $q = 1$.*

The second conclusion of Theorem 1.25 can be applied to $\mathrm{RCD}(K, N)$ spaces since, as we have already remarked, they are proper.

The next deep identification result was first proven by J. Cheeger in the seminal paper [**48**] in the context of metric measure spaces satisfying doubling and Poincaré inequalities. We refer to [**9**, Theorem 8.4] for the present formulation and for a different proof. We remark that Theorem 1.26 applies in particular to $\mathrm{RCD}(K, N)$ spaces since, they are locally doubling (see (1.16) above) and satisfy a local Poincaré inequality (see Section 2.3).

**Theorem 1.26.** *Let $(X, \mathsf{d}, \mathfrak{m})$ be a m.m.s. with $\mathfrak{m}$ doubling, $\operatorname{supp}\mathfrak{m} = X$ and supporting a weak $(1, p)$-Poincaré inequality for some $1 < p < \infty$ (see Definition 1.67). Then, for any $f \in H^{1,p}(X, \mathsf{d}, \mathfrak{m}) \cap \mathrm{Lip}_{\mathrm{loc}}(X)$, it holds $\mathrm{lip}(f) = |\nabla f|_p$ $\mathfrak{m}$-a.e. on $X$.*

**3.2. Heat flow on RCD spaces.** The infinetisimally Hilbertian condition (see (1.12)) implies that the heat semigroup $P_t$ is linear for any $t \geq 0$. In [**13, 17**] it has been proved that, for $\mathrm{RCD}(K,\infty)$ metric measure spaces, the dual semigroup $P^*_t : \mathscr{P}_2(X) \to \mathscr{P}_2(X)$ of $P_t$, defined by

$$\int f \, \mathrm{d}P^*_t \mu := \int P_t f \, \mathrm{d}\mu \qquad \forall \mu \in \mathscr{P}_2(X), \quad \forall f \in \mathrm{Lip_b}(X),$$

is well-defined, maps probability measures into probability measures absolutely continuous w.r.t. $\mathfrak{m}$, and is $K$-contractive, i.e.

$$W_2(P^*_t \mu, P^*_t \mu) \leq e^{-tK} W_2(\mu, \nu) \quad \text{for any } \mu, \nu \in \mathscr{P}_2(X) \text{ and } t \geq 0.$$

Then, for any $t > 0$, we can introduce the so called *heat kernel* $p_t : X \times X \to [0,\infty)$ by

$$p_t(x, \cdot)\mathfrak{m} := P^*_t \delta_x.$$

From now on, for any $f \in L^\infty(X, \mathfrak{m})$ we will denote by $P_t f$ the pointwise representative defined by

$$P_t f(x) = \int f(y) p_t(x, y) \, \mathrm{d}\mathfrak{m}(y) \quad \text{for any } x \in X.$$

Since $\mathrm{RCD}(K, N)$ spaces are locally doubling, as we have already remarked, and they satisfy a local Poincaré inequality the general theory of Dirichlet forms as developed in [**134**] guarantees that we can find a locally Hölder continuous representative of $p$ on $X \times X \times (0,\infty)$.

Moreover in [**106**] the following finer properties of the heat kernel over $\mathrm{RCD}(K, N)$ spaces, have been proved. There exist constants $C_1 > 1$ and $c \geq 0$ such that

(1.20)
$$\frac{1}{C_1 \mathfrak{m}(B(x, \sqrt{t}))} \exp\left\{-\frac{\mathsf{d}^2(x, y)}{3t} - ct\right\} \leq p_t(x, y) \leq \frac{C_1}{\mathfrak{m}(B(x, \sqrt{t}))} \exp\left\{-\frac{\mathsf{d}^2(x, y)}{5t} + ct\right\}$$

for any $x, y \in X$ and for any $t > 0$. Moreover it holds

$$|\nabla p_t(x, \cdot)|(y) \leq \frac{C_1}{\sqrt{t}\mathfrak{m}(B(x, \sqrt{t}))} \exp\left\{-\frac{\mathsf{d}^2(x, y)}{5t} + ct\right\} \quad \text{for } \mathfrak{m}\text{-a.e. } y \in X, \tag{1.21}$$

for any $t > 0$ and for any $x \in X$. In (1.20) and (1.21) one can take $c = 0$ whenever $(X, \mathsf{d}, \mathfrak{m})$ is an $\mathrm{RCD}(0, N)$ metric measure space.

Let us now present three foundamental regularizing properties of the heat semigroup on $\mathrm{RCD}(K, \infty)$ spaces referring again to [**13**, **17**] for the proofs of these results.

(i) *Bakry-Émery* contraction estimate:

$$|\nabla P_t f|^2 \le e^{-2Kt} P_t |\nabla f|^2 \quad \mathfrak{m}\text{-a.e.,} \quad \text{for any } t > 0 \text{ and } f \in H^{1,2}(X, \mathsf{d}, \mathfrak{m}). \tag{1.22}$$

This contraction estimate can be generalized to the whole range of exponents $1 < p < \infty$, let us focus on the strongest one, the inequality relative to $p = 1$, which has been proven in [**90**] assuming that $(X, \mathsf{d})$ is proper.

$$|DP_t f| \le e^{-Kt} P_t^* |Df| \quad \mathfrak{m}\text{-a.e.,} \quad \text{for any } t > 0 \text{ and } f \in \mathrm{BV}(X, \mathsf{d}, \mathfrak{m}). \tag{1.23}$$

(ii) $L^\infty - \mathrm{Lip}$ *regularization* of the heat flow: for any $f \in L^\infty(X, \mathfrak{m})$, we have $P_t f \in \mathrm{Lip}(X)$ with

$$\sqrt{2 I_{2K}(t)}\, \mathrm{Lip}(P_t f) \le \|f\|_{L^\infty}, \quad \text{for any } t > 0, \tag{1.24}$$

where $I_L(t) := \int_0^t e^{Lr}\, \mathrm{d}r$ and $\mathrm{Lip}(P_t f)$ denotes the Lipschitz constant of $P_t f$.

(iii) *Sobolev to Lipschitz property*: for any $f \in H^{1,2}(X, \mathsf{d}, \mathfrak{m})$ it holds

$$|\nabla f| \in L^\infty(X, \mathfrak{m}) \implies \mathrm{Lip}(f) \le \|\nabla f\|_\infty \tag{1.25}$$

up to modifying $f$ on a $\mathfrak{m}$-negligible set.

The local version of the *Sobolev to Lipschitz property* reads as follows. Any $f \in H^{1,2}_{\mathrm{loc}}(X, \mathsf{d}, \mathfrak{m})$ with $|\nabla f| \in L^\infty(B(x, 2r), \mathfrak{m})$ for some $x \in X$ and $r > 0$, admits a Lipschitz representative $\bar f$ in $B(x, r)$ such that $\mathrm{Lip}\left(\bar f_{|B(x,r)}\right) \le \|\nabla f\|_{L^\infty(B(x,2r),\mathfrak{m})}$.

**3.3. Bochner inequality.** RCD spaces can be studied both from the Lagrangian point of view, the one developed so far, and the Eulerian point of view which we are going to describe in this section. The latter is based on the $\Gamma$-calculus and aims at characterized these spaces via Bochner inequality.

In the infinite dimensional case this equivalence was studied in [**17**], then [**77**] established equivalence with the dimensional Bochner inequality for the so-called class $\mathrm{RCD}^*(K, N)$ (see also [**24**]). Equivalence between $\mathrm{RCD}^*(K, N)$ and $\mathrm{RCD}(K, N)$ has been eventually achieved in [**46**].

Let us start by recalling the Bochner inequality in the smooth setting. We consider $(M, g)$ a Riemannian manifold, for any $f \in C_c^\infty(M)$ it holds

$$\frac{1}{2}\Delta|\nabla f|^2 - \nabla f \cdot \nabla(\Delta f) = |\operatorname{Hess}(f)|^2 + \operatorname{Ric}(\nabla f, \nabla f), \tag{1.26}$$

where $\Delta$ is the Laplace-Beltrami operator and $|\operatorname{Hess}(f)|$ denotes the Hilbert-Schmidt norm of $\operatorname{Hess}(f)$. Observe now that, when $\mathrm{Ric}_g \ge K$, (1.26) gives the inequality

$$\frac{1}{2}\Delta|\nabla f|^2 - \nabla f \cdot \nabla(\Delta f) \ge K|\nabla f|^2 \quad \forall f \in C_c^\infty(M). \tag{1.27}$$

Moreover it is not difficult to see that the validity of (1.27) implies in turn the lower bound $\mathrm{Ric}_g \ge K$.

If now we take into account both the lower bound on the Ricci curvature, $\mathrm{Ric}_g \geq K$ and the upper bound on the dimension $\dim M \leq N$, from (1.26) we can deduce

$$\frac{1}{2}\Delta|\nabla f|^2 - \nabla f \cdot \nabla(\Delta f) \geq \frac{(\Delta f)^2}{N} + K|\nabla f|^2 \quad \forall f \in C_c^\infty(M), \tag{1.28}$$

where we have used the inequality

$$|\operatorname{Hess}(f)|^2 \geq \frac{(\Delta f)^2}{\dim M} \geq \frac{(\Delta f)^2}{N}.$$

Again, it is simple to verify that if (1.28) holds then $\mathrm{Ric}_g \geq K$ and $\dim M \leq N$. In other words the curvature-dimension condition is equivalent to the validity of the Bochner inequality (1.28) over smooth Riemannian manifolds. Let us now present a similar result for non-smooth spaces.

**Theorem 1.27.** *Let $K \in \mathbb{R}$ and $N \in \mathbb{N} \cup \{\infty\}$. A metric measure space $(X, \mathsf{d}, \mathfrak{m})$ is* $\mathrm{RCD}(K, N)$ *if and only if the following condition hold:*

- (i) *there exist $c_1, c_2 > 0$ such that $\mathfrak{m}(B_r(x)) \leq c_1 e^{c_2 r^2}$ for some $x \in X$ and every $r > 0$;*
- (ii) *the Sobolev-to-Lipschitz property* (1.25)*;*
- (iii) *the Bochner inequality*

$$\frac{1}{2}\Delta|\nabla f|^2 - \nabla f \cdot \nabla(\Delta f) \geq \frac{(\Delta f)^2}{N} + K|\nabla f|^2 \tag{1.29}$$

*holds in the distributional sense for any $f \in D(\Delta) \cap L^\infty(X, \mathfrak{m})$ with $\Delta f \in L^\infty(X, \mathfrak{m})$.*

We refer to [**77**] for the proof of this result, pointing out that it has been originally proven for the $\mathrm{RCD}^*$ class, and after [**46**] we know that it coincides with the RCD class (actually the equivalence has been proven under the additional assumption $\mathfrak{m}(X) < \infty$, but due to the local nature of the argument, it is thought that the equivalence holds in full generality).

## 4. Measured Gromov-Hausdorff convergence and stability results

We dedicate this section to an overview of the subject of convergence and stability for Sobolev functions defined on converging sequences of metric measure spaces. The main references for this part are [**92**] and [**19, 20**].

**Definition 1.28.** A sequence $\{(X_i, \mathsf{d}_i, \mathfrak{m}_i, x_i)\}_{i\in\mathbb{N}}$ of pointed m.m.s. is said to converge in the pmGH topology to $(Y, \varrho, \mu, y)$ if there exist a complete separable metric space $(Z, \mathsf{d}_Z)$ and isometric embeddings

$$\begin{aligned} &\Psi_i : (\operatorname{supp} \mathfrak{m}_i, \mathsf{d}_i) \to (Z, \mathsf{d}_Z) \qquad \forall i \in \mathbb{N}, \\ &\Psi : (\operatorname{supp} \mu, \varrho) \to (Z, \mathsf{d}_Z), \end{aligned}$$

such that for every $\varepsilon > 0$ and $R > 0$ there exists $i_0$ such that for every $i > i_0$

$$\Psi(B_R^Y(y)) \subset [\Psi_i(B_R^{X_i}(x_i))]_\varepsilon, \qquad \Psi_i(B_R^{X_i}(x_i)) \subset [\Psi(B_R^Y(y))]_\varepsilon,$$

where $[A]_\varepsilon := \{z \in Z \ : \mathsf{d}_Z(z, A) < \varepsilon\}$ for every $A \subset Z$. Moreover $(\Psi_i)_* \mathfrak{m}_i \rightharpoonup \Psi_* \mu$, where the convergence is understood in duality with $\mathrm{C_{bs}}(Z)$.

In the case of a sequence of uniformly locally doubling m.m.s. $(X_i, \mathsf{d}_i, \mathfrak{m}_i, x_i)$ (as in the case of $\mathrm{RCD}(K, N)$ spaces), the pointed measured Gromov-Hausdorff convergence to $(Y, \varrho, \mu, y)$ can be equivalently characterized asking for the existence of a proper metric space $(Z, \mathsf{d}_Z)$ such that all the metric spaces $(X_i, \mathsf{d}_i)$ are isometrically embedded into $(Z, \mathsf{d}_Z)$,

$x_i \to y$ and $\mathfrak{m}_i \rightharpoonup \mu$ in duality with $\mathrm{C_{bs}}(Z)$. This is the so called extrinsic approach, that we shall adopt in the rest of this thesis.

**Remark 1.29.** It is clear from Definition 1.28 that the notion of pmGH convergence regards equivalence classes of pointed metric measure spaces modulo isomorphism. Here $(X_1, \mathsf{d}_1, \mathfrak{m}_1, x_1)$ is said to be isomorphic to $(X_2, \mathsf{d}_2, \mathfrak{m}_2, x_2)$ if there exists an isometry $\Psi : (\operatorname{supp}\mathfrak{m}_1, \mathsf{d}_1) \to (\operatorname{supp}\mathfrak{m}_2, \mathsf{d}_2)$ such that $\Psi(x_1) = x_2$ and $\Psi_*\mathfrak{m}_1 = \mathfrak{m}_2$.

It is useful to introduce a distance on the space of equivalence classes of pointed metric measure spaces, which is denoted by $\mathsf{d}_{\mathrm{pmGH}}$, that metrizise the pmGH topology. We refer to [**92**] for more details on its definition.

**4.1. Stability results for RCD spaces.** In this section we provide stability and compactness results related to the $\mathrm{CD}(K,N)$ and $\mathrm{RCD}(K,N)$ condition for $K \in \mathbb{R}$ and $N \in [1,\infty]$.

Let us start by presenting the stability result for RCD spaces, which will play a central role in the sequel. We refer to [**17**, **92**] for its proof.

**Theorem 1.30.** *Let $K \in \mathbb{R}$ and $N \in [1,\infty]$ be fixed. If a sequence $((X_n, \mathsf{d}_n\mathfrak{m}_n, x_n))_{n\in\mathbb{N}}$ of $\mathrm{RCD}(K_n, N_n)$ pointed m.m.s. converge to $(Y, \varrho, \mu, y)$ in the pmGH topology, and $K_n \to K$ and $N_n \to N$, then $(Y, \varrho, \mu, y)$ is an $\mathrm{RCD}(K,N)$ metric measure space.*

Let us remark in passing that the same statement of Theorem 1.30 holds replacing $\mathrm{RCD}(K,N)$ with $\mathrm{CD}(K,N)$ (Cf. [**92**, **116**, **136**]). Nevertheless the stability of RCD spaces is not an easy consequence of the one for CD spaces since the infinetisimally Hilbertian assumption is not stable in general. This is actually not surprising since it is a first order notion. The crucial point in Theorem 1.30 is that, when the infinetisimally Hilbertian condition is combined with the CD assumption then the resulting notion is stable.

The compactness of the classes of $\mathrm{CD}(K,N)$ and $\mathrm{RCD}(K,N)$ spaces is related to the local doubling property.

**Definition 1.31.** We say that a family of metric measure spaces $\{(X_n, \mathsf{d}_n, \mathfrak{m}_n)\}_{n\in\mathbb{N}}$ is uniformly locally doubling if there exists a nondecreasing function $C : [0,\infty) \to [0,\infty)$ such that for any $R > 0$ it holds

$$\mathfrak{m}_n(B_{2r}(x_n)) \le C(R)\mathfrak{m}_n(B_r(x_n)) \quad \text{for any } x_n \in X_n,\ r \in (0,R) \text{ and } n \in \mathbb{N}.$$

We say that a pointed metric measure space $(X, \mathsf{d}, \mathfrak{m}, x)$ is normalized if

$$\int_{B_1(x)} (1 - \mathsf{d}(x,y))\,\mathrm{d}\mathfrak{m}(y) = 1.$$

Sequence of normalized uniformly locally doubling m.m.s. are precompact in the pmGH topology. It can be shown by the standard argument of Gromov: the measures are uniformly doubling, hence balls of given radius around the reference points are uniformly totally bounded and thus compact in the GH-topology. Then weak compactness of the measures follows using the doubling condition again and the fact that they are normalized.

Since, as we have already observed in (1.16), $\mathrm{CD}(K,N)$ spaces are uniformly locally doubling when $N < \infty$, we conclude that they are precompact in the pmGH topology (and therefore compact in view of Theorem 1.30.

**Theorem 1.32.** *Let $K \in \mathbb{R}$ and $N \in [1,\infty)$ be fixed. Any sequence $((X_n, \mathsf{d}_n, \mathfrak{m}_n, x_n))_{n\in\mathbb{N}}$ of $\mathrm{CD}(K,N)$ pointed m.m.s. admits a convergent subsequence in the pmGH topology.*

**4.2. Convergence of functions defined on varying spaces.** This section aims at studying convergence results for sequence of functions defined on varying spaces.

From now until the end of the section we always assume that $(X_i, \mathsf{d}_i, \mathfrak{m}_i)$ are $\mathrm{RCD}(K,N)$ metric measure spaces for any $i \in \mathbb{N}$.

**Definition 1.33.** Let $(X_i, \mathsf{d}_i, \mathfrak{m}_i, x_i)$ be pointed m.m.s. converging in the pmGH topology to $(Y, \varrho, \mu, y)$ and let $f_i : X_i \to \mathbb{R}$, $f : Y \to \mathbb{R}$. Assume the convergence to be realized into a common metric space $(Z, \mathsf{d}_Z)$ as above. Then we say that $f_i \to f$ pointwise if $f_i(x_i) \to f(x)$ for every sequence of points $x_i \in X_i$ such that $x_i \to x$ in $Z$. If moreover for every $\varepsilon > 0$ there exists $\delta > 0$ such that $|f_i(x_i) - f(x)| \le \varepsilon$ for every $i \ge \delta^{-1}$ and every $x_i \in X_i$, $x \in Y$ with $\mathsf{d}_Z(x_i, x) \le \delta$, then we say that $f_i \to f$ uniformly.

The next proposition is a version of the Ascoli–Arzelà compactness theorem. We omit the proof, that can be obtained arguing as in the case of a fixed space.

**Proposition 1.34.** *Let $(X_i, \mathsf{d}_i, \mathfrak{m}_i, x_i)$ and $(Y, \rho, \mu, y)$ be as above and $R > 0$, $L > 0$ fixed. Then, for any sequence of equi-continuous functions $f_i : B_R(x_i) \to \mathbb{R}$, with $\sup_i \left|f_i(x_i)\right| < \infty$ there exists a subsequence that converges uniformly to some function $f : B_R(y) \to \mathbb{R}$.*

**Remark 1.35.** If the sequence $(f_i)$ in Proposition 1.34 is uniformly $L$-Lipschitz, then the limit function is $L$-Lipschitz too. Moreover it is not difficult to see (Cf. [**21**, Proposition 3.2]) that sequences of equi-continuous, uniformly bounded functions converge also in the weak/strong $L^2$ sense, according to the definitions below.

We recall below the notions of convergence in $L^p$ and Sobolev spaces for functions defined over converging sequences of metric measure spaces. We will be concerned only with the cases $p = 2$ and $p = 1$ in the rest of the thesis. We refer again to [**19**, **92**] for a more general treatment and the proofs of the results we state below.

**Definition 1.36.** We say that $f_i \in L^2(X_i, \mathfrak{m}_i)$ converge in $L^2$-weak to $f \in L^2(Y, \mu)$ if $f_i \mathfrak{m}_i \rightharpoonup f\mu$ in duality with $\mathrm{C_{bs}}(Z)$ and $\sup_i \|f_i\|_{L^2(X_i, \mathfrak{m}_i)} < \infty$.
We say that $f_i \in L^2(X_i, \mathfrak{m}_i)$ converge in $L^2$-strong to $f \in L^2(Y, \mu)$ if $f_i \mathfrak{m}_i \rightharpoonup f\mu$ in duality with $\mathrm{C_{bs}}(Z)$ and $\lim_i \|f_i\|_{L^2(X_i, \mathfrak{m}_i)} = \|f\|_{L^2(Y,\mu)}$.

**Definition 1.37.** We say that $f_i \in H^{1,2}(X_i, \mathsf{d}_i, \mathfrak{m}_i)$ are weakly convergent to $f \in H^{1,2}(Y, \varrho, \mu)$ if they converge in $L^2$-weak and $\sup_i \mathsf{Ch}^i(f_i) < \infty$. Strong $H^{1,2}$-converge is defined asking that $f_i$ converge to $f$ in $L^2$-strong and $\lim_i \mathsf{Ch}^i(f_i) = \mathsf{Ch}(f)$.

Let us present the notions of $L^1$-strong and BV-strong convergence for sequences of functions $f_i : X_i \to \mathbb{R}$, as introduced in [**19**].

**Definition 1.38.** We say that a sequence $(f_i) \subset L^1(X_i, \mathfrak{m}_i)$ converges $L^1$-strongly to $f \in L^1(Y, \mu)$ if

$$\sigma \circ f_i \mathfrak{m}_i \rightharpoonup \sigma \circ f\mu \qquad \text{and} \qquad \int_{X_i} |f_i| \, \mathrm{d}\mathfrak{m}_i \to \int_Y |f| \, \mathrm{d}\mu,$$

where $\sigma(z) := \mathrm{sign}(z)\sqrt{|z|}$ and the weak convergence is understood in duality with $\mathrm{C_{bs}}(Z)$. Equivalently, if $\sigma \circ f_i$ $L^2$-strongly converge to $\sigma \circ f$.

We say that $f_i \in \mathrm{BV}(X_i, \mathfrak{m}_i)$ converge in energy in BV to $f \in \mathrm{BV}(Y, \mu)$ if $f_i$ converge $L^1$-strongly to $f$ and

$$\lim_{i\to\infty} |Df_i|(X_i) = |Df|(Y).$$

**Remark 1.39.** The presence of the function $\sigma$ in the definition of $L^1$-strong convergence is necessary due to the lack of reflexivity of $L^1$. Indeed the counterpart of Definition 1.36 in the case $p = 1$ is easily seen to be not equivalent to convergence in $L^1$ norm when all the spaces coincide.

The following useful stability result is part of [**19**, Proposition 3.3].

**Proposition 1.40.** *Let $p \in \{1, 2\}$. If $f_i \in L^p(X_i, \mathfrak{m}_i)$ converge in $L^p$-strong to $f \in L^p(Y, \mu)$ then $\varphi \circ f_i$ converge to $\varphi \circ f$ in $L^p$-strong for any $\varphi \in \mathrm{Lip}(\mathbb{R})$ such that $\varphi(0) = 0$. In particular, if $g_i$ are uniformly bounded in $L^\infty$ and $L^1$-strongly convergent to $g$ then*

$$\lim_{i\to\infty} \|g_i\|_{L^p(X_i,\mathfrak{m}_i)} = \|g\|_{L^p(Y,\mu)} .$$

*Moreover,*

(i) *For any $f_i, g_i \in L^p(X_i, \mathfrak{m}_i)$ such that $f_i \to f \in L^p(Y, \mu)$ and $g_i \to g \in L^p(Y, \mu)$ strongly in $L^p$ one has $f_i + g_i \to f + g$ strongly in $L^p$.*

(ii) *If $f_i \to f$ and $g_i \to g$ in $L^2$-strong then $f_i g_i \to fg$ in $L^1$-strong.*

(ii) *If $f_i \to f$ in $L^1$-strong and $\sup_{i\in\mathbb{N}} \|f_i\|_{L^\infty(X_i,\mathfrak{m}_i)} < \infty$ then $\|f_i\|_{L^2(X_i,\mathfrak{m}_i)} \to \|f\|_{L^2(Y,\mu)}$. In particular $f_i \to f$ in $L^2$-strong.*

The following localized lower semicontinuity result is taken from [**19**, Lemma 5.8].

**Proposition 1.41.** *Let $f_i \in H^{1,2}(X_i, \mathsf{d}_i, \mathfrak{m}_i)$ be weakly converging in $H^{1,2}$ to $f \in H^{1,2}(Y, \varrho, \mu)$. Then*

$$\liminf_{i\to\infty} \int_Z g\, |\nabla f_i|\, \mathrm{d}\mathfrak{m}_i \geq \int_Z g\, |\nabla f|\, \mathrm{d}\mu, \qquad \textit{for any nonnegative } g \in \mathrm{Lip}_{\mathrm{bs}}(Z).$$

Below we quote a compactness criterion borrowed from [**92**, Theorem 6.3] (see also [**19**, Theorem 7.4]).

**Theorem 1.42.** *Let $f_i \in H^{1,2}(X_i, \mathsf{d}_i, \mathfrak{m}_i)$ be such that*

$$\sup_i \left\{ \int_Z |f_i|^2\, \mathrm{d}\mathfrak{m}_i + \mathsf{Ch}^i(f_i) \right\} < \infty$$

*and*

$$\lim_{R\to\infty} \limsup_{i\to\infty} \int_{Z\setminus B_R(\bar z)} |f_i|^2\, \mathrm{d}\mathfrak{m}_i = 0,$$

*for some (and thus for all) $\bar z \in Z$. Then $(f_i)$ has a $L^2$-strongly convergent subsequence to $f \in H^{1,2}(Y, \varrho, \mu)$.*

Next we pass to a stability/compactness criterion in $H^{1,2}$. The two statements below are taken from [**19**, Corollary 5.5], [**19**, Theorem 5.7].

**Proposition 1.43.** (a) *If $f_i \in H^{1,2}(X_i, \mathsf{d}_i, \mathfrak{m}_i)$, $f_i \in \mathcal{D}(\Delta_i)$ converge in $L^2$-strong to $f$ and $\Delta_i f_i$ are uniformly bounded in $L^2$, then $f \in \mathcal{D}(\Delta)$, $\Delta_i f_i$ converge in $L^2$-weak to $\Delta f$ and $f_i$ converge in $H^{1,2}$-strong to $f$;*

(b) *for all $t > 0$, $P^i_t f_i$ converge in $H^{1,2}$-strong to $P_t f$ whenever $f_i$ converge in $L^2$-strong to $f$.*

**Theorem 1.44.** *Let $v_i, w_i \in H^{1,2}(X_i, \mathsf{d}_i, \mathfrak{m}_i)$ be strongly convergent in $H^{1,2}$ to, respectively, $v, w \in H^{1,2}(Y, \varrho, \mu)$. Then $\nabla v_i \cdot \nabla w_i$ converge $L^1$-strongly to $\nabla v \cdot \nabla w$.*

4.2.1. *Stability/compactness for functions defined on balls.* Let us now introduce the notion of local $H^{1,2}$-convergence.

**Definition 1.45.** We say that $f_i \in H^{1,2}(B_R(x_i), \mathsf{d}_i, \mathfrak{m}_i)$ are weakly convergent in $H^{1,2}$ to $f \in H^{1,2}(B_R(y), \varrho, \mu)$ on $B_R(y)$ if $f_i$ are $L^2$-weakly (or $L^2$-strongly, equivalently) to $f$ on $B_R(y)$ with $\sup_{i\in\mathbb{N}} \|f_i\|_{H^{1,2}} < \infty$. Strong convergence in $H^{1,2}$ on $B_R(y)$ is defined by requiring

$$\lim_{i\to\infty} \int_{B_R(x_i)} |\nabla f_i|^2 \, \mathrm{d}\mathfrak{m}_i = \int_{B_R(y)} |\nabla f|^2 \, \mathrm{d}\mu.$$

Let us now collect results from [**20**] that will play a role in this note.

**Lemma 1.46** ([**20**, Lemma 2.10])**.** *For any $f \in \mathrm{Lip}_{\mathrm{bs}}(B_R(y), \varrho)$ there exist $f_i \in \mathrm{Lip}_c(B_R(x_i))$ satisfying*

$$\sup_{i\in\mathbb{N}} \||\nabla f_i|\|_{L^\infty(X_i,\mathfrak{m}_i)} < \infty$$

*and strongly convergent to $f$ in $H^{1,2}$.*

**Theorem 1.47** ([**20**, Theorem 4.4])**.** *Let $f_i \in D(\Delta, B_R(x_i))$ with*

$$\sup_{i\in\mathbb{N}} \int_{B_R(x_i)} (|f_i|^2 + |\nabla f_i|^2 + (\Delta f_i)^2) \, \mathrm{d}\mathfrak{m}_i < \infty,$$

*and let $f$ be an $L^2$-strong limit function of $f_i$ on $B_R(y)$. Then:*

(i) *$f \in D(\Delta, B_R(y))$;*
(ii) *$\Delta f_i \to \Delta f$ on $B_R(y)$ weakly in $L^2$;*
(iii) *$|\nabla f_i|^2 \to |\nabla f|^2$ on $B_R(y)$ strongly in $L^1$.*

**Proposition 1.48** ([**20**, Corollary 4.12])**.** *Let $f \in H^{1,2}(B_R(y), \varrho, \mu)$ be a harmonic function (i.e., $f \in D(\Delta, B_R(y))$ with $\Delta f = 0$). Then, for any $0 < r < R$ there exist $f_i \in H^{1,2}(B_r(x_i), \mathsf{d}_i, \mathfrak{m}_i)$ harmonic such that $f_i \to f$ on $B_r(y)$ strongly in $H^{1,2}$.*

Let us conclude this section by presenting a stability result for heat kernels taken from [**21**, Theorem 3.3].

**Theorem 1.49.** *The heat kernel $p^i$ of $(X_i, \mathsf{d}_i, \mathfrak{m}_i)$ satisfy*

$$\lim_{i\to\infty} p^i_{t_i}(x_i, y_i) = p^Y_t(x, y),$$

*whenever* $\mathrm{supp}\,\mathfrak{m}_i \times \mathrm{supp}\,\mathfrak{m}_i \times [0,\infty) \ni (x_i, y_i, t_i) \to (x, y, t) \in \mathrm{supp}\,\mu \times \mathrm{supp}\,\mu \times [0, \infty)$.

## 5. Normed modules

We begin by briefly recalling the definitions of normed module over $(X, \mathsf{d}, \mathfrak{m})$, which have been introduced in [**87**] and are in turn inspired by the theory developed in [**139**].

Let $R$ be either $L^\infty(\mathfrak{m})$ or $L^0(\mathfrak{m})$. Let $\mathscr{M}$ be a module over the commutative ring $R$. Then an *$L^p$-pointwise norm* on $\mathscr{M}$, for some $p \in \{0\} \cup [1, \infty)$, is any mapping $|\cdot| : \mathscr{M} \to L^p(\mathfrak{m})$ such that

$$\begin{aligned} |v| &\geq 0 && \text{for every } v \in \mathscr{M}, \text{ with equality if and only if } v = 0, \\ |v + w| &\leq |v| + |w| && \text{for every } v, w \in \mathscr{M}, \\ |fv| &= |f||v| && \text{for every } f \in R \text{ and } v \in \mathscr{M}, \end{aligned} \tag{1.30}$$

where all (in)equalities are in the $\mathfrak{m}$-a.e. sense. We shall consider two classes of normed modules:

- $L^p(\mathfrak{m})$-NORMED $L^\infty(\mathfrak{m})$-MODULES, WITH $p \in [1,\infty)$. A module $\mathscr{M}^p$ over $L^\infty(\mathfrak{m})$ endowed with an $L^p$-pointwise norm $|\cdot|$ such that $\|v\|_{\mathscr{M}^p} := \left\||v|\right\|_{L^p(\mathfrak{m})}$ is a complete norm on $\mathscr{M}^p$.
- $L^0(\mathfrak{m})$-NORMED $L^0(\mathfrak{m})$-MODULES. A module $\mathscr{M}^0$ over $L^0(\mathfrak{m})$ endowed with an $L^0$-pointwise norm $|\cdot|$ such that $\mathsf{d}_{\mathscr{M}^0}(v,w) := \int \min\left\{|v-w|, 1\right\} \,\mathrm{d}\mathfrak{m}'$ (where $\mathfrak{m}'$ is any probability measure that is mutually absolutely continuous with $\mathfrak{m}$) is a complete distance on $\mathscr{M}^0$.

We refer to [**88**] for an account of the abstract normed modules theory on metric measure spaces.

Assume $(X,\mathsf{d},\mathfrak{m})$ is *infinitesimally Hilbertian*, i.e., its Sobolev space $H^{1,2}(X,\mathsf{d},\mathfrak{m})$ is Hilbert (see Definition 1.12). Then a key example of normed module on $X$ is represented by the *tangent module* $L^0(TX)$, which is characterized as follows: there is a unique couple $\big(L^0(TX),\nabla\big)$, where $L^0(TX)$ is an $L^0(\mathfrak{m})$-normed $L^0(\mathfrak{m})$-module and $\nabla : H^{1,2}(X) \to L^0(TX)$ is a linear *gradient* map, such that the following hold:

$$|\nabla f| \text{ coincides with the minimal relaxed slope of } f \text{ for every } f \in H^{1,2}(X),$$
$$\left\{ \sum_{i=1}^n \chi_{E_i}\nabla f_i \,\middle|\, (E_i)_{i=1}^n \text{ Borel partition of } X,\ (f_i)_{i=1}^n \subset H^{1,2}(X) \right\} \text{ is dense in } L^0(TX).$$

For any exponent $p \in [1,\infty]$, we set $L^p(TX) := \big\{v \in L^0(TX) \ : \ |v| \in L^p(\mathfrak{m})\big\}$. It can be readily checked that the space $L^p(TX)$ has a natural $L^p(\mathfrak{m})$-normed $L^\infty(\mathfrak{m})$-module structure (for $p<\infty$).

**Definition 1.50** (Hilbert modules)**.** We say that an $L^2(\mathfrak{m})$-module $\mathscr{M}$ is a Hilbert module provided that $\mathscr{M}$ seen as Banach space is a Hilbert space.

Over Hilbert modules it is defined the *pointwise scalar product*

$$v\cdot w := \frac{1}{2}\left(|v+w|^2 - |v|^2 - |w|^2\right)$$

and a version of Riesz theorem holds.

**Proposition 1.51** (Riesz theorem for Hilbert modules)**.** *Let $\mathscr{M}$ be an $L^2(\mathfrak{m})$-module. Then the dual module*

$$\mathscr{M}^* := \Big\{L : \mathscr{M} \to L^1(\mathfrak{m}) \textit{ linear, continuous and } L(fv) = fv \ \forall f \in L^\infty(\mathfrak{m}),\ v \in \mathscr{M}\Big\}$$

*endowed with*

$$|L|_* := \operatorname{ess\,sup}\left\{|L(v)| \mid v \in \mathscr{M},\ |v| \le 1 \ \mathfrak{m}\text{-}a.e.\right\}$$

*is isomorphic to $\mathscr{M}$ trough $A : \mathscr{M} \to \mathscr{M}^*$, where $A_v : \mathscr{M}^* \to L^1(\mathfrak{m})$ is defined as $A_v(w) := v\cdot w$ for any $v, w \in \mathscr{M}$. Moreover $|A(v)|_* = |v|$.*

The module $L^2(TX)$ inherits the Hilbertian structure from $H^{1,2}(X,\mathsf{d},\mathfrak{m})$ and therefore it is isomorphic to its dual $L^2(T^*X)$, the so-called *cotangent module* over $(X,\mathsf{d},\mathfrak{m})$.

Exploiting the Hilbertian structure we can define the following notions of divergence.

**Definition 1.52.** We declare that $v \in L^2(TX)$ belongs to $D(\mathrm{div})$ provided there exists a (uniquely determined) function $\mathrm{div}(v) \in L^2(X,\mathfrak{m})$ such that

$$\int \nabla f \cdot v \,\mathrm{d}\mathfrak{m} = -\int f\,\mathrm{div}(v)\,\mathrm{d}\mathfrak{m} \quad \text{for every } f \in H^{1,2}(X,\mathsf{d},\mathfrak{m}).$$

The domain $D(\mathrm{div})$ is a vector subspace of $L^2(TX)$ and the operators $\mathrm{div} : D(\mathrm{div}) \to L^2(X, \mathfrak{m})$ is linear.

It can be readily checked that a given function $f \in H^{1,2}(X, \mathsf{d}, \mathfrak{m})$ belongs to $D(\Delta)$ if and only if its gradient $\nabla f$ belongs to $D(\mathrm{div})$. In this case, it also holds that $\Delta f = \mathrm{div}(\nabla f)$.

**5.1. Tangent vectors and derivations.** We begin by introducing derivations over metric measure spaces.

**Definition 1.53** (Derivation)**.** Let $(X, \mathsf{d}, \mathfrak{m})$ be a metric measure space. Then a *derivation* on $X$ is a linear map $b : \mathrm{Lip}_{\mathrm{bs}}(X) \to L^0(\mathfrak{m})$ such that the following properties are satisfied:

i) LEIBNIZ RULE. $b(fg) = b(f)g + fb(g)$ for every $f, g \in \mathrm{Lip}_{\mathrm{bs}}(X)$.

ii) WEAK LOCALITY. There exists $G \in L^0(\mathfrak{m})$ such that

$$|b(f)| \leq G \; \mathrm{lip}_a(f)^1 \quad \mathfrak{m}\text{-a.e.} \quad \text{for every } f \in \mathrm{Lip}_{\mathrm{bs}}(X). \tag{1.31}$$

The least function $G$ (in the $\mathfrak{m}$-a.e. sense) with this property is denoted by $|b|$.

The space of all derivations on $X$ is denoted by $\mathrm{Der}(X)$. Given any derivation $b \in \mathrm{Der}(X)$, we define its *support* $\mathrm{supp}(b) \subset X$ as the essential closure of $\{|b| \neq 0\}$. For any open set $U \subset X$, we write $\mathrm{supp}(b) \Subset U$ if $\mathrm{supp}(b)$ is bounded and $\mathrm{dist}(\mathrm{supp}(b), X \setminus U) > 0$. Given any $b \in \mathrm{Der}(X)$ with $|b| \in L^1_{\mathrm{loc}}(X)$, we say that $\mathrm{div}(b) \in L^p(\mathfrak{m})$ – for some exponent $p \in [1, \infty]$ – provided there exists a function $h \in L^p(\mathfrak{m})$ such that

$$-\int b(f) \; \mathrm{d}\mathfrak{m} = \int fh \; \mathrm{d}\mathfrak{m} \quad \text{for every } f \in \mathrm{Lip}_{\mathrm{bs}}(X). \tag{1.32}$$

The function $h$ is uniquely determined, thus it can be unambiguously denoted by $\mathrm{div}(b)$. We set

$$\mathrm{Der}^p(X) := \Big\{ b \in \mathrm{Der}(X) \;\Big|\; |b| \in L^p(\mathfrak{m}) \Big\},$$
$$\mathrm{Der}^{p,p}(X) := \Big\{ b \in \mathrm{Der}^p(X) \;\Big|\; \mathrm{div}(b) \in L^p(\mathfrak{m}) \Big\}$$

for any $p \in [1, \infty]$. The set $\mathrm{Der}^p(X)$ is a Banach space if endowed with the norm $\|b\|_p := \||b|\|_{L^p(\mathfrak{m})}$.

**Remark 1.54.** We claim that for every $b \in \mathrm{Der}^{p,p}(X)$ – where $p \in [1, \infty]$ – it holds that

$$\mathrm{supp}\Big(\mathrm{div}(b)\Big) \subset \mathrm{supp}(b). \tag{1.33}$$

In order to prove it, fix any open bounded subset $U$ of $X \setminus \mathrm{supp}(b)$. Then formula (1.32) guarantees that $\int f \, \mathrm{div}(b) \; \mathrm{d}\mathfrak{m} = -\int b(f) \; \mathrm{d}\mathfrak{m} = 0$ for every $f \in \mathrm{Lip}_{\mathrm{bs}}(U)$, whence accordingly $\mathrm{div}(b) = 0$ holds $\mathfrak{m}$-a.e. on $U$. By arbitrariness of $U$, we conclude that the claim (1.33) is verified.

We often use the notation $b \cdot \nabla f$ in place of $b(f)$. The following identification result is due to Gigli [**87**].

**Theorem 1.55** (Derivation and vector fields)**.** *For any vector field $v \in L^2(TX)$ the map*

$$A(v) : \mathrm{Lip}_{\mathrm{bs}}(X) \to L^2(X, \mathfrak{m}), \qquad A(v)(f) := v \cdot \nabla f$$

*is a derivation. Viceversa, given any derivation $b$ there exists $v \in L^2(TX)$ such that $b(f) := v \cdot \nabla f$ for any $f \in \mathrm{Lip}_{\mathrm{bs}}(X)$.*

---

[1]where $\mathrm{lip}_a(f)(x) := \lim_{r \to 0} \sup_{\mathsf{d}(x,y) < r} \frac{|f(x) - f(y)|}{\mathsf{d}(x,y)}$ is the so-called asymptotic Lipschitz constant.

The following technical result, taken from [**75**, Proposition 6.5], will play a role in the sequel.

**Proposition 1.56.** *Let $(X, \mathsf{d}, \mathfrak{m})$ be an infinitesimally Hilbertian metric measure space. Let us denote by $\overline{\mathbb{D}}$ the closure in $\mathrm{Der}^2(X)$ of the pre-Hilbert space $\mathbb{D} := \Big(\mathrm{Der}^{2,2}(X), \|\cdot\|_2\Big)$. Then $\overline{\mathbb{D}}$ has a natural structure of Hilbert $L^2(\mathfrak{m})$-normed $L^\infty(\mathfrak{m})$-module and the map $A\colon L^2(TX) \to \overline{\mathbb{D}}$, defined as*

$$A(v)(f) \coloneqq v \cdot \nabla f \quad \text{for every } v \in L^2(TX) \text{ and } f \in \mathrm{Lip}_{bs}(X),$$

*is a normed module isomorphism between $L^2(TX)$ and $\overline{\mathbb{D}}$.*

*Moreover, it holds $A\Big(D(\mathrm{div})\Big) = \mathbb{D}$ and*

$$\mathrm{div}(A(v)) = \mathrm{div}(v) \quad \text{for every } v \in D(\mathrm{div}).$$

**5.2. Second order calculus over** RCD **spaces.** Gigli in [**87**] developed a second order calculus for $\mathrm{RCD}(K,\infty)$ metric measure spaces. The notions of Hessian and covariant derivative have been introduced as bilinear forms on $L^2(TX)$. It is worth pointing out that the RCD condition plays a central role in this theory, it provides good regularization properties of the heat semigroup that gives in turn a rich class of test objects to work with.

Let us start by introducing the algebra of *test functions*.

$$\mathrm{Test}(X, \mathsf{d}, \mathfrak{m}) := \Big\{ f \in D(\Delta) \cap L^\infty(\mathfrak{m}) \,\Big|\, |\nabla f| \in L^\infty(\mathfrak{m}),\ \Delta f \in H^{1,2}(X, \mathsf{d}, \mathfrak{m}) \cap L^\infty(\mathfrak{m}) \Big\}.$$

Thanks to the Sobolev-to-Lipschitz property (1.25), we know that any test function admits a Lipschitz representative. Moreover, it holds that $\mathrm{Test}(X, \mathsf{d}, \mathfrak{m})$ is dense in $H^{1,2}(X, \mathsf{d}, \mathfrak{m})$ and that $\nabla f \cdot \nabla g \in H^{1,2}(X, \mathsf{d}, \mathfrak{m})$ for every $f, g \in \mathrm{Test}(X, \mathsf{d}\mathfrak{m})$. We refer to [**87**, Section 3.2] for a proof of these results.

Let us present a useful approximation result.

**Lemma 1.57.** *For any $f \in \mathrm{Lip}(X)$ there exists a sequence $(f_n)$ with $|f_n| + |\nabla f_n| \le C(K, \mathrm{Lip}(f))$ converging to $f$ in $H^{1,2}(X, \mathsf{d}, \mathfrak{m})$. Moreover if $f$ is nonnegative then $f_n$ can be taken nonnegative.*

We are now ready to introduce the Sobolev spaces $W^{2,2}(X, \mathsf{d}, \mathfrak{m})$, $H^{2,2}(X, \mathsf{d}, \mathfrak{m})$.

**Definition 1.58.** We say that $f \in H^{1,2}(X, \mathsf{d}, \mathfrak{m})$ belongs to $W^{2,2}(X, \mathsf{d}, \mathfrak{m})$ if there exists an element of the tensor product $L^2(T^*X) \otimes L^2(T^*X)$ (cf. [**87**, Section 1.5]), denoted by $\mathrm{Hess}(f)$, such that

$$\begin{aligned} &2\int h \,\mathrm{Hess}(f)(\nabla g_1 \otimes \nabla g_2)\, \mathrm{d}\mathfrak{m} \\ &= -\int \nabla f \cdot \nabla g_1 \,\mathrm{div}(h \nabla g_2) + \nabla f \cdot \nabla g_2 \,\mathrm{div}(h\nabla g_1) + \nabla f \cdot \nabla(\nabla g_1 \cdot \nabla g_2)\, \mathrm{d}\mathfrak{m} \end{aligned}$$

holds for every $h, g_1, g_2 \in \mathrm{Test}(X, \mathsf{d}, \mathfrak{m})$. The pointwise norm $\Big|\mathrm{Hess}(f)\Big|$ of $\mathrm{Hess}(f)$ belongs to $L^2(\mathfrak{m})$. We endow $W^{2,2}(X, \mathsf{d}, \mathfrak{m})$ with the norm

$$\|f\|^2_{W^{2,2}} := \|f\|^2_{H^{1,2}} + \||\,\mathrm{Hess}(f)|\|^2_{L^2}\,.$$

The space $H^{2,2}(X, \mathsf{d}, \mathfrak{m}) \subset W^{2,2}(X, \mathsf{d}, \mathfrak{m})$ is defined as the $W^{2,2}$-closure of $\mathrm{Test}(X, \mathsf{d}, \mathfrak{m})$.

Let us now present the improved version of the Bochner inequality, taking into account the Hessian, in the $\mathrm{RCD}(K,N)$ setting, referring to [**99**, Theorem 3.3] for its proof.

**Theorem 1.59.** *Let* $(X, \mathsf{d}, \mathfrak{m})$ *be an* $\mathrm{RCD}(K, N)$ *metric measure space. Then*

$$\frac{1}{2}\Delta|\nabla f|^2 - \nabla f \cdot \nabla(\Delta f) \geq |\operatorname{Hess}(f)|^2 + K|\nabla f|^2 \quad \forall f \in \mathrm{Test}(X, \mathsf{d}, \mathfrak{m}). \tag{1.34}$$

As proved in [**87**, Proposition 3.3.18], we have the inclusion

$$D(\Delta) \subset H^{2,2}(X, \mathsf{d}, \mathfrak{m}). \tag{1.35}$$

Moreover, when $(X, \mathsf{d}, \mathfrak{m})$ is an $\mathrm{RCD}(K, N)$ m.m.s., one has the local estimate

$$\begin{aligned} \int_{B_1(x)} |\operatorname{Hess}(f)|^2 \, \mathrm{d}\mathfrak{m} \\ \leq C_{N,K} \left( \int_{B_2(x)} |\Delta f|^2 \, \mathrm{d}\mathfrak{m} + \inf_{m \in \mathbb{R}} \int_{B_2(x)} \left| |\nabla f|^2 - m \right| \, \mathrm{d}\mathfrak{m} \right) - K \int_{B_2(x)} |\nabla f|^2 \, \mathrm{d}\mathfrak{m}, \end{aligned} \tag{1.36}$$

that can be checked integrating the improved Bochner inequality (1.34) against a good cut-off function.

**Lemma 1.60** (Good cut-off functions [**24**, **121**])**.** *Let* $(X, \mathsf{d}, \mathfrak{m})$ *be an* $\mathrm{RCD}(K, N)$ *space. Let* $\Omega \subseteq X$ *be an open set and* $K \subseteq \Omega$ *a compact set. Then there exists* $\eta \in \mathrm{Test}(X, \mathsf{d}, \mathfrak{m})$ *such that* $0 \leq \eta \leq 1$ *on* $X$*, the support of* $\eta$ *is compactly contained in* $\Omega$*, and* $\eta = 1$ *on some open neighbourhood of* $K$*.*

Let us recall that the Hessian enjoys the following locality property that has been proved in [**87**, Proposition 3.3.24].

**Proposition 1.61.** *Given* $f_1, f_2 \in H^{2,2}(X, \mathsf{d}, \mathfrak{m})$ *it holds*

$$|\operatorname{Hess}(f)_1| = |\operatorname{Hess}(f)_2| \quad \mathfrak{m}\text{-}a.e.\ in\ \{f_1 = f_2\}\,.$$

In addition we shall use the following inequality that has been proved in [**87**, Proposition 3.3.22].

$$|\nabla(\nabla f \cdot \nabla g)| \leq |\operatorname{Hess}(f)|\,|\nabla g| + |\operatorname{Hess} g|\,|\nabla f| \quad \text{for any } f, g \in H^{2,2}(X, \mathsf{d}, \mathfrak{m}). \tag{1.37}$$

The next definition introduced the Hessian of functions defined on open sets.

**Definition 1.62.** Given an open set $\Omega \subseteq X$ and a function $f \in D(\Omega, \Delta) \cap \mathrm{Lip}(X)$ one can define (the modulus of) its Hessian as follows:

$$\left|\operatorname{Hess}(f)\right| := \left|\operatorname{Hess}(\eta f)\right| \ \mathfrak{m}\text{-a.e. on } \{\eta = 1\}, \quad \text{for every } \eta \in \mathrm{Test}(X) \text{ with } \mathrm{spt}(\eta) \subseteq \Omega. \tag{1.38}$$

The function $\left|\operatorname{Hess}(f)\right| : \Omega \to [0, \infty)$ is well defined thanks to Proposition 1.61 and the fact that $\eta f \in \mathrm{Test}(X, \mathsf{d}, \mathfrak{m})$ for every $\eta$ as in (1.38).

Below we introduce the class of Sobolev vector fields with covariant derivative in $L^2$.

**Definition 1.63.** The Sobolev space $W^{1,2}_C(TX) \subset L^2(TX)$ is the space of all $v \in L^2(TX)$ for which there exists an element of the tensor product $L^2(T^*X) \otimes L^2(T^*X)$, denoted by $\nabla v$, such that it holds

$$\int h \nabla v(\nabla g_1, \nabla g_2) \, \mathrm{d}\mathfrak{m} = -\int v \cdot \nabla g_2 \, \operatorname{div}(h g_1) + h \operatorname{Hess}(g_2)(v, \nabla g_1) \, \mathrm{d}\mathfrak{m}$$

for any choice of $h, g_1, g_2 \in \mathrm{Test}(X, \mathsf{d}, \mathfrak{m})$. We endow the space $W^{1,2}_C(TX)$ with the norm

$$\|v\|^2_{W^{1,2}_C} := \|v\|^2_{L^2} + \||\nabla v|\|^2_{L^2}\,.$$

The space $H^{1,2}_C(X,\mathsf{d},\mathfrak{m}) \subset W^{1,2}_C(X,\mathsf{d},\mathfrak{m})$ is defined as the $W^{1,2}_C$-closure of $\mathrm{TestV}(X,\mathsf{d},\mathfrak{m})$.

Here $\mathrm{TestV}(X,\mathsf{d},\mathfrak{m})$ is the class of test vector fields defined as follows.

$$\mathrm{TestV}(X,\mathsf{d},\mathfrak{m}) := \left\{ \sum_{i=1}^n g_i \nabla f_i \;\middle|\; g_i, f_i \in \mathrm{Test}(X,\mathsf{d},\mathfrak{m}), n \in \mathbb{N} \right\} \tag{1.39}$$

**Remark 1.64.** In the framework of RCD spaces it is still unknown whether $H^{2,2}(X,\mathsf{d},\mathfrak{m}) = W^{2,2}(X,\mathsf{d},\mathfrak{m})$ and $H^{1,2}_C(X,\mathsf{d},\mathfrak{m}) = W^{1,2}_C(X,\mathsf{d},\mathfrak{m})$. We refer to [**87**] for a discussion on this question.

Let us finally introduce the class of velocity fields with divergence and symmetric covariant derivative in $L^2$.

**Definition 1.65.** The Sobolev space $W^{1,2}_{C,s}(TX) \subset L^2(TX)$ is the space of all $v \in L^2(TX)$ with $\operatorname{div} v \in L^2(X,\mathfrak{m})$ for which there exists an element of the tensor product $L^2(T^*X) \otimes L^2(T^*X)$, denoted by $\nabla_{\mathrm{sym}} v$, such that it holds

$$\begin{aligned}\int h \nabla_{\mathrm{sym}} v(\nabla g_1, \nabla g_2)\, \mathrm{d}\mathfrak{m} \\ = \frac{1}{2} \int \{ -v \cdot \nabla g_2 \operatorname{div}(h \nabla g_1) - v \cdot \nabla g_1 \operatorname{div}(h \nabla g_2) + \operatorname{div}(hv) \nabla g_1 \cdot \nabla g_2 \}\, \mathrm{d}\mathfrak{m},\end{aligned} \tag{1.40}$$

for any choice of $h, g_1, g_2 \in \mathrm{Test}(X,\mathsf{d},\mathfrak{m})$. We endow the space $W^{1,2}_{C,s}(TX)$ with the norm

$$\|v\|^2_{W^{1,2}_{C,s}} := \|v\|^2_{L^2} + \||\nabla_{\mathrm{sym}} v|\|^2_{L^2}.$$

**Remark 1.66.** It easily follows from the definition that the symmetric covariant derivative of any vector field in $W^{1,2}_{C,s}(TX)$ is a symmetric tensor. Moreover, one can prove that any $v \in W^{1,2}_C(TX)$ with $\operatorname{div} v \in L^2(X,\mathfrak{m})$ belongs to $W^{1,2}_{C,s}(TX)$, and $\nabla_{\mathrm{sym}} v$ is the symmetric part of the covariant derivative $\nabla v$.

**5.3. Module with respect to the capacity measure.** We present a variant of the notion of $L^0$-normed $L^0$-module – where the Borel measure $\mathfrak{m}$ is replaced by the capacity – which has been proposed in [**67**]. This technical construction has the aim to go toward a “more pointwise notion” of vector field. Observe indeed that, as we have seen in Section 5.1, vector fields over a metric measure space $(X,\mathsf{d},\mathfrak{m})$ are elements of the tangent module $L^2(TX)$, therefore they are defined up to $\mathfrak{m}$-negligible sets. The capacity introduced below, under suitable assumptions on the ambient space, behaves as a codimension-2 measure (Cf. Section 5.3.1). Hence, element of a tangent module with respect to the capacity are defined up to codimension-2 subsets.

Our interest on this theory is motivated by the study of codimension-1 structures, such as boundaries of sets with finite perimeter. See Chapter 4 and Chapter 5

5.3.1. *Capacity.* Let us begin by recalling some basic result concerning the capacity on metric measure spaces.

Fix a metric measure space $(X,\mathsf{d},\mathfrak{m})$. The *capacity* of a given set $E \subset X$ is defined as

$$\mathrm{Cap}(E) := \inf \left\{ \|f\|^2_{H^{1,2}(X)} \;\middle|\; f \in H^{1,2}(X,\mathsf{d},\mathfrak{m}),\ f \geq 1 \ \mathfrak{m}\text{-a.e. on some neighbourhood of } E \right\}.$$

It turns out that Cap is a submodular outer measure on $X$, finite on all bounded sets, such that the inequality $\mathfrak{m}(E) \leq \mathrm{Cap}(E)$ holds for any Borel set $E \subset X$. Any function

$f : X \to [0,\infty]$ can be integrated with respect to the capacity via Cavalieri's formula:

$$\int f \,\mathrm{dCap} := \int_0^\infty \mathrm{Cap}\Big(\{f > t\}\Big)\,\mathrm{d}t.$$

The function $t \mapsto \mathrm{Cap}\Big(\{f > t\}\Big)$ is non-increasing, thus in particular it is Lebesgue measurable. The integral operator $f \mapsto \int f \,\mathrm{dCap}$ is subadditive as a consequence of the submodularity of Cap. Given any set $E \subset X$, we shall use the shorthand notation $\int_E f \,\mathrm{dCap} := \int \chi_E f \,\mathrm{dCap}$.

The natural setting to study properties of Cap is the one of *PI spaces*.

**Definition 1.67.** $(X, \mathsf{d}, \mathfrak{m})$ satisfies a weak local $(1,p)$-Poincaré inequality with constants $C_P > 0$ and $\lambda \geq 1$ if it holds
(1.41)

$$\fint_{B_r(x)} \big|f - (f)_{x,r}\big| \,\mathrm{d}\mathfrak{m} \leq C_P\, r \left(\fint_{B_{\lambda r}(x)} |\nabla f|_p^p \,\mathrm{d}\mathfrak{m}\right)^{\frac{1}{p}} \qquad \text{for all } f \in H^{1,p}(X),\ x \in X,\ r > 0,$$

where

$$(f)_{x,r} := \fint_{B_r(x)} f \,\mathrm{d}\mathfrak{m}. \tag{1.42}$$

A *PI space* is a locally doubling metric measure space (see (1.16)) supporting a weak local $(1,p)$-Poincaré inequality.

Let us now introduce the *codimension-$\alpha$ Hausdorff measure*.

**Definition 1.68.** Given a locally doubling metric measure space $(X, \mathsf{d}, \mathfrak{m})$, for any $\alpha > 0$ we set

$$h_\alpha(B_r(x)) := \frac{\mathfrak{m}(B_r(x))}{r^\alpha} \qquad \text{for any } x \in X,\ r \in (0,\infty).$$

The codimension-$\alpha$ Hausdorff measure $\mathscr{H}^{h_\alpha}$ is the Borel regular outer measure built from $h_\alpha$ with the Carathéodory construction. We will denote by $\mathscr{H}^{h_\alpha}_\delta$ the pre-measure with parameter $\delta$.

Let us now prove two results connecting the *codimension-$\alpha$ Hausdorff measure* and the capacity. Their proofs are inspired by those given for the analogous results in the Euclidean setting in [**78**].

**Lemma 1.69.** *Let $(X, \mathsf{d}, \mathfrak{m})$ be a locally doubling m.m.s.. Let $f \in L^1(X, \mathfrak{m})$, $f \geq 0$ be given. Then for any exponent $\alpha > 0$ it holds that*

$$\mathscr{H}^{h_\alpha}(\Lambda_\alpha) = 0, \qquad \textit{where we set } \Lambda_\alpha := \left\{x \in X \;\middle|\; \limsup_{r \searrow 0} r^\alpha (f)_{x,r} > 0\right\}.$$

*Proof.* By Lebesgue differentiation theorem we know that the limit $\lim_{r\searrow 0}(f)_{x,r}$ exists and is finite for $\mathfrak{m}$-a.e. $x \in X$, thus for any $\alpha > 0$ we have that $\limsup_{r\searrow 0} r^\alpha (f)_{x,r} = 0$ holds for $\mathfrak{m}$-a.e. $x \in X$. This means that $\mathfrak{m}(\Lambda_\alpha) = 0$. Calling

$$\Lambda_\alpha^k := \left\{x \in X \;\middle|\; \limsup_{r \searrow 0} r^\alpha (f)_{x,r} \geq 1/k\right\} \qquad \text{for every } k \in \mathbb{N},$$

we see that $\Lambda_\alpha = \bigcup_k \Lambda_\alpha^k$, thus in particular $\mathfrak{m}(\Lambda_\alpha^k) = 0$ for every $k \in \mathbb{N}$. Given that $f \in L^1(X, \mathfrak{m})$, for any $\varepsilon > 0$ there exists $\delta > 0$ such that $\int_A f \,\mathrm{d}\mathfrak{m} \leq \varepsilon$ for any Borel set

$A \subset X$ satisfying $\mathfrak{m}(A) < \delta$. Fix $k \in \mathbb{N}$ and pick an open set $U \subset X$ such that $\Lambda_\alpha^k \subset U$ and $\mathfrak{m}(U) < \delta$. Let us define

$$\mathcal{F} := \left\{ B_r(x) \,\middle|\, x \in \Lambda_\alpha^k,\ r \in (0,\varepsilon),\ B_r(x) \subset U,\ \int_{B_r(x)} f \,\mathrm{d}\mathfrak{m} \geq \mathfrak{m}\big(B_r(x)\big)/(r^\alpha k) \right\}.$$

Therefore by applying the Vitali covering theorem we can find a sequence $(B_i)_{i\in\mathbb{N}} \subset \mathcal{F}$ of pairwise disjoint balls $B_i = B_{r_i}(x_i)$ such that $\Lambda_\alpha^k \subset \bigcup_i B_{5r_i}(x_i)$. Being $\mathfrak{m}$ locally doubling, there exists a constant $C_D \geq 1$ such that $\mathfrak{m}\big(B_{5r}(x)\big) \leq C_D\, \mathfrak{m}\big(B_r(x)\big)$ holds for every $x \in X$ and $r < \varepsilon$. Consequently, one has that

$$\begin{aligned}\mathscr{H}^{h_\alpha}_{10\varepsilon}(\Lambda_\alpha^k) &\leq \frac{1}{5^\alpha} \sum_{i=1}^\infty \frac{\mathfrak{m}\big(B_{5r_i}(x_i)\big)}{r_i^\alpha} \leq \frac{C_D}{5^\alpha} \sum_{i=1}^\infty \frac{\mathfrak{m}(B_i)}{r_i^\alpha} \leq \frac{C_D k}{5^\alpha} \sum_{i=1}^\infty \int_{B_i} f \,\mathrm{d}\mathfrak{m} \leq \frac{C_D k}{5^\alpha} \int_U f \,\mathrm{d}\mathfrak{m} \\ &\leq \frac{C_D k}{5^\alpha} \varepsilon.\end{aligned}$$

By letting $\varepsilon \searrow 0$ we conclude that $\mathscr{H}^{h_\alpha}(\Lambda_\alpha^k) = 0$, whence $\mathscr{H}^{h_\alpha}(\Lambda_\alpha) = \lim_k \mathscr{H}^{h_\alpha}(\Lambda_\alpha^k) = 0$. □

**Theorem 1.70.** *Let $(X, \mathsf{d}, \mathfrak{m})$ be a PI space. Then it holds that $\mathscr{H}^{h_\alpha} \ll \mathrm{Cap}$ for every $\alpha \in (0,2)$.*

*Proof.* Fix $\alpha \in (0,2)$ and a set $A \subset X$ with $\mathrm{Cap}(A) = 0$. We aim to prove that $\mathscr{H}^{h_\alpha}(A) = 0$. By definition of capacity, we can find a sequence $(f_i)_i \subset H^{1,2}(X)$ such that $f_i \geq 1$ on some neighbourhood of $A$ and $\|f_i\|_{H^{1,2}(X)} \leq 1/2^i$ for every $i \in \mathbb{N}$. Since $\sum_{i=1}^\infty \|f_i\|_{H^{1,2}(X)} < \infty$, one has that $g := \sum_{i=1}^\infty f_i$ is a well-defined element of the Banach space $H^{1,2}(X)$. For any $k \in \mathbb{N}$ it clearly holds that $g \geq k$ on some neighbourhood of $A$, whence for any $x \in A$ we have $(g)_{x,r} \geq k$ for every $r < \mathrm{dist}\big(x, \{g < k\}\big)$ and accordingly

$$\lim_{r\searrow 0} (g)_{x,r} = \infty \quad \text{for every } x \in A. \tag{1.43}$$

Furthermore, we claim that

$$\limsup_{r\searrow 0} r^\alpha \fint_{B_r(x)} |Dg|^2 \,\mathrm{d}\mathfrak{m} = \infty \quad \text{for every } x \in A. \tag{1.44}$$

In order to prove it, we argue by contradiction: suppose that

$$\limsup_{r\searrow 0} r^\alpha \fint_{B_r(x)} |Dg|^2 \,\mathrm{d}\mathfrak{m} < \infty$$

for some $x \in A$, so that there exists a constant $M > 0$ such that

$$r^\alpha \fint_{B_r(x)} |Dg|^2 \,\mathrm{d}\mathfrak{m} \leq M \quad \text{for every } r \in (0,1). \tag{1.45}$$

Call $C_D$ the doubling constant of $\mathfrak{m}$ (for $r < 1/2$). Therefore, for every $r < 1/(2\lambda)$ we have that

$$\begin{aligned}\Big|(g)_{x,r} - (g)_{x,2r}\Big| &= \frac{1}{\mathfrak{m}\Big(B_r(x)\Big)}\bigg|\int_{B_r(x)} g - (g)_{x,2r}\,\mathrm{d}\mathfrak{m}\bigg| \\ &\le C_D ⨍_{B_{2r}(x)} \Big|g - (g)_{x,2r}\Big|\,\mathrm{d}\mathfrak{m} \\ &\overset{(1.41)}{\le} 2\,C_D\,C_P\,r\left(⨍_{B_{2\lambda r}(x)} |Dg|^2\,\mathrm{d}\mathfrak{m}\right)^{\frac{1}{2}} \\ &\overset{(1.45)}{\le} (2^{1-\frac{\alpha}{2}}\,C_D\,C_P\,\lambda^{-\frac{\alpha}{2}}\,M^{\frac{1}{2}})\,r^{1-\frac{\alpha}{2}}.\end{aligned}$$

Let us set $C := 2^{1-\frac{\alpha}{2}}\,C_D\,C_P\,\lambda^{-\frac{\alpha}{2}}\,M^{\frac{1}{2}}$ and $\theta := 1-\alpha/2 \in (0,1)$. Then the previous computation gives $\sum_{i=2}^{\infty}\Big|(g)_{x,2^{-i}} - (g)_{x,2^{-i+1}}\Big| \le C\sum_{i=2}^{\infty}(2^{\theta})^{-i} < \infty$, contradicting (1.43). This proves (1.44).

Finally, it immediately follows from (1.44) that $A$ is contained in the set of all points $x \in X$ that satisfy $\limsup_{r\searrow 0} r^{\alpha}⨍_{B_r(x)} |Dg|^2\,\mathrm{d}\mathfrak{m} > 0$, which is $\mathscr{H}^{h_\alpha}$-negligible by Lemma 1.69. Therefore, we conclude that $\mathscr{H}^{h_\alpha}(A) = 0$, thus completing the proof of the statement. □

The space of all Borel functions on $X$ – considered up to Cap-a.e. equality – is denoted by $L^0(\mathrm{Cap})$. If continuous functions are strongly dense in $H^{1,2}(X,\mathsf{d},\mathfrak{m})$ (this condition is met, for instance, if the space is infinitesimally Hilbertian), then there exists a unique "quasi-continuous representative" map $\mathsf{QCR} : H^{1,2}(X) \to L^0(\mathrm{Cap})$ that is characterized as follows: $\mathsf{QCR}$ is a continuous map, and for any $f \in H^{1,2}(X)$ it holds that $\mathsf{QCR}(f)$ is (the equivalence class of) a quasi-continuous function that is $\mathfrak{m}$-a.e. coinciding with $f$ itself. Let us recall that a function $f : X \to \mathbb{R}$ is said to be quasi-continuous if for any $\varepsilon > 0$ there exists a set $E \subset X$ with $\mathrm{Cap}(E) < \varepsilon$ such that $f : X \setminus E \to \mathbb{R}$ is continuous. We refer to [**67**, Theorem 1.20] for a proof of this result.

5.3.2. *Module w.r.t.* Cap. Given a module $\mathscr{M}_{\mathrm{Cap}}$ over the ring $L^0(\mathrm{Cap})$, we say that a mapping $|\cdot| : \mathscr{M}_{\mathrm{Cap}} \to L^0(\mathrm{Cap})$ is a *pointwise norm* provided it satisfies the (in)equalities in (1.30) in the Cap-a.e. sense for any choice of $v, w \in \mathscr{M}_{\mathrm{Cap}}$ and $f \in L^0(\mathrm{Cap})$. Then the space $\mathscr{M}_{\mathrm{Cap}}$ is said to be an $L^0(\mathrm{Cap})$-*normed* $L^0(\mathrm{Cap})$-*module* if it is complete when endowed with the distance

$$\mathsf{d}_{\mathscr{M}_{\mathrm{Cap}}}(v,w) := \sum_{k\in\mathbb{N}} \frac{1}{2^k \max\Big\{\mathrm{Cap}(A_k),1\Big\}}\int_{A_k} \min\Big\{|v-w|,1\Big\}\,\mathrm{d}\mathrm{Cap},$$

where $(A_k)_k$ is any increasing sequence of open subsets of $X$ having finite capacity that is chosen in such a way that any bounded set $B \subset X$ is contained in $A_k$ for some $k \in \mathbb{N}$ sufficiently big.

Let us recall, since this fact plays a crucial role in the discussion below, that $|\nabla f|^2 \in H^{1,2}(X)$ for any $f \in \mathrm{Test}(X)$, and thus $|\nabla f| \in H^{1,2}(X)$ as well (see [**67**]). In particular, for any $f \in \mathrm{Test}(X)$, $|\nabla f|$ admits a quasi-continuous representative.

**Theorem 1.71** (Tangent $L^0(\mathrm{Cap})$-module [**67**])**.** *Let* $(X,\mathsf{d},\mathfrak{m})$ *be an* $\mathrm{RCD}(K,\infty)$ *space. Then there exists a unique couple* $\Big(L^0_{\mathrm{Cap}}(TX),\tilde{\nabla}\Big)$*, where* $L^0_{\mathrm{Cap}}(TX)$ *is an* $L^0(\mathrm{Cap})$*-normed* $L^0(\mathrm{Cap})$*-module and* $\tilde{\nabla} : \mathrm{Test}(X) \to L^0_{\mathrm{Cap}}(TX)$ *is a linear operator, such that the following* *hold:*

$$|\tilde{\nabla} f| = \mathsf{QCR}(|\nabla f|) \quad \textit{in the } \mathrm{Cap}\textit{-a.e. sense} \qquad \textit{for every } f \in \mathrm{Test}(X),$$

$$\Big\{ \sum_{n\in\mathbb{N}} \chi_{E_n} \tilde{\nabla} f_n \;\Big|\; (E_n)_n \textit{ Borel partition of } X,\ (f_n)_n \subset \mathrm{Test}(X) \Big\} \qquad \textit{is dense in } L^0_{\mathrm{Cap}}(TX).$$

*The space* $L^0_{\mathrm{Cap}}(TX)$ *is called* capacitary tangent module *on* $X$*, while* $\tilde{\nabla}$ *is the* capacitary gradient.

**Remark 1.72.** The tangent $L^0(\mathrm{Cap})$-module $L^0_{\mathrm{Cap}}(TX)$ is a Hilbert module; cf. [**67**, Proposition 2.8].

Fix any Radon measure $\mu$ on a m.m.s. $(X, \mathsf{d}, \mathfrak{m})$ and suppose that $\mu \ll \mathrm{Cap}$. Then there is a natural projection $\pi_\mu : L^0(\mathrm{Cap}) \to L^0(\mu)$. Given an $L^0(\mathrm{Cap})$-normed $L^0(\mathrm{Cap})$-module $\mathscr{M}_{\mathrm{Cap}}$, we define an equivalence relation $\sim_\mu$ on $\mathscr{M}_{\mathrm{Cap}}$ as follows: given any $v, w \in \mathscr{M}_{\mathrm{Cap}}$, we declare that

$$v \sim_\mu w \quad \Longleftrightarrow \quad |v - w| = 0 \ \text{ holds } \mu\text{-a.e. on } X.$$

Then the quotient $\mathscr{M}^0_\mu := \mathscr{M}_{\mathrm{Cap}}/\sim_\mu$ inherits a natural structure of $L^0(\mu)$-normed $L^0(\mu)$-module. Call $\bar{\pi}_\mu : \mathscr{M}_{\mathrm{Cap}} \to \mathscr{M}^0_\mu$ the canonical projection. Moreover, for any exponent $p \in [1, \infty)$ we define

$$\mathscr{M}^p_\mu := \Big\{ v \in \mathscr{M}^0_\mu \;\Big|\; |v| \in L^p(\mu) \Big\}. \tag{1.46}$$

It turns out that $\mathscr{M}^p_\mu$ is an $L^p(\mu)$-normed $L^\infty(\mu)$-module. Notice that $|\bar{\pi}_\mu(v)| = \pi_\mu(|v|)$ holds in the $\mu$-a.e. sense for every $v \in \mathscr{M}_{\mathrm{Cap}}$.

**Lemma 1.73.** *Let* $(X, \mathsf{d}, \mathfrak{m})$ *be a m.m.s.,* $\mathscr{M}_{\mathrm{Cap}}$ *an* $L^0(\mathrm{Cap})$*-normed* $L^0(\mathrm{Cap})$*-module. Fix a finite Borel measure* $\mu \geq 0$ *on* $X$ *such that* $\mu \ll \mathrm{Cap}$*. Let* $V$ *be a linear subspace of* $\mathscr{M}_{\mathrm{Cap}}$ *such that* $|v|$ *admits a bounded* $\mathrm{Cap}$*-a.e. representative for every* $v \in V$ *and*

$$\mathcal{V} := \Big\{ \sum_{n\in\mathbb{N}} \chi_{E_n} v_n \;\Big|\; (E_n)_{n\in\mathbb{N}} \textit{ Borel partition of } X,\ (v_n)_{n\in\mathbb{N}} \subset V \Big\} \qquad \textit{is dense in } \mathscr{M}_{\mathrm{Cap}}.$$

*Then for any* $p \in [1, \infty)$ *it holds that*

$$\mathcal{W} := \Big\{ \sum_{i=1}^n \chi_{E_i} \bar{\pi}_\mu(v_i) \;\Big|\; n \in \mathbb{N},\ (E_i)_{i=1}^n \textit{ Borel partition of } X,\ (v_i)_{i=1}^n \subset V \Big\}$$

*is dense in* $\mathscr{M}^p_\mu$.

*Proof.* Fix $v \in \mathscr{M}^p_\mu$ and $\varepsilon > 0$. Since $|v|^p \in L^1(\mu)$, there is $\delta > 0$ such that $\Big( \int_E |v|^p \, \mathrm{d}\mu \Big)^{\frac{1}{p}} \leq \varepsilon/3$ holds for any Borel set $E \subset X$ with $\mu(E) < \delta$. Choose any $\bar{v} \in \mathscr{M}_{\mathrm{Cap}}$ such that $\bar{\pi}_\mu(\bar{v}) = v$. We can find $(\bar{v}_k)_k \subset \mathcal{V}$ so that $|\bar{v}_k - \bar{v}| \to 0$ in $L^0(\mathrm{Cap})$. Hence $\Big|\bar{\pi}_\mu(\bar{v}_k) - \bar{\pi}_\mu(\bar{v})\Big| = \pi_\mu\Big(|\bar{v}_k - \bar{v}|\Big) \to 0$ in $L^0(\mu)$. Thanks to Egorov theorem, there exists a compact set $K \subset X$ with $\mu(X \setminus K) < \delta$ such that (possibly taking a not relabeled subsequence) it holds that $\Big|\bar{\pi}_\mu(\bar{v}_k) - v\Big| \to 0$ uniformly on $K$. Consequently, by dominated convergence theorem we see that $\chi_K \bar{\pi}_\mu(\bar{v}_k) \to \chi_K v$ in $\mathscr{M}^p_\mu$. Then we can pick $k \in \mathbb{N}$ so that the element $\bar{w} := \bar{v}_k$ satisfies $\Big\| \chi_K \bar{\pi}_\mu(\bar{w}) - \chi_K v \Big\|_{\mathscr{M}^p_\mu} \leq \varepsilon/3$. If $\bar{w}$ is written as $\sum_{n\in\mathbb{N}} \chi_{E_n} \bar{w}_n$, then we have $\chi_K \bar{\pi}_\mu(\bar{w}) = \sum_{n\in\mathbb{N}} \chi_{K\cap E_n} \bar{\pi}_\mu(\bar{w}_n)$. By dominated convergence theorem we know that for $N \in \mathbb{N}$

sufficiently big the element $z := \sum_{n=1}^N \chi_{K\cap E_n}\bar\pi_\mu(\bar w_n) \in \mathcal{W}$ satisfies $\left\|z - \chi_K\bar\pi_\mu(\bar w)\right\|_{\mathscr{M}^p_\mu} \le \varepsilon/3$. Therefore, we conclude that

$$\|z - v\|_{\mathscr{M}^p_\mu} \le \left\|z - \chi_K\bar\pi_\mu(\bar w)\right\|_{\mathscr{M}^p_\mu} + \left\|\chi_K\bar\pi_\mu(\bar w) - \chi_K v\right\|_{\mathscr{M}^p_\mu} + \|\chi_{X\setminus K} v\|_{\mathscr{M}^p_\mu} \le \varepsilon,$$

thus proving the statement. □

In Chapter 5 we apply the above presented construction with $\mu = \mathrm{Per}(E,\cdot)$, when $E \subset X$ is of finite perimeter. This is possible thanks to the fact that $\mathrm{Per}(E,\cdot)$ is absolutely continuous with respect to $\mathscr{H}^{h_1}$, and that $\mathscr{H}^{h_1} \ll \mathrm{Cap}$ as a consequence of the results in Section 5.3.1.

## 6. Flows of Sobolev velocity fields

**6.1. Regular Lagrangian flows and Sobolev vector fields.** In this section we present the notion of regular Lagrangian flow (RLF for short), firstly introduced in the Euclidean setting by Ambrosio in [**5**], inspired by the earlier work of Di Perna and Lions [**76**]. It was defined as a generalized notion of flow in order to study ordinary differential equations associated to weakly differentiable vector fields. It is indeed well-known that, in general, it is not possible to define in a unique way a flow associated to a non Lipschitz vector field, since the trajectories starting from a given point might be non unique.

The theory of existence and uniqueness for regular Lagrangian flows in the context of $\mathrm{RCD}(K,\infty)$ metric measure spaces was developed by Ambrosio and Trevisan in [**27**]. The authors work with a very weak notion of symmetric covariant derivative for a vector field (see [**28**, Definition 5.4]) and prove existence and uniqueness of the RLF associated to any bounded vector field $b$ with symmetric derivative in $L^2$ and bounded divergence, over an $\mathrm{RCD}(K,\infty)$ space (actually the results in [**27**] cover also more general settings).

Having the notion of derivation (see Section 5.1) at hand, time dependent vector fields over $(X,\mathsf{d},\mathfrak{m})$ can be introduced in the natural way.

**Definition 1.74.** Let us fix $T > 0$ and $p \in [1,\infty]$. We say that $b : [0,T] \to L^p(TX)$ is a time dependent vector field if, for every $f \in H^{1,q}(X,\mathsf{d},\mathfrak{m})$ (where $q$ is the dual exponent of $p$), the map

$$(t,x) \mapsto b_t \cdot \nabla f(x)$$

is measurable with respect to the product sigma-algebra $\mathscr{L}^1 \otimes \mathscr{B}(X)$. We say that $b$ is bounded if

$$\|b\|_{L^\infty} := \||b|\|_{L^\infty([0,T]\times X)} < \infty,$$

and that $b \in L^1((0,T), L^p(TX))$ if

$$\int_0^T \|b_s\|_{L^p}\,\mathrm{d}s < \infty.$$

In the sequel we shall stress the dependence of a vector field $b$ on the time variable only in case it is relevant for the sake of clarity.

In the context of $\mathrm{RCD}(K,\infty)$ spaces the definition of Regular Lagrangian flow reads as follows (see [**27**, **28**]).

**Definition 1.75.** Let us fix a time dependent vector field $b$ (see Definition 1.74). We say that $\boldsymbol{X} : [0,T] \times X \to X$ is a Regular Lagrangian flow associated to $b$ if the following conditions hold true:

(i) $\boldsymbol{X}(0,x)=x$ and $X(\cdot,x)\in C([0,T],X)$ for every $x\in X$;
(ii) there exists $L\geq 0$, called compressibility constant, such that

$$\boldsymbol{X}(t,\cdot)_*\mathfrak{m}\leq L\mathfrak{m},\qquad \text{for every } t\in[0,T];$$

(iii) for every $f\in \mathrm{Test}(X,\mathsf{d},\mathfrak{m})$ the map $t\mapsto f(\boldsymbol{X}(t,x))$ belongs to $AC([0,T])$ for $\mathfrak{m}$-a.e. $x\in X$ and

$$\frac{\mathrm{d}}{\mathrm{d}t}f(\boldsymbol{X}(t,x))=b_t\cdot\nabla f(\boldsymbol{X}(t,x))\qquad \text{for a.e. } t\in(0,T). \tag{1.47}$$

The selection of "good" trajectories is encoded in condition (ii), which is added to ensure that the RLF does not concentrate too much the reference measure $\mathfrak{m}$.

We remark that the notion of RLF is stable under modification in a negligible set of initial conditions, but we prefer to work with a pointwise defined map in order to avoid technical issues.

Here and in the following we use the shortened notation $\boldsymbol{X}_t(x)=\boldsymbol{X}(t,x)$.

**Remark 1.76.** Under the additional assumption $b\in L^1((0,T);L^2(TX))$, equality (1.47) holds true for every $g\in H^{1,2}(X,\mathsf{d},\mathfrak{m})$ (where it is understood that in this case the map $t\mapsto g(\boldsymbol{X}_t(x))$ belongs to $H^{1,1}((0,T))$ for $\mathfrak{m}$-a.e. $x\in X$) if and only if it holds for every $h\in D$ with $D\subset H^{1,2}(X,\mathsf{d},\mathfrak{m})$ dense with respect to the strong topology. Indeed, if this is the case, for any $g\in H^{1,2}(X,\mathsf{d},\mathfrak{m})$ and every $\varepsilon>0$ we can find $h\in D$ such that $\|g-h\|_{H^{1,2}(X,\mathsf{d},\mathfrak{m})}<\varepsilon$. Hence, since (1.47) holds true for $h$, we can estimate

$$\begin{aligned}
\int\left|g(\boldsymbol{X}(t,x))-g(x)-\int_0^t b_s\cdot\nabla g(\boldsymbol{X}(s,x))\,\mathrm{d}s\right|^2\mathrm{d}\mathfrak{m}(x)&\\
\leq 2\int|g(\boldsymbol{X}(t,x))-h(\boldsymbol{X}(t,x))|^2\,\mathrm{d}\mathfrak{m}(x)+2\int|g(x)-h(x))|^2\,\mathrm{d}\mathfrak{m}(x)&\\
+2\int\left|\int_0^t b_s\cdot\nabla(g-h)(\boldsymbol{X}(s,x))\,\mathrm{d}s\right|^2\mathrm{d}\mathfrak{m}(x)&\\
\leq 2(L+1)\,\|g-h\|^2_{L^2(X,\mathfrak{m})}+2L\,\|g-h\|^2_{H^{1,2}(X,\mathsf{d},\mathfrak{m})}\sqrt{t}\int_0^t\|b_s\|^2_{L^2}\,\mathrm{d}s&\\
\leq \varepsilon^2C(L,t,\|b\|_{L^1((0,T);L^2(TX))}),&
\end{aligned}$$

that, together with an application of Fubini's theorem, implies the validity of (1.47) for $g$.

Moreover, one can easily prove, via a localization procedure, that also functions in the class $H^{1,2}_{\mathrm{loc}}(X,\mathsf{d},\mathfrak{m})$ are admissible tests in (1.47).

**6.2. Existence and uniqueness of regular Lagrangian flows.** It is well known that to obtain an existence and uniqueness theory for regular Lagrangian flows it is necessary to restrict to a class of sufficiently regular vector fields, even in the case of a smooth ambient space. Below we introduce a very weak notion of Sobolev vector field with symmetric covariant derivative in $L^2$, following [**27**]. This definition is sufficient to show existence and uniqueness of RLF, and it is weaker than the notion of vector fields with symmetric covariant derivative and divergence in $L^2$ presented in Definition 1.65.

**Definition 1.77.** Let $b\in L^\infty(TX)$ with $\operatorname{div}b\in L^\infty(X,\mathfrak{m})$. We write $|D_{\mathrm{sym}}b|\in L^2(X,\mathfrak{m})$ if there exists a constant $c>0$ such that

$$\left|\int D_{\mathrm{sym}}b(\nabla\varphi,\nabla\psi)\,\mathrm{d}\mathfrak{m}\right|\leq c\,\|\nabla\varphi\|_{L^4}\,\|\nabla\psi\|_{L^4}\qquad\forall\varphi,\,\psi\in\mathrm{Test}(X,\mathsf{d},\mathfrak{m}), \tag{1.48}$$

where

$$\int D_{\rm sym}b(\nabla\varphi,\nabla\psi)\,\mathrm{d}\mathfrak{m} := -\frac{1}{2}\int \{b\cdot\nabla\varphi\ \Delta\psi + b\cdot\nabla\psi\ \Delta\varphi - \mathrm{div}\, b\ \nabla\varphi\cdot\nabla\psi\}\,\mathrm{d}\mathfrak{m}.$$

We let $\|D_{\rm sym}b\|_{L^2}$ be the smallest $c$ in (1.48). In particular we write $D_{\rm sym}b = 0$ if $\|D_{\rm sym}b\|_{L^2} = 0$.

In the next theorem we resume some general result concerning Regular Lagrangian flows that will be used in the sequel.

**Theorem 1.78.** *Let $(X,\mathsf{d},\mathfrak{m})$ be an* $\mathrm{RCD}(K,\infty)$ *space for some $K\in\mathbb{R}$. For any $b\in L^\infty((0,T),L^\infty(TX))$ with $\mathrm{div}\, b\in L^\infty((0,T),L^\infty(X,\mathfrak{m}))$ and $|D_{\rm sym}b|\in L^1((0,T),L^2(X,\mathfrak{m}))$, according to* (1.48)*, there exists a unique RLF $\boldsymbol{X}:[0,T]\times X\to X$ satisfying the bound*

$$e^{-t\|\mathrm{div}\, b\|_{L^\infty}}\mathfrak{m}\le (\boldsymbol{X}_t)_*\,\mathfrak{m}\le e^{t\|\mathrm{div}\, b\|_{L^\infty}}\mathfrak{m}. \tag{1.49}$$

*Uniqueness is understood in the following sense: if $\boldsymbol{X}$ and $\bar{\boldsymbol{X}}$ are Regular Lagrangian flows associated to $b$, then for $\mathfrak{m}$-a.e. $x\in X$ one has $\boldsymbol{X}_t(x)=\bar{\boldsymbol{X}}_t(x)$ for any $t\in[0,T]$.*

*Moreover, if $b\in L^\infty(TX)$ satisfies $\mathrm{div}\, b\in L^\infty(X,\mathfrak{m})$ and $|D_{sym}b|\in L^2(X,\mathfrak{m})$ the unique regular Lagrangian flow is defined for any $t\in\mathbb{R}$,* [2] *and we have:*

(i) *$\boldsymbol{X}$ satisfies the semigroup property: for any $s\in\mathbb{R}$ it holds that, for $\mathfrak{m}$-a.e. $x\in X$,*

$$\boldsymbol{X}(t,\boldsymbol{X}(s,x)) = \boldsymbol{X}(t+s,x)\qquad \forall t\in\mathbb{R}. \tag{1.50}$$

(ii) *For any $\bar u\in L^1(X,\mathfrak{m})\cap L^\infty$ there exists $u\in L^\infty_{\rm loc}(\mathbb{R},L^1(X,\mathfrak{m})\cap L^\infty)$ such that $(\boldsymbol{X}_t)_*(u\mathfrak{m}) = u_t\mathfrak{m}$ and it solves the continuity equation, i.e. for any $\varphi\in\mathrm{Test}(X)$ the map $t\mapsto\int\varphi u_t\,\mathrm{d}\mathfrak{m}$ is locally absolutely continuous with distributional derivative*

$$\frac{\mathrm{d}}{\mathrm{d}t}\int\varphi u_t\,\mathrm{d}\mathfrak{m} = \int(b\cdot\nabla\varphi)u_t\,\mathrm{d}\mathfrak{m};$$

(iii) *if $\mathrm{div}\, b=0$ and $D_{\rm sym}b=0$ then $\boldsymbol{X}_t$ admits a representative which is a measure-preserving isometry, i.e.*

$$\mathsf{d}(\boldsymbol{X}_t(x),\boldsymbol{X}_t(y)) = \mathsf{d}(x,y)\qquad\forall x,y\in X\quad\textit{and}\quad(\boldsymbol{X}_t)_*\,\mathfrak{m}=\mathfrak{m},$$

*for any $t\in\mathbb{R}$. Furthermore in this case the semigroup property* (1.50) *is satisfied pointwise.*

*Proof.* (i) and (ii) immediately follow from the results in [**27**] (see Theorem 8.3 together with Theorem 4.3 and Theorem 4.4). Let us prove (iii). From (1.49) we conclude that $(X_t)_*\,\mathfrak{m}=\mathfrak{m}$ for any $t\in\mathbb{R}$. Let us now take $\bar u\in H^{1,2}(X,\mathsf{d},\mathfrak{m})\cap L^\infty$ and $u$ as in (ii). Thanks to [**27**, Lemma 5.8] we get that $P_\alpha u_t\in\mathrm{Test}(X,\mathsf{d},\mathfrak{m})$ is still a solution of the continuity equation for any $\alpha\in(0,1)$. Then we can compute

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\int|\nabla P_\alpha u_t|^2\,\mathrm{d}\mathfrak{m} = -\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\int P_\alpha u_t\Delta P_\alpha u_t\,\mathrm{d}\mathfrak{m} = -\int b\cdot\nabla\Delta P_\alpha u_t\ P_\alpha u_t\,\mathrm{d}\mathfrak{m}.$$

Since $\mathrm{div}\, b=0$ and $\nabla_{\rm sym}b=0$, we deduce

$$-\int b\cdot\nabla\Delta P_\alpha u_t\ P_\alpha u_t\,\mathrm{d}\mathfrak{m} = \int b\cdot\nabla P_\alpha u_t\ P_\alpha\Delta u_t\,\mathrm{d}\mathfrak{m} = 0,$$

[2] To be more precise, there exist unique Regular Lagrangian flows $\boldsymbol{X}^+,\boldsymbol{X}^-:[0,\infty)\times X\to X$ associated to $b$ and $-b$ respectively and we let $\boldsymbol{X}_t=\boldsymbol{X}^+_t$ for $t\ge 0$ and $\boldsymbol{X}_t=\boldsymbol{X}^-_{-t}$ for $t\le 0$.

therefore

$$\int |\nabla P_\alpha u_t|^2 \, \mathrm{d}\mathfrak{m} = \int |\nabla P_\alpha \bar{u}|^2 \, \mathrm{d}\mathfrak{m} \qquad \forall t \in \mathbb{R}, \quad \forall \alpha \in (0,1). \tag{1.51}$$

Taking the limit in (1.51) as $\alpha \to 0$ it easily follows that $u_t \in H^{1,2}(X, \mathsf{d}, \mathfrak{m})$ for any $t \in \mathbb{R}$ and that $\int |\nabla u_t|^2 \, \mathrm{d}\mathfrak{m}$ does not depend on $t \in \mathbb{R}$. Using the identity $u_t(x) = \bar{u}(\boldsymbol{X}(-t,x))$ (which can be checked using the semigroup property (1.50) and $\left(\boldsymbol{X}_t\right)_* \mathfrak{m} = \mathfrak{m}$) we deduce that, for any $t \in \mathbb{R}$,

$$\mathsf{Ch}(\bar{u} \circ \boldsymbol{X}_t) = \mathsf{Ch}(\bar{u}) \qquad \forall \bar{u} \in H^{1,2}(X, \mathsf{d}, \mathfrak{m}) \cap L^\infty,$$

and (iii) follows from arguments that have been used several times in the literature, as in [**84**, Proposition 4.20]. □

## 7. Splitting theorem and $\delta$-plitting maps

In [**84**] Gigli proved that in $\mathrm{RCD}(0,N)$ spaces the *splitting theorem* still holds, extending to this abstract framework the results obtained by Cheeger-Gromoll [**54**] and Cheeger-Colding [**50**] for smooth Riemannian manifolds and Ricci limit spaces, respectively.

**Theorem 1.79.** *Let $(X, \mathsf{d}, \mathfrak{m})$ be an $\mathrm{RCD}(0,N)$ m.m.s. containing a line, that is to say a curve $\gamma : \mathbb{R} \to X$ such that*

$$\mathsf{d}(\gamma(s), \gamma(t)) = |t - s|, \quad \forall s, t \in \mathbb{R}.$$

*Then there exists a m.m.s. $(X', \mathsf{d}', \mathfrak{m}')$ such that $(X, \mathsf{d}, \mathfrak{m})$ is isomorphic as a m.m.s. to*

$$(X', \mathsf{d}', \mathfrak{m}') \times (\mathbb{R}, \mathsf{d}_{Eucl}, \mathscr{L}^1).$$

*Furthermore:*

(i) *If $n \geq$ then $(X', \mathsf{d}', \mathfrak{m}')$ is an $\mathrm{RCD}(0, N-1)$;*
(ii) *if $N \in [1,2)$ then $X'$ is a point.*

*Moreover, $\gamma(t) = (x', t)$ for any $t \in \mathbb{R}$, for some $x' \in X'$.*

### 7.1. Splitting maps on RCD spaces.

This section is devoted to the study of $\delta$-splitting maps. Let us recall that their introduction in the study of spaces with lower Ricci curvature bounds dates back to [**50**]. Here we follow [**41**].

**Definition 1.80.** Let $(X, \mathsf{d}, \mathfrak{m})$ be an $\mathrm{RCD}(-1,N)$ metric measure space, $x \in X$ and $\delta > 0$ be fixed. We say that $u := (u_1, \dots, u_k) : B_r(x) \to \mathbb{R}^k$ is a $\delta$-splitting map provided it is harmonic (meaning that $u_a \in D(\Delta, B_r(x))$ with $\Delta u_a = 0$ for any $a = 1, \dots, k$) and satisfies:

(i) $u_a$ is $C_N$-Lipschitz for any $a = 1, \dots, k$;
(ii) $r^2 \fint_{B_r(x)} |\operatorname{Hess} u_a|^2 \, \mathrm{d}\mathfrak{m} < \delta$ for any $a = 1, \dots, k$;
(iii) $\fint_{B_r(x)} |\nabla u_a \cdot \nabla u_b - \delta_{a,b}| \, \mathrm{d}\mathfrak{m} < \delta$ for any $a, b = 1, \dots, k$.

**Remark 1.81.** Let us clarify the meaning of $|\operatorname{Hess} u|$ when $u : B_r(x) \to \mathbb{R}$ is harmonic and not necessarily globally defined. For any ball $B_{2s}(y) \subset B_r(x)$ we take a good cut-off function $\eta$ according to Lemma 1.60 that satisfies $\eta = 1$ in $B_s(y)$ and $\eta = 0$ in $X \setminus B_{2s}(y)$. As we already remarked after Definition 1.14, one has $\eta u \in D(\Delta)$, therefore $\eta u \in H^{2,2}(X, \mathsf{d}, \mathfrak{m})$ as a consequence of (1.35). We can now set $|\operatorname{Hess} u| := |\operatorname{Hess}(\eta u)|$ in $B_s(y)$. Observe that this is a good definition thanks to the locality of the Hessian (see Proposition 1.61).

**Remark 1.82.** With respect to the definition of $\delta$-splitting map which is nowadays adopted within the theory of Ricci limits (see for instance [**57**, Definition 1.20]) the main difference is condition (i). Therein the sharper bound $|\nabla u| \le 1 + \delta$ is imposed in the definition though, as they observe, it can be obtained as a consequence of the bound $|\nabla u| \le C_N$ and of the other defining properties (when working in the smooth framework).

The power of $\delta$-splitting maps in the theory of lower Ricci bounds is that, roughly speaking, they allow to pass from analysis to geometry and vice-versa. Namely, the existence of a $\delta$-splitting map with $k$ components on a Riemannian manifold with Ricci bounded below by $-\delta$ can be turned into $\varepsilon$-GH closeness (in the scale invariant sense) to a space which splits a factor $\mathbb{R}^k$ and vice-versa (see [**50**] and [**57**, Lemma 1.21]).

The first result presented below, Proposition 1.83, corresponds to the rough statement "the existence of a $\delta$-splitting map with $k$ components implies that the m.m.s. is $\varepsilon$-close to a product $\mathbb{R}^k \times Z$". The second one, Proposition 1.85, ensures that, over an $\mathrm{RCD}(-\varepsilon, N)$ space $\varepsilon$-close to a product $\mathbb{R}^k \times Z$, one can build a $\delta$-splitting map with $k$ components.

In order to shorten the notation for the rest of the paper we write $(\mathbb{R}^k \times Z, (0^k, z))$ to denote the pointed metric measure space $(\mathbb{R}^k \times Z, \mathsf{d}_{\mathrm{Eucl}} \times \mathsf{d}_Z, \mathscr{L}^k \times \mathfrak{m}_Z, (0^k, z))$.

**Proposition 1.83.** *Let $N > 1$ be fixed. Then, for any $\varepsilon > 0$, there exists $\delta = \delta_{N,\varepsilon} > 0$ such that, for any $\mathrm{RCD}(-\delta, N)$ m.m.s. $(X, \mathsf{d}, \mathfrak{m})$ and for any $x \in X$, if there exists a map $u : B_{\delta^{-1}}(x) \to \mathbb{R}^k$ such that $u$ is a $\delta$-splitting map over $B_s(x)$ for any $0 < s < \delta^{-1}$, then*

$$\mathsf{d}_{pmGH}\Big((X, \mathsf{d}, \mathfrak{m}, x), (\mathbb{R}^k \times Z, (0^k, z))\Big) < \varepsilon$$

*for some pointed $\mathrm{RCD}(0, N-k)$ metric measure space $(Z, \mathsf{d}_Z, \mathfrak{m}_Z, z)$.*

*Proof.* We wish to prove the sought conclusion arguing by contradiction. To this aim let us suppose that, for any $n \ge 1$, there exist an $\mathrm{RCD}(-1/n, N)$ m.m.s. $(X_n, \mathsf{d}_n, \mathfrak{m}_n)$, a point $x_n \in X_n$ and a map $u_n : B_n(x_n) \to \mathbb{R}^k$ which is a $1/n$-splitting map when restricted to $B_s(x_n)$ for any $0 < s < n$. Up to extracting a subsequence, that we do not relabel, we can assume that $(X_n, \mathsf{d}_n, \mathfrak{m}_n, x_n)$ converge in the pmGH-topology to an $\mathrm{RCD}(0, N)$ p.m.m.s. $(X_\infty, \mathsf{d}_\infty, \mathfrak{m}_\infty, x_\infty)$. Here we have used the stability and compactness property of $\mathrm{RCD}(K, N)$ spaces, cf. Section 4.1. We claim that $X_\infty$ splits off a factor $\mathbb{R}^k$. Observe that, if this is the case, then we reach the sought contradiction. The rest of this proof is dedicated to establishing the claim.

We wish to prove that there exists a function $v : X_\infty \to \mathbb{R}^k$ such that, letting $v := (v^1, \dots, v^k)$, it holds that $v^i$ is Lipschitz, harmonic and with vanishing Hessian for any $i = 1, \dots, k$ and $\nabla v^i \cdot \nabla v^j = \delta_{ij}$ $\mathfrak{m}_\infty$-a.e. for any $i, j = 1, \dots, k$. The function $v$ will be obtained as a limit function of the $1/n$-splitting maps $u_n : B_n(x_n) \to \mathbb{R}^k$. Indeed, since by the assumption in the defining condition of a $\delta$-splitting map the $u_n$ are $C_N$-Lipschitz for any $n \in \mathbb{N}$ and we can assume without loss of generality that $u_n(x_n) = 0^k$ for any $n \in \mathbb{N}$, by a generalized version of the Ascoli–Arzelà theorem (Proposition 1.34) we can infer the existence of $v : X_\infty \to \mathbb{R}^k$ such that $u_n$ converge to $v$ locally uniformly on $B_R(x_n)$ for any $R > 0$. As a consequence, it is easy to check that $u_n$ converge strongly in $L^2$ (see Definition 1.36) to $v$ on $B_R(x_n)$ for any $R > 0$. Since the functions $u_n$ are harmonic on $B_{2R}(x_n)$, at least for $n$ sufficiently large, by Theorem 1.47 and Proposition 1.40 it follows that $v$ is harmonic and that, for any $R > 0$ and $i, j = 1, \dots, k$,

$$\fint_{B_R(x_\infty)} \left|\nabla v^i \cdot \nabla v^j - \delta_{ij}\right| \mathrm{d}\mathfrak{m}_\infty = \lim_{n\to\infty} \fint_{B_R(x_n)} \left|\nabla u_n^i \cdot \nabla u_n^j - \delta_{ij}\right| \mathrm{d}\mathfrak{m}_n = 0.$$

Hence $\nabla v^i \cdot \nabla v^j = \delta_{ij}$ $\mathfrak{m}_\infty$-a.e. on $X_\infty$.

Since $(X_\infty, \mathsf{d}_\infty, \mathfrak{m}_\infty)$ is an $\mathrm{RCD}(0,N)$ m.m.s., from $\Delta v^i = 0$ and $|\nabla v^i|^2 = 1$ we infer by (1.36) that $\operatorname{Hess} v^i = 0$, for any $i = 1, \dots, k$. All in all we get by a standard argument (cf. the proof of [**29**, Lemma 1.21]) that $X_\infty$ splits a factor $\mathbb{R}^k$, as we claimed. □

**Corollary 1.84.** *Let $N > 1$ and $K \in \mathbb{R}$ be fixed. For any $\varepsilon > 0$ there exists $\delta > 0$ such that, for any $r > 0$, for any $\mathrm{RCD}(K,N)$ m.m.s. $(X, \mathsf{d}, \mathfrak{m})$ and for any $x \in X$, if there exists $u : B_r(x) \to \mathbb{R}^k$ such that $u : B_s(x) \to \mathbb{R}^k$ is a $\delta$-splitting map for any $0 < s < r$, then for any $(Y, \varrho, \mu, y) \in \mathrm{Tan}_x(X, \mathsf{d}, \mathfrak{m})$ there exists an $\mathrm{RCD}(0, N-k)$ p.m.m.s. $(Z, \mathsf{d}_Z, \mathfrak{m}_Z, z)$ such that*

$$\mathsf{d}_{pmGH}\left((Y, \varrho, \mu, y), (Z \times \mathbb{R}^k, (z, 0^k))\right) < \varepsilon.$$

*Proof.* Choose $\delta = \delta(K, N, \varepsilon/2)$ given by Proposition 1.83. If $(Y, \varrho, \mu, y) \in \mathrm{Tan}_x(X, \mathsf{d}, \mathfrak{m})$ then there exists $t > 0$ such that $t^{-1}r > \delta^{-1}$, $t^2 |K| \le \delta$ and

$$\mathsf{d}_{pmGH}\left((X, t^{-1}\mathsf{d}, \mathfrak{m}_x^t, x), (Y, \varrho, \mu, y)\right) < \varepsilon/2. \tag{1.52}$$

Thanks to Proposition 1.83, applied to $(X, t^{-1}\mathsf{d}, \mathfrak{m}_x^t, x)$, there exists an $\mathrm{RCD}(0, N-k)$ p.m.m.s. $(Z, \mathsf{d}_Z, \mathfrak{m}_Z, z)$ such that

$$\mathsf{d}_{pmGH}\left((X, t^{-1}\mathsf{d}, \mathfrak{m}_x^t, x), (Z \times \mathbb{R}^k, (z, 0^k))\right) < \varepsilon/2. \tag{1.53}$$

The conclusion follows from (1.52) and (1.53) by the triangle inequality. □

**Proposition 1.85.** *Let $N > 1$ be fixed. For any $\delta > 0$ there exists $\varepsilon = \varepsilon_{N,\delta} > 0$ such that, if $(X, \mathsf{d}, \mathfrak{m})$ is an $\mathrm{RCD}(-\varepsilon, N)$ m.m.s., $x \in X$ and*

$$\mathsf{d}_{pmGH}\left((X, \mathsf{d}, \mathfrak{m}, x), \left(\mathbb{R}^k \times Z, (0^k, z)\right)\right) < \varepsilon$$

*for some pointed $\mathrm{RCD}(0, N-k)$ metric measure space $(Z, \mathsf{d}_Z, \mathfrak{m}_Z, z)$, then there exists a $\delta$-splitting map $u : B_5(x) \to \mathbb{R}^k$.*

*Proof.* Suppose the conclusion to be false, then we could find a sequence $(X_n, \mathsf{d}_n, \mathfrak{m}_n, x_n)$ of pointed $\mathrm{RCD}(-1/n, N)$ m.m. spaces such that, for some $\mathrm{RCD}(0, N-k)$ pointed m.m.s. $(Z, \mathsf{d}_Z, \mathfrak{m}_Z, z)$ it holds that

$$\mathsf{d}_{pmGH}\left((X_n, \mathsf{d}_n, \mathfrak{m}_n, x_n), \left(\mathbb{R}^k \times Z, (0^k, z)\right)\right) < 1/n$$

for any $n \ge 1$. Furthermore there should be $\delta_0 > 0$ such that there is no $\delta_0$-splitting map over $B_5(x_n)$ for any $n \ge 1$.

Let $v : Z \times \mathbb{R}^k \to \mathbb{R}^k$ be defined by $v(p, x) = x$ and denote by $v^1, \dots, v^k$ its components (they are the coordinate functions of the split factor). Observe that $\Delta v^i = 0$ for any $i = 1, \dots, k$ and $\nabla v^i \cdot \nabla v^j = \delta_{ij}$ for any $i, j = 1, \dots, k$. In particular, $v^i$ is harmonic on $B_{10}((z, 0^k))$. Hence we can apply Proposition 1.48 to get harmonic functions $v_n^i : B_9(x_n) \to \mathbb{R}$ that converge strongly in $H^{1,2}$ to $v^i$ on $B_9((z, 0^k))$.

Observe that, thanks to [**105**, Theorem 1.1], we can assume that $v_n^i$ is $C_N$-Lipschitz for any $n \in \mathbb{N}$ and for any $i = 1, \dots, k$. We wish to prove that $v_n = (v_n^1, \dots, v_n^k)$ is a $\delta_0$-splitting map on $B_5(x_n)$ for $n$ sufficiently big.

To this aim let us recall that Theorem 1.47 yields strong $L^1$-convergence of $\nabla v_n^i \cdot \nabla v_n^j$ to $\delta_{ij}$ on $B_9((z, 0^k))$ and on $B_5((z, 0^k))$ for any $i, j = 1, \dots, k$ (as a consequence of the $L^1$

convergence of $\nabla v_n^i \cdot \nabla v_n^i$ and of $\nabla(v_n^i + v_n^j) \cdot \nabla(v_n^i + v_n^j)$). In particular, due to the uniform boundedness of the gradients we obtained above, we get

$$\lim_{n\to\infty} \fint_{B_R(x_n)} \left|\nabla v_n^i \cdot \nabla v_n^j - \delta_{ij}\right| \mathrm{d}\mathfrak{m}_n = 0,$$

for any $i, j = 1, \dots, k$ and for any $R = 5, 9$. The choice $R = 5$ gives that the second defining condition of $\delta$-splitting map is satisfied for $n$ sufficiently large and we are left with the verification of the third one. We wish to prove that

$$\lim_{n\to\infty} \int_{B_5(x_n)} \left|\mathrm{Hess}\, v_n^i\right|^2 \mathrm{d}\mathfrak{m}_n = 0$$

for any $i = 1, \dots, k$. To this aim we choose cut-off functions $\eta_n$ for the pairs $B_5(x_n) \subset B_9(x_n)$ as in Lemma 1.60 and, taking into account (1.36)

$$\int_{B_9(x_n)} \Delta\eta_n \left(\left|\nabla v_n^i\right|^2 - 1\right) \mathrm{d}\mathfrak{m}_n + C_N \frac{\mathfrak{m}_n(B_9(x_n))}{n} \geq \int_{B_5(x_n)} \left|\mathrm{Hess}\, v_n^i\right|^2 \mathrm{d}\mathfrak{m}_n \tag{1.54}$$

for any $i = 1, \dots, k$ and for any $n \geq 1$. Since, $|\Delta\eta_n| \leq C_N$ by construction and as we already observed, $|\nabla v_n^i|^2 - 1$ converge to 0 in $L^1(B_9)$ and they are uniformly bounded, we get that the left-hand side in (1.54) converges to 0 as $n \to \infty$. Hence

$$\lim_{n\to\infty} \fint_{B_5(x_n)} \left|\mathrm{Hess}\, v_n^i\right|^2 \mathrm{d}\mathfrak{m}_n = 0,$$

as we claimed. □

Arguing by scaling starting from Proposition 1.85, it is possible to obtain the following statement.

**Corollary 1.86.** *If* $(X, \mathsf{d}, \mathfrak{m})$ *is an* $\mathrm{RCD}(K, N)$ *m.m.s.,* $r^2 |K| \leq \varepsilon$ *and*

$$\mathsf{d}_{pmGH}\left(\left(X, r^{-1}\mathsf{d}, \mathfrak{m}_x^r, x\right), \left(\mathbb{R}^k \times Z, (0^k, z)\right)\right) < \varepsilon$$

*for some pointed* $\mathrm{RCD}(0, N-k)$ *metric measure space* $(Z, \mathsf{d}_Z, \mathfrak{m}_Z, z)$*, then there exists a* $\delta$*-splitting map* $u : B_{5r}(x) \to \mathbb{R}^k$*.*

CHAPTER 2

# Structure of RCD spaces: rectifiability

In the last ten years many efforts have been aimed at understanding the structure of $\mathrm{RCD}(K,N)$ spaces. After [**121**] by Mondino-Naber, we know that they are rectifiable as metric spaces and later, in the three independent works by De Philippis-Marchese-Rindler, Kell-Mondino and Gigli-Pasqualetto [**72, 95, 109**], the analysis was sharpened taking into account the behaviour of the reference measure and getting rectifiability as metric measure spaces. The development of this theory was inspired in turn by the results obtained for Ricci limit spaces in the seminal papers by Cheeger-Colding [**51**–**53**]. In this chapter we give an overview of these rectifiability results following the simple approach proposed in [**40**].

**Definition 2.1.** Given an $\mathrm{RCD}(K,N)$ space $(X,\mathsf{d},\mathfrak{m})$, a point $x\in X$ and a radius $r\in(0,1)$, we define the normalised measure $\mathfrak{m}_r^x$ on $X$ as

$$\mathfrak{m}_r^x := \frac{\mathfrak{m}}{C(x,r)}, \qquad \text{where} \qquad C(x,r) := \int_{B_r(x)} 1-\frac{\mathsf{d}(\cdot,x)}{r}\,\mathrm{d}\mathfrak{m}.$$

The *tangent cone* $\mathrm{Tan}_x(X,\mathsf{d},\mathfrak{m})$ is defined as the family of all those spaces $(Y,\varrho,\mathfrak{n},y)$ such that

$$\lim_{n\to\infty} \mathsf{d}_{\mathrm{pmGH}}\Big((X,\mathsf{d}/r_n,\mathfrak{m}_{r_n}^x,x),(Y,\varrho,\mathfrak{n},y)\Big)=0$$

for some sequence $(r_n)_n\subseteq(0,1)$ of radii with $r_n\searrow 0$.

We recall the following scaling property: if $(X,\mathsf{d},\mathfrak{m})$ is an $\mathrm{RCD}(K,N)$ space, then $(X,\mathsf{d}/r,\lambda\mathfrak{m})$ is an $\mathrm{RCD}(r^2K,N)$ space for any choice of $r,\lambda>0$. Hence, it follows from the compactness and stability property of RCD spaces (see Theorem 1.32, Theorem 1.30) that $\mathrm{Tan}_x(X,\mathsf{d},\mathfrak{m})\neq\emptyset$ and any element is a pointed $\mathrm{RCD}(0,N)$ space.

## 1. Existence of Euclidean tangent cones

In this section we briefly outline the argument presented in [**91**] leading to the following existence result.

**Theorem 2.2** (Euclidean tangents to RCD spaces)**.** *Let $(X,\mathsf{d},\mathfrak{m})$ be an $\mathrm{RCD}(K,N)$ space. Then for $\mathfrak{m}$-a.e. point $x\in X$ there exists $k\in\mathbb{N}$ with $k\le N$ such that*

$$\Big(\mathbb{R}^k,\mathsf{d}_{\mathrm{Eucl}},c_k\mathscr{L}^k,0^k\Big)\in\mathrm{Tan}_x(X,\mathsf{d},\mathfrak{m}),$$

*where we set $c_k:=\mathscr{L}^k\Big(B_1(0^k)\Big)/(k+1)$ for every $k\in\mathbb{N}$.*

The proof builds upon two ingredients. The first one (see [**91**, Lemma 3.1]) is a result ensuring that for $\mathfrak{m}$-a.e. $x\in X$ there exists a geodesic $\gamma\in\mathrm{Geo}(X)$ such that $x=\gamma(1/2)$, provided $X$ is not a singleton.

The second ingredient is a version of the iterated tangent theorem by Preiss [**128**], we refer to [**91**, Theorem 3.5] for its proof.

**Theorem 2.3** (Iterated tangent property). *Let $(X, \mathsf{d}, \mathfrak{m})$ be an* $\mathrm{RCD}(K, N)$ *space. Then for* $\mathfrak{m}$*-a.e. point $x \in X$ it holds that*

$$\mathrm{Tan}_z(Y, \varrho, \mathfrak{n}) \subseteq \mathrm{Tan}_x(X, \mathsf{d}, \mathfrak{m}) \quad \textit{for every } (Y, \varrho, \mathfrak{n}, y) \in \mathrm{Tan}_x(X, \mathsf{d}, \mathfrak{m}) \textit{ and } z \in Y.$$

Let us briefly explain how the combination of these ingredients leads to the proof of Theorem 2.2. First we recall that given any point $x \in X$ and any $k \in \mathbb{N}$, we say that an element $(Y, \varrho, \mathfrak{n}, y) \in \mathrm{Tan}_x(X, \mathsf{d}, \mathfrak{m})$ *splits off a factor* $\mathbb{R}^k$ provided

$$(Y, \varrho, \mathfrak{n}, y) \cong \Big(\mathbb{R}^k \times Z, \mathsf{d}_{\mathrm{Eucl}} \times \mathsf{d}_Z, \mathscr{L}^k \otimes \mathfrak{m}_Z, (0^k, z)\Big)$$

for some pointed $\mathrm{RCD}(0, N-k)$ space $(Z, \mathsf{d}_Z, \mathfrak{m}_Z, z)$.

The first observation is that at any $x \in X$ such that $\gamma(1/2) = x$ for some $\gamma \in \mathrm{Geo}(X)$, every element of the tangent cone $(Y, \varrho, \mathfrak{n}, y) \in \mathrm{Tan}_x(X, \mathsf{d}, \mathfrak{m})$ contains a line, and therefore splits off a factor $\mathbb{R}$ (see Section 7 for the statement of the splitting theorem). This can be checked as follows. Given $(Y, \varrho, \mathfrak{n}, y) \in \mathrm{Tan}_x(X, \mathsf{d}, \mathfrak{m})$ we consider a sequence of radii $r_n \downarrow 0$ such that

$$\lim_{n \to \infty} \mathsf{d}_{\mathrm{pmGH}}\Big((X, \mathsf{d}/r_n, \mathfrak{m}^x_{r_n}, x), (Y, \varrho, \mathfrak{n}, y)\Big) = 0. \tag{2.1}$$

Let $\gamma_n : [-\frac{1}{2r_n}, \frac{1}{2r_n}] \to (X, \mathsf{d}/r_n)$ be defined as $\gamma_n(t) := \gamma(r_n t + 1/2)$. Notice that $\gamma_n$ is a constant speed geodesic in $(X, \mathsf{d}/r_n)$ with velocity independent of $n$, and satisfying $\gamma_n(0) = x$. Let $(Z, \mathsf{d}_Z)$ be a metric space realizing (2.1). Thanks Proposition 1.34 it is not difficult to show that $\gamma_n \to \gamma_\infty$ locally uniformly in $(Z, \mathsf{d}_Z)$, where $\gamma_\infty : \mathbb{R} \to (Y, \varrho)$ is a line.

In particular we have shown that, at $\mathfrak{m}$-a.e. $x \in X$ each tangent space splits of an Euclidean factor. Now the sought conclusion follows by iterating this procedure exploiting Theorem 2.3. We refer to [**91**] for details.

## 2. Uniqueness of tangent cones

Let $(X, \mathsf{d}, \mathfrak{m})$ be an $\mathrm{RCD}(K, N)$ space. Then we define

$$\mathcal{R}_k := \left\{ x \in X \;\middle|\; \mathrm{Tan}_x(X, \mathsf{d}, \mathfrak{m}) = \left\{(\mathbb{R}^k, \mathsf{d}_{\mathrm{Eucl}}, c_k \mathscr{L}^k, 0^k)\right\} \right\} \quad \forall\, k \in \mathbb{N},\ k \le N.$$

The elements of $\mathcal{R}_k$ are said to be the *$k$-regular points* in $X$.

The main result of this section is the following.

**Theorem 2.4** (Uniqueness of tangents). *Let $(X, \mathsf{d}, \mathfrak{m})$ be an* $\mathrm{RCD}(K, N)$ *space. Then it holds*

$$\mathfrak{m}\Bigg(X \setminus \bigcup_{k \le N} \mathcal{R}_k\Bigg) = 0.$$

*In other words $\mathfrak{m}$-a.e. $x \in X$ is a regular point.*

The proof of Theorem 2.4 heavily relies on analytical and geometrical properties of $\delta$-splitting maps discussed in Section 7.1. The crucial technical ingredient is the following observation: the property of being $\delta$-splitting at some scale and location automatically enforces to many other scales and locations. This can be proven by exploiting a maximal argument, see Proposition 2.5 below.

**Proposition 2.5** (Propagation of the $\delta$-splitting property). *Let $N > 1$ be given. Then there exists a constant $C_N > 0$ such that the following property holds. If $(X, \mathsf{d}, \mathfrak{m})$ is an* $\mathrm{RCD}(K, N)$ *space and $u \colon B_{2r}(p) \to \mathbb{R}^k$ is a $\delta$-splitting map for some $p \in X$, $r > 0$ with $r^2|K| \le 1$, and*

$\delta \in (0,1)$, *then there exists a Borel set* $G \subseteq B_r(p)$ *such that* $\mathfrak{m}\Big(B_r(p)\setminus G\Big) \leq C_N\sqrt{\delta}\,\mathfrak{m}\Big(B_r(p)\Big)$ *and*

$$u\colon B_s(x) \to \mathbb{R}^k \text{ is a } \sqrt{\delta}\text{-splitting map}, \quad \text{for every } x \in G \text{ and } s \in (0,r).$$

*Proof.* Thanks to a scaling argument, it is sufficient to prove the claim in the case in which $r = 1$ and $|K| \leq 1$. Let us define $G \subseteq B_1(p)$ as $G := \bigcap_{a=1}^k G_a \cap \bigcap_{a,b=1}^k G_{a,b}$, where we set

$$G_a := \left\{ x \in B_1(p) \;\middle|\; \sup_{s\in(0,1)} ⨍_{B_s(x)} \Big|\mathrm{Hess}(u_a)\Big|^2 \,\mathrm{d}\mathfrak{m} \leq \sqrt{\delta} \right\},$$

$$G_{a,b} := \left\{ x \in B_1(p) \;\middle|\; \sup_{s\in(0,1)} ⨍_{B_s(x)} \Big|\nabla_{\mathfrak{m}} u_a \cdot \nabla_{\mathfrak{m}} u_b - \delta_{ab}\Big|^2 \,\mathrm{d}\mathfrak{m} \leq \sqrt{\delta} \right\}.$$

It holds that $u\colon B_s(x) \to \mathbb{R}^k$ is a $\sqrt{\delta}$-splitting map for all $x \in G$ and $s \in (0,1)$. To prove the claim, it remains to show that $\mathfrak{m}\Big(B_1(p)\setminus G_a\Big), \mathfrak{m}\Big(B_1(p)\setminus G_{a,b}\Big) \leq C_N\sqrt{\delta}\,\mathfrak{m}\Big(B_1(p)\Big)$ for all $a, b = 1, \dots, k$.

Given any $x \in B_1(p) \setminus G_a$, we can choose $s_x \in (0,1)$ such that

$$⨍_{B_{s_x}(x)} \Big|\mathrm{Hess}(u_a)\Big|^2 \,\mathrm{d}\mathfrak{m} > \sqrt{\delta}.$$

By using Vitali covering lemma, we can find a sequence $(x_i)_i \subseteq B_1(p) \setminus G_a$ such that $\Big\{B_{s_{x_i}}(x_i)\Big\}_i$ are pairwise disjoint and $B_1(p)\setminus G_a \subseteq \bigcup_i B_{5s_{x_i}}(x_i)$. Therefore, it holds that

$$\begin{aligned}
\mathfrak{m}\Big(B_1(p)\setminus G_a\Big) &\leq \sum_{i\in\mathbb{N}} \mathfrak{m}\Big(B_{5s_{x_i}}(x_i)\Big) \\
&\leq C_N \sum_{i\in\mathbb{N}} \mathfrak{m}\Big(B_{s_{x_i}}(x_i)\Big) \leq \frac{C_N}{\sqrt{\delta}} \sum_{i\in\mathbb{N}} \int_{B_{s_{x_i}}(x_i)} \Big|\mathrm{Hess}(u_a)\Big|^2 \,\mathrm{d}\mathfrak{m} \\
&\leq \frac{C_N \mathfrak{m}\Big(B_2(p)\Big)}{\sqrt{\delta}} ⨍_{B_2(p)} \Big|\mathrm{Hess}(u_a)\Big|^2 \,\mathrm{d}\mathfrak{m} \leq C_N\sqrt{\delta}\,\mathfrak{m}\Big(B_1(p)\Big),
\end{aligned}$$

where we used the doubling property of $\mathfrak{m}$, the defining property of $s_{x_i}$, and the fact that $u$ is a $\delta$-splitting map on $B_2(p)$. An analogous argument shows that

$$\mathfrak{m}\Big(B_1(p)\setminus G_{a,b}\Big) \leq C_N\sqrt{\delta}\,\mathfrak{m}\Big(B_1(p)\Big)$$

for all $a, b = 1, \dots, k$, thus the statement is achieved. $\square$

**2.1. Proof of Theorem 2.4.** The proof is divided in three steps.

**Step 1.** Fix any $k \in \mathbb{N}$ with $k \leq N$. We define the auxiliary sets $A_k, A_k' \subseteq X$ as follows:

i) $A_k$ is the collection of all points $x \in X$ such that

$$(\mathbb{R}^k, \mathsf{d}_{\mathrm{Eucl}}, c_k\mathscr{L}^k, 0^k) \in \mathrm{Tan}_x(X,\mathsf{d},\mathfrak{m}),$$

but no other element of $\mathrm{Tan}_x(X,\mathsf{d},\mathfrak{m})$ splits off a factor $\mathbb{R}^k$.

ii) $A_k'$ is the family of all points $x \in X$ which satisfy

$$(\mathbb{R}^k, \mathsf{d}_{\mathrm{Eucl}}, c_k\mathscr{L}^k, 0^k) \in \mathrm{Tan}_x(X,\mathsf{d},\mathfrak{m})$$

and $(\mathbb{R}^\ell, \mathsf{d}_{\mathrm{Eucl}}, c_\ell\mathscr{L}^\ell, 0^\ell) \notin \mathrm{Tan}_x(X,\mathsf{d},\mathfrak{m})$ for every $\ell \in \mathbb{N}$ with $\ell > k$.

Observe that $\mathcal{R}_k \subseteq A_k \subseteq A'_k$. The $\mathfrak{m}$-measurability of the sets $\mathcal{R}_k, A_k, A'_k$ can be proven by adapting the proof of [**121**, Lemma 6.1]. It also follows from Theorem 2.2 that

$$\mathfrak{m}\Big(X \setminus \bigcup_{k\le N} A'_k\Big) = 0.$$

**Step 2.** We aim to prove that $\mathfrak{m}(A'_k \setminus A_k) = 0$. We argue by contradiction: suppose $\mathfrak{m}(A'_k \setminus A_k) > 0$. Then we can find a point $x \in A'_k \setminus A_k$ where the iterated tangent property of Theorem 2.3 holds. Since $x \notin A_k$, there exists a pointed RCD$(0, N-k)$ space $(Y, \varrho, \mathfrak{n}, y)$ with $\mathrm{diam}(Y) > 0$ such that

$$\Big(\mathbb{R}^k \times Y, \mathsf{d}_{\mathrm{Eucl}} \times \varrho, \mathscr{L}^k \otimes \mathfrak{n}, (0^k, y)\Big) \in \mathrm{Tan}_x(X, \mathsf{d}, \mathfrak{m}).$$

Theorem 2.2 yields the existence of $z \in Y$ such that $(\mathbb{R}^\ell, \mathsf{d}_{\mathrm{Eucl}}, c_\ell \mathscr{L}^\ell, 0^\ell) \in \mathrm{Tan}_z(Y, \varrho, \mathfrak{n})$, for some $\ell \in \mathbb{N}$ with $0 < \ell \le N-k$. This implies that

$$(\mathbb{R}^{k+\ell}, \mathsf{d}_{\mathrm{Eucl}}, c_{k+\ell}\mathscr{L}^{k+\ell}, 0^{k+\ell}) \in \mathrm{Tan}_{(0^k,z)}(\mathbb{R}^k \times Y, \mathsf{d}_{\mathrm{Eucl}} \times \varrho, \mathscr{L}^k \otimes \mathfrak{n}).$$

Therefore, Theorem 2.3 guarantees that

$$(\mathbb{R}^{k+\ell}, \mathsf{d}_{\mathrm{Eucl}}, c_{k+\ell}\mathscr{L}^{k+\ell}, 0^{k+\ell}) \in \mathrm{Tan}_x(X, \mathsf{d}, \mathfrak{m}),$$

which contradicts the fact that $x \in A'_k$. Consequently, we have proven that $\mathfrak{m}(A'_k \setminus A_k) = 0$, as desired.

**Step 3.** In order to complete the proof of the statement, it suffices to show that

$$\mathfrak{m}\Big(B_R(p) \cap (A_k \setminus \mathcal{R}_k)\Big) = 0 \qquad \text{for every } p \in X \text{ and } R > 0. \tag{2.2}$$

Let $p \in X$ and $R, \eta > 0$ be fixed. Choose any $\delta \in (0, \eta)$ associated with $\eta$ as in Corollary 1.84. Moreover, choose any $\varepsilon \in (0, 1/7)$ associated with $\delta^2$ as in Corollary 1.86. Given a point $x \in A_k$, we can find $r_x \in (0,1)$ such that $4r_x^2|K| \le \varepsilon$ and

$$\mathsf{d}_{\mathrm{pmGH}}\Big(\Big(X, \mathsf{d}/(2r_x), \mathfrak{m}^x_{2r_x}, x\Big), (\mathbb{R}^k, \mathsf{d}_{\mathrm{Eucl}}, c_k\mathscr{L}^k, 0^k)\Big) \le \varepsilon.$$

By applying Vitali covering lemma to the family $\Big\{B_{r_x}(x) \,:\, x \in A_k \cap B_R(p)\Big\}$, we obtain a sequence $(x_i)_i \subseteq A_k \cap B_R(p)$ such that $\Big\{B_{r_{x_i}}(x_i)\Big\}_i$ are pairwise disjoint and $A_k \cap B_R(p) \subseteq \bigcup_i B_{5r_{x_i}}(x_i)$. For any $i \in \mathbb{N}$, we know from Corollary 1.86 that there exists a $\delta^2$-splitting map $u^i \colon B_{10r_{x_i}}(x_i) \to \mathbb{R}^k$. Furthermore, Proposition 2.5 guarantees the existence of a Borel set $G^i_\eta \subseteq B_{5r_{x_i}}(x_i)$ such that $\mathfrak{m}\Big(B_{5r_{x_i}}(x_i) \setminus G^i_\eta\Big) \le C_N \delta\, \mathfrak{m}\Big(B_{5r_{x_i}}(x_i)\Big)$ and $u^i \colon B_s(x) \to \mathbb{R}^k$ is a $\delta$-splitting map for every $x \in G^i_\eta$ and $s \in (0, 5r_{x_i})$. Hence, Corollary 1.84 guarantees that for any $x \in G^i_\eta$ the following property holds:

$$\begin{aligned}&\text{Given any element } (Y, \varrho, \mathfrak{n}, y) \in \mathrm{Tan}_x(X, \mathsf{d}, \mathfrak{m}), \text{ there exists}\\ &\text{a pointed RCD}(0, N-k) \text{ space } (Z, \mathsf{d}_Z, \mathfrak{m}_Z, z) \text{ such that}\\ &\mathsf{d}_{\mathrm{pmGH}}\Big((Y, \varrho, \mathfrak{n}, y), \Big(\mathbb{R}^k \times Z, \mathsf{d}_{\mathrm{Eucl}} \times \mathsf{d}_Z, \mathscr{L}^k \otimes \mathfrak{m}_Z, (0^k, z)\Big)\Big) \le \eta.\end{aligned} \tag{2.3}$$

Then let us define $G_\eta \coloneqq \bigcup_i G^i_\eta$. Clearly, each element of $G_\eta$ satisfies (2.3). Moreover, it holds

$$\begin{aligned}\mathfrak{m}\Big(B_R(p)\cap(A_k\setminus G_\eta)\Big) &\leq \sum_{i\in\mathbb{N}}\mathfrak{m}\Big(B_{5r_{x_i}}(x_i)\setminus G^i_\eta\Big)\leq C_N\delta\sum_{i\in\mathbb{N}}\mathfrak{m}\Big(B_{5r_{x_i}}(x_i)\Big)\\ &\leq C_N\eta\sum_{i\in\mathbb{N}}\mathfrak{m}\Big(B_{r_{x_i}}(x_i)\Big)\leq C_N\eta\,\mathfrak{m}\Big(B_{R+1}(p)\Big).\end{aligned}\tag{2.4}$$

Now consider the Borel set

$$G \coloneqq \bigcap_i\bigcup_j G_{1/2^{i+j}}.$$

It follows from (2.4) that $\mathfrak{m}\Big(B_R(p)\cap(A_k\setminus G)\Big)=0$.

Moreover, let $x\in A_k\cap G$ and $(Y,\varrho,\mathfrak{n},y)\in\mathrm{Tan}_x(X,\mathsf{d},\mathfrak{m})$ be fixed. Then by using (2.3) we can find a sequence $\Big\{(Z_i,\mathsf{d}_{Z_i},\mathfrak{m}_{Z_i},z_i)\Big\}_i$ of pointed $\mathrm{RCD}(0,N-k)$ spaces such that

$$\Big(\mathbb{R}^k\times Z_i,\mathsf{d}_{\mathrm{Eucl}}\times\mathsf{d}_{Z_i},\mathscr{L}^k\otimes\mathfrak{m}_{Z_i},(0^k,z_i)\Big)\overset{\mathrm{pmGH}}{\longrightarrow}(Y,\varrho,\mathfrak{n},y)\quad\text{ as } i\to\infty.\tag{2.5}$$

Up to a not relabelled subsequence, we can suppose that

$$(Z_i,\mathsf{d}_{Z_i},\mathfrak{m}_{Z_i},z_i)\to(Z,\mathsf{d}_Z,\mathfrak{m}_Z,z)\quad\text{in the pmGH-topology}$$

for some pointed $\mathrm{RCD}(0,N-k)$ space $(Z,\mathsf{d}_Z,\mathfrak{m}_Z,z)$. Consequently, (2.5) ensures that $(Y,\varrho,\mathfrak{n},y)$ is isomorphic to $\Big(\mathbb{R}^k\times Z,\mathsf{d}_{\mathrm{Eucl}}\times\mathsf{d}_Z,\mathscr{L}^k\otimes\mathfrak{m}_Z,(0^k,z)\Big)$. Given that $x\in A_k$, we deduce that $Z$ must be a singleton. In other words, we have proven that any element of $\mathrm{Tan}_x(X,\mathsf{d},\mathfrak{m})$ is isomorphic to $(\mathbb{R}^k,\mathsf{d}_{\mathrm{Eucl}},c_k\mathscr{L}^k,0^k)$, so that $x\in\mathcal{R}_k$. This shows that $A_k\cap G\subseteq\mathcal{R}_k$, whence the claim (2.2) follows. The statement is finally achieved.

**2.2. Lower semi-continuity of the dimension.** By combining Theorem 2.4 with the properties of $\delta$-splitting maps discussed in Section 7.1, we can give a direct proof of the following result, that was proved for the first time in [**110**]:

**Theorem 2.6.** *Let $(X,\mathsf{d},\mathfrak{m})$ be an $\mathrm{RCD}(K,N)$ space. Let $k\in\mathbb{N}$, $k\leq N$ be the maximal number such that $\mathfrak{m}(\mathcal{R}_k)>0$. Then for any $x\in X$ and $\ell>k$ we have that no element of $\mathrm{Tan}_x(X,\mathsf{d},\mathfrak{m})$ splits off a factor $\mathbb{R}^\ell$. In particular, it holds that $\mathcal{R}_\ell=\emptyset$ for every $\ell>k$.*

*Proof.* First of all, we claim that for any given $\ell>k$ there exists $\varepsilon>0$ such that

$$\mathsf{d}_{\mathrm{pmGH}}\bigg((\mathbb{R}^j,\mathsf{d}_{\mathrm{Eucl}},c_j\mathscr{L}^j,0^j),\Big(\mathbb{R}^\ell\times Z,\mathsf{d}_{\mathrm{Eucl}}\times\mathsf{d}_Z,\mathscr{L}^\ell\otimes\mathfrak{m}_Z,(0^\ell,z)\Big)\bigg)>\varepsilon\tag{2.6}$$

for every $j\leq k$ and for every pointed $\mathrm{RCD}(0,N-\ell)$ space $(Z,\mathsf{d}_Z,\mathfrak{m}_Z,z)$. This can be easily checked arguing by contradiction.

We prove the main statement by contradiction: suppose there exist $x\in X$ and $\ell>k$ such that

$$\Big(\mathbb{R}^\ell\times Z,\mathsf{d}_{\mathrm{Eucl}}\times\mathsf{d}_Z,\mathscr{L}^\ell\otimes\mathfrak{m}_Z,(0^\ell,z)\Big)\in\mathrm{Tan}_x(X,\mathsf{d},\mathfrak{m})\tag{2.7}$$

for some pointed $\mathrm{RCD}(0,N-\ell)$ space $(Z,\mathsf{d}_Z,\mathfrak{m}_Z,z)$. Consider $\varepsilon>0$ associated with $\ell$ as in (2.6). Choose $\delta>0$ associated with $\varepsilon$ as in Corollary 1.84, then $\eta>0$ associated with $\delta^2$ as in Corollary 1.86. It follows from (2.7) that there is $r>0$ such that $r^2|K|\leq\eta$ and

$$\mathsf{d}_{\mathrm{pmGH}}\bigg((X,\mathsf{d}/r,\mathfrak{m}^x_r,x),\Big(\mathbb{R}^\ell\times Z,\mathsf{d}_{\mathrm{Eucl}}\times\mathsf{d}_Z,\mathscr{L}^\ell\otimes\mathfrak{m}_Z,(0^\ell,z)\Big)\bigg)\leq\eta.$$

Then Corollary 1.86 guarantees the existence of a $\delta^2$-splitting map $u\colon B_{5r}(x) \to \mathbb{R}^\ell$. Therefore, by Propositions 2.5 and Corollary 1.84 we know that there exists a Borel set $G \subseteq B_r(x)$ with $\mathfrak{m}(G) > 0$ satisfying the following property: for any point $y \in G$, it holds that each element of $\mathrm{Tan}_y(X, \mathsf{d}, \mathfrak{m})$ is $\varepsilon$-close (with respect to the distance $\mathsf{d}_{\mathrm{pmGH}}$) to some space that splits off a factor $\mathbb{R}^\ell$. Given that $X \setminus (\mathcal{R}_1 \cup \cdots \cup \mathcal{R}_k)$ has null $\mathfrak{m}$-measure by Theorem 2.4, there must exist $y \in G$ and $j \leq k$ for which $(\mathbb{R}^j, \mathsf{d}_{\mathrm{Eucl}}, c_j \mathscr{L}^j, 0^j)$ is the only element of $\mathrm{Tan}_y(X, \mathsf{d}, \mathfrak{m})$. Consequently, we have that

$$\mathsf{d}_{\mathrm{pmGH}}\Big((\mathbb{R}^j, \mathsf{d}_{\mathrm{Eucl}}, c_j \mathscr{L}^j, 0^j), \Big(\mathbb{R}^\ell \times Z', \mathsf{d}_{\mathrm{Eucl}} \times \mathsf{d}_{Z'}, \mathscr{L}^\ell \otimes \mathfrak{m}_{Z'}, (0^\ell, z')\Big)\Big) \leq \varepsilon$$

for some pointed $\mathrm{RCD}(0, N-\ell)$ space $(Z', \mathsf{d}_{Z'}, \mathfrak{m}_{Z'}, z')$. This is in contradiction with (2.6). □

Actually, Theorem 2.6 above is an instance of a more general result that can be proved arguing in a similar manner: the essential dimension of $\mathrm{RCD}(K, N)$ spaces is lower semicontinuous with respect to pointed measured Gromov-Hausdorff convergence. This statement has been proved for the first time in [**110**, Theorem 4.10]. Below we just sketch how our techniques can provide a slightly more direct proof, still based on the same ideas and on the theory of convergence of Sobolev functions on varying spaces developed in [**19**, **20**].

**Definition 2.7.** The essential dimension $\dim(X, \mathsf{d}, \mathfrak{m})$ of an $\mathrm{RCD}(K, N)$ metric measure space is the maximal $n \in \mathbb{N}$ such that $\mathfrak{m}(\mathcal{R}_n) > 0$.

As we will se in Chapter 3 the essential dimension can be characterized as the unique $n \in \mathbb{N}$ such that $\mathfrak{m}(\mathcal{R}_n) \neq 0$.

**Theorem 2.8.** *Let $(X_n, \mathsf{d}_n, \mathfrak{m}_n, x_n)$ and $(X, \mathsf{d}, \mathfrak{m}, x)$ be pointed* $\mathrm{RCD}(K, N)$ *metric measure spaces and assume that $(X_n, \mathsf{d}_n, \mathfrak{m}_n, x_n)$ converge to $(X, \mathsf{d}, \mathfrak{m}, x)$ in the pointed measured Gromov-Hausdorff sense. Then*

$$\dim(X, \mathsf{d}, \mathfrak{m}) \leq \liminf_{n \to \infty} \dim(X_n, \mathsf{d}_n, \mathfrak{m}_n).$$

*Proof.* Let $k := \dim(X, \mathsf{d}, \mathfrak{m})$. We need to prove that, for $n$ sufficiently large, it holds $k \leq \dim(X_n, \mathsf{d}_n, \mathfrak{m}_n)$.

Up to scaling of the distance $\mathsf{d}$ on $X$, we can assume that $K \geq -1$ and by Corollary 1.86 we find $y \in X$ and a $\delta$-splitting map $u : B_2(y) \to \mathbb{R}^k$. Arguing as in the proof of Proposition 1.85, relying on the convergence and stability results of Section 4.2, we can find $1 < r < 2$, points $X_n \ni y_n \to y \in X$ and $2\delta$-splitting maps $u_n : B_r(y_n) \to \mathbb{R}^k$, for any $n$ sufficiently large (it suffices to approximate the components of $u$ in the strong $H^{1,2}$-sense with harmonic functions).

Next, Proposition 2.5 provides sets $G_n \subset B_{r/2}(y_n)$ such that

$$\mathfrak{m}_n(B_r(y_n) \setminus G_n) \leq C_N \sqrt{2\delta}\, \mathfrak{m}_n(B_{r/2}(y_n))$$

and

$$u_n\colon B_s(x) \to \mathbb{R}^k \text{ is a } \sqrt{2\delta}\text{-splitting map, for every } x \in G_n \text{ and } s \in (0, r/2),$$

for any $n$ sufficiently large.

Now it suffices to choose $\delta$ such that $\sqrt{2\delta} \leq \delta_\varepsilon$ given by Corollary 1.84 to get that, at any point in $G_n$, any tangent is $\varepsilon$-close to a space splitting a factor $\mathbb{R}^k$. Choosing $\varepsilon$ small enough and arguing as in the proof of Theorem 2.6 above we obtain that $\dim(X_n, \mathsf{d}_n, \mathfrak{m}_n) \geq k$ for sufficiently large $n$. □

## 3. Metric rectifiability of RCD spaces

Aim of this section is to exploit $\delta$-splitting maps to show that $\mathrm{RCD}(K,N)$ spaces are metrically rectifiable, in the following sense.

**Definition 2.9.** Given a metric measure space $(X,\mathsf{d},\mathfrak{m})$, $k\in\mathbb{N}$ and $\varepsilon>0$, we say that a Borel set $E\subseteq X$ is $(\mathfrak{m},k,\varepsilon)$-*rectifiable* provided there exists a sequence $\big\{(G_n,u_n)\big\}_n$, where $G_n\subseteq X$ are Borel sets satisfying $\mathfrak{m}\Big(X\setminus\bigcup_n G_n\Big)=0$ and the maps $u_n\colon G_n\to\mathbb{R}^k$ are $(1+\varepsilon)$-biLipschitz with their images.

Rectifiability of $\mathrm{RCD}(K,N)$ spaces in the above sense was first proved in [**121**, Theorem 1.1]. Below we provide a different proof, more in the spirit of the Cheeger-Colding theory for Ricci limits (cf. [**53**]) and relying on the connection between $\delta$-splitting maps and $\varepsilon$-isometries.

**Lemma 2.10.** *Let $N>1$ be given. Then for any $\eta>0$ there exists $\delta=\delta_{N,\eta}>0$ such that the following property holds. If $(X,\mathsf{d},\mathfrak{m})$ is an $\mathrm{RCD}(K,N)$ space and $u\colon B_r(x)\to\mathbb{R}^k$ is a $\delta$-splitting map for some radius $r>0$ with $r^2|K|\le 1$ and some point $x\in X$ satisfying*

$$\mathsf{d}_{\mathrm{pmGH}}\Big((X,\mathsf{d}/r,\mathfrak{m}_r^x,x),(\mathbb{R}^k,\mathsf{d}_{\mathrm{Eucl}},c_k\mathscr{L}^k,0^k)\Big)<\delta^2,$$

*then it holds that $u\colon B_r(x)\to\mathbb{R}^k$ is an $\eta r$-*GH isometry*, meaning that*

$$\Big|\big|u(y)-u(z)\big|-\mathsf{d}(y,z)\Big|\le\eta r\qquad\textit{for every } y,z\in B_r(x).$$

*Proof.* Thanks to a scaling argument, it suffices to prove the statement for $r=1$ and $|K|\le 1$. We argue by contradiction: suppose there exist $\eta>0$, a sequence of spaces $(X_n,\mathsf{d}_n,\mathfrak{m}_n,x_n)$ and a sequence of maps $u^n\colon B_1(x_n)\to\mathbb{R}^k$, such that the following properties are satisfied.

i) $(X_n,\mathsf{d}_n,\mathfrak{m}_n)$ is an $\mathrm{RCD}(K,N)$ space.
ii) $u^n$ is a $1/n$-splitting map with $u^n(x_n)=0^k$.
iii) It holds that $\mathsf{d}_{\mathrm{pmGH}}\Big((X_n,\mathsf{d}_n,\mathfrak{m}_n,x_n),(\mathbb{R}^k,\mathsf{d}_{\mathrm{Eucl}},c_k\mathscr{L}^k,0^k)\Big)\le 1/n$.
iv) $u^n$ is not an $\eta$-GH isometry, so that there exist points $y_n,z_n\in B_1(x_n)$ such that

$$\Big|\big|u^n(y_n)-u^n(z_n)\big|-\mathsf{d}_n(y_n,z_n)\Big|>\eta. \tag{2.8}$$

Observe that item iii) guarantees that $(X_n,\mathsf{d}_n,\mathfrak{m}_n,x_n)\to(\mathbb{R}^k,\mathsf{d}_{\mathrm{Eucl}},c_k\mathscr{L}^k,0^k)$ in the pmGH-topology. Possibly taking a not relabelled subsequence, it holds that $u^n\to u^\infty$ strongly in $H^{1,2}$ on $B_1(0^k)$, for some limit map $u^\infty\colon B_1(0^k)\to\mathbb{R}^k$ (cf. Section 4.2 for the theory of convergence on varying spaces). We also deduce from item ii) above that $\mathrm{Hess}(u^\infty_a)=0$ and $\nabla u^\infty_a\cdot\nabla u^\infty_b=\delta_{ab}$ on $B_1(0^k)$ for all $a,b=1,\dots,k$ (further details are discussed in the proof of Proposition 1.83), whence $u^\infty$ is the restriction to $B_1(0^k)$ of an orthogonal transformation of $\mathbb{R}^k$. This gives a contradiction since, by letting $n\to\infty$ in (2.8), we obtain that

$$\Big|\big|u^\infty(y_\infty)-u^\infty(z_\infty)\big|-|y_\infty-z_\infty|\Big|\ge\eta,$$

where $y_\infty,z_\infty\in B_1(0^k)$ stand for the limit points of $(y_n)_n$ and $(z_n)_n$, respectively (notice that $x_\infty\neq y_\infty$ as a consequence of (2.8) and (i) in Definition 1.80). $\square$

Let $(X, \mathsf{d}, \mathfrak{m})$ be an $\mathrm{RCD}(K, N)$ space. Let $k \in \mathbb{N}$ be such that $k \leq N$. Then we define

$$(\mathcal{R}_k)_{r,\delta} \coloneqq \left\{ x \in \mathcal{R}_k \;\middle|\; \mathsf{d}_{\mathrm{pmGH}}\Big((X, \mathsf{d}/s, \mathfrak{m}^x_s, x), (\mathbb{R}^k, \mathsf{d}_{\mathrm{Eucl}}, c_k\mathscr{L}^k, 0^k)\Big) < \delta \quad \forall s < r \right\}$$

for every $r, \delta > 0$. Observe that for any given $\delta > 0$ it holds that $(\mathcal{R}_k)_{r,\delta} \nearrow \mathcal{R}_k$ as $r \searrow 0$.

**Theorem 2.11** (Rectifiability of RCD spaces)**.** *Let $(X, \mathsf{d}, \mathfrak{m})$ be an $\mathrm{RCD}(K, N)$ space. Let $k \in \mathbb{N}$ be such that $k \leq N$. Then the $k$-regular set $\mathcal{R}_k$ of $X$ is $(\mathfrak{m}, k, \varepsilon)$-rectifiable for every $\varepsilon > 0$.*

*Proof.* We claim that for any $\varepsilon > 0$ there exists an $(\mathfrak{m}, k, \varepsilon)$-rectifiable set $G^\varepsilon \subset \mathcal{R}_k$ such that $\mathfrak{m}(\mathcal{R}_k \setminus G^\varepsilon) < \varepsilon$. Notice that the statement follows from the claim above observing that

$$\mathfrak{m}\Big(\mathcal{R}_k \setminus \bigcup_{n=1}^{\infty} G^{\varepsilon/n}\Big) = 0.$$

Let us prove the claim in two steps.

**Step 1.** We claim that for any $\eta > 0$ there exists $\delta = \delta_{N,\eta} \in (0, 1)$ such that the following property holds: if $(X, \mathsf{d}, \mathfrak{m})$ is an $\mathrm{RCD}(K, N)$ space and $u\colon B_{5r}(p) \to \mathbb{R}^k$ is a $\delta$-splitting map for some radius $r > 0$ satisfying $r^2|K| \leq 1$ and some point $p \in (\mathcal{R}_k)_{2r,\delta}$, then there exists a Borel set $G \subseteq B_r(p)$ such that $\mathfrak{m}\big(B_r(p) \setminus G\big) \leq C_N \eta\, \mathfrak{m}\big(B_r(p)\big)$ and

$$\Big|\big|u(x) - u(y)\big| - \mathsf{d}(x, y)\Big| \leq \eta\, \mathsf{d}(x, y) \quad \text{for every } x, y \in (\mathcal{R}_k)_{2r,\delta} \cap G. \tag{2.9}$$

To prove it, choose any $\delta \in (0, \eta^2)$ so that $\sqrt{\delta}$ is associated with $\eta$ as in Lemma 2.10. Now let us consider an $\mathrm{RCD}(K, N)$ space $(X, \mathsf{d}, \mathfrak{m})$ and a $\delta$-splitting map $u\colon B_{5r}(p) \to \mathbb{R}^k$, for some $r > 0$ with $r^2|K| \leq 1$ and $p \in (\mathcal{R}_k)_{2r,\delta}$. By Proposition 2.5, we can find a Borel set $G \subseteq B_r(p)$ such that $\mathfrak{m}\big(B_r(p) \setminus G\big) \leq C_N \eta\, \mathfrak{m}\big(B_r(p)\big)$ and $u\colon B_s(x) \to \mathbb{R}^k$ is a $\sqrt{\delta}$-splitting map for all $x \in G$ and $s \in (0, 2r)$. Then Lemma 2.10 guarantees that the map $u\colon B_s(x) \to \mathbb{R}^k$ is an $\eta s$-GH isometry for every $x \in (\mathcal{R}_k)_{2r,\delta} \cap G$ and $s \in (0, 2r)$ (here we used the fact that $x \in (\mathcal{R}_k)_{2r,\delta} \subseteq (\mathcal{R}_k)_{s,\delta}$).

Fix any $x, y \in (\mathcal{R}_k)_{2r,\delta} \cap G$. Since $\mathsf{d}(x, y) < 2r$, we know that $u\colon B_{\mathsf{d}(x,y)}(x) \to \mathbb{R}^k$ is an $\eta\, \mathsf{d}(x, y)$-GH isometry, thus in particular $\Big||u(x) - u(y)| - \mathsf{d}(x, y)\Big| \leq \eta\, \mathsf{d}(x, y)$. This yields (2.9).

**Step 2.** Fix $\bar{x} \in X$, $R > 0$, $\varepsilon > 0$. We aim to build an $(\mathfrak{m}, k, \varepsilon)$-rectifiable set $G$ satisfying $\mathfrak{m}(B_R(\bar{x}) \cap \mathcal{R}_k \setminus G) < \varepsilon$. Note that this easily implies our claim.

Let $\eta < \varepsilon$ to be chosen later, $\delta = \delta_{N,\eta}$ according to Step 1, $\bar{\varepsilon} \in (0, \delta)$ associated to $\delta$ as in Corollary 1.86 and $r > 0$ satisfying $r^2|K| \leq 1$ and $\mathfrak{m}(B_R(\bar{x}) \cap (\mathcal{R}_k \setminus (\mathcal{R})_{2r,\bar{\varepsilon}})) \leq \varepsilon/2$. By Vitali covering lemma, we find points $x_1, \ldots, x_\ell \in B_R(\bar{x}) \cap (\mathcal{R}_k)_{2r,\bar{\varepsilon}}$ for which $\big\{B_{r/5}(x_i)\big\}_{i=1}^{\ell}$ are pairwise disjoint and $B_R(\bar{x}) \cap (\mathcal{R}_k)_{2r,\bar{\varepsilon}} \subseteq B_r(x_1) \cup \cdots \cup B_r(x_\ell)$. Corollary 1.86 guarantees the existence of a $\delta$-splitting map $u^i\colon B_{5r}(x_i) \to \mathbb{R}^k$ for every $i = 1, \ldots, \ell$. Therefore Step 1 yields Borel sets $G_i \subseteq B_r(x_i)$ such that $\mathfrak{m}\big(B_r(x_i) \setminus G_i\big) \leq C_N \eta\, \mathfrak{m}\big(B_r(x_i)\big)$ and

$$\Big||u^i(x) - u^i(y)| - \mathsf{d}(x, y)\Big| \leq \eta\, \mathsf{d}(x, y) \quad \text{for every } x, y \in (\mathcal{R}_k)_{2r,\bar{\varepsilon}} \cap G_i, \text{ for every } i = 1, \ldots, \ell.$$

Since $\eta < \varepsilon$, we deduce that $u^i$ is $(1+\varepsilon)$-biLipschitz with its image when restricted to $(\mathcal{R}_k)_{2r,\bar\varepsilon} \cap G_i$, whence $G := (\mathcal{R}_k)_{2r,\bar\varepsilon} \cap \bigcup_{i=1}^{\ell} G_i$ is $(\mathfrak{m}, k, \varepsilon)$-rectifiable. Observe that

$$\begin{aligned}\mathfrak{m}\Big(\big(B_R(\bar x) \cap (\mathcal{R}_k)_{2r,\bar\varepsilon}\big) \setminus G\Big) &\le \sum_{i=1}^{\ell} \mathfrak{m}\Big(B_r(x_i) \setminus G_i\Big) \le C_N \eta \sum_{i=1}^{\ell} \mathfrak{m}\Big(B_r(x_i)\Big)\\ &\le C_N \eta \sum_{i=1}^{\ell} \mathfrak{m}\Big(B_{r/5}(x_i)\Big) \le C_N \eta\, \mathfrak{m}\Big(B_{R+1}(\bar x)\Big).\end{aligned}$$

Choosing $\eta > 0$ such that $C_N \eta\, \mathfrak{m}\Big(B_{R+1}(\bar x)\Big) < \varepsilon/2$ we get the sought conclusion. □

## 4. Behaviour of the reference measure under charts

Aim of this section is to prove absolute continuity of the reference measure $\mathfrak{m}$ of an $\mathrm{RCD}(K,N)$ metric measure space $(X, \mathsf{d}, \mathfrak{m})$ with respect to the relevant Hausdorff measure. This result was first proved in the three independent works [**72**, **95**, **109**], heavily relying on [**73**]. The strategy of our proof is essentially taken from [**95**], the main technical simplification being that the charts providing rectifiability in our case are harmonic (indeed they are $\delta$-splitting maps), while in [**95**] they were distance functions.

We refer to Section 5 for the notation regarding normed modules. Let $X, Y$ be Polish spaces. Fix a finite Borel measure $\mu \ge 0$ on $X$ and a Borel map $\phi\colon X \to Y$. We shall denote by $\phi_*$ the pushforward operator, which sends finite Borel measures on $X$ into finite Borel measures on $Y$. Then we define

$$\mathsf{Pr}_\phi(f) := \frac{\mathrm{d}\phi_*(f\mu)}{\mathrm{d}\phi_*\mu} \quad \text{for every } f \in L^1(\mu), \tag{2.10}$$

where we adopted the usual notation of geometric measure theory for the density of a measure absolutely continuous with respect to another measure. The resulting map $\mathsf{Pr}_\phi\colon L^1(\mu) \to L^1(\phi_*\mu)$ is linear and continuous. Given any $p \in (1,\infty]$, it holds that $\mathsf{Pr}_\phi$ maps continuously $L^p(\mu)$ to $L^p(\phi_*\mu)$. The *essential image* of a Borel set $E \subseteq X$ is defined as $\mathsf{Im}_\phi(E) := \Big\{\mathsf{Pr}_\phi(\chi_E) > 0\Big\} \subseteq Y$.

Let us also recall that in the framework of weighted Euclidean spaces, we have another notion of tangent module at our disposal. Given a Radon measure $\nu \ge 0$ on $\mathbb{R}^k$, we denote by $L^2(\mathbb{R}^k, \mathbb{R}^k; \nu)$ the space of all $L^2(\nu)$-maps from $\mathbb{R}^k$ to itself. It turns out that $L^2(\mathbb{R}^k, \mathbb{R}^k; \nu)$ is an $L^2(\nu)$-normed $L^\infty(\nu)$-module generated by $\Big\{\nabla f \,:\, f \in C_c^\infty(\mathbb{R}^k)\Big\}$, where $\nabla f\colon \mathbb{R}^k \to \mathbb{R}^k$ stands for the "classical" gradient of $f$.

In the statement below, for the sake of clarity, we emphasise the dependence on the reference measure of differential operator.

**Proposition 2.12** (Differential of an $\mathbb{R}^k$-valued Lipschitz map). *Let $(X, \mathsf{d}, \mu)$ be an infinitesimally Hilbertian m.m.s. such that $\mu$ is finite. Let $\phi\colon X \to \mathbb{R}^k$ be a Lipschitz map. Then there exists a unique linear and continuous operator $\mathsf{D}_\phi\colon L^2_\mu(TX) \to L^2(\mathbb{R}^k, \mathbb{R}^k; \phi_*\mu)$ such that*

$$\int_F \nabla f \cdot \mathsf{D}_\phi(v)\, \mathrm{d}\phi_*\mu = \int_{\phi^{-1}(F)} \nabla_\mu(f \circ \phi) \cdot v \, \mathrm{d}\mu \tag{2.11}$$

*for every $f \in C_c^\infty(\mathbb{R}^k)$, $v \in L^2_\mu(TX)$, $F \subseteq \mathbb{R}^k$ Borel. In particular, if $v \in D(\mathrm{div}_\mu)$, then the distributional divergence of $\mathsf{D}_\phi(v)$ is given by $\mathsf{Pr}_\phi\Big(\mathrm{div}_\mu(v)\Big)$.*

*Moreover, if the map $\phi$ is biLipschitz with its image when restricted to some Borel set $E \subseteq X$ and $v_1, \ldots, v_k \in L^2_\mu(TX)$ are independent on $E$, then the vectors*

$$\mathsf{D}_\phi(\chi_E\, v_1)(y), \ldots, \mathsf{D}_\phi(\chi_E\, v_k)(y)$$

*constitute a basis of $\mathbb{R}^k$ for $\phi_*\mu$-a.e. point $y \in \mathsf{Im}_\phi(E)$.*

*Proof.* Existence of the map $\mathsf{D}_\phi$ is proven in [**95**]: with the terminology used therein, it suffices to define $\mathsf{D}_\phi := \iota \circ \mathsf{Pr}_\phi \circ \mathrm{d}\phi$. The fact that this map satisfies (2.11) follows from [**95**, Proposition 2.7] and the very definition of $\iota$ (we do not need to require properness of $\phi$, as $\mu$ is a finite measure). Uniqueness of $\mathsf{D}_\phi$ follows from the fact that $\left\{\nabla f \,:\, f \in C^\infty_c(\mathbb{R}^k)\right\}$ generates $L^2(\mathbb{R}^k, \mathbb{R}^k; \phi_*\mu)$. Now suppose $v \in D(\mathrm{div}_\mu)$. Then for every $f \in C^\infty_c(\mathbb{R}^k)$ it holds that $f \circ \phi \in H^{1,2}(X, \mathsf{d}, \mu)$, whence

$$\begin{aligned}\int \nabla f \cdot \mathsf{D}_\phi(v)\, \mathrm{d}\phi_*\mu &\overset{(2.11)}{=} \int \nabla_\mu(f \circ \phi) \cdot v \,\mathrm{d}\mu = -\int f \circ \phi \,\mathrm{div}_\mu(v)\, \mathrm{d}\mu \\ &= -\int f \,\mathrm{d}\phi_*\Big(\mathrm{div}_\mu(v)\mu\Big) \overset{(2.10)}{=} -\int f\, \mathsf{Pr}_\phi\Big(\mathrm{div}_\mu(v)\Big)\, \mathrm{d}\phi_*\mu.\end{aligned}$$

This shows that the distributional divergence of $\mathsf{D}_\phi(v)$ is represented by $\mathsf{Pr}_\phi\Big(\mathrm{div}_\mu(v)\Big)$. Finally, the last claim of the statement follows from [**95**, Proposition 2.2] and [**95**, Proposition 2.10]. □

**Theorem 2.13** (Behaviour of $\mathfrak{m}$ under charts)**.** *Let $(X, \mathsf{d}, \mathfrak{m})$ be an* RCD$(K, N)$ *space. Consider a $\delta$-splitting map $u\colon B_r(p) \to \mathbb{R}^k$ which is $(1+\varepsilon)$-bi-Lipschitz with its image (for some $\varepsilon < 1/k$) when restricted to some compact set $K \subseteq B_r(p)$. Then it holds that*

$$u_*(\mathfrak{m}|_K) \ll \mathscr{L}^k.$$

*In particular, for any $k \in \mathbb{N}$, $k \leq N$, $\mathfrak{m}|_{\mathcal{R}_k}$ is absolutely continuous with respect to the $k$-dimensional Hausdorff measure on $(X, \mathsf{d})$.*

**Remark 2.14.** Denote by

$$\mathcal{R}^*_k := \{x \in \mathcal{R}_k : \quad \exists \lim_{r \to 0^+} \frac{\mathfrak{m}(B(x,r))}{\omega_k r^k} \in (0, \infty)\}. \tag{2.12}$$

By means of Theorem 2.13 and standard tools from geometric measure theory one can prove that $\mathfrak{m}(\mathcal{R}_k \setminus \mathcal{R}^*_k) = 0$. Moreover $\mathfrak{m} \llcorner \mathcal{R}^*_k$ and $\mathcal{H}^k \llcorner \mathcal{R}^*_k$ are mutually absolutely continuous and

$$\lim_{r \to 0^+} \frac{\mathfrak{m}(B(x,r))}{\omega_k r^k} = \frac{\mathrm{d}\mathfrak{m} \llcorner \mathcal{R}^*_k}{\mathrm{d}\mathcal{H}^k \llcorner \mathcal{R}^*_k}(x),$$

for $\mathfrak{m}$-a.e. $x \in \mathcal{R}^*_k$.

We refer to [**21**, Theorem 4.1] for details on this.

*Proof of Theorem 2.13.* First of all, fix a good cut-off function $\eta\colon X \to \mathbb{R}$ for the pair $K \subseteq B_r(p)$, in the sense of Lemma 1.60. Define $\mu := \mathfrak{m}|_{B_r(p)}$ and $\phi := \eta u\colon X \to \mathbb{R}^k$. Observe that the components $\phi_1, \ldots, \phi_k$ of $\phi$ are test functions and $\phi|_K$ is $(1+\varepsilon)$-biLipschitz with its image. Consider the differential $\mathsf{D}_\phi\colon L^2_\mu(TX) \to L^2(\mathbb{R}^k, \mathbb{R}^k; \phi_*\mu)$ defined in Proposition 2.12. Fix a sequence $(\psi_i)_i$ of compactly-supported, Lipschitz functions $\psi_i\colon X \to [0,1]$ that pointwise converge to $\chi_K$. We then set

$$v^i_a := \mathsf{D}_\phi(\psi_i \nabla_\mu \phi_a) \in L^2(\mathbb{R}^k, \mathbb{R}^k; \phi_*\mu) \quad \text{for every } i \in \mathbb{N} \text{ and } a = 1, \ldots, k.$$

Note that $\psi_i \nabla_\mu \phi_a \in D(\mathrm{div}_\mu)$ by the Leibniz rule for divergence and the fact that $\phi_a \in D(\Delta_\mu)$, whence Proposition 2.12 ensures that the distributional divergence of each vector field $v_a^i$ is an $L^2(\phi_*\mu)$-function. Hence, it holds that $\mathcal{I}_{ia} := v_a^i\,\phi_*\mu$ is a normal 1-current in $\mathbb{R}^k$ (see [**95**, Corollary 2.12]). Note also that

$$\overrightarrow{\mathcal{I}_{ia}} = \chi_{\{|v_a^i|>0\}} \frac{v_a^i}{|v_a^i|} \quad \text{and} \quad \|\mathcal{I}_{ia}\| = |v_a^i|\,\phi_*\mu \qquad \text{for every } i \in \mathbb{N} \text{ and } a = 1, \ldots, k.$$

Call $A_i$ the set of $y \in \mathbb{R}^k$ such that $v_1^i(y), \ldots, v_k^i(y)$ form a basis of $\mathbb{R}^k$. Since $(\phi_*\mu)|_{A_i} \ll \|\mathcal{I}_{ia}\|$ holds for all $a = 1, \ldots, k$, by applying [**73**, Corollary 1.12] we deduce that

$$(\phi_*\mu)|_{A_i} \ll \mathscr{L}^k \qquad \text{for every } i \in \mathbb{N}. \tag{2.13}$$

Now define $v_a := \mathsf{D}_\phi(\chi_K \nabla_\mu \phi_a) \in L^2(\mathbb{R}^k, \mathbb{R}^k; \phi_*\mu)$ for every $a = 1, \ldots, k$. It can readily checked that $\nabla_\mu\phi_1, \ldots, \nabla_\mu\phi_k$ are independent on $K$ (here the assumption $\varepsilon < 1/k$ plays a role), whence the vectors $v_1(y), \ldots, v_k(y)$ are linearly independent for $\phi_*\mu$-a.e. $y \in \mathsf{Im}_\phi(K)$ by Proposition 2.12.

Furthermore, for any given $j = 1, \ldots, k$, we can see (by using dominated convergence theorem) that $\psi_i \nabla_\mu \phi_a \to \chi_K \nabla_\mu \phi_a$ in $L^2_\mu(TX)$ as $i \to \infty$, thus $v_a^i \to v_a$ in $L^2(\mathbb{R}^k, \mathbb{R}^k; \phi_*\mu)$ as $i \to \infty$ by continuity of $\mathsf{D}_\phi$. In particular, possibly passing to a not relabelled subsequence, we can assume that $\lim_i v_a^i(y) = v_a(y)$ for $\phi_*\mu$-a.e. $y \in \mathbb{R}^k$. This implies that $(\phi_*\mu)\Big(\mathsf{Im}_\phi(K) \setminus \bigcup_i A_i\Big) = 0$, thus (2.13) yields $(\phi_*\mu)|_{\mathsf{Im}_\phi(K)} \ll \mathscr{L}^k$. Since $\mathsf{Im}_\phi(K) = \Big\{\mathsf{Pr}_\phi(\chi_K) > 0\Big\}$ by definition, we conclude that

$$u_*(\mathfrak{m}|_K) = \phi_*(\mu|_K) = \frac{\mathrm{d}\phi_*(\chi_K\mu)}{\mathrm{d}\phi_*\mu}\,\phi_*\mu = \mathsf{Pr}_\phi(\chi_K)\,\phi_*\mu \ll \mathscr{L}^k.$$

Therefore, the first part of the statement is finally achieved.

The second part of the statement follows from the first one, the inner regularity of $\mathfrak{m}$ and (the proof of) Theorem 2.11. $\square$

CHAPTER 3

# Constancy of the dimension

In Chapter 2 we proved that any $\mathrm{RCD}(K,N)$ metric measure space can be covered with regular sets $\mathcal{R}_k$ of dimension $k \in \mathbb{N} \cap [1,N]$, up to a $\mathfrak{m}$-negligible set. A very natural question left open is whether two non negligible regular sets of different dimension may exist. Let us mention that in the framework of Ricci limit spaces Cheeger and Colding in [**51**] conjectured that there should be exactly one $k$-dimensional regular set $\mathcal{R}_k$ having positive measure. However, it took more than ten years before the work [**58**], where Colding-Naber affirmatively solved this conjecture.

The analogous problem in the abstract framework of $\mathrm{RCD}(K,N)$ metric measure spaces remained open since the work of Mondino-Naber and has been recently solved in [**43**] by the author in collaboration with D. Semola.

**Theorem 3.1** (Constancy of the dimension)**.** *Let* $(X,\mathsf{d},\mathfrak{m})$ *be an* $\mathrm{RCD}(K,N)$ *for some* $K \in \mathbb{R}$ *and* $1 < N < \infty$*. Then, there is exactly one regular set* $\mathcal{R}_k$ *having positive* $\mathfrak{m}$*-measure in the Mondino-Naber decomposition of* $(X,\mathsf{d},\mathfrak{m})$.

Let us first explain why it seems hard to adapt the strategy pursued by Colding-Naber to the case of $\mathrm{RCD}(K,N)$ spaces and after present the heuristic standing behind our new approach.

The technique developed in [**58**] is based on fine estimates on the behaviour of balls of small radii centered along the interior of a minimizing geodesic over a smooth Riemannian manifold that are stable enough to pass through the possibly singular Gromov-Hausdorff limits. When dealing with an abstract $\mathrm{RCD}(K,N)$ space there is no smooth approximating sequence one can appeal on. Nevertheless, one could try to reproduce their main estimate (see [**58**, Theorem 1.3]) directly at the level of the given metric measure space but, up to our knowledge, the calculus tools available at this stage, although being quite powerful (see for instance [**87**]), are still not sufficient to such an issue. The study of the flow of a suitably chosen vector field, which was at the technical core of the proof in the case of Ricci limit spaces, will be the starting point in our approach too. A key idea is the following: one would expect the geometry to change continuously along a flow and that, as a consequence, flow maps might be a useful tool to prove that the space has a certain "homogeneity" property. As we have already seen in Chapter 1, any metric measure space has a first order differential structure. Furthermore in [**87**], Gigli proved that any $\mathrm{RCD}(K,\infty)$ metric measure space has also a second order differential structure. Roughly speaking, this turns into the possibility of finding many functions with second order derivatives in $L^2$. At the level of vector fields, which are by themselves first order differential objects, this has the outcome that one can define a notion of covariant derivative and that the class of Sobolev vector fields with covariant derivative in $L^2$ is a rich one.

Moreover, as we have already seen in Section 6.1, after [**27**] we have a notion of flow associated to Sobolev vector fields over RCD spaces. The following weak transitivity property of flow maps is an instance of the fact that such a class of vector fields is rich. Given any two probability measures $\mu_0, \mu_1$ absolutely continuous and with bounded densities with respect to $\mathfrak{m}$, we can find a Sobolev vector field such that, calling $F$ its regular Lagrangian flow at $t = 1$, it holds $F_*\mu_0 = \mu_1$. To be precise, we are able to prove that $F$ maps a "big portion" of $\mu_0$ to $\mu_1$, that is still enough for our purposes. In Section 2.2 below, we will see how the Lewy-Stampacchia inequality, proved in this abstract framework by Gigli-Mosconi in [**93**], leads to the proof of this weak transitivity result.

With the weak transitivity of flows at hand it is natural to investigate regularity properties of flow maps. This is a natural strategy to prove "homogeneity"properties of the space that may rule out the possibility of having pieces of different dimensions.

Unfortunately, the study of regularity for Lagrangian flows associated to Sobolev vector fields is far from being trivial also in the Euclidean setting. The first result in this direction was obtained by Crippa-De Lellis in [**63**] building upon some ideas that have previously appeared in [**23**].

Let us explain the reasons behind these difficulties and why they required some new ideas to be settled in the case of a Riemannian manifold and a change of perspective in the case of an $\mathrm{RCD}(K,N)$ metric measure space.

Let $b : \mathbb{R}^d \to \mathbb{R}^d$ be a vector field and denote by $\boldsymbol{X} : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ its flow map, that we assume to be well-defined for every $t \in [0,T]$ and for every $x \in \mathbb{R}^d$. A natural way to measure the regularity of $\boldsymbol{X}$ is in terms of Lipschitz continuity and it is a rather elementary fact that, whenever $b$ is Lipschitz, the flow map $\boldsymbol{X}_t$ is Lipschitz as well. Indeed, willing to control the distance between trajectories starting from different points $x, y \in \mathbb{R}^d$, it is sufficient to compute

$$\frac{\mathrm{d}}{\mathrm{d}t}|\boldsymbol{X}_t(x) - \boldsymbol{X}_t(y)| \le |b(\boldsymbol{X}_t(x)) - b(\boldsymbol{X}_t(y))| \le \mathrm{Lip}(b)|\boldsymbol{X}_t(x) - \boldsymbol{X}_t(y)|, \tag{3.1}$$

to obtain that

$$|\boldsymbol{X}_t(x) - \boldsymbol{X}_t(y)| \le e^{t\,\mathrm{Lip}\,b}|x-y|, \qquad \text{for any } t \in [0,T].$$

Lowering the regularity assumption on the vector field from Lipschitz to Sobolev, the second inequality in (3.1) fails and we cannot expect Lipschitz regularity for the regular Lagrangian flow $\boldsymbol{X}_t$ that, in general, might even be discontinuous. However, in the aforementioned paper, Crippa-De Lellis obtained a Lusin-Lipschitz regularity result for Lagrangian flows associated to vector fields $b \in H^{1,p}(\mathbb{R}^d;\mathbb{R}^d)$ for $p > 1$. That is to say, they proved that, for every bounded $K \subset \mathbb{R}^d$ and for every $\varepsilon > 0$, there exist $C = C(\varepsilon, \|b\|_{H^{1,p}}, K) > 0$ and $E \subset K$ with $\mathscr{L}^d(K \setminus E) < \varepsilon$ such that $\boldsymbol{X}_t$ is $C$-Lipschitz over $E$, for any $t \in [0,T]$.

The key tool exploited by Crippa-De Lellis seeking for an analogue of (3.1) is the so-called maximal estimate for Sobolev functions: there exists $C_d > 0$, such that any $f \in H^{1,p}(\mathbb{R}^d;\mathbb{R})$ admits a representative, still denoted by $f$, such that

$$|f(x) - f(y)| \le C_d(M|\nabla f|(x) + M|\nabla f|(y))\,|x-y|\,, \qquad \text{for any } x, y \in X, \tag{3.2}$$

where $M|\nabla f|$ is the maximal operator applied to $|\nabla f|$. Observe that, if $p > 1$, then $\|M|\nabla f|\|_{L^p} \le C_{p,d}\,\||\nabla f|\|_{L^p}$ for some constant $C_{p,d} > 0$. Moreover, since on $\mathbb{R}^d$ a vector field is Sobolev if and only if its components are so, (3.2) holds true also for any $b \in H^{1,p}(\mathbb{R}^d;\mathbb{R}^d)$.

This being said, the replacement of (3.1) in the Sobolev case is

$$\frac{\mathrm{d}}{\mathrm{d}t}|\boldsymbol{X}_t(x) - \boldsymbol{X}_t(y)| \le C\left\{M|\nabla b|(\boldsymbol{X}_t(x)) + M|\nabla b|(\boldsymbol{X}_t(y))\right\}|\boldsymbol{X}_t(x) - \boldsymbol{X}_t(y)|. \tag{3.3}$$

The sought regularity for $\boldsymbol{X}_t$ does not follow any more applying Gronwall lemma to (3.3). However, one might think of (3.3) as a quantitative infinitesimal version of the regularity result for the Lagrangian flow.

Having such a perspective in mind, the situation changes significantly passing from the Euclidean space to an $\mathrm{RCD}(K, N)$ metric measure space or, more simply, to a smooth Riemannian manifold. Indeed, while the maximal estimate for real valued Sobolev functions (3.2) is a very robust result, which holds true in every doubling metric measure space satisfying the Poincaré inequality and in a relevant class of non-doubling spaces (see [**8**, **9**]), we are not aware of any intrinsic way to lift it to the level of vector fields.

Therefore we chose for an alternative approach to the problem. Let us introduce the more appealing notation $\mathsf{d}$ for the distance function but still think, for sake of simplicity to the Euclidean case. Trying to turn the Sobolev regularity of the vector field into some bound for the right hand side in the expression

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathsf{d}(\boldsymbol{X}_t(x), \boldsymbol{X}_t(y)) = b \cdot \nabla \mathsf{d}_{\boldsymbol{X}_t(x)}(\boldsymbol{X}_t(y)) + b \cdot \nabla \mathsf{d}_{\boldsymbol{X}_t(y)}(\boldsymbol{X}_t(x)), \tag{3.4}$$

a natural attempt could be to appeal to the interpolation

$$b \cdot \nabla \mathsf{d}_x(y) + b \cdot \nabla \mathsf{d}_y(x) = \int_0^1 \nabla_{\mathrm{sym}} b(\gamma'(s), \gamma'(s))\, \mathrm{d}s, \tag{3.5}$$

where $\gamma : [0,1] \to \mathbb{R}^d$ is the geodesic joining $x$ to $y$ and $\nabla_{\mathrm{sym}} b$ is the symmetric part of the covariant derivative of $b$. However, when the bounds on $\nabla_{\mathrm{sym}} b$ are only of integral type, it is not clear how to obtain useful estimates from (3.5) without deeply involving the Euclidean structure, that is something to be avoided in view of the extensions to the metric setting.

The starting point of the study in our previous paper [**42**] was, instead, the following observation: suppose that $d \geq 3$, then, calling $G$ the Green function of the Laplacian on $\mathbb{R}^d$, it holds $G(x,y) = c_d \mathsf{d}(x,y)^{2-d}$. This implies in turn that controlling the distance between two trajectories of the flow is the same as controlling the Green function along them. Moreover, computing the rate of change of the Green function along the flow, we end up with the necessity to find bounds for the quantity

$$b \cdot \nabla G_x(y) + b \cdot \nabla G_y(x),$$

that, assuming $\operatorname{div} b = 0$ for sake of simplicity, we can formally rewrite as

$$\begin{aligned} b \cdot \nabla G_x(y) + b \cdot \nabla G_y(x) &= -\int_{\mathbb{R}^d} b(w) \cdot \nabla G_x(w)\, \mathrm{d}\Delta G_y(w) - \int_{\mathbb{R}^d} b(w) \cdot \nabla G_y(w)\, \mathrm{d}\Delta G_x(w) \\ &= 2\int_{\mathbb{R}^d} \nabla_{\mathrm{sym}} b(\nabla G_x, \nabla G_y)\, \mathrm{d}\mathfrak{m}. \end{aligned} \tag{3.6}$$

Observe that, being (3.6) in integral form, we can expect it to fit better than (3.5) with the assumption $\nabla_{\mathrm{sym}} b \in L^2$ and this expectation is confirmed by the validity, for some $C > 0$, of the key estimate

$$\int_{\mathbb{R}^d} f |\nabla G_x| |\nabla G_y|\, \mathrm{d}\mathfrak{m} \leq C G(x,y)(Mf(x) + Mf(y)), \quad \text{for any } x, y \in X \tag{3.7}$$

and for any Borel function $f : \mathbb{R}^d \to [0, \infty)$, see Proposition 3.11 below.

Starting from this idea, in [**42**] we obtained the first extension of the Lusin-Lipschitz regularity estimate for Lagrangian flows of Sobolev vector fields outside from the Euclidean setting, covering the case of Ahlfors regular, compact $\mathrm{RCD}(K, N)$ spaces. In that case, which includes for instance Riemannian manifolds, Alexandrov spaces and non-collapsed $\mathrm{RCD}(K,N)$ spaces (see [**71**]), a uniform control over the volume growth of balls turns into a global comparison between the Green function, that was introduced as an intermediate tool, and a negative power of the distance, by means of which we wanted to measure regularity.

In the general case we prove that a version of Crippa-De Lellis' result holds true if we understand Lusin-Lipschitz regularity, with respect to a newly defined quasi-metric $\mathsf{d}_G = 1/G$, $G$ being the minimal positive Green function of the Laplacian over $(X,\mathsf{d},\mathfrak{m})$.

Let us point out, just at a speculative level, that having at our disposal a perfect extension of Crippa-De Lellis' result to the metric setting, it would have been rather easy to exclude the possibility of regular sets of different dimensions with positive measure in the Mondino-Naber decomposition of an $\mathrm{RCD}(K,N)$ metric measure space, just building on the transitivity result and the observation that, given $k<n$, it is impossible to find a Lipschitz map $\Phi:\mathbb{R}^k\to\mathbb{R}^n$ such that $\Phi_*\mathscr{L}^k \ll \mathscr{L}^n$.

Here we exploit a modification of this idea. We start observing that it is possible to catch, in a quantitative way, the asymptotic behaviour of the Green function near to regular points of the metric measure space in terms of distance, measure and dimension (see Lemma 3.32). This allows to find a counterpart for the "preservation of the Hausdorff dimension via biLipschitz maps" formulated just in terms of Green functions (see Theorem 3.33) and to complete the proof of Theorem 3.1, the spirit being that a control over two among distance, reference measure and Green function gives in turn a control over the remaining one.

## 1. $G$-egularity of Lagrangian flows

This section is dedicated to establish a regularity results for Lagrangian flows of Sobolev vector fields. Regularity is understood with respect to a newly defined quasi-metric $\mathsf{d}_G = 1/G$, where $G$ is the Green function of the Laplacian. A natural setting to have existence of a positive Green function is that one of $\mathrm{RCD}(0,N)$ metric measure spaces satisfying suitable volume growth conditions (see assumptions Assumption 1 and Assumption 2 below). Under these assumptions, in Section 1.3 we prove that $(X,\mathsf{d}_G,\mathfrak{m})$ is a doubling quasi-metric space and in Section 1.4 we exploit this structural result, together with the maximal estimate (3.37), to implement Crippa-De Lellis' scheme.

In Section 1.5 we show how this strategy can be adapted to cover the case of a possibly negative lower Ricci curvature bound.

**1.1. Key properties of the Green function.** In this section we study the properties of the Green function, that is a central object in our work. From now on we assume that $(X,\mathsf{d},\mathfrak{m})$ is an $\mathrm{RCD}(0,N)$ metric measure space. Further assumptions on the space will be added in the sequel.

We set

$$G(x,y) := \int_0^\infty p_t(x,y)\,\mathrm{d}t$$

and, for every $\varepsilon>0$,

$$G^\varepsilon(x,y) := \int_\varepsilon^\infty p_t(x,y)\,\mathrm{d}t. \tag{3.8}$$

We shall adopt in the sequel also the notation $G_x(\cdot) := G(x,\cdot)$ (and analogously for $G^\varepsilon$).

Before going on let us observe that, at least at a formal level, the Green function is the fundamental solution of the Laplace operator. Indeed

$$\begin{aligned}\Delta_y G_x(\cdot) &= \Delta_y \left( \int_0^\infty p_t(x, \cdot)\, \mathrm{d}t \right) = \int_0^\infty \Delta_y p_t(x, \cdot)\, \mathrm{d}t \\ &= \int_0^\infty \frac{\mathrm{d}}{\mathrm{d}t} p_t(x, \cdot)\, \mathrm{d}t = [p_t(x, \cdot)]_0^\infty = -\delta_x.\end{aligned}$$

In order to get the good definition of both $G$ and $G^\varepsilon$, up to the end of this section, unless otherwise stated, we will work under the following assumption.

**Assumption 1.** There exists $x \in X$ such that

$$\int_1^\infty \frac{s}{\mathfrak{m}(B(x,s))}\, \mathrm{d}s < \infty. \tag{3.9}$$

Recall that, for a non compact Riemannian manifold with nonnegative Ricci curvature, it was proved by Varopoulos that (3.9) is a necessary and sufficient condition for the existence of a positive Green function of the Laplacian (and this condition is known as *non-parabolicity* in the literature).

**Remark 3.2.** Let us observe that all the metric measure spaces obtained as tensor products between an arbitrary $\mathrm{RCD}(0, N)$ m.m.s. $(X, \mathsf{d}, \mathfrak{m})$ and an Euclidean factor $(\mathbb{R}^k, \mathsf{d}_{\mathbb{R}^k}, \mathscr{L}^k)$ for $k \geq 3$ do satisfy Assumption 1.

We now introduce functions $F,\ H : X \times (0, \infty) \to (0, \infty)$ by

$$F(x, r) := \int_r^\infty \frac{s}{\mathfrak{m}(B(x,s))}\, \mathrm{d}s \tag{3.10}$$

and

$$H(x, r) := \int_r^\infty \frac{1}{\mathfrak{m}(B(x,s))}\, \mathrm{d}s. \tag{3.11}$$

They are the objects we will use to estimate the Green function and its gradient (see [**97**] for analogous results in the smooth setting). As for the Green function, we will often write $F_x(r)$ or $H_x(r)$ in place of $F(x, r)$ and $H(x, r)$.

**Remark 3.3.** Let us remark that both $F$ and $H$ are continuous w.r.t. the first variable. It can be seen recalling that spheres are negligible on doubling m.m.s and using the continuity of the function $x \mapsto \mathfrak{m}(B(x, r))$ (with $r > 0$ fixed).

The next proposition has the aim to provide estimates for the Green function and its gradient, in terms of $F_x(\mathsf{d}(x, y))$ and $H_x(\mathsf{d}(x, y))$ that are simpler objects to work with.

**Proposition 3.4** (Main estimates for $G$)**.** *Let* $(X, \mathsf{d}, \mathfrak{m})$ *be an* $\mathrm{RCD}(0, N)$ *m.m.s. satisfying Assumption 1. Then there exists a constant* $C_2 \geq 1$*, depending only on* $N$*, such that, for any* $x \in X$*,*

$$\frac{1}{C_2} F_x(\mathsf{d}(x, y)) \leq G_x(y) \leq C_2 F_x(\mathsf{d}(x, y)) \quad \textit{for any } y \in X. \tag{3.12}$$

*Moreover for any* $x \in X$ *it holds that* $G_x \in H^{1,1}_{\mathrm{loc}}(X, \mathsf{d}, \mathfrak{m})$ *and*

$$|\nabla G_x|(y) \leq \int_0^\infty |\nabla p_t(x, \cdot)|(y)\, \mathrm{d}t \leq C_2 H_x(\mathsf{d}(x, y)), \quad \textit{for } \mathfrak{m}\textit{-a.e. } y \in X. \tag{3.13}$$

Before giving the proof of Proposition 3.4 let us state and prove some technical lemmas. The first one deals with the integrability properties of the maps $y \mapsto F_x(\mathsf{d}(x,y))$ and $y \mapsto H_x(\mathsf{d}(x,y))$. Since its formulation and its proof do not require any regularity assumption for the metric measure space, apart from the validity of Assumption 1, we state it in this great generality.

**Lemma 3.5.** *Let $(X, \mathsf{d}, \mathfrak{m})$ be a m.m.s. satisfying Assumption 1. Then for every $x \in X$, the functions $y \mapsto F_x(\mathsf{d}(x,y))$ and $y \mapsto H_x(\mathsf{d}(x,y))$ belong to $L^1_{\mathrm{loc}}(X, \mathfrak{m})$. Moreover the map $(w,z) \mapsto H(w, \mathsf{d}(w,z))$ belongs to $L^1_{\mathrm{loc}}(X \times X, \mathfrak{m} \times \mathfrak{m})$.*

*Proof.* Let $g : \mathbb{R} \to [0, \infty)$ be a Borel function, define $f(r) := \int_r^\infty g(s)\,\mathrm{d}s$. Observe that

$$\int_{B(x,R)} f(\mathsf{d}_x(w))\,\mathrm{d}\mathfrak{m}(w) = \int_0^R g(s)\mathfrak{m}(B(x,s))\,\mathrm{d}s + f(R)\mathfrak{m}(B(x,R)), \quad \text{for any } R > 0, \tag{3.14}$$

as an application of Fubini's theorem shows. Fix now any $x \in X$. Applying (3.14), first with $g(s) = \frac{s}{\mathfrak{m}(B(x,s))}$ and then with $g(s) = \frac{1}{\mathfrak{m}(B(x,s))}$, we get

$$\int_{B(x,R)} F_x(\mathsf{d}_x(w))\,\mathrm{d}\mathfrak{m}(w) = \frac{R^2}{2} + F_x(R)\mathfrak{m}(B(x,R)), \tag{3.15}$$

and

$$\int_{B(x,R)} H_x(\mathsf{d}_x(w))\,\mathrm{d}\mathfrak{m}(w) = R + H_x(R)\mathfrak{m}(B(x,R)), \tag{3.16}$$

that imply in turn that $y \mapsto F_x(\mathsf{d}(x,y))$ and $y \mapsto H_x(\mathsf{d}(x,y))$ belong to $L^1_{\mathrm{loc}}(X, \mathfrak{m})$.

We now prove the local integrability of $(w,z) \mapsto H(w, \mathsf{d}(w,z))$. It suffices to show that

$$\int_{B(\bar{x},R)} \int_{B(\bar{x},R)} H(w, \mathsf{d}(w,z))\,\mathrm{d}\mathfrak{m}(z)\,\mathrm{d}\mathfrak{m}(w) < \infty, \qquad \forall R > 0,\ \forall \bar{x} \in X.$$

Observe that for every $w \in B(\bar{x}, R)$ it holds $B(\bar{x}, R) \subset B(w, 2R)$. Hence

$$\begin{aligned}
&\int_{B(\bar{x},R)} \int_{B(\bar{x},R)} H(w, \mathsf{d}(w,z))\,\mathrm{d}\mathfrak{m}(z)\,\mathrm{d}\mathfrak{m}(w) \\
&\qquad \leq \int_{B(\bar{x},R)} \int_{B(w,2R)} H(w, \mathsf{d}(w,z))\,\mathrm{d}\mathfrak{m}(z)\,\mathrm{d}\mathfrak{m}(w) \\
&\qquad = \int_{B(\bar{x},R)} [2R + \mathfrak{m}(B(w,2R))H_w(2R)]\,\mathrm{d}\mathfrak{m}(w) \\
&\qquad \leq 2R\mathfrak{m}(B(\bar{x},R)) + \mathfrak{m}(B(\bar{x},3R)) \int_{B(\bar{x},R)} H_w(2R)\,\mathrm{d}\mathfrak{m}(w),
\end{aligned}$$

where we used (3.16) passing from the second to the third line above. Since $B(\bar{x}, s/2) \subset B(w,s)$ for every $w \in B(\bar{x}, R)$ and $s > 2R$, we obtain

$$\begin{aligned}
\int_{B(\bar{x},R)} H_w(2R)\,\mathrm{d}\mathfrak{m}(w) &= \int_{2R}^\infty \int_{B(\bar{x},R)} \frac{1}{\mathfrak{m}(B(w,s))}\,\mathrm{d}\mathfrak{m}(w)\,\mathrm{d}s \\
&\leq \int_{2R}^\infty \int_{B(\bar{x},R)} \frac{1}{\mathfrak{m}(B(\bar{x},s/2))}\,\mathrm{d}\mathfrak{m}(w)\,\mathrm{d}s \\
&= \mathfrak{m}(B(\bar{x},R)) \int_{2R}^\infty \frac{1}{\mathfrak{m}(B(\bar{x},s/2))}\,\mathrm{d}s < \infty.
\end{aligned}$$

□

The following lemma deals with the regularity properties of $G_x^\varepsilon$, that is a regular approximation of $G_x$.

**Lemma 3.6.** *Let* $(X, \mathsf{d}, \mathfrak{m})$ *be an* $\mathrm{RCD}(0,N)$ *space satisfying Assumption 1 and fix* $x \in X$. *For every* $0 < \varepsilon < 1$ *the function* $G_x^\varepsilon$ *belongs to* $\mathrm{Lip}_b(X) \cap D_{\mathrm{loc}}(\Delta)$ *and it holds* $\Delta G_x^\varepsilon = -p_\varepsilon(x,\cdot)$. *Moreover* $G_x \in H^{1,1}_{\mathrm{loc}}(X, \mathsf{d}, \mathfrak{m})$ *and*

$$\lim_{\varepsilon \to 0} G_x^\varepsilon = G_x \qquad \text{in } H^{1,1}_{\mathrm{loc}}(X, \mathsf{d}, \mathfrak{m}). \tag{3.17}$$

*Proof.* First of all let us prove that $G_x^\varepsilon \in L^\infty(X, \mathfrak{m})$. Using (1.20) and Assumption 1 we have

$$G_x^\varepsilon(y) = \int_\varepsilon^\infty p_t(x,y)\,\mathrm{d}y \le \int_\varepsilon^\infty \frac{C_1}{\mathfrak{m}(B(x,\sqrt{t}))}\,\mathrm{d}t = 2C_1 \int_{\sqrt{\varepsilon}}^\infty \frac{t}{\mathfrak{m}(B(x,t))}\,\mathrm{d}t < \infty.$$

The proof of the regularity statement $G_x^\varepsilon \in \mathrm{Lip_b}(X)$ will follow after proving that the identity $G_x^{\alpha+t} = P_t G_x^\alpha$ holds true for any $\alpha, t \in (0,\infty)$ by the regularization properties of the heat semigroup (since we proved that $G^\alpha \in L^\infty$). To this aim, for any $x, y \in X$ and for any $t, \alpha > 0$, we compute

$$\begin{aligned} P_t G_x^\alpha(y) &= \int p_t(y,z) G_x^\alpha(z)\,\mathrm{d}\mathfrak{m}(z) = \int_\alpha^\infty \int p_t(y,z) p_s(x,z)\,\mathrm{d}\mathfrak{m}(z)\,\mathrm{d}s \\ &= \int_\alpha^\infty p_{t+s}(x,y)\,\mathrm{d}s = \int_{\alpha+t}^\infty p_s(x,y)\,\mathrm{d}s = G_x^{\alpha+t}(y). \end{aligned}$$

In order to prove that $G_x^\varepsilon \in D_{\mathrm{loc}}(\Delta)$ and $\Delta G_x^\varepsilon = p_\varepsilon(x,\cdot)$ we consider a function $f \in \mathrm{Test}(X, \mathsf{d}, \mathfrak{m})$ and we compute

$$\int G_x^\varepsilon(w) \Delta f(w)\,\mathrm{d}\mathfrak{m}(w) = \int_\varepsilon^\infty P_t \Delta f(x)\,\mathrm{d}t = -P_\varepsilon f(x),$$

where the last equality follows from the observation that $P_r f \to 0$ pointwise as $r \to \infty$ for any $f \in L^1 \cap L^2(X, \mathfrak{m})$, that is a consequence of the estimates for the heat kernel (1.20) and the fact that $\mathfrak{m}(X) = \infty$.

Let us prove (3.17). We preliminary observe that $G_x^\varepsilon \to G_x$ in $L^1_{\mathrm{loc}}(X, \mathfrak{m})$, since $G_x - G_x^\varepsilon \ge 0$ and

$$\int \{G_x(y) - G_x^\varepsilon(y)\}\,\mathrm{d}\mathfrak{m}(y) = \int \int_0^\varepsilon p_t(x,y)\,\mathrm{d}t\,\mathrm{d}\mathfrak{m}(y) = \int_0^\varepsilon \int p_t(x,y)\,\mathrm{d}\mathfrak{m}(y)\,\mathrm{d}t = \varepsilon.$$

To conclude the proof it suffices to show that $G_x^\varepsilon$ is a Cauchy sequence in $W^{1,1}_{\mathrm{loc}}(X, \mathsf{d}, \mathfrak{m})$. We claim that, for every $0 < \varepsilon_1 < \varepsilon_2 < 1$,

$$|\nabla(G_x^{\varepsilon_1} - G_x^{\varepsilon_2})|(y) = \mathrm{Lip}(G_x^{\varepsilon_1} - G_x^{\varepsilon_2})(y) \le \int_{\varepsilon_2}^{\varepsilon_1} \mathrm{Lip}\, p_t(x,\cdot)(y)\,\mathrm{d}t, \quad \text{for } \mathfrak{m}\text{-a.e. } y \in X. \tag{3.18}$$

As a consequence of the Bishop-Gromov inequality (1.15) we get

$$\begin{aligned} \sup_{t>0} \int \frac{e^{-\frac{\mathsf{d}^2(x,y)}{5t}}}{\mathfrak{m}(B(x,\sqrt{t}))}\,\mathrm{d}\mathfrak{m}(y) &= \sup_{t>0} \frac{1}{\mathfrak{m}(B(x,\sqrt{t}))} \int \int_{\mathsf{d}^2(x,y)/t}^\infty \frac{e^{-s/5}}{5}\,\mathrm{d}s\,\mathrm{d}\mathfrak{m}(y) \\ &= \sup_{t>0} \int_0^\infty \frac{e^{-s/5}}{5} \frac{\mathfrak{m}(B(x,\sqrt{st}))}{\mathfrak{m}(B(x,\sqrt{t}))}\,\mathrm{d}s \\ &\le \int_0^\infty \frac{e^{-s/5}}{5} \max\{s,1\}^{N/2}\,\mathrm{d}s < \infty, \end{aligned}$$

that, together with the estimates for the gradient of the heat kernel (1.21), implies

$$\int_0^1 \int |\nabla p_t(x,\cdot)|(y)\,\mathrm{d}\mathfrak{m}(y)\,\mathrm{d}t \le \int_0^1 \frac{C_2}{\sqrt{t}} \int \frac{e^{-\frac{\mathsf{d}^2(x,y)}{5t}}}{\mathfrak{m}(B(x,\sqrt{t}))}\,\mathrm{d}\mathfrak{m}(y)\,\mathrm{d}t < \infty,$$

therefore (3.18) will yield the desired conclusion. This being said let us pass to the verification of (3.18). Observe that the $\mathfrak{m}$-a.e. identifications between slopes and minimal weak upper gradients above follow from the local Lipschitz regularity of the heat kernel and $G_x^\varepsilon$ for $\varepsilon > 0$ thanks to Theorem 1.26. Observe that the very definition of $G^\varepsilon$ grants that

$$\operatorname{Lip}(G_x^{\varepsilon_1} - G_x^{\varepsilon_2})(y) \le \limsup_{z\to y} \int_{\varepsilon_2}^{\varepsilon_1} \frac{|p_t(x,y) - p_t(x,z)|}{\mathsf{d}(y,z)}\,\mathrm{d}t, \quad \text{for every } y \in X. \tag{3.19}$$

Moreover, for any $r < \frac{1}{2}d(x,y)$, the gradient estimate for the heat kernel (1.21) yields

$$|\nabla p_t(x,\cdot)|(w) \le \frac{C_1 e^{-\frac{r^2}{5t}}}{\sqrt{t}\mathfrak{m}(B(x,\sqrt{t}))} \quad \text{for } \mathfrak{m}\text{-a.e } w \in B(y,r). \tag{3.20}$$

Hence $p_t(x,\cdot)$ is Lipschitz in $B(y,r/2)$ with Lipschitz constant bounded from above by the right hand side of (3.20), thanks to a local version of the *Sobolev to Lipschitz property*. Summarizing we obtain the bound

$$\frac{|p_t(x,y) - p_t(x,z)|}{\mathsf{d}(y,z)} \le \frac{C_1 e^{-\frac{r^2}{5t}}}{\sqrt{t}\mathfrak{m}(B(x,\sqrt{t}))},$$

for every $z \in B(y,r/2)$ and every $t \in (0,\infty)$. Hence we can apply Fatou's lemma and pass from (3.19) to (3.18). ☐

**Remark 3.7.** Proceeding as in the proof of Lemma 3.6 above, one can prove that, for any $\eta \in \operatorname{Test}(X,\mathsf{d},\mathfrak{m})$ with compact support, it holds that $\eta G_x^\varepsilon \in \operatorname{Test}(X,\mathsf{d},\mathfrak{m})$ for any $x \in X$ and for any $\varepsilon > 0$.

We state another technical lemma, its elementary proof can be obtained with minor modifications to the proof of [**97**, Lemma 5.50].

**Lemma 3.8.** *Let $\varphi : (0,\infty) \to (0,\infty)$ be monotone increasing and set*

$$\psi(r) := \int_0^\infty \frac{1}{\varphi(\sqrt{t})} \exp\left(-\frac{r^2}{t}\right)\mathrm{d}t.$$

*If $\varphi$ satisfies the local doubling property*

$$\varphi(2r) \le C(R)\varphi(r) \quad \textit{for any } 0 < r < R,$$

*for some non decreasing function $C : (0,\infty) \to (0,\infty)$, then there exists a non decreasing function $\Lambda : (0,\infty) \to (0,\infty)$, whose values depend only on the function $C$, such that*

$$\frac{1}{\Lambda(R)} \int_r^\infty \frac{s}{\varphi(s)}\,\mathrm{d}s \le \psi(r) \le \Lambda(R) \int_r^\infty \frac{s}{\varphi(s)}\,\mathrm{d}s, \tag{3.21}$$

*for any $0 < r < R$ and for any $R \in (0,\infty)$. Moreover, when $C$ is constant, we can choose $\Lambda$ to be constant.*

*Proof of Proposition 3.4.* The proof of (3.12) follows from the estimates for the heat kernel (1.20) applying Lemma 3.8 with $\varphi(r) := \mathfrak{m}(B(x,r))$.

In order to prove (3.13) we observe that, arguing exactly as in the proof of (3.17), one can prove that, for any $\varepsilon > 0$ and any $x \in X$,

$$|\nabla G_x^\varepsilon|(y) \leq \int_\varepsilon^\infty |\nabla p_t(x,\cdot)|(y) \quad \text{for } \mathfrak{m}\text{-a.e. } y \in X. \tag{3.22}$$

The sought conclusion follows from (3.17). The proof of the inequality

$$\int_0^\infty |\nabla p_t(x,\cdot)|(y)\,\mathrm{d}t \leq C_2 H_x(\mathsf{d}(x,y)), \qquad \text{for } \mathfrak{m}\text{-a.e. } y \in X$$

follows from the gradient estimate for the heat kernel (1.21), applying Lemma 3.8 with choice $\varphi(r) := r\mathfrak{m}(B(x,r))$. □

**Remark 3.9.** It is clear from the proof of Proposition 3.4 that the regularized functions $G^\varepsilon$ satisfy

$$|\nabla G_x^\varepsilon|(y) \leq \int_\varepsilon^\infty |\nabla p_t(x,\cdot)|(y)\,\mathrm{d}t \leq C_2 H_x(\mathsf{d}(x,y)), \qquad \text{for } \mathfrak{m}\text{-a.e. } y \in X.$$

**Remark 3.10.** As a consequence of (3.13) and of the continuity of the map $x \mapsto H_x(r)$, by exploiting the monotonicity w.r.t. $r$ of $H$ and a local version of the *Sobolev to Lipschitz* property, one can prove that $G_x$ is continuous in $X \setminus \{x\}$.

**1.2. Green maximal estimate.** Let us state and prove a maximal estimate that, as we anticipated in the introduction, is a key tool to bound the rate of change of the Green function along trajectories of a Lagrangian flow. It will be crucial in the proof of the vector-valued maximal estimate Proposition 3.19.

**Proposition 3.11** (Maximal estimate, scalar version)**.** *Let $(X,\mathsf{d},\mathfrak{m})$ be an* $\mathrm{RCD}(0,N)$ *metric measure space satisfying Assumption 1. Then there exists $C_M > 0$, depending only on $N$, such that, for any Borel function $f : X \to [0,\infty)$, it holds*

$$\int f(w)\,|\nabla G_x(w)|\,|\nabla G_y(w)|\,\mathrm{d}\mathfrak{m}(w) \leq C_M G(x,y)\left(Mf(x) + Mf(y)\right), \tag{3.23}$$

*for every $x, y \in X$.*

*Proof.* Fix two different points in $x, y \in X$. Thanks to (3.13) we can estimate the left hand side of (3.23) with

$$C_2^2 \int_0^\infty \int_0^\infty \int f(w) \frac{\mathbf{1}_{B(x,r)}(w)}{\mathfrak{m}(B(x,r))} \frac{\mathbf{1}_{B(y,s)}(w)}{\mathfrak{m}(B(y,s))}\,\mathrm{d}\mathfrak{m}(w)\,\mathrm{d}s\,\mathrm{d}r.$$

By splitting the domain $(0,\infty) \times (0,\infty)$ into $A_1, A_2$ and $A_3$, with $A_1 := \{(s,r)|\ \mathsf{d}(x,y) + s \leq r\}$, $A_2 := \{(s,r)|\ \mathsf{d}(x,y) + r \leq s\}$ and $A_3 := \{(s,r)|\ \mathsf{d}(x,y) > |r-s|\}$ we are left with the estimates of the following quantities:

$$I_1 := \int_{A_1} \int f(w) \frac{\mathbf{1}_{B(x,r)}(w)}{\mathfrak{m}(B(x,r))} \frac{\mathbf{1}_{B(y,s)}(w)}{\mathfrak{m}(B(y,s))}\,\mathrm{d}\mathfrak{m}(w)\,\mathrm{d}s\,\mathrm{d}r,$$

$$I_2 := \int_{A_2} \int f(w) \frac{\mathbf{1}_{B(x,r)}(w)}{\mathfrak{m}(B(x,r))} \frac{\mathbf{1}_{B(y,s)}(w)}{\mathfrak{m}(B(y,s))}\,\mathrm{d}\mathfrak{m}(w)\,\mathrm{d}s\,\mathrm{d}r$$

and

$$I_3 := \int_{A_3} \int f(w) \frac{\mathbf{1}_{B(x,r)}(w)}{\mathfrak{m}(B(x,r))} \frac{\mathbf{1}_{B(y,s)}(w)}{\mathfrak{m}(B(y,s))} \, \mathrm{d}\mathfrak{m}(w) \, \mathrm{d}s \, \mathrm{d}r.$$

In order to estimate $I_1$, we observe that $B(y,s) \subset B(x,r)$ for every $(s,r) \in A_1$, thus

$$\begin{aligned} I_1 =& \int_{A_1} \frac{1}{\mathfrak{m}(B(x,r))} \fint_{B(y,s)} f(w) \, \mathrm{d}\mathfrak{m}(w) \, \mathrm{d}s \, \mathrm{d}r \\ \leq & Mf(y) \int_{\mathsf{d}(x,y)}^{\infty} \int_0^{r-\mathsf{d}(x,y)} \frac{1}{\mathfrak{m}(B(x,r))} \, \mathrm{d}s \, \mathrm{d}r \\ \leq & Mf(y) \int_{\mathsf{d}(x,y)}^{\infty} \frac{r}{\mathfrak{m}(B(x,r))} \, \mathrm{d}r \\ \leq & C_2 G(x,y) Mf(y). \end{aligned}$$

By symmetry we get

$$I_2 \leq C_2 G(x,y) Mf(x).$$

To estimate $I_3$ let us observe that, if $r + s < \mathsf{d}(x,y)$, then $B(x,r) \cap B(y,s) = \emptyset$. Thus the integration can be restricted to the smaller domain $B := \{(s,r) |\ \mathsf{d}(x,y) > |r-s|,\ r+s \geq \mathsf{d}(x,y)\}$ that we split once more into $B_1 := \{(s,r) |\ \mathsf{d}(x,y) > r-s,\ r+s \geq \mathsf{d}(x,y),\ r \geq s\}$ and $B_2 := \{(s,r) |\ \mathsf{d}(x,y) > s-r,\ r+s \geq \mathsf{d}(x,y),\ r < s\}$. Therefore we have

$$\begin{aligned} I_3 =& \int_{B} \int f(w) \frac{\mathbf{1}_{B(x,r)}(w)}{\mathfrak{m}(B(x,r))} \frac{\mathbf{1}_{B(y,s)}(w)}{\mathfrak{m}(B(y,s))} \, \mathrm{d}\mathfrak{m}(w) \, \mathrm{d}s \, \mathrm{d}r \\ =& \int_{B_1} \int f(w) \frac{\mathbf{1}_{B(x,r)}(w)}{\mathfrak{m}(B(x,r))} \frac{\mathbf{1}_{B(y,s)}(w)}{\mathfrak{m}(B(y,s))} \, \mathrm{d}\mathfrak{m}(w) \, \mathrm{d}s \, \mathrm{d}r \\ &+ \int_{B_2} \int f(w) \frac{\mathbf{1}_{B(x,r)}(w)}{\mathfrak{m}(B(x,r))} \frac{\mathbf{1}_{B(y,s)}(w)}{\mathfrak{m}(B(y,s))} \, \mathrm{d}\mathfrak{m}(w) \, \mathrm{d}s \, \mathrm{d}r \\ =:& I_3^1 + I_3^2. \end{aligned}$$

We now deal with $I_3^1$. Using the rough estimate $\mathbf{1}_{B(x,r)} \leq 1$ we obtain

$$\begin{aligned} I_3^1 \leq & \int_{B_1} \frac{1}{\mathfrak{m}(B(x,r))} \fint_{B(y,s)} f(w) \, \mathrm{d}\mathfrak{m}(w) \, \mathrm{d}s \, \mathrm{d}r \\ \leq & Mf(y) \int_{\mathsf{d}(x,y)/2}^{\infty} \int_{|\mathsf{d}(x,y)-r|}^{r} \frac{1}{\mathfrak{m}(B(x,r))} \, \mathrm{d}s \, \mathrm{d}r \\ \leq & Mf(y) \int_{\mathsf{d}(x,y)/2}^{\infty} \frac{r}{\mathfrak{m}(B(x,r))} \, \mathrm{d}r \\ =& Mf(y) \frac{1}{4} \int_{\mathsf{d}(x,y)}^{\infty} \frac{r}{\mathfrak{m}(B(x,r/2))} \, \mathrm{d}r. \end{aligned}$$

With a simple application of (1.15) and (3.12) we conclude that

$$I_3^1 \leq C(C_2, N) Mf(y) G(x,y).$$

By symmetry we also have $I_3^2 \leq C(C_2, N) Mf(x) G(x,y)$. Putting all these estimates together we obtain the desired result. $\square$

**Remark 3.12.** It is clear from the proof of Proposition 3.11 and from Remark 3.9 that the same estimate holds true if one puts $\nabla G_x^\varepsilon$ and $\nabla G_y^\varepsilon$ in place of $\nabla G_x$ and $\nabla G_y$ at the left hand side of (3.23). More precisely it holds that

$$\int f(z)\,|\nabla G_x^\varepsilon(z)|\,\left|\nabla G_y^\varepsilon(z)\right|\,\mathrm{d}\mathfrak{m}(z) \le C_M G(x,y)\,(Mf(x)+Mf(y))\,, \tag{3.24}$$

for every $x, y \in X$.

**1.3. The Green quasi-metric.** This section is devoted to the study of the following function

$$\mathsf{d}_G(x,y) := \begin{cases} \frac{1}{G(x,y)} & \text{if } x \neq y, \\ 0 & \text{otherwise.} \end{cases} \tag{3.25}$$

Our aim is to prove that $\mathsf{d}_G$ is a *quasi-metric* on $X$ (i.e. it satisfies an approximated triangle inequality, see Proposition 3.14 below) and that $\mathfrak{m}$ is still a doubling measure over $(X, \mathsf{d}_G)$ (see Proposition 3.17 below for a precise statement). The terminology, quite common in the literature about analysis on metric spaces, is borrowed from [**100**, Chapter 14]. In order to do so we will need to impose an assumption stronger than Assumption 1 to the m.m.s. $(X, \mathsf{d}, \mathfrak{m})$.

**Assumption 2.** There exists an $\mathrm{RCD}(0, N-3)$ metric measure space $(\bar{X}, \bar{\mathsf{d}}, \bar{\mathfrak{m}})$ such that $(X, \mathsf{d}, \mathfrak{m})$ is the tensor product between $(\bar{X}, \bar{\mathsf{d}}, \bar{\mathfrak{m}})$ and $(\mathbb{R}^3, \mathsf{d}_{\mathbb{R}^3}, \mathscr{L}^3)$.

First of all observe that $\mathsf{d}_G$ is symmetric and positive whenever $x \neq y$. Moreover, for every $x \in X$, the map $y \mapsto \mathsf{d}_G(x,y)$ is continuous. Indeed, thanks to the continuity of $G_x$ in $X \setminus \{x\}$ (see Remark 3.10 above), we need only to show that $\mathsf{d}_G(x, \cdot)$ is continuous at $x$, and this is the content of the following lemma.

**Lemma 3.13.** *Let $(X, \mathsf{d}, \mathfrak{m})$ be an $\mathrm{RCD}(0, N)$ m.m.s. satisfying Assumption 2. Then for any $x \in X$ it holds that $\mathsf{d}_G(x,y) \to 0$ if and only if $\mathsf{d}(x,y) \to 0$.*

*Proof.* Suppose that $\mathsf{d}_G(x,y) \to 0$. Then, by the very definition of $\mathsf{d}_G$, it must be $G(x,y) \to \infty$. Hence, since we have the uniform control $G(x,y) \le C_2 F(x, \mathsf{d}(x,y))$ and $F(x, \cdot)$ is bounded away from 0, we conclude $\mathsf{d}(x,y) \to 0$.

In order to prove the converse we observe that, if $\mathsf{d}(x,y) \to 0$, then $F(x, \mathsf{d}(x,y)) \to \infty$. Indeed, under our assumptions, $s \mapsto s/\mathfrak{m}(B(x,s))$ is not integrable at 0 and to conclude we just need to exploit the bound $G(x,y) \ge 1/C_2 F(x, \mathsf{d}(x,y))$ (see Proposition 3.4 above). ☐

**Proposition 3.14** (Almost triangle inequality)**.** *Let $(X, \mathsf{d}, \mathfrak{m})$ be an $\mathrm{RCD}(0, N)$ space satisfying Assumption 1. Then there exists a constant $C_T \ge 1$, depending only on $N$, such that*

$$\mathsf{d}_G(x,y) \le C_T(\mathsf{d}_G(x,z) + \mathsf{d}_G(z,y)) \quad \textit{for any } x, y \textit{ and } z \in X. \tag{3.26}$$

The core of the proof of Proposition 3.14 is contained in the following elementary lemma.

**Lemma 3.15.** *Let $(X, \mathsf{d}, \mathfrak{m})$ be a doubling metric measure space satisfying Assumption 1. Then there exists a constant $C$, depending only on the doubling constant of $(X, \mathsf{d}, \mathfrak{m})$, such that*

$$F_x(\mathsf{d}(x,z))F_y(\mathsf{d}(y,z)) \le C(F_x(\mathsf{d}(x,y))F_y(\mathsf{d}(y,z)) + F_y(\mathsf{d}(x,y))F_x(\mathsf{d}(x,z))),$$

*for any $x, y, z$ in $X$.*

*Proof.* Let us fix $x, y, z$ in $X$. We can assume without loss of generality that they are all different. Starting from the identity

$$F_x(\mathsf{d}(x,z))F_y(\mathsf{d}(y,z)) = \int_{\mathsf{d}(x,z)}^{\infty}\int_{\mathsf{d}(y,z)}^{\infty} \frac{t}{\mathfrak{m}(B(x,t))}\frac{s}{\mathfrak{m}(B(y,s))}\,\mathrm{d}t\,\mathrm{d}s,$$

and exploiting the inclusion

$$\{\mathsf{d}(x,z)<t,\ \mathsf{d}(y,z)<s\} \subset \{\mathsf{d}(x,y)<2t,\ \mathsf{d}(y,z)<s\} \cup \{\mathsf{d}(x,y)<2s,\ \mathsf{d}(x,z)<t\},$$

we get

$$\begin{aligned} F_x(\mathsf{d}(x,z))F_y(\mathsf{d}(y,z)) \leq & \int_{\mathsf{d}(x,y)/2}^{\infty}\int_{\mathsf{d}(y,z)}^{\infty} \frac{t}{\mathfrak{m}(B(x,t))}\frac{s}{\mathfrak{m}(B(y,s))}\,\mathrm{d}t\,\mathrm{d}s \\ & + \int_{\mathsf{d}(x,z)}^{\infty}\int_{\mathsf{d}(x,y)/2}^{\infty} \frac{t}{\mathfrak{m}(B(x,t))}\frac{s}{\mathfrak{m}(B(y,s))}\,\mathrm{d}t\,\mathrm{d}s \\ = & F_x(\mathsf{d}(x,y)/2)F_y(\mathsf{d}(y,z)) + F_y(\mathsf{d}(x,y)/2)F_x(\mathsf{d}(x,z)). \end{aligned}$$

To conclude, observe that $F_w(r/2) \leq CF_w(r)$ for any $r>0$ and $w \in X$, where $C$ depends only on the doubling constant of $\mathfrak{m}$. □

*Proof of Proposition 3.14.* The desired conclusion (3.26) is equivalent to

$$G_x(z)G_y(z) \leq C_T G(x,y)(G_x(z)+G_y(z)),$$

that follows from Lemma 3.15 taking into account Proposition 3.4. □

We introduce the notation

$$B^G(x,r) := \{y \in X | \mathsf{d}_G(x,y) < r\} \tag{3.27}$$

to denote the balls with respect to the quasi-metric $\mathsf{d}_G$. The next result of this short section is about the doubling property of the measure $\mathfrak{m}$ in the quasi-metric space $(X, \mathsf{d}_G)$.

**Lemma 3.16** (Reverse Bishop-Gromov inequality)**.** *Let $(\bar{X}, \bar{\mathsf{d}}, \mu)$ be a doubling m.m.s. with doubling constant $C_\mu$ and denote by $(X, \mathsf{d}, \mathfrak{m})$ the tensor product between $(\bar{X}, \bar{\mathsf{d}}, \mu)$ and $(\mathbb{R}^k, \mathsf{d}_{\mathbb{R}^k}, \mathscr{L}^k)$ for some $k \geq 1$. Then*

$$\frac{\mathfrak{m}(B(x,R))}{\mathfrak{m}(B(x,r))} \geq \frac{1}{C_\mu\sqrt{2}^k}\left(\frac{R}{r}\right)^k, \tag{3.28}$$

*for every $0<r<R$ and for any $x \in X$.*

*Proof.* The following chain of inclusions holds true for any $x \in X$, any $v \in \mathbb{R}^k$ and any $r>0$:

$$B(x, r/\sqrt{2}) \times B(v, r/\sqrt{2}) \subset B((x,v), r) \subset B(x,r) \times B(v,r).$$

It follows that

$$\begin{aligned} \frac{\mathfrak{m}(B(x,v),R)}{\mathfrak{m}(B(x,v),r)} \geq & \frac{\mathfrak{m}(B(x,R/\sqrt{2}) \times B(v,R/\sqrt{2}))}{\mathfrak{m}(B(x,r)\times B(v,r))} = \frac{1}{\sqrt{2}^k}\frac{\mu(B(x,R/\sqrt{2}))}{\mu(B(x,r))}\left(\frac{R}{r}\right)^k \\ \geq & \frac{1}{\sqrt{2}^k}\frac{\mu(B(x,r/2))}{\mu(B(x,r))}\left(\frac{R}{r}\right)^k \geq \frac{1}{C_\mu\sqrt{2}^k}\left(\frac{R}{r}\right)^k. \end{aligned}$$

□

We finally state and prove the doubling property of $\mathfrak{m}$ with respect to the new quasi-metric $\mathsf{d}_G$.

**Proposition 3.17** (Doubling property)**.** *Assume that* $(X,\mathsf{d},\mathfrak{m})$ *is an* $\mathrm{RCD}(0,N)$ *m.m.s. satisfying Assumption 2. Then there exists a constant* $C^G>0$*, depending only on* $N$*, such that*

$$\mathfrak{m}(B^G(x,2r))\le C^G\mathfrak{m}(B^G(x,r)), \tag{3.29}$$

*for every* $x\in X$ *and every* $r>0$.

*Proof.* We begin by observing that, under Assumption 2, an application of Lemma 3.16 yields the existence of constants $a>1$ and $b<1$ such that $F_x(aR)\le bF_x(R)$ for any $x\in X$ and for any $R>0$. It follows that

$$F_x(R)=F_x(aR)+\int_R^{aR}\frac{s}{\mathfrak{m}(B(x,s))}\,\mathrm{d}s\le bF_x(R)+a\frac{R^2}{\mathfrak{m}(B(x,R))},$$

for any $x\in X$ and $R>0$, thus we get

$$F_x(R)\le\frac{a}{1-b}\frac{R^2}{\mathfrak{m}(B(x,R))}\quad\text{for any } x\in X \text{ and for any } R>0. \tag{3.30}$$

The inequality in (3.30) yields the existence of $\alpha>0$ such that $r\mapsto r^\alpha F_x(r)$ is nonincreasing on $(0,\infty)$ (just take $\alpha:=(1-b)/a$ and differentiate w.r.t. $r$). Hence we can find $\gamma<1$ such that

$$F_x(2R)\le\gamma F_x(R)\quad\text{for any } x\in X \text{ and for any } R>0. \tag{3.31}$$

Inequality (3.31) implies in turn that

$$F_x^{-1}(\gamma R)\le 2F_x^{-1}(R)\quad\text{for any } x\in X \text{ and for any } R>0. \tag{3.32}$$

The last ingredient we need are the estimates, valid for any $\lambda>0$ and for any $x\in X$:

$$\mathfrak{m}(\{G_x>\lambda\})\le\mathfrak{m}(\{F_x(\mathsf{d}_x)>\lambda/C_2\}),\quad \mathfrak{m}(\{F_x(\mathsf{d}_x)>\lambda\})\le\mathfrak{m}(\{G_x>\lambda/C_2\}), \tag{3.33}$$

where $C_2$ is the constant appearing in (3.12).

In order to conclude let us show that

$$\mathfrak{m}\left(B^G\left(x,\frac{1}{\gamma^M C_2}r\right)\right)\le 2^{M\cdot N}\mathfrak{m}(B^G(x,C_2r)), \tag{3.34}$$

for every $r>0$, $x\in X$ and $M\in\mathbb{N}$.

Using (3.33), the definition of $B^G$ and the fact that $F_x^{-1}$ is non-increasing we find

$$\begin{aligned}\mathfrak{m}\left(B^G\left(x,\frac{1}{\gamma^M C_2}r\right)\right)&=\mathfrak{m}\left(\left\{G_x>\frac{C_2\gamma^M}{r}\right\}\right)\le\mathfrak{m}\left(\left\{F_x(\mathsf{d}_x)>\frac{\gamma^M}{r}\right\}\right)\\&=\mathfrak{m}\left(B\left(x,F_x^{-1}\left(\gamma^M/r\right)\right)\right).\end{aligned}$$

Applying first $M$-times (3.32) and then the doubling inequality (1.15) in the $\mathrm{RCD}(0,N)$ m.m.s. $(X,\mathsf{d},\mathfrak{m})$, we get

$$\begin{aligned}\mathfrak{m}\left(B^G\left(x,\frac{1}{C_2\gamma^M}r\right)\right)&\le\mathfrak{m}(B(x,2^MF_x^{-1}(1/r)))\le 2^{M\cdot N}\mathfrak{m}(B(x,F_x^{-1}(1/r)))\\&=2^{M\cdot N}\mathfrak{m}(\{F_x(\mathsf{d}_x)>1/r\}).\end{aligned}$$

Using again (3.33), we obtain:

$$\mathfrak{m}\left(B^G\left(x,\frac{1}{C_2\gamma^M}r\right)\right)\le 2^{M\cdot N}\mathfrak{m}(\{G_x>\frac{1}{C_2 r}\})=2^{M\cdot N}\mathfrak{m}(B^G(x,C_2r)). \tag{3.35}$$

Setting $s:=C_2r$ in (3.35), we have that

$$\mathfrak{m}\left(B^G\left(x,\frac{1}{C_2^2\gamma^M}s\right)\right)\le 2^{(M+1)N}\mathfrak{m}(B^G(x,s)),$$

for any $s>0$, any $x\in X$ and every real number $M>1$. Choosing $M=\log_{\gamma^{-1}}(2C_2^2)$ we eventually obtain (3.29). $\square$

The last lemma of this subsection deals with the integrability properties of the maximal function associated to the quasi metric $\mathsf{d}_G$.

**Lemma 3.18.** *Assume that $(X,\mathsf{d},\mathfrak{m})$ is an* $\mathrm{RCD}(0,N)$ *m.m.s. satisfying Assumption 2. Then there exists a constant $C>0$, depending only on $N$, such that for any $f\in L^1_{\mathrm{loc}}(X,\mathfrak{m})$ it holds*

$$M^Gf(x)\le CMf(x),\qquad \forall x\in X, \tag{3.36}$$

*where $M^Gf(x):=\sup_{r>0}\fint_{B^G(x,r)}|f|\,\mathrm{d}\mathfrak{m}$ and $Mf$ is the Hardy Littlewood maximal function associated to $f$.*

*In particular $M^G$ is a bounded operator from $L^2(X,\mathfrak{m})$ into itself.*

*Proof.* Exploiting the inclusions (see (3.12))

$$\{F_x(\mathsf{d}_x)>C_2r\}\subset B^G(x,r)\subset\{F_x(\mathsf{d}_x)>r/C_2\},$$

we get

$$\fint_{B^G(x,r)}|f|\,\mathrm{d}\mathfrak{m}\le\frac{\mathfrak{m}(B(x,F_x^{-1}(r/C_2)))}{\mathfrak{m}(B(x,F_x^{-1}(C_2r)))}Mf(x).$$

Using the Bishop-Gromov inequality (1.13) and (3.31) we get

$$\frac{\mathfrak{m}(B(x,F_x^{-1}(r/C_2)))}{\mathfrak{m}(B(x,F_x^{-1}(C_2r)))}\le\left(\frac{F_x^{-1}(r/C_2)}{F_x^{-1}(C_2r)}\right)^N\le C(\gamma,C_2,N).$$

Recalling that $C_2$ and $\gamma$ are constants depending only on $N$, (3.36) follows.

Finally observe that the maximal operator $M$ maps $L^2(X,\mathfrak{m})$ into itself since $(X,\mathsf{d},\mathfrak{m})$ is a doubling m.m.s., this property is inherited by $M^G$ thanks to (3.36). $\square$

**1.4. A Lusin-type regularity result.** This section is dedicated to the study of the regularity of a flow $\boldsymbol{X}_t$ associated to a Sobolev time dependent vector field $b$. The regularity will be understood with respect to the newly introduced quasi-metric $\mathsf{d}_G$.

Let us begin with a crucial maximal estimate for vector fields.

**Proposition 3.19** (Maximal estimate, vector-valued version)**.** *Fix an* $\mathrm{RCD}(0,N)$ *m.m.s space $(X,\mathsf{d},\mathfrak{m})$ satisfying Assumption 1. Assume that $b\in W^{1,2}_{C,s}(TX)$ (Cf. Definition 1.65) is compactly supported and bounded. Then, setting $g:=|\nabla_{\mathrm{sym}}b|+|\operatorname{div}b|$, it holds*

$$|b\cdot\nabla G_x(y)+b\cdot\nabla G_y(x)|\le 2C_MG(x,y)(Mg(x)+Mg(y)), \tag{3.37}$$

*for $\mathfrak{m}\times\mathfrak{m}$-a.e. $(x,y)\in X\times X$, where $M$ stands for the maximal operator.*

*Proof.* Let us first explain the heuristic standing behind this result. Assuming that $b$ is divergence free we can formally compute

$$\begin{aligned} b\cdot\nabla G_x(y)+b\cdot\nabla G_y(x)=&-\int b\cdot\nabla G_x(w)\,\mathrm{d}\Delta G_y(w)-\int b\cdot\nabla G_y(w)\,\mathrm{d}\Delta G_x(w)\\ =&2\int\nabla_{\mathrm{sym}}b(w)(\nabla G_x(w),\nabla G_y(w))\,\mathrm{d}\mathfrak{m}(w),\end{aligned}$$

so that, taking the moduli and applying Proposition 3.11, we would reach the desired conclusion.

The proof of this result will be divided into two steps: in the first one we are going to prove an estimate for the regularized functions $G^{\varepsilon}$; in the second one the sought conclusion will be recovered by an approximation procedure.

**Step 1** We start proving that, for every $\varepsilon\in(0,1)$ and for every $x,y\in X$, it holds

$$\left|\int\left\{b\cdot\nabla G^{\varepsilon}_x\Delta G^{\varepsilon}_y+b\cdot\nabla G^{\varepsilon}_y\Delta G^{\varepsilon}_x\right\}\mathrm{d}\mathfrak{m}\right|\le 2C_M G(x,y)\left(Mg(x)+Mg(y)\right).\tag{3.38}$$

To this aim, we choose a cut-off function with compact support $\eta\in\mathrm{Test}(X,\mathsf{d},\mathfrak{m})$ such that $\eta\equiv1$ on $\operatorname{supp}b$ (the existence of such function follows from Lemma 1.60). Applying (1.40) with $h=\eta$, $f=\eta G^{\varepsilon}_x$ and $g=\eta G^{\varepsilon}_y$ (observe that they are admissible test function in the definition of symmetric covariant derivative thanks to Remark 3.7) we obtain:

$$\begin{aligned}&\left|\int\left\{b\cdot\nabla G^{\varepsilon}_x\Delta G^{\varepsilon}_y+b\cdot\,\nabla G^{\varepsilon}_y\Delta G^{\varepsilon}_x\right\}\mathrm{d}\mathfrak{m}\right|\\ &\qquad\le\left|\int\left\{b\cdot\nabla G^{\varepsilon}_x\Delta G^{\varepsilon}_y+b\cdot\nabla G^{\varepsilon}_y\Delta G^{\varepsilon}_x-\operatorname{div}b\,\nabla G^{\varepsilon}_x\cdot\nabla G^{\varepsilon}_y\right\}\mathrm{d}\mathfrak{m}\right|\\ &\qquad\quad+\left|\int\operatorname{div}b\,\nabla G^{\varepsilon}_x\cdot\nabla G^{\varepsilon}_y\,\mathrm{d}\mathfrak{m}\right|\\ &\qquad=2\left|\int\nabla_{\mathrm{sym}}b\,(\nabla G^{\varepsilon}_x,\nabla G^{\varepsilon}_y)\,\mathrm{d}\mathfrak{m}\right|+\left|\int\operatorname{div}b\,\nabla G^{\varepsilon}_x\cdot\nabla G^{\varepsilon}_y\,\mathrm{d}\mathfrak{m}\right|\\ &\qquad\le2\int g(w)|\nabla G^{\varepsilon}_x(w)||\nabla G^{\varepsilon}_y(w)|\,\mathrm{d}\mathfrak{m}(w).\end{aligned}$$

The estimate in (3.38) follows from the inequality we just obtained applying (3.24).

**Step 2** The second step of the proof aims into proving that, as $\varepsilon\to0$, it holds

$$\left|\int\left\{b\cdot\nabla G^{\varepsilon}_x\Delta G^{\varepsilon}_y+b\cdot\nabla G^{\varepsilon}_y\Delta G^{\varepsilon}_x\right\}\mathrm{d}\mathfrak{m}\right|\to\left|b\cdot\nabla G_x(y)+b\cdot\nabla G_y(x)\right|\tag{3.39}$$

in $L^1_{\mathrm{loc}}(X\times X,\mathfrak{m}\times\mathfrak{m})$. This will allow us to get (3.37) by choosing a sequence $\varepsilon_i\downarrow0$ such that the convergence in (3.39) holds true $\mathfrak{m}\times\mathfrak{m}$-a.e. on $X\times X$ and exploiting what we proved in the first step.

In order to prove (3.39), we start recalling that $\Delta G^{\varepsilon}_y(w)=-p_{\varepsilon}(y,w)$ for any $\varepsilon>0$ (see Lemma 3.6). Thus

$$\int b\cdot\nabla G^{\varepsilon}_x\Delta G^{\varepsilon}_y\,\mathrm{d}\mathfrak{m}=-P_{\varepsilon}(b\cdot\nabla G^{\varepsilon}_x)(y)\quad\text{for any }x,y\in X.\tag{3.40}$$

Moreover for our purposes it suffices to check that

$$\int_K\int_K\left|P_{\varepsilon}(b\cdot\nabla G^{\varepsilon}_x)(y)-b\cdot\nabla G_x(y)\right|\mathrm{d}\mathfrak{m}(x)\,\mathrm{d}\mathfrak{m}(y)\to0$$

as $\varepsilon \to 0$, for every compact $K \subset X$. Adding and subtracting $P_\varepsilon(b \cdot \nabla G_x)(y)$ (that is well defined since $b \cdot \nabla G_x \in L^1(X, \mathfrak{m})$), we obtain

$$\begin{aligned}\int_K \int_K &|P_\varepsilon(b \cdot \nabla G_x^\varepsilon)(y) - b \cdot \nabla G_x(y)| \, \mathrm{d}\mathfrak{m}(x) \, \mathrm{d}\mathfrak{m}(y) \\ &\leq \int_K \|P_\varepsilon(b \cdot \nabla(G_x^\varepsilon - G_x))\|_{L^1(X,\mathfrak{m})} \, \mathrm{d}\mathfrak{m}(x) + \int_K \|P_\varepsilon(b \cdot \nabla G_x) - b \cdot \nabla G_x\|_{L^1(X,\mathfrak{m})} \, \mathrm{d}\mathfrak{m}(x).\end{aligned}$$

Using the $L^1$-norm contractivity property of the semigroup $P_\varepsilon$, we deduce that

$$\|P_\varepsilon(b \cdot \nabla(G_x^\varepsilon - G_x))\|_{L^1(X,\mathfrak{m})} \leq \|b \cdot \nabla(G_x^\varepsilon - G_x)\|_{L^1(X,\mathfrak{m})} \quad \text{for any } x \in X.$$

Hence, for any $x \in X$,

$$\|P_\varepsilon(b \cdot \nabla(G_x^\varepsilon - G_x))\|_{L^1(X,\mathfrak{m})} \to 0 \quad \text{as } \varepsilon \downarrow 0,$$

since $G_x^\varepsilon \to G^x$ in $H^{1,1}_{\mathrm{loc}}(X, \mathsf{d}, \mathfrak{m})$ by Proposition 3.4 and $b$ has compact support by assumption. Also the term $\|P_\varepsilon(b \cdot \nabla G_x) - b \cdot \nabla G_x\|_{L^1(X,\mathfrak{m})}$ goes to zero for every $x \in X$ since, as just remarked, $b \cdot \nabla G_x \in L^1(X, \mathfrak{m})$. Moreover both these terms are uniformly bounded by the function $x \mapsto C \|b\|_{L^\infty} \|H_x(\mathsf{d}_x(\cdot))\|_{L^1(\mathrm{supp}(b),\mathfrak{m})}$ that is locally integrable, since the map $(x, y) \mapsto H(x, \mathsf{d}(x, y))$ belongs to $L^1_{\mathrm{loc}}(X \times X, \mathfrak{m} \times \mathfrak{m})$ in view of Lemma 3.5. The conclusion of (3.39) can now be recovered applying the dominated convergence theorem. □

Up to the end of this section we let $(X, \mathsf{d}, \mathfrak{m})$ be an $\mathrm{RCD}(0, N)$ m.m.s. satisfying Assumption 2.

Let us fix $T > 0$ and let $b_t \in L^\infty((0, T) \times X)$ be a time dependent vector field with compact support, uniformly w.r.t. time. We further assume that $b_t \in W^{1,2}_{C,s}(TX)$ (Cf. Definition 1.65) for a.e. $t \in (0, T)$ and that $|\nabla_{\mathrm{sym}} b_t| \in L^1((0, T), L^2(X, \mathfrak{m}))$ and $\operatorname{div} b_t \in L^1((0, T), L^\infty(X, \mathfrak{m}))$. Under these assumptions, the theory developed in [**27**] guarantees existence and uniqueness of the regular Lagrangian Flow $(\boldsymbol{X}_t)_{t\in[0,T]}$ of $b$. We shall denote by $L \geq 0$ its compressibility constant (Cf. Theorem 1.78)

Our aim is to implement a strategy very similar to the one adopted in [**63**] (in the Euclidean setting) and in [**42**], in order to prove a Lusin-type regularity result for RLFs in terms of the newly defined quasi-metric $\mathsf{d}_G$.

Let us spend some words to explain the very simple idea behind the just mentioned strategy. Having in mind the standard Gronwall argument explained in the introduction (see (3.1)), it is natural to try to estimate the time derivative of $\mathsf{d}_G(\boldsymbol{X}_t(x), \boldsymbol{X}_t(y))$. In order to do so, we use Corollary 3.39 and Proposition 3.19 obtaining

$$\left|\frac{\mathrm{d}}{\mathrm{d}t} G(\boldsymbol{X}_t(x), \boldsymbol{X}_t(y))\right| \leq 2C_M G(\boldsymbol{X}_t(x), \boldsymbol{X}_t(y)) \left\{Mg(\boldsymbol{X}_t(x)) + Mg(\boldsymbol{X}_t(y))\right\}, \tag{3.41}$$

for $\mathscr{L}^1$-a.e. $t \in (0, T)$ and for $\mathfrak{m} \times \mathfrak{m}$-a.e. $(x, y) \in X \times X$.

Integrating with respect to the time variable and recalling that $\mathsf{d}_G := 1/G$, we get, for any $t \in [0, T]$,

$$\mathsf{d}_G(\boldsymbol{X}_t(x), \boldsymbol{X}_t(y)) \leq \mathsf{d}_G(x, y) \exp\left\{\int_0^T Mg(\boldsymbol{X}_s(x)) \, \mathrm{d}s + \int_0^T Mg(\boldsymbol{X}_s(y)) \, \mathrm{d}s\right\} \tag{3.42}$$

for $\mathfrak{m}\times\mathfrak{m}$-a.e. $(x,y)\in X\times X$. Note that the function $g^*(x):=\int_0^T Mg(\boldsymbol{X}_s(x))\,\mathrm{d}s$ belongs to $L^2$ with

$$\|g^*\|_{L^2}\le CL\int_0^T \||\nabla b_s|+|\operatorname{div} b_s|\|_{L^2}\,\mathrm{d}s, \tag{3.43}$$

where $C$ is a universal constant and $L$ is as in Definition 1.75. Putting (3.42) and (3.43) together, we get a *weak* version of the sought Lusin-Lipschitz estimate, since the inequality in (3.42) is not point-wise but it holds only $\mathfrak{m}\times\mathfrak{m}$-almost everywhere. In order to fix this issue we adopt a slightly different approach (borrowed from [**63**]). We consider the family of functionals

$$\Phi_{t,r}(x):=\fint_{B^G(x,r)}\log\left(1+\frac{\mathsf{d}_G(\boldsymbol{X}_t(x),\boldsymbol{X}_t(y))}{r}\right)\mathrm{d}\mathfrak{m}(y), \tag{3.44}$$

for $r\in(0,\infty)$ and $t\in[0,T]$ and we bound its time derivative performing the same estimate in (3.41).

This gives an $L^2$ bound on the function

$$\Phi^*(x):=\sup_{0\le t\le T}\sup_{0<r<\infty}\Phi_{t,r}(x), \tag{3.45}$$

that will play the role of $g^*$ in (3.42). Let us remark that in order to perform such a plan we need to use the doubling and quasi-metric property of $\mathsf{d}_G$ (see Section 1.3).

**Proposition 3.20.** *Let $(X,\mathsf{d},\mathfrak{m})$ be an* RCD$(0,N)$ *m.m.s. satisfying Assumption 2. Let moreover $b$, $(\boldsymbol{X}_t)_{t\in[0,T]}$ and $L$ be as in the discussion above. Then for any compact $P\subset X$ there exists a constant $C=C(T,\mathfrak{m}(P),N)$ such that*

$$\|\Phi^*\|_{L^2(P,\mathfrak{m})}\le C\left[1+L\int_0^T\||\nabla_{\mathrm{sym}}b_s|+|\operatorname{div} b_s|\|_{L^2(X,\mathfrak{m})}\,\mathrm{d}s\right]. \tag{3.46}$$

*Proof.* By Corollary 3.39 we can say that, for $\mathfrak{m}\times\mathfrak{m}$-a.e. $(x,y)\in X\times X$, the map $t\mapsto G(\boldsymbol{X}_t(x),\boldsymbol{X}_t(y))$ belongs to $H^{1,1}((0,T))$ and its derivative is given by

$$\frac{\mathrm{d}}{\mathrm{d}t}G(\boldsymbol{X}_t(x),\boldsymbol{X}_t(y))=b_t\cdot\nabla G_{\boldsymbol{X}_t(x)}(\boldsymbol{X}_t(y))+b_t\cdot\nabla G_{\boldsymbol{X}_t(y)}(\boldsymbol{X}_t(x))\quad\text{for } \mathscr{L}^1\text{-a.e. } t\in(0,T). \tag{3.47}$$

It follows that, for $\mathfrak{m}$-a.e. $x\in X$, the map $t\mapsto\Phi_{t,r}$ belongs to $H^{1,1}((0,T))$ as well (and actually it is absolutely continuous, since it is continuous) and it holds

$$\begin{aligned}\Phi_{t,r}(x)=&\Phi_{0,r}(x)+\int_0^t\frac{\mathrm{d}}{\mathrm{d}s}\Phi_{s,r}(x)\,\mathrm{d}s\\ \le&\Phi_{0,r}(x)+\int_0^t\fint_{B^G(x,r)}\frac{|\frac{\mathrm{d}}{\mathrm{d}s}G(\boldsymbol{X}_s(x),\boldsymbol{X}_s(y))|}{G(\boldsymbol{X}_s(x),\boldsymbol{X}_s(y))}\cdot\frac{1}{G(\boldsymbol{X}_s(x),\boldsymbol{X}_s(y))r+1}\,\mathrm{d}\mathfrak{m}(y)\,\mathrm{d}s\\ \le&\Phi_{0,r}(x)+\int_0^t\fint_{B^G(x,r)}\frac{|b_s\cdot\nabla G_{\boldsymbol{X}_s(x)}(\boldsymbol{X}_s(y))+b_s\cdot\nabla G_{\boldsymbol{X}_s(y)}(\boldsymbol{X}_s(x))|}{G(\boldsymbol{X}_s(x),\boldsymbol{X}_s(y))}\,\mathrm{d}\mathfrak{m}(y)\,\mathrm{d}s.\end{aligned}$$

Applying Proposition 3.19 and observing that $\Phi_{0,r}\le\log 2$, we conclude that

$$\Phi_{t,r}(x)\le\log 2+\int_0^t\fint_{B^G(x,r)}\{Mg_s(\boldsymbol{X}_s(x))+Mg_s(\boldsymbol{X}_s(y))\}\,\mathrm{d}\mathfrak{m}(y)\,\mathrm{d}s\quad\text{for } \mathfrak{m}\text{-a.e. } x\in X,$$

where $g_s := |\nabla_{\mathrm{sym}} b_s| + |\operatorname{div} b_s|$. We can finally estimate $\Phi^*$ obtaining that, for $\mathfrak{m}$-a.e. $x \in X$, it holds

$$\begin{aligned}\Phi^*(x) &\le \log 2 + \sup_{0\le t\le T}\sup_{0<r<\infty} \int_0^t \fint_{B^G(x,r)} \{Mg_s(\boldsymbol{X}_s(x)) + Mg_s(\boldsymbol{X}_s(y))\}\,\mathrm{d}\mathfrak{m}(y)\,\mathrm{d}s \\ &\le \log 2 + \int_0^T Mg_s(\boldsymbol{X}_s(x))\,\mathrm{d}s + \int_0^T M^G Mg_s(\boldsymbol{X}_s(\cdot))(x)\,\mathrm{d}s,\end{aligned}$$

where $M^G$ is the maximal operator associated to the quasi-metric $\mathsf{d}_G$ (while $M$ still denotes the maximal operator associated to the m.m.s. $(X, \mathsf{d}, \mathfrak{m})$).

We remark that $M^G$ maps $L^2(X, \mathfrak{m})$ into itself (see Lemma 3.18).

Passing to the $L^2(\mathfrak{m})$-norms over $P$ and taking into account the assumption that the RLF has compressibility constant $L < \infty$ we obtain (3.46). □

Below we state and prove our main regularity result for Regular Lagrangian flows.

**Theorem 3.21.** *Let $(X, \mathsf{d}, \mathfrak{m})$, $b$, $(\boldsymbol{X}_t)_{t\in[0,T]}$ and $L$ be as in the assumptions of Proposition 3.20 above. Then there exists $C = C(N)$ such that, for any $x, y \in X$ and for any $t \in [0, T]$, it holds*

$$\mathsf{d}_G(\boldsymbol{X}_t(x), \boldsymbol{X}_t(y)) \le Ce^{C(\Phi^*(x)+\Phi^*(y))}\mathsf{d}_G(x, y), \tag{3.48}$$

*where $\Phi^*$ was defined in* (3.45).

*Moreover, for any compact $P \subset X$, the following property holds: for every $\varepsilon > 0$ there exists a Borel set $E \subset P$ such that $\mathfrak{m}(P \setminus E) < \varepsilon$ and*

$$\mathsf{d}_G(\boldsymbol{X}_t(x), \boldsymbol{X}_t(y)) \le C \exp\left(2C\frac{\|\Phi^*\|_{L^2(P,\mathfrak{m})}}{\sqrt{\varepsilon}}\right)\mathsf{d}_G(x, y) \quad \text{for any } x, y \in E, \tag{3.49}$$

*for every $t \in [0, T]$. We remark that this last statement is meaningful since, under our regularity assumptions on $b$, Proposition 3.20 grants that $\|\Phi^*\|_{L^2(P,\mathfrak{m})} < \infty$.*

*Proof.* Fix $x, y \in X$ such that $x \neq y$ and set $r := \mathsf{d}_G(x, y)$. Exploiting Proposition 3.14 and the subadditivity and monotonicity of $s \mapsto \log(1 + s)$, we obtain that, for any $z \in X$, it holds

$$\begin{aligned}\log\left(1 + \frac{\mathsf{d}_G(\boldsymbol{X}_t(x), \boldsymbol{X}_t(y))}{C_T r}\right) &\le \log\left(1 + \frac{\mathsf{d}_G(\boldsymbol{X}_t(x), \boldsymbol{X}_t(z))}{r}\right) \\ &\quad + \log\left(1 + \frac{\mathsf{d}_G(\boldsymbol{X}_t(z), \boldsymbol{X}_t(y))}{r}\right).\end{aligned}$$

Taking the mean value w.r.t. the $z$ variable of the above written inequality over $B^G(x, r)$ and exploiting the inclusion $B^G(x, r) \subset B^G(y, 2C_T r)$ which follows from Proposition 3.14, we obtain

$$\begin{aligned}&\log\left(1+\frac{\mathsf{d}_G(\boldsymbol{X}_t(x), \boldsymbol{X}_t(y))}{C_T r}\right) \\ &\quad\le \fint_{B^G(x,r)} \log\left(1 + \frac{\mathsf{d}_G(\boldsymbol{X}_t(x), \boldsymbol{X}_t(z))}{r}\right)\mathrm{d}\mathfrak{m}(z) \\ &\qquad + \fint_{B^G(x,r)} \log\left(1 + \frac{\mathsf{d}_G(\boldsymbol{X}_t(z), \boldsymbol{X}_t(y))}{r}\right)\mathrm{d}\mathfrak{m}(z)\end{aligned}$$

$$\leq \Phi^*(x) + \frac{\mathfrak{m}(B^G(y, 2C_T r))}{\mathfrak{m}(B^G(x, r))} \fint_{B^G(y,2C_T r)} \log\left(1 + \frac{\mathsf{d}_G(\boldsymbol{X}_t(z), \boldsymbol{X}_t(y))}{r}\right) \mathrm{d}\mathfrak{m}(z).$$

Thanks to Proposition 3.17 we can estimate $\frac{\mathfrak{m}(B^G(y,2C_T r))}{\mathfrak{m}(B^G(x,r))}$ with a constant depending only on $C^G$ and $C_T$. Hence, there exists $C = C(N)$ (recall that $C_T$ and $C^G$ depend only on $N$) such that

$$\log\left(1 + \frac{\mathsf{d}_G(\boldsymbol{X}_t(x), \boldsymbol{X}_t(y))}{C_T \mathsf{d}_G(x, y)}\right) \leq C(\Phi^*(x) + \Phi^*(y)), \tag{3.50}$$

for any $x, y \in X$ such that $x \neq y$ and for any $t \in [0, T]$ and it is easily seen that (3.50) implies (3.48).

Letting now $E := \{x \in P \ : \Phi^*(x) \leq \|\Phi^*\|_{L^2(P,\mathfrak{m})} / \sqrt{\varepsilon}\}$, by Chebyshev inequality we deduce that $\mathfrak{m}(P \setminus E) < \varepsilon$. Conclusion (3.49) directly follows now from (3.48). □

**1.5. Extension to the case of an arbitrary lower Ricci bound.** The aim of this section is to provide regularity results for regular Lagrangian Flows of Sobolev vector fields over RCD$(K, N)$ metric measure spaces in the case of a possibly negative lower Ricci bound $K$. The main difference with respect to the previously treated case of nonnegative lower Ricci curvature bound is that the regularity has to be understood in terms of the fundamental solution of an elliptic operator different from the Laplacian.

The spirit of this part will be to show how to adapt the estimates of Section 1.1 and Section 1.4 above to this more general setting up to pay the price that they become local and less intrinsic.

**Assumption 3.** Throughout this section we assume that $(X, \mathsf{d}, \mathfrak{m})$ is the tensor product between an arbitrary RCD$(K, N-3)$ m.m.s. for some $K \in \mathbb{R}$ and $4 < N < \infty$ and a Euclidean factor $(\mathbb{R}^3, \mathsf{d}_{\mathbb{R}^3}, \mathscr{L}^3)$.

Let us stress once more that, for the purposes of the upcoming Section 2, it will be not too restrictive to have a regularity result for Regular Lagrangian flows just over spaces satisfying Assumption 3.

Let $c \geq 0$ be the constant appearing in (1.20) and (1.21) and set

$$\bar{G}(x, y) := \int_0^\infty e^{-ct} p_t(x, y) \,\mathrm{d}t \quad \text{for any } x, y \in X,$$

and, in analogy with (3.8),

$$\bar{G}^\varepsilon(x, y) := \int_\varepsilon^\infty e^{-ct} p_t(x, y) \quad \text{for any } \varepsilon > 0 \text{ and any } x, y \in X.$$

As in the case of the Green function $G$, we shall adopt in the sequel also the notation $\bar{G}_x(\cdot) = \bar{G}(x, \cdot)$ (and analogously for $\bar{G}^\varepsilon$).

Observe that, assuming that $c > 0$, $\bar{G}_x$ is well defined and belongs to $L^1(X, \mathfrak{m})$ for every $x \in X$. Indeed an application of Fubini's theorem yields

$$\int \bar{G}_x(w) \,\mathrm{d}\mathfrak{m}(w) = \int_0^\infty e^{-ct} \int p_t(x, w) \,\mathrm{d}\mathfrak{m}(w) \,\mathrm{d}t = \int_0^\infty e^{-ct} \,\mathrm{d}t < \infty. \tag{3.51}$$

We can also remark that the above stated conclusion holds true without any extra hypothesis on the RCD$(K, N)$ m.m.s. $(X, \mathsf{d}, \mathfrak{m})$. Nevertheless, the validity of Assumption 3 will be crucial in order to obtain meaningful estimates for $\bar{G}$ and its gradient in terms of the functions $F$ and $H$ introduced in (3.10), (3.11).

At least at a formal level one can check that $\bar{G}$ solves the equation $\Delta\bar{G}_x = -\delta_x + c\bar{G}_x$. Indeed

$$\begin{aligned}\Delta_y\bar{G}_x(\cdot) =& \Delta_y\left(\int_0^\infty e^{-ct}p_t(x,\cdot)\,\mathrm{d}t\right)\\ =& \int_0^\infty e^{-ct}\Delta_y p_t(x,\cdot)\,\mathrm{d}t = \int_0^\infty e^{-ct}\frac{\mathrm{d}}{\mathrm{d}t}p_t(x,\cdot)\,\mathrm{d}t\\ =& [p_t(x,\cdot)]_0^\infty + c\int_0^\infty e^{-ct}p_t(x,\cdot)\,\mathrm{d}t = -\delta_x + c\bar{G}_x(\cdot).\end{aligned}$$

To let the above computation become rigorous, one can proceed as in the proof of Lemma 3.6 and check firstly that $\bar{G}_x^\varepsilon \in \mathrm{Lip}_b \cap D_{\mathrm{loc}}(\Delta)$ for any $x \in X$ and any $\varepsilon > 0$, with

$$\Delta\bar{G}_x^\varepsilon(y) = -e^{-c\varepsilon}p_\varepsilon(x,y) + c\bar{G}_x^\varepsilon(y), \qquad \text{for } \mathfrak{m}\text{-a.e. } y \in X, \tag{3.52}$$

and then that

$$\lim_{\varepsilon\to 0}\bar{G}_x^\varepsilon \to \bar{G}_x \qquad \text{in } H^{1,1}(X,\mathsf{d},\mathfrak{m}). \tag{3.53}$$

Our primary goal is now to obtain useful local estimates for $\bar{G}$ and its gradient in terms of $F$ and $H$.

**Proposition 3.22** (Main estimates for $\bar{G}$)**.** *Let $(X,\mathsf{d},\mathfrak{m})$ be an $\mathrm{RCD}(K,N)$ m.m.s. satisfying Assumption 3. Then, for any compact $P \subset X$, there exists $\bar{C} = \bar{C}(P) \geq 1$ such that*

$$\frac{1}{\bar{C}}F_x(\mathsf{d}(x,y)) \leq \bar{G}_x(y) \leq \bar{C}F_x(\mathsf{d}(x,y)) \quad \textit{for any } x,y \in P. \tag{3.54}$$

*Moreover for any $x \in X$ it holds $\bar{G}_x \in H^{1,1}_{\mathrm{loc}}(X,\mathsf{d},\mathfrak{m})$ and, for any $x \in P$,*

$$|\nabla\bar{G}_x|(y) \leq \int_0^\infty e^{-ct}|\nabla p_t(x,\cdot)|(y)\,\mathrm{d}t \leq \bar{C}H_x(\mathsf{d}(x,y)) \quad \textit{for } \mathfrak{m}\textit{-a.e. } y \in P. \tag{3.55}$$

*Proof.* Applying the estimates for the heat kernel (1.20) we find out that

$$\frac{1}{C_1}\int_0^\infty \frac{e^{-2ct}e^{-\frac{\mathsf{d}(x,y)^2}{3t}}}{\mathfrak{m}(B(x,\sqrt{t}))}\,\mathrm{d}t \leq \bar{G}_x(y) \leq C_1\int_0^\infty \frac{e^{-\frac{\mathsf{d}(x,y)^2}{5t}}}{\mathfrak{m}(B(x,\sqrt{t}))}\,\mathrm{d}t \quad \text{for any } x,y \in X. \tag{3.56}$$

Exploiting (1.16) and Lemma 3.8, we obtain from (3.56) that

$$\bar{G}_x(y) \leq C_1\Lambda(R)F_x\left(\frac{\mathsf{d}(x,y)}{\sqrt{5}}\right) \quad \text{for any } x,y \text{ such that } \mathsf{d}(x,y) < R, \tag{3.57}$$

where $\Lambda$ is the function in the statement of Lemma 3.8. The bound from above in (3.54) follows from (3.57) together with the following observation, that will play a role also in the sequel: for any compact $P \subset X$, for any $R > 0$ and for any $\lambda < 1$, there exists $C(P,R,\lambda) \geq 0$ such that

$$F_x(\lambda r) \leq C(P,R,\lambda)F_x(r) \quad \text{for any } x \in P \text{ and any } 0 < r < R. \tag{3.58}$$

Indeed (3.58) can be checked splitting

$$F_x(\lambda r) = \int_{\lambda r}^{\lambda R}\frac{s}{\mathfrak{m}(B(x,s))}\,\mathrm{d}s + \int_{\lambda R}^\infty \frac{s}{\mathfrak{m}(B(x,s))}\,\mathrm{d}s, \tag{3.59}$$

$$F_x(r) = \int_r^R \frac{s}{\mathfrak{m}(B(x,s))}\,\mathrm{d}s + \int_R^\infty \frac{s}{\mathfrak{m}(B(x,s))}\,\mathrm{d}s \tag{3.60}$$

and using the local doubling property (1.16) together with a change of variables to bound the first term in (3.59) with the first one in (3.60) and the continuity of $x \mapsto F_x(R)$ to compare the second terms (here the compactness of $P$ comes into play).

To obtain the lower bound in (3.54) we proceed as follows. Starting from the lower bound in (3.56), exploiting the elementary inequality $e^{-d^2/3t} \geq e^{-1/3}\mathbf{1}_{[d,\infty]}(\sqrt{t})$ and changing variables, we obtain

$$\int_0^\infty \frac{e^{-2ct}e^{-\frac{\mathsf{d}(x,y)^2}{3t}}}{\mathfrak{m}(B(x,\sqrt{t}))}\,\mathrm{d}t \geq e^{-1/3}\int_{\mathsf{d}(x,y)}^\infty e^{-2ct^2}\frac{t}{\mathfrak{m}(B(x,t))}\,\mathrm{d}t.$$

To conclude it suffices to observe that, splitting the integral in two parts and using a continuity argument, as in the verification of (3.58) above, it is possible to find a constant $C(P) > 0$ such that

$$\int_{\mathsf{d}(x,y)}^\infty e^{-2ct^2}\frac{t}{\mathfrak{m}(B(x,t))}\,\mathrm{d}t \geq C(P)\int_{\mathsf{d}(x,y)}^\infty \frac{t}{\mathfrak{m}(B(x,t))}\,\mathrm{d}t = C(P)F_x(\mathsf{d}(x,y)),$$

for any $x, y \in P$.

The proof of (3.55) can be obtained with arguments analogous to those one we presented above, starting from (1.21) and following the strategy we adopted to prove (3.13). $\square$

Another crucial ingredient to perform the regularity scheme by Crippa-De Lellis in the case of nonnegative lower Ricci curvature bound was the scalar maximal estimate we obtained in Proposition 3.11. In Proposition 3.23 below we prove that an analogous result holds true, in local form, also in the case of an arbitrary lower Ricci bound.

**Proposition 3.23** (Maximal estimate, scalar version). *Let $(X, \mathsf{d}, \mathfrak{m})$ be an* RCD$(K, N)$ *m.m.s. satisfying Assumption 3. For any compact $P \subset X$, there exists $C_M(P) > 0$ such that, for any Borel function $f : X \to [0, \infty)$ supported in $P$, it holds*

$$\int f(w)\left|\nabla \bar{G}_x(w)\right|\left|\nabla \bar{G}_y(w)\right|\mathrm{d}\mathfrak{m}(w) \leq \bar{C}_M(P)\bar{G}(x,y)\left(Mf(x) + Mf(y)\right), \tag{3.61}$$

*for any $x, y \in P$.*

*Proof.* We begin by recalling that, as an intermediate step in the proof of Proposition 3.11, we obtained the following inequality:

$$\begin{aligned}&\int f(w)H_x(\mathsf{d}(x,w))H_y(\mathsf{d}(y,w))\,\mathrm{d}\mathfrak{m}(w)\\ &\quad\leq C\left(F_x\left(\frac{\mathsf{d}(x,y)}{2}\right) + F_y\left(\frac{\mathsf{d}(x,y)}{2}\right) + F_x(\mathsf{d}(x,y)) + F_y(\mathsf{d}(x,y))\right)\left(Mf(x) + Mf(y)\right),\end{aligned} \tag{3.62}$$

for any $x, y \in X$, for some numerical constant $C > 0$ (the assumptions concerning the m.m.s. $(X, \mathsf{d}, \mathfrak{m})$ played no role in that part of the proof).

Let us observe then that, thanks to (3.55),

$$\int f(w)\left|\nabla \bar{G}_x(w)\right|\left|\nabla \bar{G}_y(w)\right|\mathrm{d}\mathfrak{m}(w) \leq \bar{C}(P)^2\int f(w)H_x(\mathsf{d}(x,w))H_y(\mathsf{d}(y,w))\,\mathrm{d}\mathfrak{m}(w) \tag{3.63}$$

for any $x, y \in P$. Exploiting (3.58) with $\lambda = 1/2$, (3.62) and (3.63), we obtain that, up to increasing the constant $\bar{C}(P)$, it holds

(3.64)
$$\int f(w) \left|\nabla \bar{G}_x(w)\right| \left|\nabla \bar{G}_y(w)\right| \mathrm{d}\mathfrak{m}(w) \le \bar{C}(P) \left(F_x(\mathsf{d}(x,y)) + F_y(\mathsf{d}(x,y))\right) \left(Mf(x) + Mf(y)\right)$$

for any $x, y \in P$. The sought conclusion (3.61) follows from (3.64) and the lower bound in (3.54). ☐

**Remark 3.24.** It follows from the proof of Proposition 3.23 above that also the estimate

$$\int f(w) \left|\nabla \bar{G}^\varepsilon_x(w)\right| \left|\nabla \bar{G}^\varepsilon_y(w)\right| \mathrm{d}\mathfrak{m}(w) \le \bar{C}_M(P) \bar{G}(x,y) \left(Mf(x) + Mf(y)\right) \tag{3.65}$$

holds true, for any $\varepsilon > 0$ and for any $x, y \in P$.

By analogy with (3.25), we introduce a function $\mathsf{d}_{\bar{G}}$, that we will use to measure the regularity of RLFs, in the following way:

$$\mathsf{d}_{\bar{G}}(x,y) := \begin{cases} \frac{1}{\bar{G}(x,y)} & \text{if } x \neq y, \\ 0 & \text{otherwise.} \end{cases} \tag{3.66}$$

It is immediate to check that is symmetric, nonnegative and that $\mathsf{d}_{\bar{G}}(x,y) = 0$ if and only if $x = y$. Moreover, following verbatim the proof of Lemma 3.13 and exploiting the two-sided bounds in (3.54), it is easy to prove that, for any $x \in X$, the map $y \to \mathsf{d}_{\bar{G}}(x,y)$ is continuous with respect to $\mathsf{d}$.

By analogy with (3.27), we introduce the notation $B^{\bar{G}}$ for the "balls" associated to $\mathsf{d}_{\bar{G}}$, that is to say, for any $x \in X$ and for any $r > 0$, we let

$$B^{\bar{G}}(x,r) := \{y \in X \ : \mathsf{d}_{\bar{G}}(x,y) < r\}.$$

The aim of Proposition 3.25 and Proposition 3.27 below is to show that, at least locally, $\mathsf{d}_{\bar{G}}$ is a quasi-metric on $X$ and that $(X, \mathsf{d}_{\bar{G}}, \mathfrak{m})$ is a locally doubling quasi-metric measure space.

**Proposition 3.25** (Local almost triangle inequality)**.** *Let* $(X, \mathsf{d}, \mathfrak{m})$ *be an* $\mathrm{RCD}(K,N)$ *m.m.s. satisfying Assumption 3. Then, for any compact $P \subset X$, there exists a constant $\bar{C}_T(P) \ge 1$ such that*

$$\mathsf{d}_{\bar{G}}(x,y) \le \bar{C}_T(P)(\mathsf{d}_{\bar{G}}(x,z) + \mathsf{d}_{\bar{G}}(z,y)) \quad \textit{for any } x,y,z \in P. \tag{3.67}$$

*Proof.* Recall that, as an intermediate step in the proof of Lemma 3.15, we proved that, without any further assumption on the m.m.s. $(X, \mathsf{d}, \mathfrak{m})$, it holds

$$F_x(\mathsf{d}(x,z))F_y(\mathsf{d}(y,z)) \le F_x(\mathsf{d}(x,y)/2)F_y(\mathsf{d}(y,z)) + F_y(\mathsf{d}(x,y)/2)F_x(\mathsf{d}(x,z)), \tag{3.68}$$

for any $x, y, z \in X$.

Applying (3.58) with $\lambda = 1/2$ and exploiting the two-sided bounds in (3.54), we pass from (3.68) to the sought (3.67). ☐

**Remark 3.26.** A first non completely trivial consequence of Proposition 3.25 is that any compact $P \subset X$ is bounded w.r.t. the $\mathsf{d}_{\bar{G}}$ quasi-metric.

**Proposition 3.27** (Local doubling property). *Let $(X, \mathsf{d}, \mathfrak{m})$ be an* $\mathrm{RCD}(K, N)$ *m.m.s. satisfying Assumption 3. Then, for any compact $P \subset X$ and for any $R > 0$, there exists a constant $\bar{C}^{\bar{G}}(P, R) > 0$ such that*

$$\mathfrak{m}(B^{\bar{G}}(x, 2r)) \leq \bar{C}^{\bar{G}} \mathfrak{m}(B^{\bar{G}}(x, r)), \quad \textit{for any } x \in P \textit{ and for any } 0 < r < R. \tag{3.69}$$

*Proof.* The conclusion can be obtained arguing as in the proof of Proposition 3.17, exploiting the fact that $(X, \mathsf{d}, \mathfrak{m})$ is locally doubling (see (1.16)) and the local comparison between $\bar{G}$ and $F$ obtained in (3.54). We just indicate here the adjustments one has to do.

First of all, we observe that a local version of Lemma 3.16 holds true, namely if $(\bar{X}, \bar{\mathsf{d}}, \mu)$ is a locally doubling m.m.s. with function $C_\mu : (0, \infty) \to (0, \infty)$ and $(X, \mathsf{d}, \mathfrak{m})$ is the tensor product between $(\bar{X}, \bar{\mathsf{d}}, \mu)$ and $(\mathbb{R}^k, \mathsf{d}_{\mathbb{R}^k}, \mathscr{L}^k)$, it holds

$$\frac{\mathfrak{m}(B(x, R))}{\mathfrak{m}(B(x, r))} \geq \frac{1}{C_\mu(R) \sqrt{2}^k} \left( \frac{R}{r} \right)^k, \tag{3.70}$$

for any $x \in X$ and for any $0 < r < R$.

We wish to obtain a local version of (3.31), that is to say the existence of $\gamma = \gamma(P, R)$ such that

$$F_x(\frac{r}{2}) \leq \gamma F_x(r), \quad \text{for any } x \in P \text{ and for any } 0 < r < R. \tag{3.71}$$

This can be obtained arguing as in the proof of (3.31), exploiting the splitting of the integration intervals we introduced in (3.59), (3.60) and assumption Assumption 3 together with (3.70) in place of Lemma 3.16.

The validity of (3.70) implies in turn that, for any $S > 0$, we can find $\gamma < 1$ such that

$$F_x^{-1}(\gamma s) \leq F_x^{-1}(s), \quad \text{for any } x \in P \text{ and for any } s > S. \tag{3.72}$$

Having (3.72) at our disposal, we can achieve (3.69) exploiting the local version of (3.33), which is a consequence of (3.54)[1]. □

We end this introductory discussion about the properties of the modified Green function $\bar{G}$ with a vector valued maximal estimate.

**Proposition 3.28** (Maximal estimate, vector-valued version). *Let $(X, \mathsf{d}, \mathfrak{m})$ be satisfy the* $\mathrm{RCD}(K, N)$ *condition and Assumption 3. Let $P \subset X$ be a compact set. Then, for any $b \in W^{1,2}_{C,s}(TX)$ bounded and with compact support in $P$, there exists a positive function $F \in L^2(P, \mathfrak{m})$ such that*

$$\left| b \cdot \nabla \bar{G}_x(y) + b \cdot \nabla \bar{G}_y(x) \right| \leq \bar{G}(x, y)(F(x) + F(y)) \quad \textit{for } \mathfrak{m} \times \mathfrak{m}\textit{-a.e. } (x, y) \in P \times P, \tag{3.73}$$

*and*

$$\|F\|_{L^2(P, \mathfrak{m})} \leq C_V \left\| |\nabla_{\mathrm{sym}} b| + |\operatorname{div} b| \right\|_{L^2(X, \mathfrak{m})}, \tag{3.74}$$

*where $C_V = C_V(P) > 0$.*

[1] In the whole proof we tacitly exploited the fact that any compact subset of $X$ is both $\mathsf{d}$-bounded and $\mathsf{d}_{\bar{G}}$-bounded, see Remark 3.26 above.

*Proof.* The strategy we follow is the same proposed in the proof of Proposition 3.19. First we are going to prove that there exists $F$ as above such that

$$\left|\int \left\{b\cdot\nabla\bar G^\varepsilon_x(w)p_\varepsilon(y,w)+b\cdot\nabla\bar G^\varepsilon_y(w)p_\varepsilon(x,w)\right\}\mathrm{d}\mathfrak{m}(w)\right|\le \bar G(x,y)\left(F(x)+F(y)\right), \tag{3.75}$$

for any $x,y\in P$ and for any $0<\varepsilon<1$. The stated conclusion will then follow from (3.75), taking into account (3.53) and following verbatim the second step of the proof of Proposition 3.19.

Recall from (3.52) that $p_\varepsilon(x,w)=e^{c\varepsilon}[-\Delta\bar G^\varepsilon_x(w)+c\bar G^\varepsilon_x(w)]$ for $\mathfrak{m}$-a.e. $w\in X$. Hence we can estimate

$$\begin{aligned}
&\left|\int b\cdot\nabla\bar G^\varepsilon_x(w)p_\varepsilon(y,w)+b\cdot\nabla\bar G^\varepsilon_y(w)p_\varepsilon(x,w)\,\mathrm{d}\mathfrak{m}(w)\right|\\
&=e^{c\varepsilon}\left|\int \left\{b\cdot\nabla\bar G^\varepsilon_x(-\Delta\bar G^\varepsilon_y+c\bar G^\varepsilon_y)+b\cdot\nabla\bar G^\varepsilon_y(-\Delta\bar G^\varepsilon_x+c\bar G^\varepsilon_x)\right\}\mathrm{d}\mathfrak{m}\right|\\
&\le e^{c\varepsilon}\left|\int \left\{b\cdot\nabla\bar G^\varepsilon_x\Delta\bar G^\varepsilon_y+b\cdot\nabla\bar G^\varepsilon_y\Delta\bar G^\varepsilon_x\right\}\mathrm{d}\mathfrak{m}\right|+ce^{c\varepsilon}\left|\int \left\{b\cdot\nabla\bar G^\varepsilon_x\ \bar G^\varepsilon_y+b\cdot\nabla\bar G^\varepsilon_y\ \bar G^\varepsilon_x\right\}\mathrm{d}\mathfrak{m}\right|\\
&=:I^\varepsilon_1(x,y)+I^\varepsilon_2(x,y).
\end{aligned}$$

Arguing as in the first step of the proof of Proposition 3.19 and applying Remark 3.24, we obtain that

$$I^\varepsilon_1(x,y)\le e^{c\varepsilon}\bar C_M(P)\bar G(x,y)(Mg(x)+Mg(y)),\quad \text{for any } x,y\in P \text{ and for any } 0<\varepsilon<1, \tag{3.76}$$

where $g:=|\nabla_{\mathrm{sym}}b|+|\operatorname{div} b|$. Dealing with $I^\varepsilon_2$, let us integrate by parts and apply the Leibniz rule and Proposition 3.25, to obtain that

$$I^\varepsilon_2(x,y)=ce^{c\varepsilon}\left|\int \operatorname{div} b\ \bar G^\varepsilon_x\bar G^\varepsilon_y\,\mathrm{d}\mathfrak{m}\right|\le ce^{c\varepsilon}\bar C_T(P)\bar G(x,y)\left(\int g\bar G_x\,\mathrm{d}\mathfrak{m}+\int g\bar G_y\,\mathrm{d}\mathfrak{m}\right), \tag{3.77}$$

for any $x,y\in P$.

Let us set

$$F(x):=e^cC_M(P)Mg(x)+ce^c\int g\bar G_x\,\mathrm{d}\mathfrak{m},\qquad \forall x\in P.$$

It remains only to show (3.74). To this aim we recall (1.18) and we observe that,

$$\int\left(\int g\bar G_x\,\mathrm{d}\mathfrak{m}\right)^2\mathrm{d}\mathfrak{m}(x)=\int\left(\int_0^\infty e^{-ct}P_tg(x)\,\mathrm{d}t\right)^2\mathrm{d}\mathfrak{m}(x)\le c^{-1}\left\|g\right\|^2_{L^2(X,\mathfrak{m})} \tag{3.78}$$

The proof is complete. □

With Proposition 3.25, Proposition 3.27 and Proposition 3.28 at our disposal we can develop a regularity theory for Regular Lagrangian flows of Sobolev vector fields in terms of the quasi-metric $\mathsf{d}_{\bar G}$.

To this aim let us fix $T>0$ and let $b_t\in L^\infty((0,T)\times X)$ be a time dependent vector field with compact support, uniformly w.r.t. time. We further assume that $b_t\in W^{1,2}_{C,s}(TX)$ for a.e. $t\in(0,T)$, that $|\nabla_{\mathrm{sym}}b_t|\in L^1((0,T);L^2(X,\mathfrak{m}))$ and that $\operatorname{div} b_t\in L^1((0,T);L^\infty(X,\mathfrak{m}))$.

Let $(\boldsymbol{X}_t)_{t\in[0,T]}$ be the Regular Lagrangian flow of $b$, whose existence and uniqueness follow by the theory developed in [**27**]. In analogy with the case of nonnegative lower Ricci curvature bound, we set

$$\bar\Phi_{t,r}(x):=\fint_{B^{\bar G}(x,r)}\log\left(1+\frac{\mathsf{d}_{\bar G}(\boldsymbol{X}_t(x),\boldsymbol{X}_t(y))}{r}\right)\mathrm{d}\mathfrak{m}(y), \tag{3.79}$$

for $r \in (0,\infty)$ and $t \in [0,T]$ and, for any $R > 0$,

$$\bar{\Phi}^*_R(x) := \sup_{0\le t\le T} \sup_{0<r<R} \bar{\Phi}_{t,r}(x). \tag{3.80}$$

Below we state the main regularity result of this part. Its proof can be obtained from the result of this subsection, using Remark 3.40 and recalling (1.18).

**Theorem 3.29.** *Let $(X,\mathsf{d},\mathfrak{m})$ be an $\mathrm{RCD}(K,N)$ m.m.s. satisfying Assumption 3. Let $b$ and $(\boldsymbol{X}_t)_{t\in[0,T]}$ be as in the discussion above. Then, for any compact $P \subset X$ such that $P$ contains the $(T\,\|b\|_{L^\infty})$-enlargement of $\operatorname{supp} b$, there exist $\bar{C} > 0$ and $R > 0$, depending on $P$, such that for any $x,y \in P$ and for any $t \in [0,T]$, it holds*

$$\mathsf{d}_{\bar{G}}(\boldsymbol{X}_t(x),\boldsymbol{X}_t(y)) \le \bar{C} e^{\bar{C}\left(\bar{\Phi}^*_R(x)+\bar{\Phi}^*_R(y)\right)} \mathsf{d}_{\bar{G}}(x,y).$$

*Moreover, $\bar{\Phi}^*_R$ belongs to $L^2(P,\mathfrak{m})$ and the following Lusin-approximation property holds: for every $\varepsilon > 0$ there exists a Borel set $E \subset P$ such that $\mathfrak{m}(P\setminus E) < \varepsilon$ and*

$$\mathsf{d}_{\bar{G}}(\boldsymbol{X}_t(x),\boldsymbol{X}_t(y)) \le \bar{C}\exp\left(2\bar{C}\frac{\left\|\bar{\Phi}^*_R\right\|_{L^2(P,\mathfrak{m})}}{\sqrt{\varepsilon}}\right)\mathsf{d}_{\bar{G}}(x,y) \quad \forall\, x,y \in P \text{ and } t\in[0,T].$$

In analogy with the case of real valued functions (where the Lipschitz regularity is understood w.r.t. the distance $\mathsf{d}$) we introduce the following.

**Definition 3.30.** Let $(X,\mathsf{d},\mathfrak{m})$ be an $\mathrm{RCD}(0,N)$ m.m.s. satisfying Assumption 1. We say that a map $\Phi : X \to X$ is $\mathsf{d}_G$-Lusin Lipschitz if there exists a family $\{E_n : n \in \mathbb{N}\}$ of Borel subsets of $X$ such that $\mathfrak{m}(X \setminus \cup_{n\in\mathbb{N}} E_n) = 0$ and

$$\mathsf{d}_G(\Phi(x),\Phi(y)) \le n\mathsf{d}_G(x,y),$$

for any $x,y \in E_n$ and for any $n \in \mathbb{N}$.

By analogy, if $(X,\mathsf{d},\mathfrak{m})$ is an $\mathrm{RCD}(K,N)$ m.m.s. satisfying Assumption 3, we say that $\Psi : X \to X$ is $\mathsf{d}_{\bar{G}}$-Lusin Lipschitz if it satisfies the above conditions with $\mathsf{d}_{\bar{G}}$ in place of $\mathsf{d}_G$.

Let us remark that, with the terminology we just introduced, we can combine Proposition 3.20 and Theorem 3.21 above to say that the Regular Lagrangian flow of a sufficiently regular vector field over an $\mathrm{RCD}(0,N)$ m.m.s. satisfying Assumption 2 is a $\mathsf{d}_G$-Lusin Lipschitz map (the RLF of a sufficiently regular vector field over an $\mathrm{RCD}(K,N)$ m.m.s. satisfying Assumption 3 is a $\mathsf{d}_{\bar{G}}$-Lusin Lipschitz map, respectively).

## 2. Constancy of the dimension

We begin by remarking that the statement of Theorem 3.1 is not affected by taking the tensor product with Euclidean factors and, by means of this simple observation, we will put ourselves in position to apply the results of Section 1 and Section 2.1.

In Section 2.1 below we start proving that $\mathsf{d}_G/\mathsf{d}_{\bar{G}}$-Lusin Lipschitz maps with bounded compressiblity from an $\mathrm{RCD}(K,N)$ m.m.s. into itself are regular enough to carry an information about the dimension from their domain to their image. This rigidity result has to be compared with the standard fact that biLipschitz maps preserve the Hausdorff dimension. Then we are going to prove that the class of RLFs of Sobolev vector fields, that we know to be $\mathsf{d}_G/\mathsf{d}_{\bar{G}}$-Lusin Lipschitz from Section 1, is rich enough to gain “transitivity” at the level of probability measures with bounded support and bounded density w.r.t. $\mathfrak{m}$. Better said, the primary goal of Section 2.2 below will be to show that any two probability measures which are intermediate points of a $W_2$-geodesic joining probabilities with bounded support and bounded density w.r.t. $\mathfrak{m}$ can be obtained one from the other via push-forward through the RLF of a vector field satisfying the assumptions of Theorem 3.21 (or Theorem 3.29).

Eventually in Section 2.3 we will combine all the previously developed ingredients to prove that the above mentioned "transitivity" is not compatible with the "rigidity" we obtain in Section 2.1 and the possibility of having non negligible regular sets of different dimensions in the Mondino-Naber decomposition of $(X, \mathsf{d}, \mathfrak{m})$.

**2.1. A rigidity result for $\mathsf{d}_G$-Lusin Lipschitz maps.** The aim of this subsection is to prove a rigidity result for $\mathsf{d}_G$ and $\mathsf{d}_{\bar{G}}$-Lusin Lipschitz maps (see Definition 3.30) that we are going to apply later on to regular Lagrangian Flows of Sobolev vector fields.

Roughly speaking, given an $\mathrm{RCD}(K, N)$ m.m.s. satisfying Assumption 3, we are going to prove that a $\mathsf{d}_G/\mathsf{d}_{\bar{G}}$-Lusin Lipschitz map with bounded compressibility cannot move a part of dimension $n$ of $(X, \mathsf{d}, \mathfrak{m})$ into a part of dimension $k < n$ (see Theorem 3.33 below for a precise statement). Just at a speculative level, let us point out that, in the case of $\mathsf{d}$-Lusin Lipschitz maps, this conclusion would have been a direct consequence of standard geometric measure theory arguments. However, a priori, it is not clear how to build non trivial maps from the space into itself with $\mathsf{d}$-Lusin Lipschitz regularity, while in Section 1 above we were able to obtain $\mathsf{d}_G$-Lusin Lipschitz regularity for a very rich family of maps.

We begin with a Euclidean result. It can be considered in some sense as a much simplified version of Sard's lemma.

**Proposition 3.31.** *Fix $k, n \in \mathbb{N}$ such that $1 \leq k < n$. Let $A \subset \mathbb{R}^n$, $\Phi : A \to \mathbb{R}^k$ be such that*

$$\lim_{r \to 0^+} \sup_{y \in A \cap B(x,r)} \frac{|\Phi(y) - \Phi(x)|}{|y - x|^{\frac{n}{k}}} = 0, \quad \textit{for any } x \in A. \tag{3.81}$$

*Then $\mathcal{H}^k(\Phi(A)) = 0$.*

*Proof.* We wish to prove that $\mathcal{H}^k_\delta(\Phi(A)) = 0$ for any $\delta > 0$. Let us assume without loss of generality that $A \subset P$ for some compact $P \subset \mathbb{R}^n$. Fix now $\varepsilon > 0$. It follows from (3.81) that, for any $x \in A$, we can find $r_x < \delta/10$ such that, for any $y \in B(x, 5r_x) \cap A$, it holds

$$|\Phi(y) - \Phi(x)| \leq \varepsilon \, |x - y|^{\frac{n}{k}} \,. \tag{3.82}$$

Moreover, if $\varepsilon, \delta < 1$, then (3.82) grants that $\Phi(B(x, 5r_x) \cap A)$ has diameter less than $\delta$, for any $x \in A$.

Applying Vitali's covering theorem we can find a subfamily $\mathcal{F} := \{B(x_i, r_i) : i \in \mathbb{N}\}$ such that the balls $B(x_i, r_i)$ are disjoint and $A \subset \cup_{i \in \mathbb{N}} B(x_i, 5r_i)$. Hence $\{\Phi(A \cap B(x_i, 5r_i)) : i \in \mathbb{N}\}$ is an admissible covering of $\Phi(A)$ in the definition of $\mathcal{H}^k_\delta(\Phi(A))$. Therefore

$$\mathcal{H}^k_\delta(\Phi(A)) \leq \omega_k 5^n \varepsilon^k \sum_{i=0}^{\infty} r_i^n,$$

since it follows from (3.82) that $\Phi(B(x_i, 5r_i) \cap A) \subset B(\Phi(x_i), \varepsilon 5^{\frac{n}{k}} r_i^{\frac{n}{k}})$ for any $i \in \mathbb{N}$. Observing now that $\sum_{i=0}^{\infty} \omega_n r_i^n \leq \mathcal{H}^n(P^1) < \infty$, where $P^1$ is the 1-enlargement of the compact $P$, we conclude that $\mathcal{H}^k_\delta(\Phi(A)) = 0$ for any $\delta > 0$, as we claimed. $\square$

It is a rather classical fact in Riemannian geometry that on an $n$-dimensional compact Riemannian manifold with $n > 2$ the Green function behaves locally like the distance raised to the power $2-n$ (see [**30**, Chapter 4]). The comparison is also global on a non compact manifold with nonnegative Ricci curvature and Euclidean volume growth.

**Lemma 3.32.** *Let* $(X,\mathsf{d},\mathfrak{m})$ *be an* $\mathrm{RCD}(K,N)$ *m.m.s. satisfying Assumption 1. Suppose that* $x\in\mathcal{R}_k^*$ *(see* (2.12) *for the definition of* $\mathcal{R}_k^*$*) for some* $k\geq 3$ *and denote by* $\theta_k(x)\in(0,\infty)$ *the value of the limit appearing in* (2.12)*. Then*

$$\lim_{r\to 0^+}\frac{F(x,r)}{\frac{1}{r^{k-2}}}=\frac{k-2}{\omega_k\theta_k(x)}.$$

*Proof.* Let us observe that

$$\frac{F(x,r)}{\frac{1}{r^{k-2}}}=(k-2)\frac{\int_r^\infty\frac{s}{\mathfrak{m}(B(x,s))}\,\mathrm{d}s}{\int_r^\infty\frac{1}{s^{k-1}}\,\mathrm{d}s}.$$

An application of De L'Hopital's rule yields now

$$\lim_{r\to 0^+}\frac{F(x,r)}{\frac{1}{r^{k-2}}}=\lim_{r\to 0^+}(k-2)\frac{\frac{r}{\mathfrak{m}(B(x,r))}}{\frac{1}{r^{k-1}}}=\frac{k-2}{\omega_k\theta_k(x)},$$

since, by the very definition of $\theta_k(x)$, it holds $\lim_{r\to 0^+}\frac{\mathfrak{m}(B(x,r))}{\omega_k r^k}=\theta_k(x)$. □

Let us assume, up to the end of this section that $(X,\mathsf{d},\mathfrak{m})$ is an $\mathrm{RCD}(K,N)$ metric measure space satisfying Assumption 3. It is not difficult to check that, under this assumption, the regular sets $\mathcal{R}_k$ of $(X,\mathsf{d},\mathfrak{m})$ associated to $k=0,1$ and $2$ are empty.

**Theorem 3.33.** *Let* $(X,\mathsf{d},\mathfrak{m})$ *be as in the discussion above. Let* $\Phi:X\to X$ *be either a* $\mathsf{d}_G$*-Lusin Lipschitz or a* $\mathsf{d}_{\bar G}$*-Lusin Lipschitz map (see Definition 3.30 above). Fix* $\mu\in\mathscr{P}(X)$ *absolutely continuous w.r.t.* $\mathfrak{m}$ *and assume that* $\nu:=\Phi_*\mu\ll\mathfrak{m}$*. If* $\mu$ *is concentrated on* $\mathcal{R}_n$ *for some* $n\geq 3$*, then* $\nu$ *is concentrated on* $\cup_{k\geq n}\mathcal{R}_k$*.*

*Proof.* Let us begin by outlining the strategy of proof.

The first step consists in proving that, if we have a $\mathsf{d}_G/\mathsf{d}_{\bar G}$-Lusin Lipschitz map which maps a subset of $\mathcal{R}_n^*$ into $\mathcal{R}_k^*$ for some $n>k\geq 3$ and we read it after the composition with bi-Lipschtiz charts, then we essentially end up with a map from a subset of $\mathbb{R}^n$ into $\mathbb{R}^k$ which satisfies the assumptions of Proposition 3.31.

In the second step we show how this information can be used to prove that $\nu=\Phi_*\mu$ is concentrated over $\cup_{k\geq n}\mathcal{R}_k$, a formal argument being the following one: suppose that $\mathfrak{m}(\Phi(\mathcal{R}_n^*)\cap\mathcal{R}_k^*)=0$, then, neglecting the measurability issues, we could compute

$$\begin{aligned}\Phi_*\mu(\mathcal{R}_k^*)&=\mu\left(\Phi^{-1}(\mathcal{R}_k^*)\right)=\mu\left(\Phi^{-1}(\mathcal{R}_k^*)\cap\mathcal{R}_n^*\right)\\&\leq\mu\left(\Phi^{-1}(\mathcal{R}_k^*\cap\Phi(\mathcal{R}_n^*))\right)=\Phi_*\mu(\mathcal{R}_k^*\cap\mathcal{R}_n^*)=0.\end{aligned}$$

**Step 1.** Recall from Theorem 2.11 that, for any $3\leq l\leq N$, we can find $\mathcal{S}_l^*\subset\mathcal{R}_l^*$ such that $\mathfrak{m}(\mathcal{R}_l^*\setminus\mathcal{S}_l^*)=0$ and $\mathcal{S}_l^*$ is a countable union of Borel sets which are 2-biLipschitz equivalent to subsets of $\mathbb{R}^l$. We want to prove that, if $P\subset\mathcal{S}_n^*$ is such that $\Phi$ is $\mathsf{d}_G/\mathsf{d}_{\bar G}$-Lipschitz over $P$, then $\mathcal{H}^k(\Phi(P)\cap\mathcal{R}_k^*)=0$ for any $3\leq k<n$. Since $\mathcal{H}^k\llcorner\mathcal{R}_k^*$ and $\mathfrak{m}\llcorner\mathcal{R}_k^*$ are mutually absolutely continuous (see (2.12)) and $\mathfrak{m}(\mathcal{R}_k^*\setminus\mathcal{S}_k^*)=0$, it suffices to prove that $\mathcal{H}^k(\Phi(P)\cap\mathcal{S}_k^*)=0$. Therefore, to prove the claimed conclusion, we can reduce ourselves to the case when $P$ is contained into the domain of an $n$-dimensional 2-biLipschitz chart, that we shall call $\alpha$, and $\Phi(P)$ is contained in the domain of a $k$-dimensional 2-biLipschitz chart, that we shall call $\beta$.

Next, with the aid of Lemma 3.13 and Lemma 3.32 above, we wish to prove that, for any $x \in P$, it holds

$$\lim_{r \to 0^+} \sup_{y \in B(x,r) \cap P} \frac{\mathsf{d}(\Phi(x), \Phi(y))}{\mathsf{d}(x,y)^{\frac{n-2}{k-2}}} < \infty. \tag{3.83}$$

To this aim we observe that, by the very definition of $\mathsf{d}_G$ and thanks to the two-sided bounds we obtained in Proposition 3.4, the $\mathsf{d}_G$-Lipschitz regularity assumption can be turned into

$$\lim_{r \to 0^+} \sup_{y \in B(x,r) \cap P} \frac{F(x, \mathsf{d}(x,y))}{F(\Phi(x), \mathsf{d}(\Phi(x), \Phi(y)))} < \infty$$

and the same holds true in case we are working with $\mathsf{d}_{\bar{G}}$, thanks to (3.54). Observe now that Lemma 3.13 grants that, as $\mathsf{d}(x,y) \to 0$, also $\mathsf{d}_G(x,y) \to 0$ (and an analogous result holds for $\mathsf{d}_{\bar{G}}$, as we observed after (3.66)). Hence we can apply Lemma 3.32 to obtain, taking into account the fact that $x \in \mathcal{R}_n^*$ and $\Phi(x) \in \mathcal{R}_k^*$,

$$\lim_{r \to 0^+} \sup_{y \in B(x,r) \cap P} \frac{\mathsf{d}(\Phi(x), \Phi(y))^{k-2}}{\mathsf{d}(x,y)^{n-2}} < \infty,$$

which easily yields (3.83).

This being said, observe that, denoting by $\varphi := \beta \circ \Phi \circ \alpha^{-1} : \alpha(P) \to \beta(\Phi(P))$ (where we remark that $\alpha(P) \subset \mathbb{R}^n$ and $\beta(\Phi(P)) \subset \mathbb{R}^k$), the map $\varphi$ satisfies the assumptions of Proposition 3.31, since $(n-2)/(k-2) > n/k$. Therefore $\mathcal{H}^k(\beta(\Phi(P))) = 0$. Hence $\mathcal{H}^k(\Phi(P)) = 0$, since $\beta$ is 2-bi-Lipschitz.

It easily follows that, if $Q \subset \mathcal{S}_n^*$, $Q = \cup_{i \in \mathbb{N}} Q_i$ where $\Phi|_{Q_i}$ is $\mathsf{d}_G/\mathsf{d}_{\bar{G}}$-Lipschitz for any $i \in \mathbb{N}$, then $\mathcal{H}^k(\mathcal{R}_k^* \cap \Phi(Q)) = 0$ for any $3 \leq k < n$.

**Step 2.** Suppose by contradiction that

$$\nu\left(\bigcup_{k<n} \mathcal{R}_k\right) > 0.$$

Then we can find $k < n$ such that $\nu(\mathcal{R}_k) > 0$. Moreover, thanks to (2.12) and to the assumption $\nu \ll \mathfrak{m}$, we can also say that $\nu(\mathcal{R}_k^*) > 0$.

We want to prove that, if this is the case, we can find a compact $P \subset \mathcal{R}_n^*$ such that $P = \cup_{i \in \mathbb{N}} P_i$, where $\Phi\Big|_{P_i}$ is $\mathsf{d}_G/\mathsf{d}_{\bar{G}}$-Lipschitz for any $i \in \mathbb{N}$, $\Phi(P) \subset \mathcal{R}_k^*$ and $\mathfrak{m}(\Phi(P)) > 0$. This would contradict what we obtained in step 1 above, since by Theorem 2.11 we know that $\mathfrak{m} \llcorner \mathcal{R}_k^*$ is absolutely continuous w.r.t. $\mathcal{H}^k$ and $\mathcal{H}^k(\Phi(P) \cap \mathcal{R}_k^*) = 0$.

We are assuming that $\nu(\mathcal{R}_k^*) = \Phi_* \mu(\mathcal{R}_k^*) > 0$, hence $\mu(\Phi^{-1}(\mathcal{R}_k^*)) = \mu(\Phi^{-1}(\mathcal{R}_k^*) \cap \mathcal{S}_n^*) > 0$, since $\mu$ is concentrated on $\mathcal{R}_n$ and therefore it is concentrated on $\mathcal{S}_n^*$. Thus, the inner regularity of $\mu$ and the assumption on $\Phi$ grant that we can find a compact $P \subset \Phi^{-1}(\mathcal{R}_k^*) \cap \mathcal{S}_n^*$ such that $\mu(P) > 0$ and $P$ is the union of countably many subsets where $\Phi$ is $\mathsf{d}_G/\mathsf{d}_{\bar{G}}$-Lipschitz. It remains to prove that $\mathfrak{m}(\Phi(P)) > 0$. To this aim, observe that

$$0 < \mu(P \cap \Phi^{-1}(\mathcal{R}_k^*)) \leq \mu(\Phi^{-1}(\Phi(P) \cap \mathcal{R}_k^*)) = \Phi_* \mu(\Phi(P) \cap \mathcal{R}_k^*).$$

The claimed conclusion $\mathfrak{m}(\Phi(P) \cap \mathcal{R}_k^*) > 0$ follows recalling that $\nu = \Phi_* \mu \ll \mathfrak{m}$. $\square$

**2.2. Regularity of vector fields drifting $W_2$-geodesics.** In Theorem 3.34 below, which is [**93**, Theorem 3.13], we state a version of the so-called Lewy-Stampacchia inequality. It will be the key tool in order to apply the regularity theory of Lagrangian Flows we developed in Section 1 to vector fields drifting $W_2$-geodesics.

Below we will indicate by $l_{K,N}:[0,\infty)\to[0,\infty)$ the continuous function, whose explicit expression will be of no importance for our purposes, appearing in the Laplacian comparison theorem (see [**86**] and [**93**, Theorem 3.5]). $\mathcal{Q}_t$ denotes the Hopf-Lax semigroup, see (1.10).

**Theorem 3.34.** *Let $(X,\mathsf{d},\mathfrak{m})$ be an $\mathrm{RCD}(K,N)$ metric measure space for some $K\in\mathbb{R}$ and $1<N<\infty$. Let $\mu_0,\mu_1\in\mathscr{P}(X)$ be absolutely continuous w.r.t. $\mathfrak{m}$ and with bounded supports, $(\mu_t)_{t\in[0,1]}$ be the $W_2$-geodesic connecting them and $\phi:X\to\mathbb{R}$ be a Kantorovich potential inducing it (which we can assume to be Lipschitz and with compact support).*

*Then, for every $t\in(0,1)$, there exists $\eta_t\in\mathrm{Lip}(X)$ with compact support, uniformly w.r.t. time, and such that*

$$-\mathcal{Q}_t(-\phi)\le\eta_t\le\mathcal{Q}_{(1-t)}(-\phi^c),\tag{3.84}$$

$$(t\eta_t)^{cc}(x)=t\eta_t(x)\quad\text{and}\quad(-(1-t)\eta_t)^{cc}(x)=-(1-t)\eta_t(x)\quad\text{for any }x\in\operatorname{supp}\mu_t$$

*and $\eta_t\in D(\Delta)$ with*

$$\|\Delta\eta_t\|_{L^\infty}\le\max\left\{\frac{l_{K,N}(2\sqrt{t\,\|\phi\|_{L^\infty}})}{t},\frac{l_{K,N}(\sqrt{2(1-t)\,\|\phi\|_{L^\infty}})}{1-t}\right\}.\tag{3.85}$$

**Remark 3.35.** We remark that, passing from the starting potentials to the regularized potentials $\eta_t$, we gain global regularity without modifying the potential in the support of $\mu_t$, as it follows from (3.84) recalling that $-\mathcal{Q}_t(-\phi)=\mathcal{Q}_{(1-t)}(-\phi^c)$ on $\operatorname{supp}\mu_t$ (see [**93**, Proposition 3.6]).

In view of the applications of the forthcoming Section 2.3, in Proposition 3.36 below we collect some consequences of the improved regularity of Kantorovich potentials.

**Proposition 3.36.** *Let $(X,\mathsf{d},\mathfrak{m})$ be an $\mathrm{RCD}(K,N)$ m.m.s. for some $K\in\mathbb{R}$ and $1<N<\infty$. Let $\mu_0,\mu_1\in\mathscr{P}(X)$ be absolutely continuous w.r.t. $\mathfrak{m}$ with bounded densities and bounded supports. Then there exists a time dependent vector field $(b_t)_{t\in(0,1)}$ such that the following conditions are satisfied:*

(i) *for any $t\in(0,1)$ it holds $b_t\in H^{1,2}_C(TX)$ and*

$$\int_\varepsilon^{1-\varepsilon}\left\{\|\nabla_{\mathrm{sym}}b_s\|_{L^2(X,\mathfrak{m})}+\|\operatorname{div}b_s\|_{L^2(X,\mathfrak{m})}\right\}\mathrm{d}s<\infty\quad\text{for any }0<\varepsilon<1;\tag{3.86}$$

(ii) *for any $0<s<1$, denoting by $(\boldsymbol{X}^t_s)_{t\in[s,1)}$ the Regular Lagrangian flow of $(b_t)_{t\in(s,1)}$, it holds that $\left(\boldsymbol{X}^t_s\right)_*\mu_s=\mu_t$ for any $s\le t<1$.*

*Proof.* We claim that the vector field $(\nabla\eta_s)_{s\in(0,1)}$ (where $\eta_s$ are the regularized Kantorovich potentials we introduced in Theorem 3.34) does the right job.

Observe that, for any $s\in(0,1)$, it holds that $\nabla\eta_t$ is bounded with bounded support, as it was stated in Theorem 3.34, Moreover, since $\eta_s\in D(\Delta)$, (1.35) implies that $\eta_s\in H^{2,2}(X,\mathsf{d},\mathfrak{m})$ which yields, in turn, $\nabla\eta_s\in H^{1,2}_C(TX)$.

Let us check (3.86). To this aim we observe that the construction described in the proof of [**93**, Theorem 3.13] guarantees that the regularized potentials can be chosen to have all support contained in the same compact set $C \subset X$. Hence

$$\begin{aligned}&\int_\varepsilon^{1-\varepsilon} \|\mathrm{div}\, b_s\|_{L^2}\,\mathrm{d}s\\ &\le \int_\varepsilon^{1-\varepsilon} \max\left\{\frac{l_{K,N}(2\sqrt{s\,\|\phi\|_{L^\infty}})}{s}, \frac{l_{K,N}(\sqrt{2(1-s)\,\|\phi\|_{L^\infty}})}{1-s}\right\}\mathfrak{m}(C)\,\mathrm{d}s < \infty.\end{aligned}$$

Dealing with the bound of the Sobolev norm we recall that (1.36) provides the quantitative bound

$$\int |\mathrm{Hess}(f)|^2_{\mathrm{HS}}\,\mathrm{d}\mathfrak{m} \le \int \left\{(\Delta f)^2 - K\,|\nabla f|^2\right\}\mathrm{d}\mathfrak{m} \tag{3.87}$$

for any $f \in D(\Delta)$. Recalling that the regularized potentials can also be chosen uniformly Lipschitz on $(0,1)$, the sought bound for $\int_\varepsilon^{1-\varepsilon} \|\nabla_{\mathrm{sym}} b_s\|_{L^2}\,\mathrm{d}s$ follows applying (3.87) to the functions $\eta_s$, taking into account the $L^\infty$-bound for the laplacian (3.85) and the uniform boundedness of the supports.

Passing to the proof of (ii), observe that the very construction of the regularized Kantorovich potentials (see Remark 3.35) $\eta_s$ grants that $(\mu_s, b_s)_{s\in(0,1)}$ is a solution to the continuity equation with uniformly bounded density (the uniform bound for the densities is a consequence of Proposition 1.24). Moreover, (3.86) guarantees, via the results of [**27**], that, for any $0 < s < t < 1$, there exists a unique Regular Lagrangian flow $(\boldsymbol{X}_s^r)_{r\in[s,t]}$ of $(b_r)_{r\in(s,t)}$. Observe that, by the very definition of RLF, also $r \mapsto (\boldsymbol{X}_s^r)_*\mu_s$ is a solution, with uniformly bounded density and initial datum $\mu_s$, to the continuity equation induced by $(b_r)_{r\in(s,t)}$. Hence $(\boldsymbol{X}_s^t)_*\mu_s = \mu_t$ for any $s \le t < 1$, since the conclusion in (i) implies that the continuity equation induced by $(b_r)_{r\in(s,t)}$ has a unique solution with uniformly bounded density (again by the results of [**27**]). ☐

**2.3. Proof of Theorem 3.1.** As we already observed, the statement is not affected by tensorization with Euclidean factors. Thus we assume without loss of generality that $(X, \mathsf{d}, \mathfrak{m})$ satisfies either Assumption 2 or Assumption 3.

Suppose by contradiction that there exist $3 \le k < n$ such that $\mathfrak{m}(\mathcal{R}_k), \mathfrak{m}(\mathcal{R}_n) > 0$. Then we can find $\eta_0, \eta_1 \in \mathscr{P}(X)$, absolutely continuous w.r.t. $\mathfrak{m}$, with bounded densities and bounded supports, such that $\eta_0(\mathcal{R}_n) = 1$ and $\eta_1(\mathcal{R}_k) = 1$. Let $(\eta_r)_{r\in[0,1]}$ be the $W_2$-geodesic joining them and recall from Proposition 1.24 that the measures $\eta_r$ are absolutely continuous w.r.t. $\mathfrak{m}$, with uniformly bounded densities and uniformly bounded supports. Applying the second conclusion in Proposition 1.24, we can also conclude that there exist $0 < s < t < 1$ such that $\eta_s(\mathcal{R}_n) > 1/2$ and $\eta_t(\mathcal{R}_k) > 1/2$. Calling $\Pi \in \mathscr{P}(\mathrm{Geo}(X))$ the unique geodesic plan lifting $(\eta_r)_{r\in[0,1]}$, it follows from what we just observed that

$$\Pi(\{\gamma \in \mathrm{Geo}(X) : \gamma(s) \in \mathcal{R}_n \text{ and } \gamma(t) \in \mathcal{R}_k\}) > 0.$$

Hence, setting

$$A := \{\gamma \in \mathrm{Geo}(X) : \gamma(s) \in \mathcal{R}_n \text{ and } \gamma(t) \in \mathcal{R}_k\}, \quad \bar{\Pi} := \frac{1}{\Pi(A)}\Pi \llcorner A \quad \text{and } \mu_r := (e_r)_*\bar{\Pi},$$

for any $r \in [0,1]$, we obtain a $W_2$-geodesic $(\mu_r)_{r\in[0,1]}$ which joins probabilities with bounded support and bounded densities w.r.t. $\mathfrak{m}$ and such that $\mu_s$ is concentrated on $\mathcal{R}_n$ and $\mu_t$ is concentrated on $\mathcal{R}_k$.

Next we apply Proposition 3.36 to the $W_2$-geodesic $(\mu_r)_{r\in[0,1]}$ to obtain that, with the notation therein introduced, $\boldsymbol{X}_t^s$ is the RLF of a Sobolev time dependent vector field satisfying the assumptions of Theorem 3.21 (or Theorem 3.29). Hence $\boldsymbol{X}_t^s$ is a $\mathsf{d}_G/\mathsf{d}_{\bar{G}}$-Lusin Lipschitz map such that $\left(\boldsymbol{X}_s^t\right)_* \mu_s = \mu_t$ and, applying Theorem 3.33, we eventually reach a contradiction.

## 3. Differentiability of the Green function along flow lines

This section is devoted the proof of a technical result about the structure of regular Lagrangian flows associated to vector fields with product structure over product spaces. As a corollary we will obtain that, for $\mathfrak{m}\times\mathfrak{m}$-a.e. $(x,y)\in X\times X$, the map $t\mapsto G(\boldsymbol{X}_t(x),\boldsymbol{X}_t(y))$ is differentiable $\mathscr{L}^1$-a.e., with the explicit and expected formula for the derivative we used in the proof of Proposition 3.20.

Let $(X,\mathsf{d}_X,\mathfrak{m}_X)$ and $(Y,\mathsf{d}_Y,\mathfrak{m}_Y)$ be $\mathrm{RCD}(K,\infty)$ m.m. spaces. Let $Z:=X\times Y$ be endowed with the product m.m.s. structure, namely

$$\mathsf{d}_Z^2((x,y),(x',y')) := \mathsf{d}_X^2(x,x')+\mathsf{d}_Y^2(y,y') \quad \text{and} \quad \mathfrak{m}_Z := \mathfrak{m}_X\times\mathfrak{m}_Y$$

and recall from [**13**, **17**] that $(X,\mathsf{d}_Z,\mathfrak{m}_Z)$ is an $\mathrm{RCD}(K,\infty)$ m.m.s itself.

We will denote by $\pi_X$ and $\pi_Y$ the canonical projections from $Z$ onto $X$ and $Y$ respectively. This being said we introduce the so-called algebra of tensor products by

$$\mathcal{A} := \left\{\sum_{j=1}^n g_j\circ\pi_X h_j\circ\pi_Y :\ g_j\in H^{1,2}_{\mathrm{loc}}\cap L^\infty_{\mathrm{loc}}(X) \text{ and } h_j\in H^{1,2}_{\mathrm{loc}}\cap L^\infty_{\mathrm{loc}}(Y)\ \forall j=1,\dots,n\right\}.$$

Below we state and prove a useful density result concerning the density of the algebra of tensors.

**Theorem 3.37.** *Let $X,Y$ and $Z$ be as above. Then, for any $f\in H^{1,2}_{\mathrm{loc}}(Z,\mathsf{d}_Z,\mathfrak{m}_Z)\cap L^\infty_{\mathrm{loc}}(Z,\mathfrak{m})$ and for any compact $P\subset Z$, there exists a sequence $(f)_{n\in\mathbb{N}}$ in $\mathcal{A}$ with $\|f_n\|_{L^\infty(P)}$ uniformly bounded and such that $\|f_n-f\|_{L^2(P,\mathfrak{m}_Z)}+\||\nabla(f_n-f)|\|_{L^2(P,\mathfrak{m}_Z)}\to 0$ as $n\to\infty$.*

*Proof.* Let us denote by $\bar{\mathcal{A}}$ the set of functions $f\in H^{1,2}_{\mathrm{loc}}(Z,\mathsf{d}_Z,\mathfrak{m}_Z)\cap L^\infty_{\mathrm{loc}}(Z,\mathfrak{m})$ for which the statement of the theorem holds true. Let $\mathcal{A}_{\mathsf{d}}$ be the smallest subset of $\mathrm{Lip}_b(X)$ containing truncated distances from points of $Z$ and closed with respect to sum, product and lattice operations, let $\mathcal{A}_{\mathsf{dbs}}\subset H^{1,2}(Z,\mathsf{d}_Z,\mathfrak{m}_Z)\cap L^\infty(Z,\mathfrak{m}_Z)$ be the subalgebra of $\mathcal{A}_{\mathsf{d}}$ made by functions with bounded support. In [**26**, Theorem B.1] it is proved that $\mathcal{A}_{\mathsf{dbs}}$ is dense in $H^{1,2}(Z,\mathsf{d}_Z,\mathfrak{m}_Z)$ and it is straightforward to check that one can approximate any bounded function in $H^{1,2}(Z,\mathsf{d}_Z,\mathfrak{m}_Z)$ with a sequence of uniformly bounded functions in $\mathcal{A}_{\mathsf{dbs}}$. Hence, to get the stated conclusion, it is sufficient to prove that $\mathsf{d}_Z(z,\cdot)\wedge k\in\bar{\mathcal{A}}$ for any $z\in Z$, for any $k\geq 0$, and the implication $f,g\in\bar{\mathcal{A}}\implies f\wedge g\in\bar{\mathcal{A}}$.

Let us first prove that $\mathsf{d}_Z(z,\cdot)\in\bar{\mathcal{A}}$ for any $z\in Z$. For any natural $n\geq 1$ let $(h_n^k)_{k\in\mathbb{N}}$ be a sequence of polynomials converging to $t\mapsto\sqrt{1/n+t}$ in $C^1_{\mathrm{loc}}([0,\infty))$ as $k\to\infty$. Let us fix $z\in Z$, it is simple to see that $h_n^k(\mathsf{d}_Z(z,\cdot)^2)$ converges in $H^{1,2}_{\mathrm{loc}}(Z,\mathsf{d}_Z,\mathfrak{m}_Z)\cap L^\infty_{\mathrm{loc}}(Z,\mathfrak{m}_Z)$ to $\sqrt{1/n+\mathsf{d}_Z^2(z,\cdot)}$ when $k\to\infty$ and that $\sqrt{1/n+\mathsf{d}_Z^2(z,\cdot)}\to\mathsf{d}_Z(z,\cdot)$, in the same topology, when $n\to\infty$. Observe that the very definition of $\mathsf{d}_Z$ yields $\mathsf{d}_Z(z,w)^2=\mathsf{d}_X(\pi_X(z),\pi_X(w))^2+\mathsf{d}_Y(\pi_Y(z),\pi_Y(w))^2$ for any $w\in Z$, therefore $h_n^k(\mathsf{d}_Z(z,\cdot)^2)\in\mathcal{A}$.

Let us now prove the implication $g \in \bar{\mathcal{A}} \implies |g| \in \bar{\mathcal{A}}$. With this aim, let us fix $g \in \bar{\mathcal{A}}$ and a sequence $g_m \in \mathcal{A}$ converging to $g$ in $H^{1,2}_{\mathrm{loc}}(Z, \mathsf{d}_Z, \mathfrak{m}_Z) \cap L^\infty_{\mathrm{loc}}(Z, \mathfrak{m}_Z)$ when $m \to \infty$. Set $g^k_{n,m} := h^k_n \circ g^2_m$, we have $g^k_{n,m} \in \mathcal{A}$ and it is easy to check that it converges to $\sqrt{1/n + g^2_m}$ in $H^{1,2}_{\mathrm{loc}}(Z, \mathsf{d}_Z, \mathfrak{m}_Z) \cap L^\infty_{\mathrm{loc}}(Z, \mathfrak{m}_Z)$ as $k \to \infty$. Moreover $\sqrt{1/n + g^2_m} \to |g_m|$ in $H^{1,2}_{\mathrm{loc}}(Z, \mathsf{d}_Z, \mathfrak{m}_Z) \cap L^\infty_{\mathrm{loc}}(Z, \mathfrak{m}_Z)$ when $n \to \infty$ and eventually $|g_m| \to |g|$, in the same topology, when $m \to \infty$. By a diagonal argument, we recover the sought approximating sequence.

Finally we exploit the identity

$$a \wedge b = \frac{|a+b| - |a-b|}{2}, \qquad \forall a, b \in [0, \infty),$$

to deduce that $\mathsf{d}_Z(z, \cdot) \wedge k \in \bar{\mathcal{A}}$ for any $z \in Z$, for any $k \geq 0$ and the implication $f, g \in \bar{\mathcal{A}} \implies f \wedge g \in \bar{\mathcal{A}}$. □

Let us consider now $b^X_t \in L^1((0,T); L^2(TX))$ and $b^Y_t \in L^1((0,T); L^2(TY))$. We introduce the "product" vector field $b^Z_t$ by saying that, for every $f \in H^{1,2}(Z, \mathsf{d}_Z, \mathfrak{m}_Z)$,

$$b^Z_t \cdot \nabla f(x,y) := b^X_t \cdot \nabla f_y(x) + b^Y_t \cdot \nabla f_x(y),$$

for $\mathfrak{m}_Z$-a.e. $(x,y) \in Z$, where $f_x(y) := f(x,y)$, $f_y(x) := f(x,y)$ and we are implicitly exploiting the tensorization of the Cheeger energy (see [**13, 17**]). It is simple to check that $b^Z_t \in L^1((0,T); L^2_{\mathrm{loc}}(Z, \mathfrak{m}_Z))$ and

$$|b^Z_t|^2(x,y) \leq |b^X_t|^2(x) + |b^Y_t|^2(y), \qquad \text{for } \mathfrak{m} \times \mathfrak{m}\text{-a.e. } (x,y) \in X \times Y.$$

**Proposition 3.38.** *Let $b^X_t$ and $b^Y_t$ be as above and $\boldsymbol{X}^X_t$ and $\boldsymbol{X}^Y_t$ be regular Lagrangian flows associated to $b^X_t$ and $b^Y_t$, respectively. Then*

$$\boldsymbol{X}^Z_t(x,y) := (\boldsymbol{X}^X_t(x), \boldsymbol{X}^Y_t(y))$$

*is a regular Lagrangian flow associated to $b^Z_t$.*

*Proof.* We need to check the validity of the three defining conditions in Definition 1.75.

The first one is trivial and the bounded compressibility property of $\boldsymbol{X}^Z_t$ is a direct consequence of the bounded compressibility property of $\boldsymbol{X}^X_t$ and $\boldsymbol{X}^Y_t$.

Dealing with the third one, we observe that, thanks to Theorem 3.37 and Remark 1.76, it is sufficient to check its validity testing it for any $f \in \mathcal{A}$. Moreover, by the linearity of (1.47) w.r.t. the test function, we can assume without loss of generality that $f = g \circ \pi_X h \circ \pi_Y$, with $g \in H^{1,2}_{\mathrm{loc}}(X, \mathsf{d}_X, \mathfrak{m}_X) \cap L^\infty_{\mathrm{loc}}(X, \mathfrak{m}_X)$ and $h \in H^{1,2}_{\mathrm{loc}}(Y, \mathsf{d}_Y, \mathfrak{m}_Y) \cap L^\infty_{\mathrm{loc}}(Y, \mathfrak{m}_Y)$. We need to prove that for $\mathfrak{m}_Z$-a.e. $(x,y) \in X \times Y$ the map $z \mapsto f(\boldsymbol{X}^Z_t(z))$ belongs to $H^{1,1}((0,T))$ and has derivative given by

$$\frac{\mathrm{d}}{\mathrm{d}t} f(\boldsymbol{X}^Z_t(z)) = b^Z_t \cdot \nabla^Z f(\boldsymbol{X}_t(z)) \quad \text{for } \mathscr{L}^1\text{-a.e. } t \in (0,T). \tag{3.88}$$

To this aim we observe that, since $\boldsymbol{X}^X_t$ and $\boldsymbol{X}^Y_t$ are regular Lagrangian flows of $b^X_t$ and $b^Y_t$ respectively, it holds that the maps $t \mapsto g(\boldsymbol{X}^X_t(x))$ and $t \mapsto h(\boldsymbol{X}^Y_t(y))$ are bounded and belong to $H^{1,1}((0,T))$ for $\mathfrak{m}_X$-a.e. $x \in X$ and $\mathfrak{m}_Y$-a.e. $y \in Y$ respectively. Moreover

$$\frac{\mathrm{d}}{\mathrm{d}t} g(\boldsymbol{X}^X_t(x)) = b^X_t \cdot \nabla g(\boldsymbol{X}^X_t(x)) \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}t} h(\boldsymbol{X}^Y_t(y)) = b^Y_t \cdot \nabla h(\boldsymbol{X}^Y_t(y))$$

for $\mathscr{L}^1$-a.e. $t \in (0,T)$, for $\mathfrak{m}_X$-a.e. $x \in X$ and $\mathfrak{m}_Y$-a.e. $y \in Y$, respectively. Applying Fubini's theorem and the Leibniz rule we obtain that, for $\mathfrak{m}_X \times \mathfrak{m}_Y$-a.e. $(x,y) \in X \times Y$, the map $t \mapsto g(\boldsymbol{X}_t^X(x))h(\boldsymbol{X}_t^Y(y))$ belongs to $H^{1,1}((0,T))$, moreover

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(g(\boldsymbol{X}_t^X(x))h(\boldsymbol{X}_t^Y(y))\right) &= \left(\frac{\mathrm{d}}{\mathrm{d}t}g(\boldsymbol{X}_t^X(x))\right)h(\boldsymbol{X}_t^Y(y)) + g(\boldsymbol{X}_t^X(x))\left(\frac{\mathrm{d}}{\mathrm{d}t}h(\boldsymbol{X}_t^Y(y))\right)\\ &= h(\boldsymbol{X}_t^Y(y))b_t^X \cdot \nabla g(\boldsymbol{X}_t^X(x)) + g(\boldsymbol{X}_t^X(x))b_t^Y \cdot \nabla h(\boldsymbol{X}_t^Y(y))\\ &= b_t^Z \cdot \nabla f(\boldsymbol{X}_t^Z(x,y)),\end{aligned}$$

for $\mathscr{L}^1$-a.e. $t \in (0,T)$, which implies (3.88). □

The following corollary of Proposition 3.38 plays an important role in the proof of Proposition 3.20.

**Corollary 3.39.** *Let $(X,\mathsf{d},\mathfrak{m})$ be an $\mathrm{RCD}(0,N)$ m.m.s. satisfying Assumption 1. Let moreover $b \in L^1((0,T);L^2(TX))$ and $\boldsymbol{X}_t$ be a regular Lagrangian flow associated to $b$. Then, the map*

$$t \mapsto G(\boldsymbol{X}_t(x),\boldsymbol{X}_t(y))$$

*belongs to $H^{1,1}((0,T))$ for $\mathfrak{m}\times\mathfrak{m}$-a.e. $(x,y) \in X \times X$ and its derivative is given by*

$$\frac{\mathrm{d}}{\mathrm{d}t}G(\boldsymbol{X}_t(x),\boldsymbol{X}_t(y)) = b_t \cdot \nabla G_{\boldsymbol{X}_t(x)}(\boldsymbol{X}_t(y)) + b_t \cdot \nabla G_{\boldsymbol{X}_t(y)}(\boldsymbol{X}_t(x)),$$

*for $\mathscr{L}^1$-a.e. $t \in (0,T)$.*

*Proof.* Let us start observing that $G^\varepsilon \in H^{1,2}_{\mathrm{loc}}(X \times X)$ for any $\varepsilon > 0$ (actually it has locally bounded weak upper gradient as one can prove with the same techniques introduced in the proof of Proposition 3.4, taking into account also Remark 3.3).

It follows from Proposition 3.38, applied with $X = Y$ and $b^X = b^Y =: b$, that

$$G^\varepsilon(\boldsymbol{X}_t(x),\boldsymbol{X}_t(y)) - G^\varepsilon(x,y) = \int_0^t \left\{b_s \cdot \nabla G^\varepsilon_{\boldsymbol{X}_s(x)}(\boldsymbol{X}_s)(y) + b_s \cdot \nabla G^\varepsilon_{\boldsymbol{X}_s(y)}(\boldsymbol{X}_s(x))\right\}\mathrm{d}s, \tag{3.89}$$

for $\mathfrak{m}\times\mathfrak{m}$-a.e. $(x,y) \in X \times X$ for every $t \in [0,T]$. We wish to pass to the limit as $\varepsilon \downarrow 0$ in (3.89) to obtain that for any $t \in [0,T]$ it holds

$$G(\boldsymbol{X}_t(x),\boldsymbol{X}_t(y)) - G(x,y) = \int_0^t \left\{b_s \cdot \nabla G_{\boldsymbol{X}_s(x)}(\boldsymbol{X}_s(y)) + b_s \cdot \nabla G_{\boldsymbol{X}_s(y)}(\boldsymbol{X}_s(x))\right\}\mathrm{d}s, \tag{3.90}$$

for $\mathfrak{m}\times\mathfrak{m}$-a.e. $(x,y) \in X \times X$. The sought conclusion would easily follow. To this aim let us observe that the left hand side in (3.89) converges to $G(\boldsymbol{X}_t(y),\boldsymbol{X}_t(x)) - G(x,y)$ in $L^1_{\mathrm{loc}}(X \times X, \mathfrak{m}\times\mathfrak{m})$. Thus, it suffices to prove that the right hand side in (3.89) converges to

$$\int_0^t \left\{b_s \cdot \nabla G_{\boldsymbol{X}_s(x)}(\boldsymbol{X}_s(y)) + b_s \cdot \nabla G_{\boldsymbol{X}_s(y)}(\boldsymbol{X}_s(x))\right\}\mathrm{d}s \qquad \text{in } L^1_{\mathrm{loc}}(X \times X, \mathfrak{m}\times\mathfrak{m}).$$

To this aim we fix $z \in X$ such that $\mathsf{d}(\boldsymbol{X}_s(z),z) \le \|b\|_{L^\infty} t$ for every $s \in [0,t]$ (observe that this property holds true for $\mathfrak{m}$-a.e. point). The triangle inequality yields

$$\mathsf{d}(\boldsymbol{X}_s(z),\boldsymbol{X}_s(y)) \le 2t\,\|b\|_{L^\infty} + \mathsf{d}(z,y), \qquad \text{for } \mathfrak{m}\text{-a.e. } y \in X.$$

Thus, setting $B := B(z,R)$, for some $R > 0$, and $\bar{B} := B(z, R + 2t\,\|b\|_{L^\infty})$, we have

$$(\boldsymbol{X}_s)_*(\mathbf{1}_B\mathfrak{m}) \le L\mathbf{1}_{\bar{B}}\mathfrak{m}. \tag{3.91}$$

The bounded compressibility property of the RLF allows us to estimate

$$
\begin{aligned}
&\int_{B\times B}\left|\int_0^t b_s\cdot\nabla G^{\varepsilon}_{\boldsymbol{X}_s(x)}(\boldsymbol{X}_s(y))\,\mathrm{d}s-\int_0^t b_s\cdot\nabla G_{\boldsymbol{X}_s(x)}(\boldsymbol{X}_s(y))\,\mathrm{d}s\right|\mathrm{d}\mathfrak{m}(x)\,\mathrm{d}\mathfrak{m}(y)\\
&\quad\le\int_0^t\int_B\int_B|b_s|(\boldsymbol{X}_s(y))\cdot|\nabla(G^{\varepsilon}_{\boldsymbol{X}_s(x)}-G_{\boldsymbol{X}_s(x)})|(\boldsymbol{X}_s(y))\,\mathrm{d}\mathfrak{m}(y)\,\mathrm{d}\mathfrak{m}(x)\,\mathrm{d}s\\
&\quad\le L^2t\,\|b\|_{L^\infty}\int_{\bar B}\|\nabla(G^{\varepsilon}_x-G_x)\|_{L^1(\bar B)}\,\mathrm{d}\mathfrak{m}(x).
\end{aligned}
$$

The last term goes to zero, as a simple application of dominated convergence theorem shows (for more details about this step we refer to the proof of Proposition 3.19, where we dealt with a similar term). Arguing similarly for the term $\int_0^t b_s\cdot\nabla G^{\varepsilon}_{\boldsymbol{X}_s(y)}(\boldsymbol{X}_s(x))\,\mathrm{d}s$ we obtain the sought conclusion. □

**Remark 3.40.** A conclusion analogous to the one stated in Corollary 3.39 holds true with $\bar G$ in place of $G$ assuming that $(X,\mathsf{d},\mathfrak{m})$ is an $\mathrm{RCD}(K,N)$ m.m.s. satisfying assumption Assumption 3. To get this result it suffices to argue as in the proof of Corollary 3.39 using Proposition 3.22 instead of Proposition 3.4.

CHAPTER 4

# Tangents to sets of finite perimeter

In the previous chapters we have proven that $\mathrm{RCD}(K,N)$ spaces are rectifiable as metric measure spaces with a constant dimension in the almost everywhere sense. This being the state of the art, we have reached a good understanding of the structure of these spaces *up to measure zero*. It is therefore quite natural to try to push the study further, investigating their structure, both from the analytic and from the geometric points of view, up to sets of positive codimension.

In this perspective in the last two years there have been some remarkable developments. We wish to mention a few of them below, without the aim of being complete in this list.

- In the setting of non collapsed Ricci limit spaces, Cheeger-Jiang-Naber have obtained in [**55**] rectifiability for the singular sets of any codimension. Let us also mention [**57**] where deep ideas on the study of singular sets have been introduced. We point out also [**29**] where some estimates are proved for the singular strata of non collapsed RCD spaces.
- There have been some efforts aimed at defining a notion of boundary for metric measure spaces and relating it with the singular set of codimension 1. See [**108**] and the very recent [**107**].
- One of the main contributions of [**87**] was the development of the language of tensor fields defined almost everywhere (with respect to the reference measure) on RCD spaces. In [**67**] the notion of tensor field defined "2-capacity-almost everywhere" is defined and it is proved that Sobolev vector fields on RCD spaces have a representative in this class.

The most natural codimension one structures on non-smooth spaces are *boundaries of sets with finite perimeter*. Indeed, on the one hand the coarea formula (Theorem 4.2) provides plenty of sets of finite perimeter even in the non-smooth context, on the other one there is no hope to have a notion of smooth hypersurface within this setting.

One of the most fundamental results of geometric measure theory, that eventually led to the Federer-Fleming theory of currents [**80**], is De Giorgi's structure theorem for sets $E$ of finite perimeter on the Euclidean space. De Giorgi's theorem, established in [**65, 66**], provides the representation

$$|D\chi_E| = \mathscr{H}^{n-1} \llcorner \mathcal{F}E$$

of the perimeter measure $|D\chi_E|$ as the restriction of $\mathscr{H}^{n-1}$ to a suitable measure-theoretic boundary $\mathcal{F}E$ of $E$. In addition, it provides a description of $E$ on small scales, showing that for all $x \in \mathcal{F}E$ the rescaled set $r^{-1}(E-x)$ is close, for $r>0$ sufficiently small, to an halfspace orthogonal to $\nu_E(x)$.

The analysis performed in the next two chapters has the ambitious goal to provide the extension of this result to the setting of $\mathrm{RCD}(K,N)$ metric measure spaces.

As in De Giorgi's theory the first step in the analysis of fine properties of sets with finite perimeter consists in studying blow-ups. This is the main goal of the present chapter.

We follows the approach of [**7**] which builds upon a new splitting theorem for spaces admitting a rigid function in the Bakry-Émery inequality. Namely $\mathrm{RCD}(0,N)$ spaces endowed with a scalar function $f$ satisfying

$$|\nabla P_t f| = P_t|\nabla f| \qquad \mathfrak{m}\text{-a.e. in } X, \text{ for every } t \geq 0. \tag{4.1}$$

The rigidity result, stated in Theorem 4.4, shows that (4.1) implies the splitting of the m.m.s. as $Z\times\mathbb{R}$, in addition with a monotonic dependence on $f$ on the split real variable. This result could be considered as "dual" to the classical splitting theorem, since the basic assumption is not the existence of a curve with a special property (namely an entire geodesic), but rather the existence of a function satisfying (4.1). However, our proof builds on Gigli's splitting theorem and is achieved in these steps:

(i)) by first variations in (4.1) we prove that the unit vector fields $b_t = \nabla P_t f/|\nabla P_t f|$ are independent of $t$, divergence-free and with symmetric part of derivative in $L^2$, in a suitable weak sense;

(ii) because of this, the theory of flows developed in [**27**] applies, and provides a measure-preserving semigroup of isometries $\boldsymbol{X}_t$;

(iii) we use $\boldsymbol{X}_t$ to show in Proposition 4.14 that $(P_s f)\circ \boldsymbol{X}_{-t}$ is a value function, more precisely

$$P_s f(\boldsymbol{X}_{-t}(x)) = \min_{\overline{B}_t(x)} P_s f \qquad \forall x\in X,\ s>0,\ t\geq 0.$$

In the proof of this fact we have been inspired by the analysis of isotropic Hamilton-Jacobi equations made in [**122**], even though our proof is self-contained. Using this representation of $(P_s f)\circ \boldsymbol{X}_{-t}$ it is not hard to prove that all flow curves $t\mapsto \boldsymbol{X}_t(x)$ are lines and, in particular, Gigli's theorem [**84**] applies. Even though this refinement does not play a role in the second part of the paper, we also prove that the validity of $|\nabla P_t f| = P_t|\nabla f|$ for some $t>0$ implies the validity for all $t\geq 0$, namely (4.1).

Now, let us explain the relation between (4.1) and the fine structure of sets of finite perimeter. In De Giorgi's proof and its many extensions to currents and other complex objects, the normal direction $\nu_E$ coming out of the blow-up analysis is identified by looking at the polar decomposition $D\chi_E = \nu_E|D\chi_E|$ of the distributional derivative (choosing approximate continuity points of $\nu_E$, relative to $|D\chi_E|$). In turn, the polar decomposition essentially depends on the particular structure of the tangent bundle of the Euclidean space. In the RCD theory, as in Cheeger's theory of PI spaces, the tangent bundle is defined only up to $\mathfrak{m}$-negligible sets, not in a pointwise sense. So, it could in principle be used to write a polar decomposition analogous to the Euclidean theory only for vector-valued (in a suitable sense) measures absolutely continuous w.r.t. $\mathfrak{m}$. We bypass this difficulty by establishing this new principle: at $|D\chi_E|$-a.e. point $x$, any tangent set $F$ to $E$ at $x$ in any tangent, pointed, metric measure structure $(Y,\varrho,\mu,y)$ has to satisfy the condition

$$|\nabla P_t\chi_F|\mu = P_t^*|D\chi_F| \qquad \forall t\geq 0. \tag{4.2}$$

Notice that $|D\chi_F|$, the semigroup $P_t$ and its dual $P_t^*$ in (4.2) have, of course, to be understood in the tangent metric measure structure. The proof of this principle, given in Theorem 4.31, ultimately relies on the lower semicontinuity of the perimeter measure $|D\chi_E|$ (as it happens for the powerful principle that lower semicontinuity and locality imply asymptotic local minimality, see [**79**, **140**], and [**48**]) and gradient contractivity. From (4.2), gradient contractivity easily yields that all functions $f = P_s\chi_F$ satisfy (4.1), this leads to a splitting *both* of $(Y, \varrho, \mu)$ and $F$, and to the identification of a "tangent halfspace" $F$ to $E$ at $x$. Using these ideas we can prove the following structure results for sets of finite perimeter $E$ in $\mathrm{RCD}(K, N)$ m.m.s.: first, in Theorem 4.32, we prove that $E$ admits a Euclidean half-space as tangent at $x$ for $|D\chi_E|$-a.e. $x \in X$.

The chapter is organized as follows. We begin by introducing sets of finite perimeter on metric measure spaces along with their basic properties in Section 1. In Section 2 we present the rigidity result for the Bakry-Émery inequality on $\mathrm{RCD}(0, N)$ spaces. We dedicate Section 3 to the study of the behaviour of sequences of sets $E_i$ in m.m.s. $(X_i, \mathsf{d}_i, \mathfrak{m}_i)$ convergent in the measured Gromov-Hausdorff sense to $(X, \mathsf{d}, \mathfrak{m})$. In particular, using appropriate notions of compactness for sequences of functions and measures in varying metric measure structures, we focus on compactness and lower semicontinuity of the perimeter measure. We apply these results in Section 4, where we specialize our analysis to the case when $(X_i, \mathsf{d}_i, \mathfrak{m}_i)$ arise from the rescaling of a pointed metric measure space. Then, using the splitting property and the principle that "tangents to a tangent are tangent", we are able to recover the above mentioned structure results of sets of finite perimeter. Finally, Section 5 is devoted to a self-contained proof of the iterated tangents principle, adapting the argument of Preiss' seminal paper [**128**] (see also [**22**, **91**]).

## 1. Sets of finite perimeter

Let us begin by introducing the notion of sets with finite perimeter on a metric measure space $(X, \mathsf{d}, \mathfrak{m})$ and by listing a few properties.

**Definition 4.1** (Perimeter and sets of finite perimeter)**.** Given a Borel set $E \subset X$ and an open set $A$ the perimeter $\mathrm{Per}(E, A)$ is defined as

$$\mathrm{Per}(E, A) := \inf\left\{\liminf_{n\to\infty} \int_A \mathrm{lip}(u_n)\,\mathrm{d}\mathfrak{m} : u_n \in \mathrm{Lip}_{\mathrm{loc}}(A), \quad u_n \to \chi_E \quad \text{in } L^1_{\mathrm{loc}}(A, \mathfrak{m})\right\}.$$

We say that $E$ has finite perimeter if $\mathrm{Per}(E, X) < \infty$. In that case it can be proved that the set function $A \mapsto \mathrm{Per}(E, A)$ is the restriction to open sets of a finite Borel measure $\mathrm{Per}(E, \cdot)$ defined by

$$\mathrm{Per}(E, B) := \inf\{\mathrm{Per}(E, A) : B \subset A,\ A \text{ open}\}.$$

Let us remark for the sake of clarity that $E \subset X$ with finite $\mathfrak{m}$-measure is a set of finite perimeter if and only if $\chi_E \in \mathrm{BV}(X, \mathsf{d}, \mathfrak{m})$ and that $\mathrm{Per}(E, \cdot) = |D\chi_E|(\cdot)$. In the following we will say that $E \subset X$ is a set of locally finite perimeter if $\chi_E$ is a function of locally bounded variation, that is to say $\eta\chi_E \in \mathrm{BV}(X, \mathsf{d}, \mathfrak{m})$ for any $\eta \in \mathrm{Lip}_{\mathrm{bs}}(X, \mathsf{d})$.

The following coarea formula for functions of bounded variation on metric measure spaces is taken from [**119**, Proposition 4.2], dealing with locally compact spaces and its proof works in the more general setting of metric measure spaces.

**Theorem 4.2** (Coarea formula)**.** *Let $v \in \mathrm{BV}(X, \mathsf{d}, \mathfrak{m})$. Then, $\{v > r\}$ has finite perimeter for $\mathscr{L}^1$-a.e. $r \in \mathbb{R}$ and, for any Borel function $f : X \to [0, \infty]$, it holds*

$$\int f\,\mathrm{d}\,|Dv| = \int_{-\infty}^{\infty} \left(\int f\,\mathrm{d}\,\mathrm{Per}(\{v > r\}, \cdot)\right)\mathrm{d}r. \tag{4.3}$$

By applying the coarea formula to the distance function we obtain immediately that, given $x \in X$, the ball $B_r(x)$ has finite perimeter for $\mathscr{L}^1$-a.e. $r > 0$, and in the sequel this fact will also be used in the quantitative form provided by (4.3).

We also recall (see for instance [**3**, **4**]) that the perimeter is local and sets of locally finite perimeter are an algebra, more precisely:

(i) $\mathrm{Per}(E, A) = P(F, A)$ whenever $\mathfrak{m}(A \cap (E \Delta F)) = 0$;
(ii) $\mathrm{Per}(E \cup F, A) + \mathrm{Per}(E \cap F, A) \le \mathrm{Per}(E, A) + \mathrm{Per}(F, A)$;
(iii) $\mathrm{Per}(E, A) = \mathrm{Per}(X \setminus E, A)$.

We will need also the following localized version of the coarea formula, which is an easy consequence of [**119**, Remark 4.3].

**Corollary 4.3.** *Let $v \in \mathrm{BV}(X, \mathsf{d}, \mathfrak{m})$ be continuous and nonnegative. Then, for any Borel function $f : X \to [0, \infty]$, it holds*

$$\int_{\{s \le v < t\}} f \,\mathrm{d}\,|Dv| = \int_s^t \left( \int f \,\mathrm{d}\,\mathrm{Per}(\{v > r\}, \cdot) \right) \mathrm{d}r, \qquad 0 \le s < t < \infty. \tag{4.4}$$

## 2. Rigidity of the Bakry-Émery inequality and splitting theorem

Our aim in this section is to prove a rigidity result for $\mathrm{RCD}(0, N)$ spaces admitting a non constant function satisfying the equality in the Bakry-Émery inequality for exponent $p = 1$ (1.23). Our investigation of the consequences of this rigidity property was motivated by the study of blow-ups of sets of finite perimeter (see Theorem 4.31 below).

**Theorem 4.4.** *Let $(X, \mathsf{d}, \mathfrak{m})$ be an $\mathrm{RCD}(0, N)$ m.m.s.. Assume that there exist a non constant function $f \in \mathrm{Lip_b}(X)$ and $s > 0$ satisfying*

$$|\nabla P_s f| = P_s |\nabla f| \qquad \mathfrak{m}\text{-a.e. in } X. \tag{4.5}$$

*Then there exists a m.m.s. $(X', \mathsf{d}', \mathfrak{m}')$ such that $X$ is isomorphic, as a metric measure space, to $X' \times \mathbb{R}$. Furthermore:*

(i) *if $N \ge 2$ then $(X', \mathsf{d}', \mathfrak{m}')$ is an $\mathrm{RCD}(0, N-1)$ m.m.s.;*
(ii) *if $N \in [1, 2)$ then $X'$ is a point.*

*Moreover, the function $f$ written in coordinates $(x', t) \in X' \times \mathbb{R}$ depends only on the variable $t$ and it is monotone.*

**Remark 4.5.** Let us point out that the action of the heat semigroup in $L^\infty(X, \mathfrak{m})$ can be defined by means of

$$P_t f(x) := \int f(y) p_t(x, y) \,\mathrm{d}\mathfrak{m}(y),$$

where $p_t$ is the heat kernel. Using an approximation argument is it possible to see that, for any $f \in L^\infty(X, \mathfrak{m})$ and every $\varphi \in L^1(X, \mathfrak{m})$ the map $t \to \int \varphi P_t f \,\mathrm{d}\mathfrak{m}$ is absolutely continuous with derivative

$$\frac{\mathrm{d}}{\mathrm{d}t} \varphi \int P_t f \,\mathrm{d}\mathfrak{m} = \int \varphi \Delta P_t f \,\mathrm{d}\mathfrak{m},$$

in other words $P_t f$ is still a solution of the heat equation.

**Remark 4.6.** The assumption $f \in \mathrm{Lip_b}(X)$ in Theorem 1.79 can be replaced with the more general $f \in \mathrm{Lip}(X)$, provided we extend the action of the heat semigroup to the class of Borel functions with at most linear growth at infinity, i.e.

$$|f(x)| \le C(1 + \mathsf{d}(x, x_0)) \qquad \text{for any } x \in X$$

for some $x_0 \in X$ and $C \geq 0$. Even though under the $\mathrm{RCD}(0,N)$ condition the Gaussian estimates for the heat kernel provide this extension, we shall consider only the case $f \in \mathrm{Lip_b}(X)$ that is enough for our purposes.

In order to better motivate the statement of Theorem 4.4 let us spend a few words about the rigidity case in the Bakry Émery inequality for $p=2$. Assume that $(M^n, \mathsf{d}_g, e^{-V}\mathrm{Vol}_g)$ is a smooth weighted Riemannian manifold with nonnegative generalized $N$-Ricci tensor $\mathrm{Ric}_N$, where

$$\mathrm{Ric}_N := \mathrm{Ric} + \mathrm{Hess}\, V - \frac{\nabla V \otimes \nabla V}{N-n},$$

and the last term is defined to be 0 when $V$ is constant and $N=n$. Let $f : M \to \mathbb{R}$ be such that $|\nabla P_t f|^2 = P_t |\nabla f|^2$ for some $t > 0$. Then we can compute

$$\begin{aligned} 0 =& P_t |\nabla f|^2 - |\nabla P_t f|^2 = \int_0^t \frac{\mathrm{d}}{\mathrm{d}s} P_s |\nabla P_{t-s} f|^2 \,\mathrm{d}s \\ =& 2\int_0^t P_s \left( |\mathrm{Hess}\, P_{t-s} f|^2 + \mathrm{Ric}_N(\nabla P_{t-s} f, \nabla P_{t-s} f) + \frac{(\nabla V \cdot \nabla P_{t-s} f)^2}{N-n} \right) \mathrm{d}s, \end{aligned}$$

where the second equality follows from the generalized Bochner identity and $\Delta$ is the weighted Laplacian. Therefore $\mathrm{Hess}(f) \equiv 0$, $(\nabla V \cdot \nabla f)^2 \equiv 0$. Thus $\Delta f \equiv 0$ since

$$\frac{(\Delta f)^2}{N} \leq |\,\mathrm{Hess}(f)|^2 + \frac{(\nabla V \cdot \nabla f)^2}{N-n} = 0.$$

Using a standard argument we obtain that $M^n$ splits isometrically as $L \times \mathbb{R}$ for some Riemannian manifold $L$. Taking into account the fact that $\Delta f = 0$ we can prove that also the measure splits.

Furthermore, denoting by $z, t$ the coordinates on $L$ and $\mathbb{R}$ respectively, it holds that $P_s f(z,t) = f(z,t) = \alpha t$ for any $s \geq 0$ and for any $t \in \mathbb{R}$, for some constant $\alpha \neq 0$.

Passing to the study of the case $p=1$, any function $f : \mathbb{R}^n \to \mathbb{R}$ such that $|\nabla P_t f| \equiv P_t |\nabla f|$ is of the form $f(z) = \phi(z \cdot v)$ for some monotone function $\phi : \mathbb{R} \to \mathbb{R}$ and some $v \in \mathbb{R}^n$. This is due to the commutation between gradient operator and heat flow on the Euclidean space and to the characterization of the equality case in Jensen's inequality. More in general, thanks to the tensorization property of the heat flow, it is possible to check that on any product m.m.s. $X = X' \times \mathbb{R}$, any function $f$ depending only on the variable $t \in \mathbb{R}$ in a monotone way satisfies $|\nabla P_t f| = P_t |\nabla f|$ almost everywhere. Basically Theorem 4.4 is telling us that, in the setting of $\mathrm{RCD}(0,N)$ spaces, this is the only possible case.

About the strategy of the proof let us observe that, as the examples above illustrated show, in the rigidity case for $p=1$ it is not necessarily true that the rigid function has vanishing Hessian. Therefore we cannot directly use $P_s f$ as a splitting function. Still our strategy relies on the properties of the normalized gradient $\nabla P_s f / |\nabla P_s f|$. First we will prove that it has vanishing symmetric covariant derivative and then that its flow lines are metric lines. The conclusion will be eventually achieved building upon the splitting theorem.

Let us start proving that if the rigidity condition (4.5) holds for some $s > 0$ then it must hold for any $s \geq 0$.

**Lemma 4.7.** *Let $(X, \mathsf{d}, \mathfrak{m})$ be an $\mathrm{RCD}(0,N)$ metric measure space and $f \in \mathrm{Lip_b}(X)$. If there exists $s > 0$ such that*

$$|\nabla P_s f| = P_s |\nabla f| \qquad \mathfrak{m}\text{-a.e. in } X, \tag{4.6}$$

*Then* $|\nabla P_r f| = P_r|\nabla f|$ *for any* $r \geq 0$.

*Proof.* It is simple to check that $|\nabla P_r f| = P_r|\nabla f|$ for any $0 \leq r \leq s$. Indeed, using (4.6) and the Bakry-Émery inequality (1.23), we have

$$0 \leq P_{s-r}\left(P_r|\nabla f| - |\nabla P_r f|\right) = P_s|\nabla f| - P_{s-r}|\nabla P_r f| = |\nabla P_s f| - P_{s-r}|\nabla P_r f| \leq 0.$$

Let us now fix $\varphi \in \mathrm{Test}_c(X, \mathsf{d}, \mathfrak{m})$ and set

$$F(r) := \int ((P_r|\nabla f|)^2 - |\nabla P_r f|^2)\varphi \, \mathrm{d}\mathfrak{m}. \tag{4.7}$$

We claim that $F(r)$ is a real analytic function in $(0, \infty)$. Observe that the claim, together with the information $F \equiv 0$ in $(0, s)$, implies $F(r) = 0$ for any $r \geq 0$ and thus our conclusion, due to the arbitrariness of the test function.

Integrating by parts the right hand side in (4.7) and using (1.8), we can write

$$F(r) = \int (P_r|\nabla f|)^2\varphi \, \mathrm{d}\mathfrak{m} + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}r}\int (P_r f)^2\varphi \, \mathrm{d}\mathfrak{m} - \frac{1}{2}\int (P_r f)^2 \Delta\varphi \, \mathrm{d}\mathfrak{m},$$

so the claim is a consequence of Lemma 4.8 below. □

**Lemma 4.8.** *Let* $(X, \mathsf{d}, \mathfrak{m})$ *be an* $\mathrm{RCD}(K, N)$ *m.m.s.. For any* $g \in L^\infty(X, \mathfrak{m})$ *and any* $\varphi \in L^1(X, \mathfrak{m})$ *the map* $t \mapsto \int (P_t g)^2 \varphi \, \mathrm{d}\mathfrak{m}$ *is real analytic in* $(0, \infty)$.

*Proof.* Exploiting a well-known analyticity criterion for real functions, it is enough to show, for any $[a, b] \subset (0, \infty)$, the existence of a constant $C = C(K, N, a, b)$ such that

$$\left|\frac{\mathrm{d}^n}{\mathrm{d}t^n}\int (P_t g)^2 \varphi \, \mathrm{d}\mathfrak{m}\right| \leq C^n \left\|g\right\|_{L^\infty}^2 \left\|\varphi\right\|_{L^1} \qquad \forall t \in (a, b), \quad \forall n \in \mathbb{N}. \tag{4.8}$$

Observe that (4.8) can be checked commuting the operators $P_t$ and $\Delta$ and using iteratively the estimate

$$\left\|\Delta P_t g\right\|_{L^\infty} \leq C' \left\|g\right\|_{L^\infty} \qquad \forall t \in (a, b), \tag{4.9}$$

where $C' > 0$ depends only on $N$, $K$, $a$ and $b$.

Let us prove (4.9) arguing by duality. For any $\psi \in L^1 \cap L^2(X, \mathfrak{m})$, we have

$$\begin{aligned} \left|\int \Delta P_t g \ \psi \, \mathrm{d}\mathfrak{m}\right| &= \left|\int \nabla P_{t/2} g \cdot \nabla P_{t/2}\psi \, \mathrm{d}\mathfrak{m}\right| \\ &\leq \left\|\nabla P_{t/2} g\right\|_{L^\infty} \left\|\nabla P_{t/2}\psi\right\|_{L^1} \\ &\leq C'' \left\|g\right\|_{L^\infty} \ C'' \left\|\psi\right\|_{L^1}, \end{aligned}$$

where the last inequality is a consequence of the following fact: there exists $C''(N, K, a, b) > 0$ such that

$$\left\|\nabla P_t h\right\|_{L^p} \leq C'' \left\|h\right\|_{L^p} \qquad \forall t \in (a, b), \quad \forall h \in L^p(X, \mathfrak{m}) \text{ with } 1 \leq p \leq \infty. \tag{4.10}$$

In order to check (4.10) we use the Gaussian estimates for the heat kernel and its gradient (1.20), (1.21) obtaining that there exists a constant $\alpha > 1$ such that

$$|\nabla P_t h|(x) \leq C'' P_{\alpha t} \left|h\right|(x), \qquad \text{for } \mathfrak{m}\text{-a.e. } x \in X, \quad \forall t \in (a, b),$$

and we take the $L^p$ norm both sides. □

Let us introduce the most important object of our investigation. For any $s > 0$ we consider the vector field

$$b_s := \frac{\nabla P_s f}{P_s|\nabla f|}, \tag{4.11}$$

that, since $P_s\,|\nabla f| > 0$ $\mathfrak{m}$-a.e., is well defined and satisfies

$$|b_s| = 1 \quad \mathfrak{m}\text{-a.e. in } X, \quad \forall s > 0, \tag{4.12}$$

thanks to (4.5).

The first important ingredient of the proof of Theorem 4.4 is the following proposition. Its proof is inspired by an analogous result in [**84**].

**Proposition 4.9** (Variation formula, version 1)**.** *For any $s > 0$, $t \geq 0$ and any $g \in \mathrm{Test}(X, \mathsf{d}, \mathfrak{m})$ it holds*

$$b_{t+s} \cdot \nabla P_t g = P_t(b_s \cdot \nabla g), \qquad \mathfrak{m}\text{-}a.e.\ in\ X. \tag{4.13}$$

Before proving Proposition 4.9 we need to state a simple lemma.

**Lemma 4.10.** *For any $s \geq 0$ the function $P_s f$ satisfies*

$$|\nabla P_{t+s} f| = P_t|\nabla P_s f|, \qquad \mathfrak{m}\text{-}a.e.\ in\ X, \quad \forall t \geq 0. \tag{4.14}$$

*Proof.* Using first the Bakry-Émery inequality (1.23) and then twice (4.5) we get

$$|\nabla P_{t+s} f| \leq P_t|\nabla P_s f| = P_{t+s}|\nabla f| = |\nabla P_{t+s} f|,$$

that proves our claim. □

*Proof of Proposition 4.9.* Let $s > 0$, $t \geq 0$ be fixed. The idea of the proof is to obtain (4.13) as the Euler equation associated to the functional

$$\Psi(h) := \int (P_t|\nabla h| - |\nabla P_t h|)\varphi \,\mathrm{d}\mathfrak{m} \qquad h \in \mathrm{Lip}(X),$$

where $\varphi \in \mathrm{Lip_{bs}}$ is a fixed nonnegative cut-off function. Indeed, thanks to Lemma 4.10 and the Bakry-Émery contraction estimate (1.23), we know that $P_s f$ is a minimum of $\Psi$. Thus

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}|_{\varepsilon=0}\Psi(P_s f + \varepsilon g) = 0 \qquad \forall g \in \mathrm{Test}(X, \mathsf{d}, \mathfrak{m}).$$

Notice that the differentiability of $\varepsilon \mapsto \Psi(P_s f + \varepsilon g)$ at $\varepsilon = 0$ can be easily checked using $|\nabla P_s f| = P_s|\nabla f| > 0$. Then we compute

$$\begin{aligned}
0 =& \frac{\mathrm{d}}{\mathrm{d}\varepsilon}|_{\varepsilon=0}\Psi(P_s f + \varepsilon g)\\
=& \frac{\mathrm{d}}{\mathrm{d}\varepsilon}|_{\varepsilon=0}\int (P_t|\nabla P_s f + \varepsilon\nabla g| - |\nabla(P_{t+s} f + \varepsilon P_t g)|)\varphi \,\mathrm{d}\mathfrak{m}\\
=& \int \left( P_t\left(\frac{\nabla P_s f}{|\nabla P_s f|} \cdot \nabla g\right) - \frac{\nabla P_{t+s} f}{|\nabla P_{t+s} f|} \cdot \nabla P_t g\right)\varphi \,\mathrm{d}\mathfrak{m}\\
=& \int (P_t(b_s \cdot \nabla g) - b_{t+s} \cdot \nabla P_t g)\varphi \,\mathrm{d}\mathfrak{m}.
\end{aligned}$$

The conclusion follows from the arbitrariness of $\varphi$. □

As a first consequence of Proposition 4.9 we get the following.

**Proposition 4.11.** *For any $s > 0$ it holds $\operatorname{div} b_s = 0$ and $\nabla_{\rm sym} b_s = 0$ according to Definition 1.77.*

*In particular, there exists a regular Lagrangian flow $\boldsymbol{X}^s : \mathbb{R} \times X \to X$ of $b_s$ with*

$$(\boldsymbol{X}^s_t)_* \mathfrak{m} = \mathfrak{m}, \qquad \mathsf{d}(\boldsymbol{X}^s_t(x), \boldsymbol{X}^s_t(y)) = \mathsf{d}(x,y) \qquad \forall t \in \mathbb{R}, \quad \forall x,\, y \in X.$$

*Proof.* Let $g \in \mathrm{Test}_c(X, \mathsf{d}, \mathfrak{m})$ be fixed. Using (4.13) we obtain

$$\begin{aligned} \left| \int b_s \cdot \nabla g(x) \,\mathrm{d}\mathfrak{m}(x) \right| &= \left| \int P_t(b_s \cdot \nabla g)(x) \,\mathrm{d}\mathfrak{m}(x) \right| \\ &= \left| \int b_{t+s} \cdot \nabla P_t g(x) \,\mathrm{d}\mathfrak{m}(x) \right| \\ &\le \int |\nabla P_t g|(x) \,\mathrm{d}\mathfrak{m}(x). \end{aligned}$$

To get $\operatorname{div} b_s = 0$ it suffices to show that

$$\lim_{t \to \infty} \int |\nabla P_t g|(x) \,\mathrm{d}\mathfrak{m}(x) = 0, \tag{4.15}$$

for any nonnegative $g \in \mathrm{Test}_c(X, \mathsf{d}, \mathfrak{m})$. To this aim we use the Gaussian estimates for the heat kernel and its gradient (1.20), (1.21) concluding that there exist a constant $C = C(N) > 0$ and $\alpha > 1$ such that

$$|\nabla P_t g|(x) \le \frac{C}{\sqrt{t}} P_{\alpha t} g(x), \qquad \text{for } \mathfrak{m}\text{-a.e. } x \in X. \tag{4.16}$$

Let us prove that $\nabla_{\rm sym} b_s = 0$ for any $s > 0$. First observe that, since $b_s$ is divergence-free we have

$$\int b_{t+s} \cdot \nabla P_t g \; P_t g \,\mathrm{d}\mathfrak{m} = \frac{1}{2} \int b_{t+s} \cdot \nabla (P_t g)^2 \,\mathrm{d}\mathfrak{m} = 0, \tag{4.17}$$

for any $g \in \mathrm{Test}(X, \mathsf{d}, \mathfrak{m})$, for any $s > 0$ and $t \ge 0$. Using again (4.13) and (4.17) we deduce

$$\begin{aligned} 0 =& \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \int b_{t+s} \cdot \nabla P_t g \; P_t g \,\mathrm{d}\mathfrak{m} = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \int P_t\left(b_s \cdot \nabla g\right) P_t g \,\mathrm{d}\mathfrak{m} \\ =& \int \Delta(b_s \cdot \nabla g)\; g \,\mathrm{d}\mathfrak{m} + \int b_s \cdot \nabla g \Delta g \,\mathrm{d}\mathfrak{m} \\ =& 2 \int b_s \cdot \nabla g \; \Delta g \,\mathrm{d}\mathfrak{m}, \end{aligned}$$

that, by polarization, implies our claim.

The second part of the statement follows from (iii) in Theorem 1.78. □

We are now in position to show that $b_s$ does not depend on $s > 0$.

**Lemma 4.12** (Variation formula, version 2)**.** *The vector field $b := b_s$ does not depend on $s > 0$. In particular, it holds*

$$b \cdot \nabla P_t g = P_t(b \cdot \nabla g) \qquad \mathfrak{m}\text{-}a.e., \tag{4.18}$$

*for every $g \in \mathrm{Test}(X, \mathsf{d}, \mathfrak{m})$ and every $t \ge 0$.*

The most important ingredient in the proof of Lemma 4.12 is the following lemma.

**Lemma 4.13.** *Let $(X, \mathsf{d}, \mathfrak{m})$ be an $\mathrm{RCD}(K, \infty)$ m.m.s. and let $T : X \to X$ be a measure preserving isometry. Then, for any $f \in L^2(X, \mathfrak{m})$, it holds*

$$P_t(f \circ T)(x) = (P_t f) \circ T(x), \tag{4.19}$$

*for any $t > 0$ and for $\mathfrak{m}$-a.e. $x \in X$.*

*Proof.* We just provide a sketch of the proof since the result is quite standard in the field.

First we observe that, since $T$ is a measure preserving isometry, it holds that $f \in H^{1,2}(X, \mathsf{d}, \mathfrak{m})$ if and only if $f \circ T \in H^{1,2}(X, \mathsf{d}, \mathfrak{m})$ and in that case $\mathsf{Ch}(f \circ T) = \mathsf{Ch}(f)$. From this observation we deduce (4.19), since the heat flow is the gradient flow of the Cheeger energy in $L^2(X, \mathfrak{m})$.

□

*Proof of Lemma 4.12.* Let $s > 0$ and let $\boldsymbol{X}^s$, the regular Lagrangian flow associated to $b_s$, be fixed.

We know from Proposition 4.11 that for any $t \in \mathbb{R}$ the flow map $\boldsymbol{X}^s_t$ is a measure preserving isometry of $X$. Therefore, for any $r \geq 0$ and any $g \in \mathrm{Test}(X, \mathsf{d}, \mathfrak{m})$, using (4.19) with $T = \boldsymbol{X}^s_t$ and (4.13), we get

$$\begin{aligned}(b_s \cdot \nabla P_r g) \circ \boldsymbol{X}^s_t =& \frac{\mathrm{d}}{\mathrm{d}t} P_r(g) \circ \boldsymbol{X}^s_t = \frac{\mathrm{d}}{\mathrm{d}t} P_r(g \circ \boldsymbol{X}^s_t) \\ =& P_r((b_s \cdot \nabla g) \circ \boldsymbol{X}^s_t) = P_r(b_s \cdot \nabla g) \circ \boldsymbol{X}^s_t \\ =& (b_{r+s} \cdot \nabla P_r g) \circ \boldsymbol{X}^s_t.\end{aligned}$$

Since $g$ is arbitrary, the first conclusion in the statement follows. The second one is a direct consequence of Proposition 4.9. □

Let us denote by $\boldsymbol{X}$ the regular Lagrangian flow of $b$ from now on, choosing in particular the "good representative" of Theorem 1.78 (iv). Our next aim is to prove that for any $x \in X$ the curve $t \mapsto \boldsymbol{X}_t(x)$ is a line. This will yield the sought conclusion about the product structure of $(X, \mathsf{d}, \mathfrak{m})$ by means of the splitting theorem Theorem 1.79.

**Proposition 4.14.** *For all $s > 0$ the identity*

$$P_s f(\boldsymbol{X}_{-t}(x)) = \min_{\overline{B}_t(x)} P_s f \tag{4.20}$$

*holds true for any $t \geq 0$ and any $x \in X$.*

Before then passing to the proof we wish to explain the heuristic standing behind it with a formal computation:

$$\frac{\mathrm{d}}{\mathrm{d}t} P_s f(\boldsymbol{X}_{-t}(x)) = -\nabla P_s f \cdot \frac{\nabla P_s f}{|\nabla P_s f|}(\boldsymbol{X}_{-t}(x)) = -\left|\nabla P_s f\right|(\boldsymbol{X}_{-t}(x)) = -\left|\nabla\left(P_s f \circ \boldsymbol{X}_t\right)\right|(x).$$

Therefore, setting $u(t, x) := P_s f(\boldsymbol{X}_{-t}(x))$, it holds that

$$\partial_t u(t, x) + |\nabla_x u(t, x)| = 0 \tag{4.21}$$

and it is well known that the Hopf-Lax semigroup

$$\mathcal{Q}_t u_0(x) := \min_{\overline{B}_t(x)} u_0$$

provides a solution of (4.21), and the unique viscosity solution (see [**122**]). Proposition 4.14 is just telling us that $u(t, x) = P_s f(X_{-t}(x))$ is precisely the Hopf-Lax semigroup solution.

*Proof of Proposition 4.14.* Let us denote by $u(t,x)$ the left hand side in (4.20). Since

$$\mathsf{d}(\boldsymbol{X}_{-t}(x),x)\le t,$$

the inequality $\ge$ in (4.20) is obvious.

Now, we claim that for all $\gamma\in \mathrm{Lip}_1([0,\infty);X)$ the function $t\mapsto u(t,\gamma(t))$ is nonincreasing. In order to prove the claim, first we observe that $t\mapsto u(t,x)=P_sf(\boldsymbol{X}_{-t}(x))$ is of class $C^1$, since its derivative is $-P_s|\nabla f|(\boldsymbol{X}_{-t}(x))$ that is a continuous function. Indeed, the validity of this condition for $\mathfrak{m}$-a.e. $x\in X$ follows from the defining conditions of RLF and we can extend it to all $x\in X$ by continuity of the maps $(t,x)\mapsto u(t,x)$ and $(t,x)\mapsto -P_s\left|\nabla f\right|(\boldsymbol{X}_{-t}(x))$. Then taking into account the Leibniz rule in [**14**, Lemma 4.3.4], it suffices to show that

$$\limsup_{h\to 0^+}\frac{|u(t,\gamma(t+h))-u(t,\gamma(t))|}{h}\le P_s|\nabla f|(\boldsymbol{X}_{-t}(\gamma(t))).$$

This inequality follows easily from Lemma 4.15 below and the inequality $|\nabla P_sf|\le P_s|\nabla f|$, since

$$\frac{|u(t,\gamma(t+h))-u(t,\gamma(t))|}{h}\le \fint_t^{t+h} P_s|\nabla f|(\boldsymbol{X}_{-t}(\gamma(r)))\,\mathrm{d}r,$$

(here we also used that $r\mapsto \boldsymbol{X}_{-t}(\gamma(r))$ is 1-Lipschitz), by taking the limit as $h\downarrow 0$.

From the claim, the converse inequality in (4.20) follows easily, because for all $x\in X$ and all minimizers $\bar{x}$ of $P_sf$ in $\overline{B}_t(x)$ the geodesic property of $(X,\mathsf{d})$ guarantees the existence of $\gamma\in\mathrm{Lip}_1([0,\infty);X)$ with $\gamma(t)=x$ and $\gamma(0)=\bar{x}$. It follows that

$$u(t,x)=u(t,\gamma(t))\le u(0,\gamma(0))=u(0,\bar{x})=P_sf(\bar{x})=\min_{\overline{B}_t(x)}P_sf.$$

$\square$

**Lemma 4.15.** *Let $(X,\mathsf{d},\mathfrak{m})$ be an $\mathrm{RCD}(K,\infty)$ m.m.s. and $u\in\mathrm{Lip}(X)$. Assume that $|\nabla u|$ has a continuous representative in $L^\infty(X,\mathfrak{m})$. Then*

$$|u(\gamma(t))-u(\gamma(s))|\le\int_s^t|\nabla u|(\gamma(r))\left|\gamma'\right|(r)\,\mathrm{d}r, \tag{4.22}$$

*for any $s<t$ and for any Lipschitz curve $\gamma:\mathbb{R}\to X$ (where we denoted by $|\nabla u|$ the continuous representative of the minimal relaxed slope of $u$).*

*Proof.* To get the sought conclusion we argue by regularization via heat flow as in the proof of [**17**, Theorem 6.2].
Let $\left(\mu_r^\lambda\right)_{r\in\mathbb{R}}$ be defined by $\mu_r^\lambda:=(P_\lambda)^*\,\delta_{\gamma(r)}$. Contractivity yields now that

$$\begin{aligned}|P_\lambda u\left(\gamma(t)\right)-P_\lambda u(\gamma(s))|\le&\int_s^t\left(\int|\nabla u|^2\,\mathrm{d}\mu_r^\lambda\right)^{\frac12}\left|\dot{\mu}_r^\lambda\right|\mathrm{d}r\\ \le& e^{-K\lambda}\int_s^t\left(\int|\nabla u|^2\,\mathrm{d}\mu_r^\lambda\right)^{\frac12}|\dot{\gamma}_r|\,\mathrm{d}r\\ =&e^{-K\lambda}\int_s^t\left(P_\lambda\left|\nabla u\right|^2(\gamma(r))\right)^{\frac12}|\dot{\gamma}_r|\,\mathrm{d}r,\end{aligned}\tag{4.23}$$

for any $\lambda>0$ and for any $s,t\in\mathbb{R}$. Passing to the limit as $\lambda\downarrow 0$ both the first and the last expression in (4.23) and taking into account the continuity of $u$ and $|\nabla u|$, we obtain (4.22). $\square$

By means of Proposition 4.14 we can easily prove the following.

**Corollary 4.16.** *For any $x \in X$ the curve $t \mapsto \boldsymbol{X}_t(x)$ is a line, that is to say*

$$\mathsf{d}(\boldsymbol{X}_t(x), \boldsymbol{X}_s(x)) = |t-s| \qquad \forall s,t \in \mathbb{R}.$$

*Proof.* Let us start observing that any $x_t \in \overline{B}_t(x)$ such that

$$\min_{y\in \overline{B}_t(x)} P_s f(y) = P_s f(x_t)$$

has to satisfy $\mathsf{d}(x, x_t) = t$. Otherwise we might replace $x_t$ with $\boldsymbol{X}_{-\varepsilon}(x_t)$ (that belongs to $B_t(x)$ for $\varepsilon$ sufficiently small) and, since $P_s f$ is strictly increasing along the flow lines of $\boldsymbol{X}$, we would get a contradiction.

Furthermore $\boldsymbol{X}_t(x) \in \overline{B}_t(x)$ since $|b| = 1$. Thus it follows from (4.20) that $\mathsf{d}(\boldsymbol{X}_{-t}(x), x) = t$ for any $t \geq 0$. Using the semigroup property and the fact that $\boldsymbol{X}_t$ is an isometry for any $t \in \mathbb{R}$ (see Proposition 4.11) we get the sought conclusion. □

*Proof of Theorem 4.4.* As we anticipated the conclusion that $X$ is isomorphic to $X' \times \mathbb{R}$ for some $\mathrm{RCD}(0, N-1)$ m.m.s. $(X', \mathsf{d}', \mathfrak{m}')$ follows from Corollary 4.16 applying Theorem 1.79.

Let us deal with the second part of the statement.
First of all we claim that all the flow lines of $\boldsymbol{X}$ are vertical lines in $X$, that is to say, denoting by $(z, s) \in X' \times \mathbb{R}$ the coordinates on $X$, $\boldsymbol{X}_t(z, s) = (z, t + s)$ for any $z \in X'$ and for any $s, t \in \mathbb{R}$. Indeed, since we proved that all integral curves of $b$ are lines in $(X, \mathsf{d})$, the construction provided by the splitting theorem shows that this is certainly true for a fixed $\bar{z} \in X'$. Let us consider any other $z \in X'$ and call $\boldsymbol{X}_t((z, 0)) = \left(\boldsymbol{X}^1_t((z, 0)), \boldsymbol{X}^2_t(z, 0)\right)$. Taking into account the semigroup property (1.50) and the fact that $\boldsymbol{X}_t$ is an isometry for any $t \in \mathbb{R}$, for any $\tau \in \mathbb{R}$ we can compute

$$\begin{aligned}\tau^2 + \mathsf{d}^2_Z(\bar{z}, z) =&\mathsf{d}^2\left(\boldsymbol{X}_\tau((\bar{z}, 0)), (z, 0)\right) = \mathsf{d}^2\left(\boldsymbol{X}_{t+\tau}((\bar{z}, 0)), \boldsymbol{X}_t((z, 0))\right)\\ =&\mathsf{d}^2\left((\bar{z}, t+\tau), (\boldsymbol{X}^1_t((z, 0)), \boldsymbol{X}^2_t((z, 0)))\right)\\ =&\left|(\boldsymbol{X}^2_t((z, 0)) - t) - \tau\right|^2 + \mathsf{d}^2_Z\left(\bar{z}, \boldsymbol{X}^1_t((z, 0))\right).\end{aligned}$$

Since $\tau$ is arbitrary, it easily follows that $\boldsymbol{X}^2_t((z, 0)) = t$ for any $t \in \mathbb{R}$ and therefore $\boldsymbol{X}^1_t((z, 0)) = z$ for any $t \in \mathbb{R}$, as we claimed.

From what we just proved it follows that $\nabla P_s f$ is trivial in the $z$ variable and we can conclude that $P_s f$ depends only on the $t$-variable for any $s > 0$ thanks to the tensorization of the Cheeger energy (see [**17**, Theorem 6.19]). Passing to the limit as $s \downarrow 0$ we obtain that the same holds true also for $f$.

Knowing that $f$ depends only on the $t$-variable, the monotonicity in this variable can be immediately checked. □

## 3. Convergence and stability results for sets of finite perimeter

In this section we establish some useful compactness and stability results for sequences of sets of finite perimeter defined on a pmGH converging sequence of $\mathrm{RCD}(K, N)$ m.m. spaces. Most of the results we present here adapt and extend to the case of our interest those of [**19**].

Until the end of this section we fix a sequence $\{(X_i, \mathsf{d}_i, \mathfrak{m}_i, x_i)\}_{i\in\mathbb{N}}$ of pointed $\mathrm{RCD}(K,N)$ m.m. spaces converging in the pmGH topology to $(Y, \varrho, \mu, y)$ and a proper metric space $(Z, \mathsf{d}_Z)$ that realizes this convergence according to Definition 1.28.

Since in the rest of the note we will be mainly interested on the case of indicator functions, let us observe that, in that case, we can rephrase the notion of $L^1$-strong convergence introduced in Definition 1.38 in the following way.

**Definition 4.17.** We say that a sequence of Borel sets $E_i \subset X_i$ such that $\mathfrak{m}_i(E_i) < \infty$ for any $i \in \mathbb{N}$ converges in $L^1$-strong to a Borel set $F \subset Y$ with $\mu(F) < \infty$ if $\chi_{E_i}\mathfrak{m}_i \rightharpoonup \chi_F \mu$ in duality with $\mathrm{C_{bs}}(Z)$ and $\mathfrak{m}_i(E_i) \to \mu(F)$.

We also say that a sequence of Borel sets $E_i \subset X_i$ converges in $L^1_{\mathrm{loc}}$ to a Borel set $F \subset Y$ if $E_i \cap B_R(x_i) \to F \cap B_R(y)$ in $L^1$-strong for any $R > 0$.

**Remark 4.18.** The $L^1$-strong convergence implies $L^1_{\mathrm{loc}}$-strong convergence as a consequence of Lemma 4.22 and the following observation:

$$\chi_{B_R(x_i)} \to \chi_{B_R(y)} \quad \text{in } L^1\text{-strong, for any } R > 0.$$

This convergence property follows from the already remarked fact that spheres have vanishing measure on $\mathrm{RCD}(K,N)$ spaces.

**Remark 4.19.** It follows from the very definition of $L^1$-convergence that, if a sequence of sets $E_i \to F$ in $L^1$, then $\chi_{E_i} \to \chi_F$ in $L^2$-strong.

Let us begin with a compactness result which adapts [**19**, Proposition 7.5] to the case of our interest (basically, we add the uniform $L^\infty$ bound and this allows to remove the assumption on the existence of a common isoperimetric profile).

**Proposition 4.20.** *Let $(X_i, \mathsf{d}_i, \mathfrak{m}_i, x_i)$, $(Y, \varrho, \mu, y)$, and $(Z, \mathsf{d}_Z)$ be as above and fix $r > 0$. For any sequence of functions $f_i \in \mathrm{BV}(X_i, \mathfrak{m}_i)$ such that $\operatorname{supp} f_i \subset \overline{B}_r(x_i)$ for any $i \in \mathbb{N}$ and*

$$\sup_{i\in\mathbb{N}} \left\{ |Df_i|(X_i) + \|f_i\|_{L^\infty(X_i,\mathfrak{m}_i)} \right\} < \infty,$$

*there exist a subsequence $i(k)$ and $f \in L^\infty(Y,\mu) \cap \mathrm{BV}(Y, \varrho, \mu)$ with $\operatorname{supp} f \subset \overline{B}_r(y)$ such that $f_{i(k)} \to f$ in $L^1$-strong.*

As a corollary, a truncation and a diagonal argument provide a compactness result for sequences of sets with locally uniformly bounded perimeters.

**Corollary 4.21.** *For any sequence of Borel sets $E_i \subset X_i$ such that*

$$\sup_{i\in\mathbb{N}} \mathrm{Per}(E_i, B_R(x_i)) < \infty \qquad \forall R > 0 \tag{4.24}$$

*there exist a subsequence $i(k)$ and a Borel set $F \subset Y$ such that $E_{i(k)} \to F$ in $L^1_{\mathrm{loc}}$.*

We postpone the proof of Proposition 4.20 and Corollary 4.21 after a technical lemma that will play a role also in the sequel.

**Lemma 4.22.** *Let $(X_i, \mathsf{d}_i, \mathfrak{m}_i, x_i)$, $(Y, \varrho, \mu, y)$, and $(Z, \mathsf{d}_Z)$ be as above and $E_i$, $\tilde{E}_i \subset X_i$ satisfy $\mathfrak{m}_i(E_i) + \mathfrak{m}_i(\tilde{E}_i) < \infty$. If $E_i \to F$ and $\tilde{E}_i \to \tilde{F}$ in $L^1$-strong, for some Borel sets $F$, $\tilde{F} \subset Y$, then $E_i \cap \tilde{E}_i \to F \cap \tilde{F}$ in $L^1$-strong.*

*Proof.* Observing that

$$\chi_{E_i \cap \tilde{E}_i} = \chi_{E_i} \cdot \chi_{\tilde{E}_i} = \frac{1}{4}\left[ (\chi_{E_i} + \chi_{\tilde{E}_i})^2 - (\chi_{E_i} - \chi_{\tilde{E}_i})^2 \right],$$

the conclusion follows from Proposition 1.40. □

*Proof of Corollary 4.21.* We claim that, possibly extracting a subsequence that we do not relabel, there exist radii $R_\ell \uparrow \infty$ as $\ell \to \infty$ with the following property

$$\sup_{i\in\mathbb{N}} \mathrm{Per}(B_{R_\ell}(x_i), X_i) < \infty \qquad \forall l \in \mathbb{N}. \tag{4.25}$$

Indeed, applying the coarea formula in the localized version of Corollary 4.3 to the functions $\mathsf{d}(x_i,\cdot)$ and recalling that $|\nabla \mathsf{d}(x_i,\cdot)|_i = 1$ $\mathfrak{m}_i$-a.e. for any $i$, we obtain

$$\int_0^R \mathrm{Per}(B_r(x_i), X_i)\,\mathrm{d}r = \mathfrak{m}_i(B_R(x_i)) \quad \text{for any } R>0 \text{ and } i \in \mathbb{N}.$$

Observing that for any $R>0$ it holds $\mathfrak{m}_i(B_R(x_i)) \to \mu(B_R(y))$, an application of Fatou's lemma yields now

$$\int_0^R \liminf_{i\to\infty} \mathrm{Per}(B_r(x_i), X_i)\,\mathrm{d}r \le \liminf_{i\to\infty} \mathfrak{m}_i(B_R(x_i)) = \mu(B_R(y)) \quad \text{for any } R>0. \tag{4.26}$$

The claimed conclusion (4.25) can be obtained from (4.26) via a diagonal argument.

For any $\ell \in \mathbb{N}$ we can now estimate

$$\sup_{i\in\mathbb{N}} \mathrm{Per}(E_i \cap B_{R_\ell}(x_i), X) \le \sup_{i\in\mathbb{N}} \mathrm{Per}(E_i, B_{R_\ell+1}(x_i)) + \sup_{i\in\mathbb{N}} \mathrm{Per}(B_{R_\ell}(x_i), X) < \infty,$$

thanks to the locality and subadditivity of perimeters for the first inequality and to (4.24), (4.25) for the second one. Thus for any $\ell \in \mathbb{N}$ we can apply Proposition 4.20 to the functions $f_i := \chi_{E_i \cap B_{R_\ell}(x_i)}$. Observing that $L^1$-strong limits of characteristic functions are characteristic functions (as a consequence of Proposition 1.40), we can use a diagonal argument together with Lemma 4.22 to recover the global limit set. □

*Proof of Proposition 4.20.* Let us fix $t>0$. For any $i \in \mathbb{N}$ we write $f_i = P_t^i f_i + (f_i - P_t^i f_i)$ where, for any $i \in \mathbb{N}$, $P_t^i$ denotes the heat semigroup on $(X_i, \mathsf{d}_i, \mathfrak{m}_i)$. Observe that, as a consequence of the regularizing estimates (1.9), it holds that

$$\sup_{i\in\mathbb{N}} \left\{ \int_Z |P_t^i f_i|^2 \,\mathrm{d}\mathfrak{m}_i + \mathsf{Ch}^i(P_t^i f_i) \right\} < \infty, \tag{4.27}$$

where $\mathsf{Ch}^i$ is the Cheeger energy on $(X_i, \mathsf{d}_i, \mathfrak{m}_i)$. Moreover, we claim that

$$\limsup_{R\to\infty} \sup_{i\in\mathbb{N}} \int_{Z\setminus B_R(x_i)} |P_t^i f_i|^2 \,\mathrm{d}\mathfrak{m}_i = 0 \qquad \forall t>0. \tag{4.28}$$

Indeed, using both the Gaussian estimates for the heat kernel in (1.20), we get

$$\begin{aligned}
&\int_{Z\setminus B_R(x_i)} |P_t^i f_i|^2 \,\mathrm{d}\mathfrak{m}_i \\
&\qquad \le \|f_i\|_{L^\infty(X_i,\mathfrak{m}_i)} \int_{Z\setminus B_R(x_i)} P_t^i |f_i| \,\mathrm{d}\mathfrak{m}_i \\
&\qquad \le C \|f_i\|_{L^\infty(X_i,\mathfrak{m}_i)} \int_{Z\setminus B_R(x_i)} \int_{B_r(x_i)} \frac{e^{-\frac{\mathsf{d}^2(x,y)}{5t}+ct}}{\mathfrak{m}_i(B_{\sqrt{t}}(x))} |f_i(y)| \,\mathrm{d}\mathfrak{m}_i(y)\,\mathrm{d}\mathfrak{m}_i(x) \\
&\qquad \le C e^{-\frac{(R-r)^2}{10t}} \|f_i\|_{L^\infty(X_i,\mathfrak{m}_i)} \int_{Z\setminus B_R(x_i)} \int_{B_r(x_i)} \frac{e^{-\frac{\mathsf{d}^2(x,y)}{10t}+ct}}{\mathfrak{m}_i(B_{\sqrt{t}}(x))} |f_i(y)| \,\mathrm{d}\mathfrak{m}_i(y)\,\mathrm{d}\mathfrak{m}_i(x)
\end{aligned}$$

$$\leq C_t e^{-\frac{(R-r)^2}{10t}} \|f_i\|_{L^\infty(X_i,\mathfrak{m}_i)} \int_Z P^i_{\alpha t}|f_i| \,\mathrm{d}\mathfrak{m}_i$$
$$\leq C_t e^{-\frac{(R-r)^2}{10t}} \|f_i\|_{L^\infty(X_i,\mathfrak{m}_i)} \|f_i\|_{L^1(X_i,\mathfrak{m}_i)},$$

where $\alpha > 0$ is a constant depending only on $K$ and $N$.

Taking into account (4.27) and (4.28), we can apply Theorem 1.42 to get that $P^i_t f_i$ admits a subsequence converging in $L^1$-strong. In order to conclude the proof it suffices to observe that

$$\lim_{t\to 0^+} \sup_{i\in\mathbb{N}} \int_{X_i} |P^i_t f_i - f_i| \,\mathrm{d}\mathfrak{m}_i = 0,$$

as it follows from the inequality

$$\int_{X_i} |P^i_t f_i - f_i| \,\mathrm{d}\mathfrak{m}_i \leq C(K,t)|Df_i|(X_i),$$

with $C(K,t) \sim \sqrt{t}$ as $t \to 0$ (see for instance [**19**, Proposition 6.3]). □

Let us pass to a lower semicontinuity result for the total variations.

**Proposition 4.23.** *Let $(X_i, \mathsf{d}_i, \mathfrak{m}_i, x_i)$ be $\mathrm{RCD}(K,N)$ m.m. spaces converging in the pmGH topology to $(Y, \varrho, \mu, y)$ and $(Z, \mathsf{d}_Z)$ realizing the convergence as above. Let $f_i \in \mathrm{BV}(X_i, \mathfrak{m}_i)$ converge in $L^1$-strong to $f \in L^1(Y,\mu)$. If $\sup_i |Df_i|(X_i) < \infty$ then $f \in \mathrm{BV}(Y,\varrho,\mu)$ and*

$$\liminf_{i\to\infty} |Df_i|(X_i) \geq |Df|(Y). \tag{4.29}$$

*Furthermore, if*

$$\sup_{i\in\mathbb{N}} \|f_i\|_{L^\infty(X_i,\mathfrak{m}_i)} < \infty, \tag{4.30}$$

*then*

$$\liminf_{i\to\infty} \int_{X_i} g \,\mathrm{d}|Df_i| \geq \int_Y g \,\mathrm{d}|Df|, \qquad \text{for all } g \in \mathrm{Lip_{bs}}(Z) \text{ nonnegative.} \tag{4.31}$$

Before than proving Proposition 4.23 we state and prove a simple corollary of it.

**Corollary 4.24.** *Let $(X_i, \mathsf{d}_i, \mathfrak{m}_i, x_i)$ be $\mathrm{RCD}(K,N)$ m.m.s. converging in the pmGH topology to $(Y,\varrho,\mu,y)$ and $(Z,\mathsf{d}_Z)$ realizing the convergence as above. For any $f_i \in \mathrm{BV}(X_i,\mathsf{d}_i,\mathfrak{m}_i)$ convergent in energy in* BV *to $f \in \mathrm{BV}(Y,\varrho,\mu)$ such that $\sup_i \|f_i\|_{L^\infty(X_i,\mathfrak{m}_i)} < \infty$, it holds that $|Df_i| \rightharpoonup |Df|$ in duality with $\mathrm{C_{bs}}(Z)$.*

*Proof of Corollary 4.24.* From (4.31) we can deduce with a standard measure theoretic argument that

$$\liminf_{i\to\infty} |Df_i|(A) \geq |Df|(A) \qquad \forall A \subset Z \text{ open and bounded.} \tag{4.32}$$

Let $\nu$ be any weak limit point of $|Df_i|$, in the weak topology induced by $\mathrm{C_{bs}}(Z)$, along some subsequence $i(k)$ (the sequence $|Df_i|(X_i)$ is bounded and therefore the family $\{|Df_i|\}_i$ is weakly compact). For any open and bounded set $A \subset Z$ such that $\nu(\partial A) = 0$, it holds $\lim_k |Df_{i(k)}|(A) = \nu(A)$. Hence, taking into account also (4.32), we get $|Df|(A) \leq \nu(A)$. Thus $|Df| \leq \nu$, as measures in $Z$. On the other hand, since the evaluation on open sets is lower semicontinuous w.r.t. the weak convergence induced by $\mathrm{C_{bs}}(Z)$, by definition of convergence in energy in BV, we have $\nu(Z) \leq \liminf_k |Df_{i(k)}|(Z) = |Df|(Z)$ and therefore $\nu = |Df|$. □

*Proof of Proposition 4.23.* The first part of the statement has been proved in [**20**, Theorem 6.4]. Let us deal with the second one. Fix any $t > 0$ and observe that $P_t^i f_i \to P_t f$ in $H^{1,2}$ according to Definition 1.37. Indeed, the $L^1$-strong convergence of $f_i$ to $f$, combined with (4.30), yields that $f_i$ converge in $L^2$-strong to $f$ by Proposition 1.40. Therefore we can apply Proposition 1.43 to obtain the claimed conclusion. Hence Proposition 1.41 applies, yielding that

$$\liminf_{i\to\infty} \int_Z g|\nabla P_t^i f_i| \,\mathrm{d}\mathfrak{m}_i \geq \int_Z g|\nabla P_t f| \,\mathrm{d}\mu, \qquad \text{for all } g \in \mathrm{Lip_{bs}}(Z) \text{ nonnegative.} \tag{4.33}$$

In order to prove (4.31) starting from its regularized version (4.33), we argue as in the proof of [**20**, Lemma 5.8]. Taking into account the Bakry-Émery contraction estimate $|\nabla P_t h| \leq e^{-Kt} P_t^* |Dh|$ (see (1.23)) and the estimate

$$\|P_t g - g\|_{L^\infty} \leq C(K,N,t)\mathrm{Lip}(g), \qquad \text{with } C(K,N,t) \sim \sqrt{t} \text{ as } t \to 0$$

which is available over any $\mathrm{RCD}(K,N)$ m.m.s. (and can be proved using the Gaussian estimates for the heat kernel (1.20)), we obtain

(4.34)
$$\begin{aligned}
\liminf_{i\to\infty} \int_Z g \,\mathrm{d}\,|Df_i| \geq& \liminf_{i\to\infty} \int_Z P_t^i g \,\mathrm{d}\,|Df_i| - \limsup_{i\to\infty} \int_Z |P_t^i g - g| \,\mathrm{d}\,|Df_i| \\
\geq& e^{Kt} \liminf_{i\to\infty} \int_Z g \left|\nabla P_t^i f_i\right| \mathrm{d}\mathfrak{m}_i - C(K,N,t)\mathrm{Lip}(g) \limsup_{i\to\infty} |Df_i|\,(X_i) \\
\geq& e^{Kt} \int_Z g\,|\nabla P_t f| \,\mathrm{d}\mu - C(K,N,t)\mathrm{Lip}(g) \limsup_{i\to\infty} |Df_i|\,(X_i).
\end{aligned}$$

The sought conclusion (4.31) can be obtained passing to the $\liminf$ as $t \to 0$ in (4.34), recalling that $|\nabla P_t f|\,\mu \rightharpoonup |Df|$ in duality with $\mathrm{C_{bs}}(Z)$ as $t \downarrow 0$. ☐

The next result deals with the possibility of approximating in BV energy a set of finite perimeter in the limit space with a sequence of sets of finite perimeter defined on the approximating spaces.

**Proposition 4.25.** *Let $(X_i, \mathsf{d}_i, \mathfrak{m}_i, x_i)$ be $\mathrm{RCD}(K,N)$ m.m. spaces converging in the pmGH topology to $(Y, \varrho, \mu, y)$ and let $(Z, \mathsf{d}_Z)$ be realizing the convergence as above. Let $F \subset Y$ be a bounded set of finite perimeter. Then there exists a subsequence $(i_k)$ and (uniformly bounded) sets of finite perimeter $E_{i_k} \subset X_{i_k}$ such that $\chi_{E_{i_k}} \to \chi_F$ in energy in* BV *as $k \to \infty$.*

*Proof.* Let us begin observing that the first part of [**19**, Theorem 8.1] provides existence of a sequence $(g_i) \subset \mathrm{BV}(X_i, \mathfrak{m}_i)$ strongly converging in BV to $\chi_F$. Since by assumption $F \Subset B_R(y)$ for some $R > 0$, we can find a Lipschitz function $\eta : Z \to [0,1]$ with support contained in $B_{2R}(y)$ such that $\eta|_{B_R(y)} \equiv 1$ and it is easy to check that the sequence $f_i := \eta g_i$ still converges in $L^1$-weak to $\chi_F$ and satisfies $|Df_i| \to \mathrm{Per}(F)$ as $i \to \infty$. Furthermore, possibly composing with $\varphi(z) := (z \wedge 1) \vee 0$, using Proposition 1.40 and observing that $|D\varphi \circ f_i|\,(X_i) \leq |Df_i|\,(X_i)$ for any $i \in \mathbb{N}$ while $|D\varphi \circ \chi_F|\,(Y) = |D\chi_F|\,(Y)$, we can assume that $0 \leq f_i \leq 1$ for any $i \in \mathbb{N}$. In particular $\sup_{i\in\mathbb{N}} \|f_i\|_{L^\infty(X_i, \mathfrak{m}_i)} < \infty$. Therefore, Proposition 4.20 applies and we obtain that, possibly extracting a subsequence that we do not relabel, $f_i$ converge in BV energy to $\chi_F$.

Let us assume, possibly extracting one more subsequence, that $(f_i)_\#(\chi_{B_{2R}(y)}\mathfrak{m}_i)$ weakly converge to some measure $\sigma$ in $[0,1]$. Under this assumption, we claim that $\chi_{\{f_i > \lambda\}}$ still

converge to $\chi_F$ in $L^1$-strong for $\mathscr{L}^1$-a.e. $\lambda \in (0,1)$.
In order to prove this claim, we fix $\lambda \in (0,1)$ that is not an atom of $\sigma$, so that

$$\lim_{\varepsilon\to 0}\lim_{i\to\infty} \mathfrak{m}_i(\{\lambda-\varepsilon < f_i \le \lambda\}) = 0. \tag{4.35}$$

From (4.35), using Proposition 1.40, it is immediate to get the $L^1$-strong convergence of $\chi_{\{f_i>\lambda\}}$ to $\chi_F$: indeed, it suffices to observe that for all $\varepsilon \in (0,\lambda)$ the functions $\psi_\varepsilon \circ f_i$ still $L^1$-strongly converge to $\psi_\varepsilon \circ \chi_F = \chi_F$ for any $\psi$ continuous, identically equal to 0 on $[0,\lambda-\varepsilon]$ and identically equal to 1 on $[\lambda,1]$. From the $L^1$-strong convergence we get, in particular,

$$\liminf_{i\to\infty} \operatorname{Per}(\{f_i>\lambda\},X_i) \ge \operatorname{Per}(F,Y) \qquad \text{for } \mathscr{L}^1\text{-a.e. } \lambda\in(0,1). \tag{4.36}$$

On the other hand, the coarea formula Theorem 4.2 and the strong convergence of $f_i$ yield

$$\limsup_{i\to\infty}\int_0^1 \operatorname{Per}(\{f_i>\lambda\},X_i)\,\mathrm{d}\lambda = \limsup_{i\to\infty}|Df_i|(X_i) = \operatorname{Per}(F,Y). \tag{4.37}$$

Thanks to Scheffè's lemma, the combination of (4.36) and (4.37) gives that $\operatorname{Per}(\{f_i>\lambda\},X_i)$ converge in $L^1(0,1)$ to the constant $\operatorname{Per}(F,Y)$. Extracting a subsequence $(i(k))$ pointwise convergent on $(0,1)\setminus I$ with $\mathscr{L}^1(I)=0$ and setting $E_k=\{f_{i(k)}>\lambda\}\subset B_{2R}(y)$ with $\lambda\in(0,1)\setminus I$ and $\sigma(\{\lambda\})=0$, the conclusion is achieved. □

Let us conclude this section with a convergence result for quasi-minimal sets of finite perimeter. It will play a key role in the study of blow-ups of sets of finite perimeter we are going to perform in Section 4. The strategy of the proof is classical (Cf. [**2**, Theorem 4.8]) and slightly different from the one in the original paper [**7**] which has a small gap. The author is grateful to Nicola Gigli and Camillo Brena for pointing this out.

**Proposition 4.26.** *Let $(X_i,\mathsf{d}_i,\mathfrak{m}_i,x_i)$ be $\mathrm{RCD}(K,N)$ m.m. spaces converging in the pmGH topology to $(Y,\varrho,\mu,y)$ and let $(Z,\mathsf{d}_Z)$ be realizing the convergence as above. For any $i\in\mathbb{N}$, let $\lambda_i\ge 1$ and let $E_i\subset X_i$ be a set of finite perimeter satisfying the following $\lambda_i$-minimality condition: there exists $R_i>0$ such that*

$$\operatorname{Per}(E_i,B_{R_i}(x_i)) \le \lambda_i \operatorname{Per}(E',B_{R_i}(x_i)) \qquad \forall E'\subset X_i \text{ such that } E_i\Delta E' \Subset B_{R_i}(x_i).$$

*Assume that, as $i\to\infty$, $E_i\to F$ in $L^1_{\mathrm{loc}}$ for some set $F\subset Y$ of locally finite perimeter, $\lambda_i\to 1$ and $R_i\to\infty$. Then*

(i) *$F$ is an entire minimizer of the perimeter (relative to $(Y,\varrho,\mu)$), namely*
$$\operatorname{Per}(F,B_r(y)) \le \operatorname{Per}(F',B_r(y)) \quad \text{whenever } F\Delta F' \Subset B_r(y)\Subset Y \text{ and } r>0;$$
(ii) *$|D\chi_{E_i}| \rightharpoonup |D\chi_F|$ in duality with $\mathrm{C_{bs}}(Z)$.*

*Proof.* Let us fix $\bar y\in Y$ and let $F'\subset Y$ be a set of locally finite perimeter satisfying $F\Delta F'\Subset B_r(\bar y)$. Let $\bar x_i\in X_i$ converging to $\bar y$ in $Z$ and $R>0$ be such that the following properties hold true:

$$\sup_{i\in\mathbb{N}} \operatorname{Per}(B_R(x_i),X_i)<\infty \qquad \text{and} \qquad B_r(\bar x_i)\Subset B_R(x_i) \qquad \forall i\in\mathbb{N}. \tag{4.38}$$

Using Proposition 4.25 we can find a sequence of sets of finite perimeter $E_i'\subset X_i$ converging to $F\cap B_R(y)$ in BV energy (note that $F\cap B_R(y)$ is a set of finite perimeter thanks to (4.38)).

Let $\nu$ be any weak limit of the sequence of measures with uniformly bounded mass $|D\chi_{E_i}|$. We claim that there exists $r'<r$ such that

$$\nu(B_s(\bar y)) \le \operatorname{Per}(F',B_s(\bar y)) \;\; \text{for } \mathscr{L}^1\text{-a.e. } s\in(r',r). \tag{4.39}$$

Before proving (4.39) let us illustrate how to use it to conclude the proof. First of all, notice that (4.32) gives $\nu \geq |D\chi_F|$. If we apply (4.39) with $F' = F$ we conclude that $\nu = |D\chi_F|$ locally and then globally, achieving the conclusion (ii) in the statement. The validity of the local minimality condition (i) follows combining the identification $\nu = |D\chi_F|$ with (4.39), letting $s \uparrow r$.

Let us pass to the proof of (4.39). We fix a parameter $s \in (0, r)$ with $\nu(\partial B_s(\bar{y})) = 0$, $\mathrm{Per}(F', \partial B_s(\bar{y})) = 0$ and set

$$\tilde{E}_i^s := (E_i' \cap B_s(\bar{x}_i)) \cup (E_i \setminus B_s(\bar{x}_i)) .$$

Using the locality of the perimeter and the $\lambda_i$-minimality of $E_i$ (notice that $R_i \geq r$ for $i$ big enough), we get

$$\begin{aligned}
\mathrm{Per}(E_i, \overline{B}_s(\bar{x}_i)) =& \mathrm{Per}(E_i, B_r(\bar{x}_i)) - \mathrm{Per}(E_i, B_r(\bar{x}_i) \setminus \overline{B}_s(\bar{x}_i)) \\
\leq& \lambda_i \mathrm{Per}(\tilde{E}_i^s, B_r(\bar{x}_i)) - \mathrm{Per}(E_i, B_r(\bar{x}_i) \setminus \overline{B}_s(\bar{x}_i)) \\
=& \lambda_i \mathrm{Per}(\tilde{E}_i^s, B_s(\bar{x}_i)) + \lambda_i \mathrm{Per}(\tilde{E}_i^s, \partial B_s(\bar{x}_i)) \\
&+ \lambda_i \mathrm{Per}(\tilde{E}_i^s, B_r(\bar{x}_i) \setminus \overline{B}_s(\bar{x}_i)) - \mathrm{Per}(E_i, B_r(\bar{x}_i) \setminus \overline{B}_s(\bar{x}_i)) \\
=& \lambda_i \mathrm{Per}(E_i', B_s(\bar{x}_i)) + \lambda_i \mathrm{Per}(\tilde{E}_i^s, \partial B_s(\bar{x}_i)) + (\lambda_i - 1) \mathrm{Per}(E_i, B_r(\bar{x}_i) \setminus \overline{B}_s(\bar{x}_i))
\end{aligned}$$

Observe that, taking the limit as $i \to \infty$, thanks to our choice of $s$, it holds that:

$$(\lambda_i - 1) \mathrm{Per}(E_i, B_r(\bar{x}_i) \setminus \overline{B}_s(\bar{x}_i)) \to 0, \quad \mathrm{Per}(E_i, \overline{B}_s(\bar{x}_i)) \to \nu(B_s(\bar{y})),$$

and eventually

$$\lambda_i \mathrm{Per}(E_i', B_s(\bar{x}_i)) \to \mathrm{Per}(F', B_s(\bar{y})),$$

since $\chi_{E_i'} \to \chi_{F' \cap B_R(y)}$ in BV energy and therefore Corollary 4.24 applies. It remains only to prove that

$$\liminf_{i\to\infty} \mathrm{Per}(\tilde{E}_i^s, \partial B_s(\bar{x}_i)) = 0, \qquad \text{for } \mathscr{L}^1\text{-a.e. } s \in (r', r). \tag{4.40}$$

Applying (4.43) of Lemma 4.27 below with $f = \chi_{E_i'} - \chi_{E_i}$ we get

$$\mathrm{Per}(\tilde{E}_i^s, X \setminus \overline{B}_s(\bar{x}_i)) \leq \int_{X_i} |\chi_{E_i'} - \chi_{E_i}| \,\mathrm{d}|D\chi_{B_s(\bar{x}_i)}| + \mathrm{Per}(E_i, X \setminus \overline{B}_s(\bar{x}_i)) \quad \text{for } \mathscr{L}^1\text{-a.e. } s \in (0, r),$$

that, together with the strong locality the perimeter, yields

$$P(\tilde{E}_i^s, \partial B_s(\bar{x}_i)) \leq \int_{X_i} |\chi_{E_i'} - \chi_{E_i}| \,\mathrm{d}|D\chi_{B_s(\bar{x}_i)}|, \quad \text{for } \mathscr{L}^1\text{-a.e. } s \in (0, r). \tag{4.41}$$

Using Fatou's lemma, (4.41), the local version of the coarea formula of Corollary 4.3 and eventually Lemma 4.22 to prove that $\chi_{E_i'} - \chi_{E_i} \to \chi_F - \chi_{F'}$ in $L^1$-strong, we conclude that

$$\begin{aligned}
\int_{r'}^{r} \liminf_{i\to\infty} \mathrm{Per}(\tilde{E}_i^s, \partial B_s(\bar{x}_i)) \,\mathrm{d}s &\leq \liminf_{i\to\infty} \int_{r'}^{r} \mathrm{Per}(\tilde{E}_i^s, \partial B_s(\bar{x}_i)) \,\mathrm{d}s \\
&\leq \liminf_{i\to\infty} \int_{r'}^{r} \int_{X_i} |\chi_{E_i'} - \chi_{E_i}| \,\mathrm{d}|D\chi_{B_s(\bar{x}_i)}| \\
&= \liminf_{i\to\infty} \int_{B_r(\bar{x}_i)\setminus B_{r'}(\bar{x}_i)} |\chi_{E_i'} - \chi_{E_i}| \,\mathrm{d}\mathfrak{m}_i \\
&= \mu((F \Delta F') \cap B_{r'}(\bar{y}))
\end{aligned}$$

therefore, choosing $r' < r$ such that $F \Delta F' \subset B_{r'}(\bar{y})$ we get (4.40). □

**Lemma 4.27** (Leibniz rule in BV). *Let* $(X, \mathsf{d}, \mathfrak{m})$ *be an* $\mathrm{RCD}(K, \infty)$ *m.m.s. and let* $x \in X$*. For any* $f \in \mathrm{BV}(X, \mathsf{d}, \mathfrak{m}) \cap L^\infty(X, \mathfrak{m})$ *and* $\mathscr{L}^1$*-a.e.* $r \in (0, \infty)$ *it holds*

$$\left|D\Big(f\chi_{B_r(x)}\Big)\right|(X) \le \int |f| \,\mathrm{d}|D\chi_{B_r(x)}| + |Df|(B_r(x)) \tag{4.42}$$

*and therefore locality gives*

$$\left|D\Big(f\chi_{B_r(x)}\Big)\right|(X \setminus B_r(x)) \le \int |f| \,\mathrm{d}|D\chi_{B_r(x)}|, \quad \text{for } \mathscr{L}^1\text{-a.e. } r \in (0, \infty). \tag{4.43}$$

*Proof.* Let us begin observing that the stated conclusion makes sense since, in view of the coarea formula Theorem 4.2, $\int |f| \,\mathrm{d}|D\chi_{B_r(x)}|$ is well defined for $\mathscr{L}^1$-a.e. $r \in (0, \infty)$.
We divide the proof into two intermediate steps. In the first one we are going to prove that (4.42) holds true under the assumption $f \in \mathrm{Lip_b}(X, \mathsf{d})$. In the second one we prove the sought inequality passing to the limit the inequalities for regularized functions that we obtained previously.

**Step 1.** More generally in this step we are going to prove, arguing by regularization on $g$, that, for any $f \in \mathrm{Lip_b}(X, \mathsf{d})$ and for any nonnegative function $g \in \mathrm{BV}(X, \mathsf{d}, \mathfrak{m}) \cap L^\infty(X, \mathfrak{m})$, it holds

$$|D(fg)|(X) \le \int |f| \,\mathrm{d}\,|Dg| + \int |g|\,|\nabla f| \,\mathrm{d}\mathfrak{m}. \tag{4.44}$$

Observe that, if $g \in \mathrm{Lip_b}(X, \mathsf{d})$ then (4.44) follows from the Leibniz rule. Hence, by the $L^\infty - \mathrm{Lip}$ regularization of the heat semigroup it follows that, for any $t > 0$,

$$\left|D\Big(fP_tg\Big)\right|(X) \le \int |f|\,|\nabla P_t g| \,\mathrm{d}\mathfrak{m} + \int P_t g\,|\nabla f| \,\mathrm{d}\mathfrak{m}. \tag{4.45}$$

The convergence of $P_t g$ to $g$ in $L^1(X, \mathfrak{m})$ as $t \to 0$, the lower semicontinuity of the total variation and the Bakry-Émery contraction estimate allow us to pass to the $\liminf$ at the left hand-side and to the limit at the right hand-side in (4.45) to get (4.44) (see also the proof of the second step for further details on the limiting procedure).

**Step 2.** It follows from what we just proved and from the $L^\infty - \mathrm{Lip}$ regularization property of the heat flow on $\mathrm{RCD}(K, \infty)$ m.m. spaces that, for any $t > 0$,

$$\left|D\Big(P_t f\chi_{B_r(x)}\Big)\right|(X) \le \int |P_t f| \,\mathrm{d}|D\chi_{B_r(x)}| + |DP_t f|(\overline{B}_r(x)) \quad \text{for } \mathscr{L}^1\text{-a.e. } r \in (0, \infty). \tag{4.46}$$

Next we observe that $P_t f\chi_{B_r(x)} \to f\chi_{B_r(x)}$ in $L^1(X, \mathfrak{m})$ as $t \to 0^+$ and therefore, by the lower semicontinuity of the total variation w.r.t. $L^1$ convergence it holds

$$\left|D\Big(f\chi_{B_r(x)}\Big)\right|(X) \le \liminf_{t \to 0^+}\left|D(P_t f\chi_{B_r(x)})\right|(X). \tag{4.47}$$

Furthermore, the $L^1(X, \mathfrak{m})$ convergence of $P_t f$ to $f$ and the coarea formula Theorem 4.2 grant that we can find a sequence $t_i \downarrow 0$ in such a way that $P_{t_i} f$ converges in $L^1(X, |D\chi_{B_r(x)}|)$ to $f$ for $\mathscr{L}^1$-a.e. $r \in (0, \infty)$. Eventually, let us observe that, due to the Bakry-Émery contraction estimate (1.23),

$$\limsup_{t \to 0^+} |DP_t f|\,(\overline{B}_r(x)) \le \limsup_{t \to 0^+} e^{-Kt} P_t^*\,|Df|\,(\overline{B}_r(x)) \le |Df|\,(\overline{B}_r(x)), \quad \forall r \in (0, \infty).$$

Passing to the $\liminf$ as $t_i \downarrow 0$ at the left hand-side of (4.46) taking into account (4.47) and to the limit at the right hand-side taking into account what we observed above, we get the sought estimate (4.42). □

Let us now introduce the notion of strong $BV$ convergence for sets of finite perimeter and prove a compactness result which builds upon tools developed in this section.

**Definition 4.28.** We say that a sequence of sets with locally finite perimeter $E_i \subset X_i$ converges locally strongly in BV to a set of locally finite perimeter $F \subset Y$ if $E_i \to F$ in $L^1_{\mathrm{loc}}$ and $|D\chi_{E_i}| \to |D\chi_F|$ in duality with $\mathrm{C_{bs}}(Z)$.

**Proposition 4.29.** *Let $E_i \subset X_i$ be sets of finite perimeter satisfying*

$$\sup_{i\in\mathbb{N}} \mathrm{Per}(E_i, B_1(x_i)) < \infty.$$

*Then there exists $F \subset Y$ of finite perimeter such that, up to extract a subsequence, $E_i \cap B_1(x_i) \to F \cap B_1(y)$ in $L^1$-strong and*

(4.48)

$$\liminf_{i\to\infty} \int g \,\mathrm{d}|D\chi_{E_i}| \geq \int g \,\mathrm{d}|D\chi_F|, \quad \forall\, g \in C(Z), \textit{ nonnegative with } \mathrm{supp}(g) \subset \bar{B}_{1/2}(y).$$

*If we further assume that*

$$\lim_{i\to\infty} |D\chi_{E_i}|\,(B_{1/2}(x_i)) = |D\chi_F|\,(B_{1/2}(y)), \tag{4.49}$$

*then* (4.48) *improves to*

$$\lim_{i\to\infty} \int g \,\mathrm{d}|D\chi_{E_i}| = \int g \,\mathrm{d}|D\chi_F|, \quad \textit{for any } g \in C(Z) \textit{ with } \mathrm{supp}(g) \subset B_{1/2}(y). \tag{4.50}$$

*Proof.* The convergence $E_i \cap B_1(x_i) \to F \cap B_1(y)$ in $L^1$-strong up to subsequence can be obtained arguing as in the proof of Corollary 4.21.

Inequality (4.48) follows from Proposition 4.23 along with a localization argument that we sketch briefly. For any $i \in \mathbb{N}$, using Lemma 1.60 we build a good cut-off function $\eta_i \in \mathrm{Lip}(X_i, \mathsf{d}_i)$ satisfying $\eta_i = 1$ in $B_{1/2}(x_i)$ and $\eta_i = 0$ in $X_i \setminus B_{3/4}(x_i)$. By Proposition 1.34, up to extract a subsequence, we can assume that $\eta_i \to \eta_\infty \in \mathrm{Lip}(Y, \rho)$ uniformly and in $L^2$-strong. It is easily seen that $\eta_\infty = 1$ in $B_{1/2}(y)$ and $\eta_\infty = 0$ in $Y \setminus B_1(y)$. The sequence $(\eta_i \chi_{E_i})_i$ satisfies

$$\eta_i \chi_{E_i} \to \eta_\infty \chi_F \text{ in } L^1\text{-strong} \quad \text{and} \quad \sup_{i\in\mathbb{N}} |D(\eta_i \chi_{E_i})|(X_i) < \infty,$$

thanks to Proposition 1.40(ii) and standard calculus rules. Applying Proposition 4.23 to the sequence $(\eta_i \chi_{E_i})_i$ we get (4.48).

Inequality (4.50)) is a weak convergence result in the ball $B_{1/2}(y) \subset Z$, which can be proved arguing as in the proof of Corollary 4.24 taking into account (4.48) and (4.49). $\square$

## 4. Tangents to sets of finite perimeter in RCD spaces

In this section we study the structure of blow-ups of sets of finite perimeter over RCD metric measure spaces. Before stating the main results we introduce a definition of tangent for sets of finite perimeter in this abstract setting.

**Definition 4.30** (Tangents to a set of finite perimeter)**.** Let $(X, \mathsf{d}, \mathfrak{m})$ be an $\mathrm{RCD}(K,N)$ metric measure space, $x \in X$ and let $E \subset X$ be a set of locally finite perimeter. We denote by $\mathrm{Tan}_x(X, \mathsf{d}, \mathfrak{m}, E)$ the collection of quintuples $(Y, \varrho, \mu, y, F)$ satisfying the following two properties:

(a) it holds $(Y, \varrho, \mu, y) \in \mathrm{Tan}_x(X, \mathsf{d}, \mathfrak{m})$ and $r_i \downarrow 0$ are such that the rescaled spaces $(X, r_i^{-1}\mathsf{d}, \mathfrak{m}_x^{r_i}, x)$ converge to $(Y, \varrho, \mu, y)$ in the pmGH topology;

(b) $F$ is a set of locally finite perimeter in $Y$ with $\mu(F) > 0$ and, if $r_i$ are as in (a), then the sequence $f_i = \chi_E$ converges in $L^1_{\rm loc}$ to $\chi_F$ according to Definition 4.17.

It is clear that the following locality property of tangents holds:

$$\mathfrak{m}\Big(A \cap (E\Delta F)\Big) = 0 \qquad \Longrightarrow \qquad \mathrm{Tan}_x(X,\mathsf{d},\mathfrak{m},E) = \mathrm{Tan}_x(X,\mathsf{d},\mathfrak{m},F) \qquad \forall x \in A. \tag{4.51}$$

whenever $E$, $F$ are sets of locally finite perimeter and $A \subset X$ is open.

We are ready to state the main results of this section.

**Theorem 4.31.** *Let $(X,\mathsf{d},\mathfrak{m})$ be an $\mathrm{RCD}(K,N)$ m.m.s. and let $E \subset X$ be a set of locally finite perimeter. For $|D\chi_E|$-a.e. $x \in X$ the set $\mathrm{Tan}_x(X,\mathsf{d},\mathfrak{m},E)$ is not empty and for all $(Y,\varrho,\mu,y,F) \in \mathrm{Tan}_x(X,\mathsf{d},\mathfrak{m},E)$, one has*

$$|\nabla P_s\chi_F|\mu = P_s^*|D\chi_F| \qquad \forall s > 0, \tag{4.52}$$

*where $P_s = P_s^Y$ is the heat semigroup relative to $(Y,\varrho,\mu)$. In particular, for all $t \geq 0$, all functions $f = P_t\chi_F$ satisfy*

$$|\nabla P_s f| = P_s|\nabla f| \qquad \mu\text{-a.e. in } Y, \text{ for all } s > 0.$$

*Moreover, for each $x \in X$ as above there exists a pointed m.m.s. $(Z,\mathsf{d}_Z,\mathfrak{m}_Z,\bar z)$ such that*

$$(Y,\varrho,\mu,y,F) = \Big((Z\times\mathbb{R}),\mathsf{d}_Z\times\mathsf{d}_{Eucl},\mathfrak{m}_Z\times\mathscr{L}^1,(\bar z,0),\{t>0\}\Big),$$

*where we denoted by $t$ the coordinate of the Euclidean factor in $Z\times\mathbb{R}$. Furthermore:*

(i) *if $N \geq 2$ then $Z$ is an $\mathrm{RCD}(0,N-1)$ m.m.s.;*
(ii) *if $N \in [1,2)$ then $Z$ is a point.*

A suitable version of the iterated tangent theorem by Preiss (see Theorem 4.38) implies also the following.

**Theorem 4.32.** *Let $(X,\mathsf{d},\mathfrak{m})$ be an $\mathrm{RCD}(K,N)$ m.m.s. and let $E \subset X$ be a set of locally finite perimeter. Then $E$ admits a Euclidean half-space as tangent at $x$ for $|D\chi_E|$-a.e. $x \in X$, that is to say*

$$\Big(\mathbb{R}^k,\mathsf{d}_{Eucl},c_k\mathscr{L}^k,0^k,\{x_k>0\}\Big) \in \mathrm{Tan}_x(X,\mathsf{d},\mathfrak{m},E), \qquad \textit{for some } k \in [1,N].$$

*Proof of Theorem 4.32.* We claim that the stated conclusion holds true at all points $x \in X$ such that both the iterated tangent property of Theorem 4.38 and the rigidity property stated in Theorem 4.31 are satisfied (observe that $|D\chi_E|$-a.e. point satisfies these two properties). Indeed, if $(Y,\varrho,\mu,y,F) \in \mathrm{Tan}_x(X,\mathsf{d},\mathfrak{m},E)$, combining Theorem 4.31 with Theorem 4.4, we can say that $(Y,\varrho,\mu)$ is isomorphic to $Z\times\mathbb{R}$ for some $\mathrm{RCD}(0,N-1)$ m.m.s. $(Z,\mathsf{d}_Z,\mathfrak{m}_Z)$. Furthermore, another consequence of Theorem 4.4 is that $F = \{t > t_0\}$ for some $t_0 \in \mathbb{R}$, where we denoted by $t$ the coordinate on the Euclidean factor of $Y$. Up to a translation we can also assume that $y = (\overline{z},0)$ for some $\overline{z} \in Z$.

We go on observing that, if $i : Z \to Y$ denotes the canonical inclusion $i(z) := (z,0)$, it holds $|D\chi_F| = i_*\mathfrak{m}_Z$ and, for this reason, we shall identify in the sequel $|D\chi_F|$ and $\mathfrak{m}_Z$. Moreover, it is easy to check that, if $(W,\mathsf{d}_W,\mathfrak{m}_W,\bar w) \in \mathrm{Tan}_z(Z,\mathsf{d}_Z,\mathfrak{m}_Z)$, then

$$(W\times\mathbb{R},\mathsf{d}_W\times\mathsf{d}_{\rm Eucl},\mathfrak{m}_W\times\mathscr{L}^1,(\bar w,0),\{t>0\}) \in \mathrm{Tan}_{(z,0)}(Y,\varrho,\mu,F).$$

The sought conclusion can now be obtained choosing $z$ to be a regular point of $(Z,\mathsf{d}_Z,\mathfrak{m}_Z)$ (recall that $\mathfrak{m}_Z$-a.e. point of $Z$ is regular), so that $W$ is a Euclidean space of dimension

$k \in [0, N-1]$ and applying Theorem 4.38 to conclude that

$$(W \times \mathbb{R}, \mathsf{d}_W \times \mathsf{d}_{\mathrm{Eucl}}, \mathfrak{m}_W \times \mathscr{L}^1, (\bar{w}, 0), \{t > 0\}) \in \mathrm{Tan}_x(X, \mathsf{d}, \mathfrak{m}, E).$$

□

The rest of this section is devoted to the proof Theorem 4.31. First, we are going to prove that tangents are non empty almost everywhere with respect to the perimeter measure, as a consequence of the compactness results developed in Section 3 and Proposition 4.34. Then, we will prove that they are rigid, in a suitable sense. This rigidity property will be achieved building mainly on two ingredients: lower semicontinuity and locality of the perimeter and the Bakry-Émery inequality, together with the characterization of its equality cases we obtained in Section 2.

We start by stating an asymptotic minimality result that stems from the lower semicontinuity of the perimeter. It has been proved, in a slightly weaker form (namely with a smaller class of competitors $E'$), first in [**3**] under Ahlfors regularity assumption and then, in [**4**], for the general case. The basic idea originates, to the authors' knowledge, in the work of Fleming [**79**] (see also [**48**, **140**] for variants of this idea in different contexts).

**Proposition 4.33** (Asymptotic minimality and doubling)**.** *Let* $(X, \mathsf{d}, \mathfrak{m})$ *be an* $\mathrm{RCD}(K, N)$ *m.m.s. and* $E \subset X$ *be a set of locally finite perimeter. For* $|D\chi_E|$*-a.e.* $x \in X$ *there exist* $r_x > 0$ *and* $\omega_x(r) : (0, r_x) \to [0, \infty)$ *such that* $\omega_x(r) \to 0$ *as* $r \to 0^+$ *and*

$$\mathrm{Per}(E, B_r(x)) \le (1 + \omega_x(r)) \, \mathrm{Per}(E', B_r(x)) \tag{4.53}$$

*whenever* $E \Delta E' \Subset B_r(x)$*. In addition,*

$$\limsup_{r \to 0^+} \frac{|D\chi_E|(B_{2r}(x))}{|D\chi_E|(B_r(x))} < \infty. \tag{4.54}$$

Also the following density estimates are important to prove that tangents are almost everywhere non empty. We refer again to [**3**, **4**] for its proof.

**Proposition 4.34.** *Let* $(X, \mathsf{d}, \mathfrak{m})$ *be an* $\mathrm{RCD}(K, N)$ *m.m.s. and* $E \subset X$ *be a set of locally finite perimeter. For* $|D\chi_E|$*-a.e.* $x \in X$ *it holds*

$$0 < \liminf_{r \to 0^+} \frac{r|D\chi_E|(B_r(x))}{\mathfrak{m}(B_r(x))} \le \limsup_{r \to 0^+} \frac{r|D\chi_E|(B_r(x))}{\mathfrak{m}(B_r(x))} < \infty, \tag{4.55}$$

*and*

$$\liminf_{r \to 0^+} \min \left\{ \frac{\mathfrak{m}(E \cap B_r(x))}{\mathfrak{m}(B(x, r))} ; \frac{\mathfrak{m}(E^c \cap B_r(x))}{\mathfrak{m}(B(x, r))} \right\} > 0. \tag{4.56}$$

**Corollary 4.35.** *Let* $(X, \mathsf{d}, \mathfrak{m})$ *be an* $\mathrm{RCD}(K, N)$ *m.m.s. and let* $E \subset X$ *be a set of locally finite perimeter. Then, for* $|D\chi_E|$*-a.e.* $x \in X$ *one has* $\mathrm{Tan}_x(X, \mathsf{d}, \mathfrak{m}, E) \neq \emptyset$ *and, if* $(Y, \varrho, \mu, y, F)$ *is as in Definition 4.30, the following properties hold true:*

(a) *$F$ is an entire minimizer of the perimeter (relative to $(Y, \varrho, \tau)$), i.e.*

$$|D\chi_F|(B_r(y)) \le |D\chi_{F'}|(B_r(y)) \qquad \text{whenever } F \Delta F' \Subset B_r(y) \Subset Y;$$

(b) *realizing the convergence in a proper metric space $(Z, \mathsf{d}_Z)$, the perimeters $|D^i \chi_E|$ weakly converge, in duality with $\mathrm{C}_{\mathrm{bs}}(Z)$, to $|D\chi_F|$. In particular $E_i \to F$ locally strongly in BV according to Definition 4.28.*

*Proof.* Let us consider $x \in X$ such that the statements of Proposition 4.33 and Proposition 4.34 hold true and a sequence of radii $r_i \to 0$ such that $(X, r^{-1}\mathsf{d}, \mu_x^r, x) \to (Y, \varrho, \mu, y)$ in the pmGH topology. Thanks to (4.55) and Corollary 4.21 with $\chi_{E_i} = \chi_E$, possibly extracting a subsequence we can assume that there exists a set $F \subset Y$ with locally finite perimeter such that $\chi_E \to \chi_F$ in $L^1_{\mathrm{loc}}$. Note that $\mu(F) > 0$ thanks to (4.56). This implies that $(Y, \varrho, \mu, y, F) \in \mathrm{Tan}(E, x)$. To achieve (a) and (b) it is enough to apply Proposition 4.26, recalling (4.53). $\square$

The next key result to prove Theorem 4.31 is Proposition 4.37. Before stating and proving it we need a technical lemma.

**Lemma 4.36.** *Let $(X_n, \mathsf{d}_n, \mathfrak{m}_n)$ be $\mathrm{RCD}(K, N)$ m.m. spaces mGH converging to $(Y, \varrho, \mu)$ and assume that the convergence is realized into a proper metric space $(Z, \mathsf{d}_Z)$. Let $\eta_n$, $\eta$ be nonnegative Borel measures giving finite mass to bounded sets, such that $\operatorname{supp}\eta_n \subset \operatorname{supp}\mathfrak{m}_n$, $\operatorname{supp}\eta \subset \operatorname{supp}\mu$ and $\eta_n$ weakly converge to $\eta$ in duality with $\mathrm{C_{bs}}(Z)$. Then*

$$P_t^Y \eta(x) \le \liminf_{n\to\infty} P_t^n \eta_n(x_n), \quad \text{for any } t > 0 \text{ and for any } \operatorname{supp}\mathfrak{m}_n \ni x_n \to x \in \operatorname{supp}\mu. \tag{4.57}$$

*Proof.* Thanks to Theorem 1.49 we know that, denoting by $p^n$ and $p^Y$ the heat kernels of $(X_n, \mathsf{d}_n, \mathfrak{m}_n)$ and $(Y, \varrho, \mu)$ respectively, it holds

$$\lim_{n\to\infty} p_t^n(x_n, y_n) = p_t^Y(x, y), \quad \text{for any } t > 0, \tag{4.58}$$

whenever $\operatorname{supp}\mathfrak{m}_n \times \operatorname{supp}\mathfrak{m}_n \ni (x_n, y_n) \to (x, y) \in \operatorname{supp}\mu \times \mu$. Since

$$P_t^Y \eta(x) = \int p_t^Y(x, y)\,\mathrm{d}\eta(y) \quad \text{and} \quad P_t^n \eta_n(x_n) = \int p_t^n(x_n, y)\,\mathrm{d}\eta(y),$$

the validity of (4.57) follows from Lemma 1.3 and Fatou's lemma with the obvious choice for the weakly convergent sequence of measures and $f_n(\cdot) := p_t^n(x_n, \cdot)$, $f(\cdot) := p_t(x, \cdot)$, which satisfy the lower semicontinuity condition (1.2) in view of (4.58). $\square$

**Proposition 4.37.** *Let $E \subset X$ be a set of finite perimeter and let $(Y, \varrho, \mu, y, F)$ be an element of $\mathrm{Tan}_x(X, \mathsf{d}, \mathfrak{m}, E)$ for some $x \in X$. Let $r_i \downarrow 0$ be a sequence of radii realizing the convergence in Definition 4.30. Then*

$$|\nabla^i P_t^i \chi_E|\mathfrak{m}_i \rightharpoonup |\nabla^Y P_t^Y \chi_F|\mu \qquad \text{in duality with } \mathrm{C_{bs}}(Z), \text{ for any } t > 0.$$

*Proof.* Let us begin by proving that, for any $\operatorname{supp}\mathfrak{m}_i \ni x_i \to x \in \operatorname{supp}\mu$ and for any $t > 0$, it holds

$$\lim_{i\to\infty} P_t^i \chi_E(x_i) = P_t^Y \chi_F(x). \tag{4.59}$$

To this aim we first observe that, by the very definition of tangent, it holds that $\chi_E \mathfrak{m}_n \rightharpoonup \chi_F \mu$ in duality with $\mathrm{C_{bs}}(Z)$ and therefore Lemma 4.36 yields

$$P_t^Y \chi_F(x) \le \liminf_{i\to\infty} P_t^i \chi_E(x_i). \tag{4.60}$$

Moreover, since $(1 - \chi_E)\mathfrak{m}_n \rightharpoonup (1 - \chi_F)\mu$ in duality with $\mathrm{C_{bs}}(Z)$, applying Lemma 4.36 once more and with a simple algebraic manipulation, we obtain

$$\limsup_{i\to\infty} P_t^i \chi_E(x_i) \le P_t^Y \chi_F(x). \tag{4.61}$$

Combining (4.60) with (4.61) we obtain (4.59).

Let us proceed observing that, in view of the quantitative form of the $L^\infty$-Lip regularization on $\mathrm{RCD}(K,\infty)$ spaces provided by (1.24), for any $t>0$ the functions $P^i_t\chi_E$ and $P^Y_t\chi_F$ are uniformly Lipschitz.

Fix now reference points $y\in Y$ and $X_i\ni x_i\to y$. Building upon Lemma 1.60, for any $R>0$ it is possible to find Lipschitz cut-off functions $\eta_R:Y\to[0,1]$, $\eta^i_R:X_i\to[0,1]$ such that $\operatorname{supp}\eta_R\subset B^Y_{2R}(y)$, $\operatorname{supp}\eta^i_R\subset B^i_{2R}(x_i)$, $\eta_R|_{B^Y_R(y)}\equiv 1$, $\eta^i_R|_{B^i_R(x_i)}\equiv 1$, uniformly Lipschitz, with uniformly bounded laplacians and such that $\eta^i_R$ converge to $\eta_R$ both pointwise and $L^2$-strongly. We remark indeed that, in view of Remark 1.35, pointwise and $L^2$-strong convergence are equivalent for uniformly bounded, uniformly continuous and uniformly boundedly supported functions. Let us observe that, if we are able to prove that

$$f_i:=\eta^i_RP^i_t\chi_E\to\eta_RP^Y_t\chi_F=:f\qquad\text{strongly in } H^{1,2}\text{ for all } R>0,$$

the conclusion will follow from the locality of the minimal weak upper gradient and Theorem 1.44, which grants the $L^1$-strong convergence of $|\nabla^i\big(\eta^i_RP^i_t\chi_E\big)|^2$ to $|\nabla^Y\eta_RP^Y_t\chi_F|^2$ (that we can improve to $L^1$-strong convergence of $|\nabla^i\big(\eta^i_RP^i_t\chi_E\big)|$ to $|\nabla^Y\eta_RP^Y_t\chi_F|$ in view of the uniform Lipschitz bounds and of Proposition 1.40).

In order to prove the above claimed convergence, we begin by observing that $f_i$ converge pointwise to $f$ by (4.59) and the very construction of the family of cut-off functions $\eta^i_R$. Therefore, taking into account the uniform Lipschitz bounds, the uniform boundedness and the uniform bounds on the supports, $f_i\to f$ strongly in $L^2$ by Remark 1.35. To improve the convergence from $L^2$-strong to $H^{1,2}$-strong we wish to apply Proposition 1.43. In order to do so, it remains to prove that $\Delta f_i$ are uniformly bounded in $L^2$. To this aim we compute

$$\Delta f_i=\Delta\eta^i_RP^i_t\chi_E+2\nabla\eta^i_R\cdot\nabla P^i_t\chi_E+\eta^i_R\Delta P^i_t\chi_E \tag{4.62}$$

and observe that all the terms at the right hand side in (4.62) are uniformly bounded in $L^2$ in view of the uniform $L^\infty$ bounds on values, minimal weak upper gradients and laplacians of the cut-off functions, the uniform $L^\infty$ and Lipschitz bounds on $P^i_t\chi_E$ and the regularizing estimate for the Laplacian under heat flow in (1.9). □

*Proof of Theorem 4.31.* Let us consider the case when $E$ has finite perimeter. The generalization to sets of locally finite perimeter can be obtained building upon Lemma 4.27 and (4.51), arguing in a standard way.

Recall that the $BV$-version (1.23) of the 1-Bakry-Émery contraction estimate gives

$$|\nabla P_t\chi_E|\mathfrak{m}\le e^{-Kt}P^*_t|D\chi_E|\qquad\forall t>0.$$

Let $h_t:X\to[0,1]$ be the density of $e^{Kt}|\nabla P_t\chi_E|\mathfrak{m}$ with respect to $P^*_t|D\chi_E|$. Then, one has

$$\int(1-P_th_t)\,\mathrm{d}|D\chi_E|=|D\chi_E|(X)-\int h_tdP^*_t|D\chi_E|=|D\chi_E|(X)-e^{Kt}\int|\nabla P_t\chi_E|\,\mathrm{d}\mathfrak{m},$$

here and in the sequel we identify the measure $P^*_t|D\chi_E|$ with its density w.r.t. the ambient measure $\mathfrak{m}$. By lower semicontinuity, this proves that $g_t:=1-P_th_t$ converges to 0 strongly in $L^1(|D\chi_E|)$.

Now, setting for simplicity of notation $\nu=|D\chi_E|$, we claim that

$$\lim_{t\to0}\frac{1}{\nu(B_{R\sqrt{t}}(x))}\int_{B_{R\sqrt{t}}(x)}g_t\,\mathrm{d}\nu=0\qquad\forall R>0,\ \text{for }\nu\text{-a.e. }x\in X. \tag{4.63}$$

Thanks to the asymptotic doubling property (4.54), it is sufficient to prove the result $\nu$-a.e. on a Borel set $F$ with this property: for some $L > 0$, for all $x \in F$ and $0 < r < 1/L$ one has $\nu(B_{5r}(x)) \le L\nu(B_r(x))$. By Vitali's theorem, it follows that the localized maximal function

$$M|g|(x) := \begin{cases} \sup\limits_{r\in(0,1/L)} \dfrac{\int_{B_r(x)} |g|\,\mathrm{d}\nu}{\nu(B_r(x))} & \text{if } x \in F; \\ 0 & \text{if } x \in X \setminus F; \end{cases}$$

satisfies

$$\nu(\{M|g| > \tau\}) \le \frac{L}{\tau} \int |g|\,\mathrm{d}\nu \qquad \forall \tau > 0.$$

Let us apply this estimate to the functions $g_t = 1 - P_t h_t$: given $\varepsilon > 0$, for $t < t(\varepsilon)$ one has $\int g_t\,\mathrm{d}\nu < \varepsilon^2$, and then $\nu(\{Mg_t > \varepsilon\}) \le L\varepsilon$. We obtain that

$$\int_{B_r(x)} g_t\,\mathrm{d}\nu \le \varepsilon\nu(B_r(x)) \qquad \text{for } r < \frac{1}{L},\ t < t(\varepsilon)$$

for all $x \in F_\varepsilon \subset F$, with $\mu(F \setminus F_\varepsilon)$ smaller than $L\varepsilon$. In particular, on $F_\varepsilon$ one has

$$\limsup_{t\downarrow 0} \frac{1}{\nu(B_{R\sqrt{t}}(x))} \int_{B_{R\sqrt{t}}(x)} g_t\,d\nu \le \varepsilon \qquad \forall R > 0.$$

Since $\varepsilon$ is arbitrary, we have proved that (4.63) holds $\nu$-a.e. on $F$.

The claimed conclusion (4.52) will be achieved through two intermediate steps starting from (4.63). First, let us observe that, for any $R$, $s$, $t > 0$ and for any $x \in X$, it holds

$$\begin{aligned} &\frac{1}{\nu(B_{R\sqrt{t}}(x))} \int_{B_{R\sqrt{t}}(x)} g_{ts}\,\mathrm{d}\nu \\ &\qquad = \frac{1}{|D^t\chi_E|\,(B^t_R(x))} \int_{B^t_R(x)} P^t_s \left(1 - e^{Kt} \frac{|\nabla^t P^t_s \chi_E|}{(P^t_s)^*\,|D^t\chi_E|}\right) \mathrm{d}|D^t\chi_E|, \end{aligned} \tag{4.64}$$

where we denoted by $P^t$, $\nabla^t$, $D^t$ and $B^t$ the heat semigroup, the minimal weak upper gradients, the total variation measure and the balls associated to the rescaled metric measure structure $(X, \sqrt{t}^{-1}\mathsf{d}, \mathfrak{m}^{\sqrt{t}}_x, x)$ and we are identifying measures absolutely continuous w.r.t. the reference one with their densities.

**Step 1.** We claim that, if $(Y, \varrho, \mu, y, F) \in \mathrm{Tan}_x(X, \mathsf{d}, \mathfrak{m}, E)$ and $t_i \downarrow 0$ is a sequence realizing the convergence in Definition 4.30, then

$$\int P_s \left(1 - \frac{|\nabla P_s \chi_F|}{P^*_s|D\chi_F|}\right) \mathrm{d}\eta_R \le \liminf_{i\to\infty} \int P^{t_i}_s \left(1 - e^{Kst_i} \frac{|\nabla^{t_i} P^{t_i}_s \chi_E|}{\left(P^{t_i}_s\right)^*|D^{t_i}\chi_E|}\right) \mathrm{d}\eta^i_R, \tag{4.65}$$

for $\mathscr{L}^1$-a.e. $R > 0$, where

$$\begin{cases} \eta_R := \dfrac{1}{|D\chi_F|\,(B_R(y))} |D\chi_F| \llcorner B_R(y), \\[2ex] \eta^i_R := \dfrac{1}{|D^{t_i}\chi_E|\,(B^{t_i}_R(x))} |D^{t_i}\chi_E| \llcorner B^{t_i}_R(x). \end{cases}$$

In order to prove (4.65), we begin observing that $\eta_R^i$ weakly converges to $\eta_R$ for $\mathscr{L}^1$-a.e. $R > 0$. Therefore, the validity of (4.65) will follow from Lemma 1.3 if we prove that

$$P_s\left(1 - \frac{|\nabla P_s\chi_F|}{\left(P_s\right)^*|D\chi_F|}\right)(w) \le \liminf_{i\to\infty} P_s^{t_i}\left(1 - e^{Kst_i}\frac{|\nabla^{t_i}P_s^{t_i}\chi_E|}{\left(P_s^{t_i}\right)^*|D^{t_i}\chi_E|}\right)(w_i), \tag{4.66}$$

whenever $w_i \in X_i \to w \in Y$. Let us observe that, for any $\phi \in \mathrm{C_{bs}}(Z)$, it holds

$$\limsup_{i\to\infty} e^{Kst_i}\int \phi\frac{|\nabla^{t_i}P_s^{t_i}\chi_E|}{\left(P_s^{t_i}\right)^*|D^{t_i}\chi_E|}\,\mathrm{d}\mathfrak{m}_i \le \int \phi\frac{|\nabla P_s\chi_F|}{P_s^*|D\chi_F|}\,\mathrm{d}\mu. \tag{4.67}$$

Indeed, by Proposition 4.37, $|\nabla^{t_i}P_s^{t_i}\chi_E|\mathfrak{m}_i$ weakly converge to $|\nabla P_s\chi_F|\mu$ in duality with $\mathrm{C_{bs}}(Z)$, and the functions

$$f_i := \frac{\phi}{\left(P_s^{t_i}\right)^*|D^{t_i}\chi_E|} \qquad \text{and} \qquad f := \frac{\phi}{P_s^*|D\chi_F|}$$

are continuous, have uniformly bounded supports and satisfy the upper semicontinuity property (1.1) thanks to Lemma 4.36 (recall that $|D^{t_i}\chi_E|$ weakly converge to $|D\chi_F|$ in duality with $\mathrm{C_{bs}}(Z)$). Hence (4.66) and then (4.65) follow from Lemma 1.1, taking into account also Remark 1.2.

**Step 2.** We can now prove (4.52). If we choose $x \in X$ such that (4.63) holds true (we proved above that $|D\chi_E|$-a.e. $x \in X$ has this property), combining (4.64) with (4.65), we obtain

$$\int_{B_R(y)} P_s\left(1 - \frac{|\nabla P_s\chi_F|}{P_s^*|D\chi_F|}\right)\mathrm{d}|D\chi_F| \le 0. \tag{4.68}$$

Observing that, by gradient contractivity on the $\mathrm{RCD}(0, N)$ space $(Y, \varrho, \mu)$, it holds

$$1 - \frac{|\nabla P_s\chi_F|}{P_s^*|D\chi_F|} \ge 0 \qquad \mu\text{-a.e. on } Y, \tag{4.69}$$

we can let $R \to \infty$ in (4.68) to get

$$\int P_s\left(1 - \frac{|\nabla P_s\chi_F|}{P_s^*|D\chi_F|}\right)\mathrm{d}|D\chi_F| = 0. \tag{4.70}$$

Then, using once more the sign property (4.69), we obtain (4.52).

Combining the just proved rigidity (4.52) with Theorem 4.4, we can say that $(Y, \varrho, \mu)$ is isomorphic to $Z \times \mathbb{R}$ for some $\mathrm{RCD}(0, N-1)$ m.m.s. $(Z, \mathsf{d}_Z, \mathfrak{m}_Z)$. Furthermore, another consequence of Theorem 4.4 is that $F = \{t > t_0\}$ for some $t_0 \in \mathbb{R}$, where we denoted by $t$ the coordinate on the Euclidean factor of $Y$. Up to a translation we can also assume that $y = (\bar{z}, 0)$ for some $\bar{z} \in Z$. □

## 5. Iterated tangent theorem for perimeter measures

In this appendix we prove a version of the iterated tangent theorem by Preiss (see [**128**]). The proof is inspired by those of [**91**, Theorem 3.2] and [**22**, Theorem 6.4], dealing with pmGH tangents to $\mathrm{RCD}(K, N)$ spaces and tangents to sets of finite perimeters over Carnot groups, respectively (see also [**111**] for a previous result regarding pGH-tangents of metric spaces equipped with a doubling measure).

**Theorem 4.38.** *Let $(X, \mathsf{d}, \mathfrak{m})$ be an $\mathrm{RCD}(K, N)$ m.m.s. and let $E \subset X$ be a set of finite perimeter. Then for $|D\chi_E|$-a.e. $x \in X$ the following property holds true: for every $(Y, \varrho, \mu, y, F) \in \mathrm{Tan}_x(X, \mathsf{d}, \mathfrak{m}, E)$ one has*

$$\mathrm{Tan}_{y'}(Y, \varrho, \mu, F) \subset \mathrm{Tan}_x(X, \mathsf{d}, \mathfrak{m}, E) \qquad \textit{for every } y' \in \operatorname{supp}|D\chi_F|.$$

Thanks to Corollary 4.35 we need only to prove the result at $|D\chi_E|$-a.e. $x \in X$ for all $(Y, \varrho, \mu, y, F) \in \mathrm{Tan}_x^*(X, \mathsf{d}, \mathfrak{m}, E)$, where $\mathrm{Tan}_x^*(X, \mathsf{d}, \mathfrak{m}, E)$ is defined adding to the conditions in Definition 4.30 the condition (b) of Corollary 4.35, namely that the perimeter measures of the rescaled spaces weakly converge, in the duality with $\mathrm{C_{bs}}(X)$, to the perimeter measure of $F$.

Let us briefly recall the notion of outer measure and its main properties. Given a positive measure $\mu$ over a metric space $(X, \mathsf{d})$ we set

$$\mu^*(A) := \inf\{\mu(B) : \ B \text{ Borel}, \ A \subset B\}, \qquad \forall A \subset X. \tag{4.71}$$

It is immediate to see that $\mu^*$ is countably sub-additive. Let us remark that if $\mu$ is asymptotically doubling then

$$\lim_{r\downarrow 0} \frac{\mu^*(A \cap B_r(x))}{\mu(B_r(x))} = 1 \qquad \text{for } \mu^*\text{-a.e. } x \in A. \tag{4.72}$$

Indeed, we can find a set $B \in \mathscr{B}(X)$ containing $A$ such that $\mu(B) = \mu^*(A)$, so that $\mu^*(C \cap A) = \mu(C \cap B)$ for every $C \in \mathscr{B}(X)$. In particular, taking $C = B_r(x)$, we have

$$\lim_{r\downarrow 0} \frac{\mu^*(A \cap B_r(x))}{\mu(B_r(x))} = \lim_{r\downarrow 0} \frac{\mu(B \cap B_r(x))}{\mu(B_r(x))} = 1,$$

for every $x \in B$ of density 1 for the measure $\mu$. Since $\mu$ is asymptotically doubling, $\mu$-a.e $x \in B$ has this property and (4.72) follows.

**Lemma 4.39.** *Let $(X, \mathsf{d}, \mathfrak{m})$ and let $E \subset X$ be as in the assumptions of Theorem 4.38. Let $A \subset X$ and $x \in A$ be such that*

$$\lim_{r\downarrow 0} \frac{|D\chi_E|^*(A \cap B_r(x))}{|D\chi_E|(B_r(x))} = 1,$$

*where $|D\chi_E|^*$ is the outer measure associated to $|D\chi_E|$ according to (4.71). Assume that $(Y, \varrho, \mu, F) \in \mathrm{Tan}_x^*(X, \mathsf{d}, \mathfrak{m}, E)$ and consider*

$$\Psi_i : (X, r_i^{-1}\mathsf{d}) \to (Z, \mathsf{d}_Z) \qquad \forall i \in \mathbb{N},$$
$$\Psi : (Y, \mathsf{d}_Y) \to (Z, \mathsf{d}_Z),$$

*a family of isometries realizing the pmGH convergence as in Definition 1.28. Then, for any $y' \in \operatorname{supp}|D\chi_F|$, there exists a sequence $(x_i) \subset A$ such that*

$$\lim_{i\to\infty} \mathsf{d}_Z(\Psi_i(x_i), \Psi(y')) = 0.$$

Roughly speaking, Lemma 4.39 tells us that it is possible to approximate every point in the support of any tangent by means of points in $A$, whenever $A$ is "large" in a measure-theoretic sense.

*Proof of Lemma 4.39.* As a first step we show the existence of an auxiliary sequence $(x_i) \subset X$, satisfying $\lim_i \mathsf{d}_Z(\Psi_i(x_i), \Psi(y')) = 0$ and

$$\lim_{i\to\infty} \frac{r_i|D\chi_E|(B_{rr_i}(x_i))}{C(x,r_i)} = |D\chi_F|(B_r(y')), \qquad \text{for } \mathscr{L}^1\text{-a.e. } r > 0, \tag{4.73}$$

where $C(x,r_i)$ was introduced in Definition 2.1.

Let us set $X_i := \Psi_i(X)$, $E_i := \Psi_i(E)$ and, with a slight abuse of notation, identity $F$ to $\Psi(F)$ and $y'$ to $\Psi(y')$. Since by assumption it holds that $|D\chi_{E_i}| \rightharpoonup |D\chi_F|$, we have

$$\lim_{i\to\infty} |D\chi_{E_i}|(B_r^Z(y')) = |D\chi_F|(B_r^Z(y')), \qquad \text{for } \mathscr{L}^1\text{-a.e. } r > 0.$$

This implies that the distance of $y'$ from $X_i$ is infinitesimal as $i \to \infty$, hence we can find points $z_i \in X_i$ converging to $y'$ in $Z$ satisfying

$$\lim_{i\to\infty} |D\chi_{E_i}|(B_r^Z(z_i)) = |D\chi_F|(B_r^Z(y')), \qquad \text{for } \mathscr{L}^1\text{-a.e. } r > 0.$$

Let us set $x_i := \Psi_i^{-1}(z_i)$. Observe that $|D\chi_F|(B_r^Z(y')) = |D\chi_F|(B_r^Y(y'))$ and

$$|D\chi_{E_i}|(B_r^Z(z_i)) = \frac{r_i|D\chi_E|(B_{rr_i}(x_i))}{C(x,r_i)},$$

so that we get (4.73).

Let us now argue by contradiction. Assuming the conclusion of the lemma to be false we might find $\varepsilon > 0$ such that the limit in (4.73) holds with $r = \varepsilon$ and

$$B_{\varepsilon r_i}(x_i) \cap A = \emptyset \qquad \text{for } i \text{ sufficiently large,}$$

with $x_i$ and $r_i$ as in (4.73). Let $M > 0$ be large enough to grant that

$$B_{\varepsilon r_i}(x_i) \subset B_{Mr_i}(x) \tag{4.74}$$

(it is simple to see that such a constant exists, since the convergence in $Z$ of $z_i = \Psi(x_i)$ ensures $\mathsf{d}(x, x_i) = O(r_i)$). Arguing as in the first part of the proof it is possible to see that

$$\lim_{i\to\infty} \frac{r_i|D\chi_E|(B_{Mr_i}(x))}{C(x,r_i)} = |D\chi_F|(B_M(y')) \qquad \text{for } \mathscr{L}^1\text{-a.e. } M > 0 \tag{4.75}$$

and from now on we assume, possibly increasing $M$, that both (4.74) and (4.75) hold true. Then, in view of (4.74), we have

$$\frac{|D\chi_E|^*(A \cap B_{Mr_i}(x))}{|D\chi_E|(B_{Mr_i}(x))} = \frac{|D\chi_E|^*(A \cap (B_{Mr_i}(x) \setminus B_{\varepsilon r_i}(x_i)))}{|D\chi_E|(B_{Mr_i}(x))} \leq 1 - \frac{|D\chi_E|(B_{\varepsilon r_i}(x_i))}{|D\chi_E|(B_{Mr_i}(x))}.$$

Observe that the left hand side converges to 1 as $i \to \infty$, since $x$ is of density 1 for $A$. Therefore, to get the sought contradiction, it suffices to show that

$$\liminf_{i\to\infty} \frac{|D\chi_E|(B_{\varepsilon r_i}(x_i))}{|D\chi_E|(B_{Mr_i}(x))} > 0.$$

Using (4.73) and (4.75), we get

$$\liminf_{i\to\infty} \frac{|D\chi_E|(B_{\varepsilon r_i}(x_i))}{|D\chi_E|(B_{Mr_i}(x))} = \frac{\lim_i \frac{r_i|D\chi_E|(B_{\varepsilon r_i}(x_i))}{C(x,r_i)}}{\lim_i \frac{r_i|D\chi_E|(B_{Mr_i}(x))}{C(x,r_i)}} \geq \frac{|D\chi_F|(B_\varepsilon(y'))}{|D\chi_F|\,(B_M(y'))} > 0,$$

where the last inequality holds true since we are assuming that $y' \in \operatorname{supp}|D\chi_F|$. $\square$

Before passing to the proof of Theorem 4.38 we need to introduce a definition and a lemma.

**Definition 4.40.** We shall denote by $\mathcal{F}(K,N)$ the set of equivalence classes of quintuples $\mathfrak{X} = (X, \mathsf{d}, \mathfrak{m}, x, \nu)$ where $(X, \mathsf{d}, \mathfrak{m}, x)$ is a pointed $\mathrm{RCD}(K,N)$ m.m.s and $\nu$ is a nonnegative and locally finite Borel measure with $\operatorname{supp}\nu \subset \operatorname{supp}\mathfrak{m}$, modulo the equivalence relation $\sim$ defined as follows. We say that $(X_1, \mathsf{d}_1, \mathfrak{m}_1, x_1, \nu_1) \sim (X_2, \mathsf{d}_2, \mathfrak{m}_2, x_2, \nu_2)$ if there exists an isometry $T : (\operatorname{supp}\mathfrak{m}_1, \mathsf{d}_1) \to (\operatorname{supp}\mathfrak{m}_2, \mathsf{d}_2)$ such that $T_*\mathfrak{m}_1 = \mathfrak{m}_2$, $T(x_1) = x_2$ and $T_*\nu_1 = \nu_2$. We shall denote by $\mathcal{F}$ the union of the sets $\mathcal{F}(K,N)$ for $K \in \mathbb{R}$, $1 \le N < \infty$. Observe that $\mathcal{F}$ can be realized as a countable union of sets $\mathcal{F}(K,N)$.

Let us introduce a distance in $\mathcal{F}$. Fix $\mathfrak{X}_1 = (X_1, \mathsf{d}_1, \mathfrak{m}_1, x_1, \nu_1)$, $\mathfrak{X}_2 = (X_2, \mathsf{d}_2, \mathfrak{m}_2, x_2, \nu_2)$ in $\mathcal{F}$, a proper metric measure space $(Z, \mathsf{d}_Z)$ and isometric embeddings $\Psi_i : (X_i, \mathsf{d}_i) \to (Z, \mathsf{d}_Z)$, $i = 1, 2$. For any integer $n \ge 1$ we define

$$\begin{aligned}
&\mathcal{D}_{n,\Psi_1,\Psi_2}(\mathfrak{X}_1, \mathfrak{X}_2) := \\
&\qquad \mathsf{d}_H(\Psi_1(X_1 \cap \overline{B}(x_1,n)), \Psi_2(X_2 \cap \overline{B}(x_2,n))) \wedge 1 \\
&\qquad + \left|\log\left(\frac{\mathfrak{m}_1(B(x_1,n))}{\mathfrak{m}_2(B(x_2,n))}\right)\right| \wedge 1 + W_1^Z\left((\Psi_1)_*\frac{\chi_{B(x_1,n)}}{\mathfrak{m}_1(B(x_1,n))}\mathfrak{m}_1, (\Psi_2)_*\frac{\chi_{B(x_2,n)}}{\mathfrak{m}_2(B(x_2,n))}\mathfrak{m}_2\right) \\
&\qquad + \left|\log\left(\frac{\nu_1(B(x_1,n))}{\nu_2(B(x_2,n))}\right)\right| \wedge 1 + W_1^Z\left((\Psi_1)_*\frac{\chi_{B(x_1,n)}}{\nu_1(B(x_1,n))}\nu_1, (\Psi_2)_*\frac{\chi_{B(x_2,n)}}{\nu_2(B(x_2,n))}\nu_2\right),
\end{aligned}$$

where $\mathsf{d}_H$ is the Hausdorff distance between compact subsets of $Z$ and $W_1^Z$ is defined as

$$W_1^Z(\mu,\nu) := \inf\left\{\int_Z \mathsf{d}_Z(x,y) \wedge 1 \,\mathrm{d}\pi(x,y) :\ \pi \in \Gamma(\mu,\nu)\right\},$$

with $\Gamma(\mu,\nu) \subset \mathscr{P}(X \times X)$ the set of probability measures having $\mu$ and $\nu$ as marginals. We finally define

$$\mathcal{D}(\mathfrak{X}_1, \mathfrak{X}_2) := \inf_{\Psi_1,\Psi_2}\left\{\mathsf{d}_Z(\Psi_1(x_1), \Psi_2(x_2)) + \sum_{n=1}^{\infty}\frac{1}{2^n}\mathcal{D}_{n,\Psi_1,\Psi_2}(\mathfrak{X}_1, \mathfrak{X}_2)\right\}, \tag{4.76}$$

the infimum being taken among all possible proper metric spaces $(Z, \mathsf{d}_Z)$ and all isometric embeddings $\Psi_i : (X_i, \mathsf{d}_i) \to (Z, \mathsf{d}_Z)$ for $i = 1, 2$.

**Lemma 4.41.** *$\mathcal{D}$ is a distance over $\mathcal{F}$ and a sequence $(X_i, \mathsf{d}_i, \mathfrak{m}_i, x_i, \nu_i) \subset \mathcal{F}$ converges to $(Y, \varrho, \mu, y, \nu)$ in the topology induced by $\mathcal{D}$ if and only if $(X_i, \mathsf{d}_i, \mathfrak{m}_i, x_i) \to (Y, \varrho, \mu, y)$ in the pmGH topology and $\nu_i \rightharpoonup \nu$ in duality with $\mathrm{C_{bs}}(Z)$, where $(Z, \mathsf{d}_Z)$ is a metric space where the pmGH convergence is realized. Moreover the subspace*

$$\overline{\mathcal{F}} := \{(X, \mathsf{d}, \mathfrak{m}, x, \nu) \in \mathcal{F} :\ \nu = h\mathfrak{m},\ \textit{with } h \in L^\infty(X, \mathfrak{m})\} \tag{4.77}$$

*is separable.*

*Proof.* The verification that $\mathcal{D}$ is a distance is quite standard, see for instance [**92**] . The equivalence between the two notions of convergence can be proved following the same strategy in the proof of [**92**, Theorem 3.15], the only difference here being the addition to the quadruple of the measure $\nu$. Let us prove that $\overline{\mathcal{F}}$ is separable. It is enough to prove that, given $K$ and $N$, for any $k > 0$ the set

$$\overline{\mathcal{F}}_k(K,N) := \{(X, \mathsf{d}, \mathfrak{m}, x, \nu) \in \mathcal{F}(K,N) :\ \nu = h\mathfrak{m},\ \text{with } \|h\|_{L^\infty(X,\mathfrak{m})} \le k\} \tag{4.78}$$

is compact. Let us fix a sequence $(X_i, \mathsf{d}_i, \mathfrak{m}_i, x_i, \nu_i) \subset \overline{\mathcal{F}}_k(K, N)$. We can assume, up to extract a subsequence, that $(X_i, \mathsf{d}_i, \mathfrak{m}_i, x_i) \to (Y, \varrho, \mu, y)$ in the pmGH topology. Let us fix a proper metric space $(Z, \mathsf{d}_Z)$ realizing this convergence. Since $\nu_i \le k\mathfrak{m}_i$ and $\mathfrak{m}_i \rightharpoonup \mu$ in duality with $\mathrm{C_{bs}}(Z)$ we deduce that the measures $\nu_i$ are locally bounded in $Z$, uniformly in $i \in \mathbb{N}$. Therefore, possibly extracting a subsequence, there exists a positive measure $\nu$ in $Z$ such that $\nu_i \rightharpoonup \nu$ in duality with $\mathrm{C_{bs}}(Z)$. It is immediate to check that $\nu \ll \mu$, with density uniformly bounded by $k$. This concludes the proof. □

*Proof of Theorem 4.38.* Since tangents are invariant w.r.t. rescaling and closed w.r.t. $\mathcal{D}$-convergence, it is enough to prove that the set of points $x \in X$ such that there exist $(Y, \varrho, \mu, y, F) \in \mathrm{Tan}^*_x(X, \mathsf{d}, \mathfrak{m}, E)$ and $y' \in \operatorname{supp}|D\chi_F|$ such that

$$(Y, \varrho, \mu^1_{y'}, y', F) \notin \mathrm{Tan}_x(X, \mathsf{d}, \mathfrak{m}, E)$$

is $|D\chi_E|^*$-negligible, where $\mu^{y'}_1 := C(y', 1)^{-1}\mu$ (see Definition 4.30).

Let us fix positive integers $k$, $m$ and a closed subset $\mathcal{U} \subset \overline{\mathcal{F}}$ with diameter, measured w.r.t. the distance $\mathcal{D}$ in (4.76), smaller than $(2k)^{-1}$. Since, according to Lemma 4.41, $\overline{\mathcal{F}}$ is separable, it is enough to prove that

$$\begin{aligned} A_{k,m} := \Big\{ x \in X \ :\ \exists\ (Y, \varrho, \mu, y, F) \in \mathrm{Tan}^*_x(X, \mathsf{d}, \mathfrak{m}, E) \cap \mathcal{U} \text{ and } y' \in \operatorname{supp}|D\chi_F| \text{ such that} \\ \mathcal{D}((Y, \varrho, \mu^1_{y'}, y', F), (X, r^{-1}\mathsf{d}, \mathfrak{m}^r_x, x, E)) \ge 2k^{-1} \quad \forall r \in (0, 1/m) \Big\} \end{aligned}$$

is $|D\chi_E|^*$-negligible, where we identified the set $F$ with the measure $\chi_F\mu$.

If, by contradiction, $|D\chi_E|^*(A_{k,m}) > 0$, then, since $|D\chi_E|$ is asymptotically doubling by Proposition 4.33, we can find $x \in A_{k,m}$ such that

$$\lim_{r \downarrow 0} \frac{|D\chi_E|^*(A_{k,m} \cap B_r(x))}{|D\chi_E|(B_r(x))} = 1,$$

see (4.72). Since $x \in A_{k,m}$ there exist $(Y, \varrho, \mu, y, F) \in \mathrm{Tan}^*_x(X, \mathsf{d}, \mathfrak{m}, E) \cap \mathcal{U}$ and $y' \in \operatorname{supp}|D\chi_F|$ such that $\mathcal{D}((Y, \varrho, \mu^1_{y'}, y', F), (X, r^{-1}\mathsf{d}, \mathfrak{m}^r_x, x, E)) \ge 2k^{-1}$ for any $r \in (0, 1/m)$ and Lemma 4.39 grants the existence of a sequence $(x_i) \subset A_{k,m}$ such that

$$\lim_{i \to \infty} \mathsf{d}_Z(\Psi_i(x_i), \Psi(y')) = 0,$$

where $\Psi_i$, $\Psi$ are the embedding maps of Definition 1.28. Then, by definition of pmGH convergence, using the space $(Z, \mathsf{d}_Z)$ we deduce

$$(X, r_i^{-1}\mathsf{d}, \mathfrak{m}^{r_i}_x, x_i) \to (Y, \varrho, \mu, y').$$

Since $\chi_{B^Z(\bar z, 1)}(1 - \mathsf{d}_Z(\cdot, \bar z))$ belongs to $\mathrm{C_b}(Z)$ for every $\bar z \in Z$, it is immediate to check that

$$(X, r_i^{-1}\mathsf{d}, \mathfrak{m}^{r_i}_{x_i}, x_i) \to (Y, \varrho, \mu^1_{y'}, y'), \qquad \text{in the pmGH topology,}$$

and $(\Psi)_*\chi_E\mathfrak{m}^{r_i}_{x_i} \rightharpoonup \Psi_*\chi_F\mu^1_{y'}$ in duality with $\mathrm{C_{bs}}(Z)$, that, thanks to (4.41), is equivalent to

$$\mathcal{D}((X, r_i^{-1}\mathsf{d}, \mathfrak{m}^{r_i}_{x_i}, x_i, E), (Y, \varrho, \mu^1_{y'}, y', F)) \to 0, \tag{4.79}$$

see Definition 4.40. Since $x_i \in A_{k,m}$ we can find $(Y_i, \varrho_i, \mu_i, y_i, F_i) \in \mathrm{Tan}^*_{x_i}(X, \mathsf{d}, \mathfrak{m}, E) \cap \mathcal{U}$ and $y'_i \in \operatorname{supp}|D\chi_{F_i}|$ such that $\mathcal{D}((Y_i, \varrho_i, (\mu_i)^1_{y'_i}, y'_i, F_i), (X, r^{-1}\mathsf{d}, \mathfrak{m}^r_{x_i}, x_i, E)) \ge 2k^{-1}$ for any $r \in (0, 1/m)$.

Using (4.79) and taking into account that by construction $\operatorname{diam} \mathcal{U} < (2k)^{-1}$, we find the sought contradiction

$$\begin{aligned}
2k^{-1} &\leq \mathcal{D}((Y_i, \varrho_i, (\mu_i)^1_{y'_i}, y'_i, F_i), (X, r_i^{-1}\mathsf{d}, \mathfrak{m}^{r_i}_{x_i}, x_i, E)) \\
&\leq \mathcal{D}((Y, \varrho, \mu^1_{y'}, y', F), (X, r_i^{-1}\mathsf{d}, \mathfrak{m}^{r_i}_{x_i}, x_i, E)) + \mathcal{D}((Y_i, \varrho_i, (\mu_i)^1_{y'_i}, y'_i, F_i), (Y, \varrho, \mu^1_{y'}, y', F)) \\
&\leq \mathcal{D}((Y, \varrho, \mu^1_{y'}, y', F), (X, r_i^{-1}\mathsf{d}, \mathfrak{m}^{r_i}_{x_i}, x_i, E)) + (2k)^{-1} \\
&\leq k^{-1},
\end{aligned}$$

for $i$ large enough. □

CHAPTER 5

# Rectifiability of the reduced boundary

One of the main results of Chapter 4 was the existence of a Euclidean half-space as tangent space to a set of finite perimeter at almost every point (with respect to the perimeter measure). The state of the theory of sets with finite perimeter at this stage is comparable to that of the structure theory after Theorem 2.2, where the $\mathfrak{m}$-a.e. existence of Euclidean tangent spaces was proved.

Aim of this chapter is to provide the counterpart in codimension 1 of Theorem 2.11 and of De Giorgi's theorem in this setting. Together with uniqueness of tangents (cf. Theorem 5.15) and rectifiability (cf. Theorem 5.21) we also establish a representation formula for the perimeter measure in terms of the codimension 1 Hausdorff measure (cf. Corollary 5.20). As an intermediate tool which is of independent interest we prove in Theorem 5.6 a Gauss–Green integration-by-parts formula for Sobolev vector fields. In this chapter we follow closely [**40**]. The proof of uniqueness for blow-ups of sets of finite perimeter follows a strategy quite similar to that of the uniqueness theorem for tangents to $\mathrm{RCD}(K, N)$ spaces adopted in Theorem 2.4. As in that case, closeness to a rigid configuration (half-space in Euclidean space) at a certain location and scale, can be turned into closeness to the same configuration at almost any location and at any scale, yielding uniqueness.

To encode the "closeness information" in analytic terms we rely on the use of harmonic $\delta$-splitting maps which we have already introduced in Section 7.1 and used in Chapter 2.

In order to explain the strategy and the difficulties in the proof of rectifiability for the reduced boundary, let us recall how things work on $\mathbb{R}^n$. Therein a crucial role is played by the exterior normal to the set of finite perimeter, which is an almost everywhere unit valued vector field providing the representation $D\chi_E = \nu_E\,|D\chi_E|$ for the distributional derivative of the set of finite perimeter $E$. Relying on the properties of the exterior normal one can obtain a characterization of blow-ups and even get rectifiability of the boundary, proving that sets where the unit normal is not oscillating too much are bi-Lipschitz to subsets of $\mathbb{R}^{n-1}$.

When trying to reproduce the Euclidean approach in the *non smooth* and *non flat* setting of RCD spaces, one faces two main difficulties. The first one is due to the fact that the theory of tangent modules, as developed in [**87**], allows to talk about vector fields only up to negligible sets with respect to the reference measure (as the reduced boundary of a set of finite perimeter is not). The second one is that controlling the behaviour of the normal vector cannot be enough to control the behaviour of the set in this framework, since the space itself might "oscillate". This is a common feature of geometry on metric measure spaces (see also the introduction of [**55**]), which can be understood looking at the following example: let $(X, \mathsf{d}, \mathfrak{m})$ be any $\mathrm{RCD}(K, N)$ m.m.s. and take its product with the Euclidean line. Then consider the "generalized half-space" $\{t < 0\}$, where $t$ denotes the coordinate along the line: it is easily seen that it is a set of locally finite perimeter and one can identify its reduced boundary with $X$. Moreover, whatever notion of unit normal we have in mind, this will be non oscillating in this case. Still, rectifiability of $(X, \mathsf{d}, \mathfrak{m})$ is highly non trivial.

To handle the first difficulty we mentioned above, we rely on the very recent [**67**], where a notion of cotangent module with respect to the 2-capacity is introduced and studied. Building upon the fact that the 2-capacity controls the perimeter measure in great generality, we introduce the notion of tangent module over the boundary of a set of finite perimeter (cf. Theorem 5.3). Furthermore we prove that there is a well-defined unit normal to a set of finite perimeter as an element of this module, that it satisfies the Gauss–Green integration-by-parts formula and, relying on functional analysis tools, that it can be approximated by regular vector fields (cf. Theorem 5.6 for a rigorous statement).

The results obtained in the study of the unit normal are then combined with the theory of $\delta$-splitting maps to prove uniqueness of tangents as well as rectifiability of the reduced boundary for sets of finite perimeter. Let us mention that the uniqueness results does not rely on the study of the unit normal to a set of finite perimeter, it just uses analytical properties of $\delta$-splitting maps. The hardest part of the work, i.e. the rectifiability result, exploits instead all the ingredients and the notion of $\delta$-orthogonality to the unit normal for $\delta$-splitting maps. We prove on the one hand that $\delta$-splitting maps $\delta$-orthogonal to the unit normal control both the geometry of the space and that of the boundary of the set of finite perimeter (and vice-versa). On the other hand the combination of $\delta$-orthogonality and $\delta$-splitting is seen to be suitable for propagation at many locations and any scale with maximal function arguments (cf. Proposition 5.24 and Proposition 5.27).

This chapter is organized as follows. We dedicate Section 1 to constructing the tangent module over the boundary of a set of finite perimeter and to establishing a Gauss–Green integration-by-parts formula. Uniqueness of blow-ups is the main outcome of Section 2, while rectifiability for the reduced boundary is obtained in Section 3.

## 1. A Gauss-Green formula on RCD spaces

Let us begin by recalling that the codimension-1 measure (Cf. Theorem 1.68) plays a crucial role in the theory of sets of finite perimeter over $\mathrm{RCD}(K,N)$ spaces, since $\mathrm{Per}(E,\cdot) \ll \mathscr{H}^{h_1}$ for any set of finite perimeter $E$. This result was proved by Ambrosio in [**4**, Lemma 5.2].

**Lemma 5.1.** *Let $(X, \mathsf{d}, \mathfrak{m})$ be a PI space. For any set of locally finite perimeter $E \subset X$ it holds*

$$\mathscr{H}^{h_1}(B) = 0 \implies \mathrm{Per}(E, B) = 0 \quad \textit{for any Borel set } B \subset X.$$

Let us now consider an $\mathrm{RCD}(K,N)$ metric measure space $(X, \mathsf{d}, \mathfrak{m})$ and $E \subset X$ a set of finite perimeter. In view of Lemma 5.1 and the discussion in Section 5.3 it makes sense to consider the projection

$$\pi_{|D\chi_E|} : L^0(\mathrm{Cap}) \to L^0(|D\chi_E|).$$

Recall also that $\mathsf{QCR} : H^{1,2}(X) \to L^0(\mathrm{Cap})$ stands for the "quasi-continuous representative" operator. Then let us define

$$\mathrm{tr}_E : H^{1,2}(X) \to L^0(|D\chi_E|), \quad \mathrm{tr}_E := \pi_{|D\chi_E|} \circ \mathsf{QCR},$$

the trace operator over the boundary of $E$. Observe that $\mathrm{tr}_E(f) \in L^\infty(|D\chi_E|)$ holds for every test function $f \in \mathrm{Test}(X, \mathsf{d}, \mathfrak{m})$.

**Remark 5.2.** When $(X, \mathsf{d}, \mathfrak{m})$ is the Euclidean space of dimension $n$ and $E \subset \mathbb{R}^n$ is open and smooth, then $\mathrm{tr}_E : H^1(\mathbb{R}^n) \to L^0(|D\chi_E|)$ coincides with the canonical trace operator. Indeed the two operators coincide on smooth functions and they are continuous. In the case of the canonical trace this is a standard result, while for $\mathrm{tr}_E$ this is a consequence of [**67**, Proposition 1.19] and the continuity of $\pi_{|D\chi_E|} : L^0(\mathrm{Cap}) \to L^0(|D\chi_E|)$.

This being said, let us state the two main results of this section. The first one gives existence and uniqueness of the tangent module over the boundary of a set of finite perimeter. The second theorem provides a Gauss–Green formula tailored for finite-dimensional RCD spaces along with a strong approximation result for the exterior normal of sets with finite perimeter. This approximation result, whose proof heavily relies on the abstract machinery of normed modules and on functional-analytic tools, plays a key role in the study of rectifiability properties for boundaries of sets with finite perimeter that we are going to perform in the last section of this chapter.

**Theorem 5.3** (Tangent module over $\partial E$)**.** *Let $(X, \mathsf{d}, \mathfrak{m})$ be an $\mathrm{RCD}(K, N)$ space. Let $E \subset X$ be a set of finite perimeter. Then there exists a unique couple $\left(L^2_E(TX), \bar{\nabla}\right)$ – where $L^2_E(TX)$ is an $L^2(|D\chi_E|)$-normed $L^\infty(|D\chi_E|)$-module and $\bar{\nabla} : \mathrm{Test}(X) \to L^2_E(TX)$ is linear – such that:*

(i) *The equality $|\bar{\nabla} f| = \mathrm{tr}_E(|\nabla f|)$ holds $|D\chi_E|$-a.e. for every $f \in \mathrm{Test}(X, \mathsf{d}, \mathfrak{m})$.*
(ii) $\left\{ \sum_{i=1}^n \chi_{E_i} \bar{\nabla} f_i \;\middle|\; (E_i)_{i=1}^n \text{ Borel partition of } X,\ (f_i)_{i=1}^n \subset \mathrm{Test}(X, \mathsf{d}, \mathfrak{m}) \right\}$ *is dense in $L^2_E(TX)$.*

*Uniqueness is intended up to unique isomorphism: given another couple $(\mathscr{M}, \bar{\nabla}')$ satisfying (i), (ii) above, there exists a unique normed module isomorphism $\Phi : L^2_E(TX) \to \mathscr{M}$ such that $\Phi \circ \bar{\nabla} = \bar{\nabla}'$. The space $L^2_E(TX)$ is called* tangent module over the boundary of $E$ *and $\bar{\nabla}$ is the* gradient*.*

Following [**67**, Definition 2.2] we introduce a notion of "quasi-continuous vector field" suitable for this context.

**Definition 5.4.** The class $\mathcal{QC}(TX) \subset L^0_{\mathrm{Cap}}(TX)$ is the $\mathsf{d}_{\mathscr{M}_{\mathrm{Cap}}}$-closure of

$$\left\{ \sum_{i=0}^n \mathsf{QCR}(g_i)\, \bar{\nabla} f_i \;\middle|\; n \in \mathbb{N},\ (f_i)_{i=1}^n, (g_i)_{i=1}^n \subset \mathrm{Test}(X, \mathsf{d}, \mathfrak{m}) \right\}$$

It has been proven in [**67**, Proposition 2.12] that $|v| \in \mathcal{QC}(X)$ whenever $v \in \mathcal{QC}(TX)$. As in the scalar case it is well-defined the "quasi-continuous representative" map

$$\mathsf{Q\bar{C}R} : H^{1,2}_C(TX) \to L^0_{\mathrm{Cap}}(TX),$$

Cf. [**67**, Theorem 2.14]. Moreover, with a slight abuse of notation we define

$$\mathrm{tr}_E : H^{1,2}_C(TX) \cap L^\infty(TX) \to L^2_E(TX), \quad \mathrm{tr}_E := \bar{\pi}_{|D\chi_E|} \circ \mathsf{Q\bar{C}R}.$$

Notice that $|\mathrm{tr}_E(v)| = \mathrm{tr}_E(|v|)$ holds $|D\chi_E|$-a.e. for every $v \in H^{1,2}_C(TX) \cap L^\infty(TX)$.

**Remark 5.5.** Arguing as in Remark 5.2 one can prove that the above defined operator $\mathrm{tr}_E$ coincides with the canonical trace in the case of smooth domains in $\mathbb{R}^n$.

**Theorem 5.6** (Gauss–Green formula on RCD spaces)**.** *Let $(X, \mathsf{d}, \mathfrak{m})$ be an $\mathrm{RCD}(K, N)$ space and $E \subset X$ be a set of finite perimeter such that $\mathfrak{m}(E) < \infty$. Then there exists a unique* *vector field $\nu_E \in L^2_E(TX)$ such that $|\nu_E| = 1$ holds $|D\chi_E|$-a.e. and*

(5.1)
$$\int_E \mathrm{div}(v)\,\mathrm{d}\mathfrak{m} = -\int \Big\langle \mathrm{tr}_E(v), \nu_E \Big\rangle\,\mathrm{d}|D\chi_E| \quad \text{for all } v \in H^{1,2}_C(TX) \cap D(\mathrm{div}) \text{ with } |v| \in L^\infty(\mathfrak{m}).$$

*Moreover, there exists a sequence $(v_n)_n \subset \mathrm{TestV}_E(X)$ of test vector fields over the boundary of $E$ (see Lemma 5.11 below for the precise definition of this class) such that $v_n \to \nu_E$ in the strong topology of $L^2_E(TX)$.*

**Remark 5.7.** In the case in which $X$ is a Riemannian manifold and $E \subset X$ is a domain with smooth boundary, it holds that $L^2_E(TX)$ is the space of all Borel vector fields over $X$ which are concentrated on the boundary of $E$ and 2-integrable with respect to the surface measure and, in this case, $\bar\nabla$ is the classical gradient for smooth functions.

**Remark 5.8.** As we have already remarked in Remark 1.72, the tangent $L^0(\mathrm{Cap})$-module $L^0_{\mathrm{Cap}}(TX)$ is a Hilbert module. Therefore, it is immediate to see by passing to the quotient that $L^2_E(TX)$ is a Hilbert module as well.

The remaining part of this section is dedicated to the proofs of Theorem 5.3 and Theorem 5.6.

### 1.1. Proof of Theorem 5.3.

Uniqueness. Call $\mathcal{W}$ the family of elements of $L^2_E(TX)$ considered in item ii). Given any $\omega = \sum_{i=1}^n \chi_{E_i}\bar\nabla f_i \in \mathcal{W}$, we are forced to set $\Phi(\omega) := \sum_{i=1}^n \chi_{E_i}\bar\nabla' f_i$. Well-posedness of such definition stems from the $|D\chi_E|$-a.e. identity

$$\left|\sum_{i=1}^n \chi_{E_i}\bar\nabla' f_i\right| = \sum_{i=1}^n \chi_{E_i}|\bar\nabla' f_i| = \sum_{i=1}^n \chi_{E_i}\,\mathrm{tr}_E(|\nabla f_i|) = \sum_{i=1}^n \chi_{E_i}|\bar\nabla f_i| = |\omega|,$$

which also shows that $\Phi$ preserves the pointwise norm. Then $\Phi$ is linear continuous, thus it can be uniquely extended to a linear continuous map $\Phi : L^2_E(TX) \to \mathscr{M}$ by density of $\mathcal{W}$ in $L^2_E(TX)$. By an approximation argument, it is easy to see that the extended $\Phi$ preserves the pointwise norm and is an $L^\infty(|D\chi_E|)$-module morphism. Finally, the map $\Phi$ is surjective, because its image is dense (as $\mathscr{M}$ satisfies ii)) and closed (as $\Phi$ is an isometry). Consequently, we have proved that there exists a unique normed module isomorphism $\Phi : L^2_E(TX) \to \mathscr{M}$ such that $\Phi \circ \bar\nabla = \bar\nabla'$.

Existence. Let us consider the tangent $L^0(\mathrm{Cap})$-module $L^0_{\mathrm{Cap}}(TX)$ and the relative capacitary gradient operator $\tilde\nabla : \mathrm{Test}(X) \to L^0_{\mathrm{Cap}}(TX)$ associated to the space $(X, \mathsf{d}, \mathfrak{m})$; cf. Theorem 1.71. We define $L^0_E(TX)$ as $L^0_{\mathrm{Cap}}(TX)/\sim_{|D\chi_E|}$ and the $L^2(|D\chi_E|)$-normed $L^\infty(|D\chi_E|)$-module $L^2_E(TX)$ as in (1.46). Moreover, we define the differential $\bar\nabla : \mathrm{Test}(X) \to L^2_E(TX)$ as $\bar\nabla := \bar\pi_{|D\chi_E|} \circ \tilde\nabla$. Clearly, the map $\bar\nabla$ is linear by construction. Given any function $f \in \mathrm{Test}(X)$, it $|D\chi_E|$-a.e. holds

$$|\bar\nabla f| = \left|\bar\pi_{|D\chi_E|}(\tilde\nabla f)\right| = \pi_{|D\chi_E|}(|\tilde\nabla f|) = \pi_{|D\chi_E|}\Big(\mathsf{QCR}(|\nabla f|)\Big) = \mathrm{tr}_E(|\nabla f|),$$

which shows that i) is satisfied. We also set $V := \mathrm{Test}(X)$ and the associated space $\mathcal{V} \subset L^0_{\mathrm{Cap}}(TX)$ as in the statement of Lemma 1.73. By the defining property of the cotangent Cap-module we know that $\mathcal{V}$ is dense in $L^0_{\mathrm{Cap}}(TX)$, whence Lemma 1.73 ensures that $\mathcal{W}$ is dense in $L^2_E(TX)$. This means that property ii) holds. Therefore, the existence part of the statement is proven.

**1.2. Proof of the Gauss-Green formula.** In this section we develop all the tools needed for the proof of Theorem 5.6. We begin by presenting three technical lemmas and we go on briefly explaining how the heat equation associated to the Hodge Laplacian can be used to regularise vector fields. We then apply this result to prove a representation formula for the total variation via integration by parts. We eventually prove Theorem 5.6.

The first lemma is a byproduct of the proof of Theorem 4.31.

**Lemma 5.9.** *Let $(X, \mathsf{d}, \mathfrak{m})$ be an $\mathrm{RCD}(K, N)$ space. Let $E \subset X$ be a set of finite perimeter. Then*

$$\lim_{t\searrow 0}\int \left|1 - e^{Kt}\frac{|\nabla P_t\chi_E|}{P_t^*|D\chi_E|}\right| P_t^*|D\chi_E| \,\mathrm{d}\mathfrak{m} = 0. \tag{5.2}$$

Here $P_t^*|D\chi_E|$ is understood as its density w.r.t. $\mathfrak{m}$.

**Lemma 5.10.** *Let $(X, \mathsf{d}, \mathfrak{m})$ be an $\mathrm{RCD}(K, N)$ space. Let $E \subset X$ be a set of finite perimeter. Then*

$$\int f\, P_t^*|D\chi_E| \,\mathrm{d}\mathfrak{m} = \int \mathrm{tr}_E(P_t f) \,\mathrm{d}|D\chi_E| \quad \textit{for every } f \in H^{1,2}(X)\cap L^\infty(\mathfrak{m}) \textit{ and } t > 0. \tag{5.3}$$

*Moreover, it holds that*

$$\lim_{t\searrow 0}\int \mathrm{tr}_E(P_t f) \,\mathrm{d}|D\chi_E| = \int \mathrm{tr}_E(f) \,\mathrm{d}|D\chi_E| \quad \textit{for every } f \in H^{1,2}(X)\cap L^\infty(\mathfrak{m}). \tag{5.4}$$

*Proof.* First of all, let us prove (5.3). Fix any $f \in H^{1,2}(X) \cap L^\infty(\mathfrak{m})$ and $t > 0$. We claim that

$$\exists\, (f_n)_n \subset \mathrm{Lip_{bs}}(X, \mathsf{d}) \text{ bounded in } L^\infty(\mathfrak{m}) : \ f_n \to f \text{ strongly in } H^{1,2}(X), \text{ weakly}^* \text{ in } L^\infty(\mathfrak{m}). \tag{5.5}$$

Let us prove it. Given any $s > 0$, the function $P_s f$ has a Lipschitz representative (still denoted by $P_s f$) thanks to the $L^\infty$-Lip regularisation of the heat flow. Since $\{P_s f\}_{s>0}$ is bounded in $L^\infty(\mathfrak{m})$ by the weak maximum principle and $P_s|\nabla f|^2 \to |\nabla f|^2$ strongly in $L^1(\mathfrak{m})$, we can find a function $G \in L^1(\mathfrak{m})$ and a sequence $s_n \searrow 0$ such that $P_{s_n}|\nabla f|^2 \le G$ holds $\mathfrak{m}$-a.e. for all $n$ and $P_{s_n} f \to f$ weakly* in $L^\infty(\mathfrak{m})$. Fix $\bar x \in X$ and for any $n \in \mathbb{N}$ choose a compactly-supported 1-Lipschitz function $\eta_n : X \to [0, 1]$ such that $\eta_n = 1$ on $B_n(\bar x)$. Therefore, standard computations (based on the Leibniz rule $\nabla(\eta_n P_{s_n} f) = \eta_n \nabla P_{s_n} f + P_{s_n} f \nabla \eta_n$, the dominated convergence theorem, and the Bakry-Émery contraction estimate) show that $f_n := \eta_n P_{s_n} f \in \mathrm{Lip_{bs}}(X, \mathsf{d})$ satisfy (5.5). Now observe that $P_t : H^{1,2}(X) \to H^{1,2}(X)$ is continuous, as a consequence of the Bakry-Émery contraction estimate and the continuity of $P_t : L^2(\mathfrak{m}) \to L^2(\mathfrak{m})$. This ensures that $P_t f_n \to P_t f$ strongly in $H^{1,2}(X)$ as $n \to \infty$, whence we know from [**67**, Propositions 1.12, 1.17 and 1.19] that (possibly passing to a not relabeled subsequence) $\mathsf{QCR}(P_t f_n) \to \mathsf{QCR}(P_t f)$ holds Cap-a.e., and accordingly $\mathrm{tr}_E(P_t f_n) \to \mathrm{tr}_E(P_t f)$ holds $|D\chi_E|$-a.e.. Moreover, since $|P_t f_n| \le \sup_k \|f_k\|_{L^\infty(\mathfrak{m})} =: C$ in the $\mathfrak{m}$-a.e. sense for all $n \in \mathbb{N}$, we deduce that $\left|\mathsf{QCR}(P_t f_n)\right| \le C$ holds Cap-a.e. for all $n \in \mathbb{N}$, and thus $\mathrm{tr}_E(P_t f_n) \le C$ holds $|D\chi_E|$-a.e. for all $n \in \mathbb{N}$. All in all, we obtain (5.3) by letting $n \to \infty$ in $\int f_n\, P_t^*|D\chi_E| \,\mathrm{d}\mathfrak{m} = \int \mathrm{tr}_E(P_t f_n) \,\mathrm{d}|D\chi_E|$, which is satisfied thanks to the defining property of $P_t^*|D\chi_E|$; here we use the dominated convergence theorem and the $L^\infty$-weak* convergence $f_n \to f$.

Let us now pass to the proof of (5.4). Fix $f \in H^{1,2}(X)\cap L^\infty(\mathfrak{m})$. By arguing as above, we see that $\left|\mathrm{tr}_E(P_t f)\right| \le \|f\|_{L^\infty(\mathfrak{m})}$ holds $|D\chi_E|$-a.e. for all $t > 0$, and that any given sequence

$t_n \searrow 0$ admits a subsequence $t_{n_i} \searrow 0$ such that $\mathrm{tr}_E(P_{t_{n_i}} f) \to \mathrm{tr}_E(f)$ holds $|D\chi_E|$-a.e.. Therefore, by dominated convergence theorem we conclude that $\lim_i \int \mathrm{tr}_E(P_{t_{n_i}} f)\,\mathrm{d}|D\chi_E| = \int \mathrm{tr}_E(f)\,\mathrm{d}|D\chi_E|$, which yields (5.4). $\square$

**Lemma 5.11** (Test vector fields over $\partial E$). *Let $(X, \mathsf{d}, \mathfrak{m})$ be an* $\mathrm{RCD}(K, N)$ *space. Let $E \subset X$ be a set of finite perimeter and finite mass. We define the class* $\mathrm{TestV}_E(X) \subset L^2_E(TX)$ *of* test vector fields over the boundary of $E$ *as*

$$\mathrm{TestV}_E(X) := \mathrm{tr}_E\Big(\mathrm{TestV}(X)\Big) = \left\{ \sum_{i=1}^{n} \mathrm{tr}_E(g_i)\bar\nabla f_i \;\middle|\; n \in \mathbb{N},\ (f_i)_{i=1}^n, (g_i)_{i=1}^n \subset \mathrm{Test}(X) \right\}.$$

*Then* $\mathrm{TestV}_E(X)$ *is dense in* $L^2_E(TX)$.

*Proof.* By item (ii) of Theorem 5.3, it suffices to show that each $v \in L^2_E(TX)$ of the form $v = \chi_E \bar\nabla f$ – where $E \subset X$ is a Borel set and $f \in \mathrm{Test}(X)$ – can be approximated by elements of $\mathrm{TestV}_E(X)$ with respect to the strong topology of $L^2_E(TX)$. Fix $\varepsilon > 0$ and choose a function $h \in \mathrm{Lip}_c(X)$ such that $\|h - \chi_E\|_{L^2(|D\chi_E|)} \le \varepsilon/(2\,\mathrm{Lip}(f))$. Moreover, by exploiting Lemma 1.57 we can find a sequence $(g_n)_n \subset \mathrm{Test}(X)$ such that $\sup_n \|g_n\|_{L^\infty(\mathfrak{m})} < \infty$ and $g_n \to h$ in $H^{1,2}(X)$. Hence, by using the results in [**67**] we see that (up to a not relabeled subsequence) it holds $\mathrm{tr}_E(g_n)(x) \to h(x)$ for $|D\chi_E|$-a.e. $x \in X$. Accordingly, by applying the dominated convergence theorem we conclude that $\big|(\mathrm{tr}_E(g_n) - h)\bar\nabla f\big| \to 0$ in $L^2(|D\chi_E|)$. Now choose $n \in \mathbb{N}$ so big that $g := g_n$ satisfies $\big\|(\mathrm{tr}_E(g) - h)\bar\nabla f\big\|_{L^2_E(TX)} < \varepsilon/2$. Hence, one has that

$$\begin{aligned}\big\|\mathrm{tr}_E(g)\bar\nabla f - v\big\|_{L^2_E(TX)} &\le \big\|(\mathrm{tr}_E(g) - h)\bar\nabla f\big\|_{L^2_E(TX)} + \big\|(h - \chi_E)\bar\nabla f\big\|_{L^2_E(TX)} \\ &\le \frac{\varepsilon}{2} + \|h - \chi_E\|_{L^2(|D\chi_E|)}\,\mathrm{Lip}(f) < \varepsilon.\end{aligned}$$

Given that $\mathrm{tr}_E(g)\bar\nabla f \in \mathrm{TestV}_E(X)$, the statement is achieved. $\square$

1.2.1. *Hodge Laplacian of vector fields on* RCD *spaces.* In this subsection we study the heat flow associated to the Hodge Laplacian being a fundamental tool to regularise vector fields in $L^2(TX)$.

Let $(X, \mathsf{d}, \mathfrak{m})$ be an $\mathrm{RCD}(K, \infty)$ space. Consider the space $H^{1,2}_{\mathrm{H}}(TX) \subset H^{1,2}_C(X, \mathsf{d}, \mathfrak{m})$ and the Hodge Laplacian $\Delta_{\mathrm{H}} : D(\Delta_{\mathrm{H}}) \subset H^{1,2}_{\mathrm{H}}(TX) \to L^2(TX)$, which have been defined in [**87**, Definition 3.5.13] and [**87**, Definition 3.5.15], respectively.

It follows from its definition that the Hodge Laplacian is self-adjoint, namely that

$$\int \langle \Delta_{\mathrm{H}} v, w\rangle\,\mathrm{d}\mathfrak{m} = \int \langle v, \Delta_{\mathrm{H}} w\rangle\,\mathrm{d}\mathfrak{m} \quad \text{for every } v, w \in D(\Delta_{\mathrm{H}}). \tag{5.6}$$

Let us consider the *augmented Hodge energy functional* $\tilde{\mathcal{E}}_{\mathrm{H}} : L^2(TX) \to [0, \infty]$, which is defined in [**87**, eq. (3.5.16)] (up to identifying $L^2(T^*X)$ with $L^2(TX)$ via the musical isomorphism). Then we denote by $(\mathsf{h}_{\mathrm{H},t})_{t\ge 0}$ the gradient flow in $L^2(TX)$ of the functional $\tilde{\mathcal{E}}_{\mathrm{H}}$. This means that for any vector field $v \in L^2(TX)$ it holds that $t \mapsto \mathsf{h}_{\mathrm{H},t}(v) \in L^2(TX)$ is the unique continuous curve on $[0, \infty)$ with $\mathsf{h}_{\mathrm{H},0}(v) = v$, which is locally absolutely continuous on $(0, \infty)$ and satisfies

$$\mathsf{h}_{\mathrm{H},t}(v) \in D(\Delta_{\mathrm{H}}) \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}t}\mathsf{h}_{\mathrm{H},t}(v) = -\Delta_{\mathrm{H}}\mathsf{h}_{\mathrm{H},t}(v) \quad \text{for every } t > 0.$$

Cf. the discussion that precedes [**87**, Proposition 3.6.10]. It also holds that

$$\mathsf{h}_{\mathrm{H},t}(\nabla f) = \nabla P_t f \quad \text{for every } f \in H^{1,2}(X) \text{ and } t \geq 0. \tag{5.7}$$

Finally, we recall that vector fields satisfy the following Bakry-Émery contraction estimate (see [**87**, Proposition 3.6.10]):

$$|\mathsf{h}_{\mathrm{H},t}(v)|^2 \leq e^{-2Kt} P_t(|v|^2) \quad \mathfrak{m}\text{-a.e.} \quad \text{for every } v \in L^2(TX) \text{ and } t \geq 0. \tag{5.8}$$

**Lemma 5.12** ($\mathsf{h}_{\mathrm{H},t}$ is self-adjoint)**.** *Let $(X, \mathsf{d}, \mathfrak{m})$ be an $\mathrm{RCD}(K, \infty)$ space. Then it holds that*

$$\int \langle \mathsf{h}_{\mathrm{H},t}(v), w \rangle \, \mathrm{d}\mathfrak{m} = \int \langle v, \mathsf{h}_{\mathrm{H},t}(w) \rangle \, \mathrm{d}\mathfrak{m} \quad \textit{for every } v, w \in L^2(TX) \textit{ and } t \geq 0. \tag{5.9}$$

*Proof.* Fix $v, w \in L^2(TX)$ and $t > 0$. We define the function $\phi \colon [0, t] \to \mathbb{R}$ as

$$\phi(s) := \int \langle \mathsf{h}_{\mathrm{H},s}(v), \mathsf{h}_{\mathrm{H},t-s}(w) \rangle \, \mathrm{d}\mathfrak{m} \quad \text{for every } s \in [0, t].$$

Therefore, the function $\phi$ is absolutely continuous and satisfies

$$\phi'(s) = -\int \langle \Delta_{\mathrm{H}} \mathsf{h}_{\mathrm{H},s}(v), \mathsf{h}_{\mathrm{H},t-s}(w) \rangle \, \mathrm{d}\mathfrak{m} + \int \langle \mathsf{h}_{\mathrm{H},s}(v), \Delta_{\mathrm{H}} \mathsf{h}_{\mathrm{H},t-s}(w) \rangle \, \mathrm{d}\mathfrak{m} \overset{(5.6)}{=} 0 \quad \text{for a.e. } t > 0.$$

Then $\phi$ is constant, thus in particular $\int \langle \mathsf{h}_{\mathrm{H},t}(v), w \rangle \, \mathrm{d}\mathfrak{m} = \phi(t) = \phi(0) = \int \langle v, \mathsf{h}_{\mathrm{H},t}(w) \rangle \, \mathrm{d}\mathfrak{m}$. □

**Proposition 5.13.** *Let $(X, \mathsf{d}, \mathfrak{m})$ be an $\mathrm{RCD}(K, \infty)$ space. Then for any $v \in D(\mathrm{div})$ it holds that*

$$\mathsf{h}_{\mathrm{H},t}(v) \in H^{1,2}_C(TX) \cap D(\mathrm{div}) \quad \textit{and} \quad \mathrm{div}(\mathsf{h}_{\mathrm{H},t}(v)) = P_t(\mathrm{div}(v)) \quad \textit{for every } t > 0.$$

*Proof.* First of all, observe that $\mathsf{h}_{\mathrm{H},t}(v) \in H^{1,2}_{\mathrm{H}}(TX) \subset H^{1,2}_C(TX)$ by [**87**, Corollary 3.6.4]. Moreover, let $f \in H^{1,2}(X)$ be given. Then it holds that

$$\begin{aligned} \int \langle \nabla f, \mathsf{h}_{\mathrm{H},t}(v) \rangle \, \mathrm{d}\mathfrak{m} &\overset{(5.9)}{=} \int \langle \mathsf{h}_{\mathrm{H},t}(\nabla f), v \rangle \, \mathrm{d}\mathfrak{m} \overset{(5.7)}{=} \int \langle \nabla P_t f, v \rangle \, \mathrm{d}\mathfrak{m} = -\int P_t f \, \mathrm{div}(v) \, \mathrm{d}\mathfrak{m} \\ &= -\int f P_t(\mathrm{div}(v)) \, \mathrm{d}\mathfrak{m}. \end{aligned}$$

By arbitrariness of $f$, we conclude that $\mathsf{h}_{\mathrm{H},t}(v) \in D(\mathrm{div})$ and $\mathrm{div}(\mathsf{h}_{\mathrm{H},t}(v)) = P_t(\mathrm{div}(v))$. □

1.2.2. *Total variation of* BV *functions via integration by parts.* The last ingredient we need is an improvement of the representation formula obtained by Di Marino (see [**74**, Theorem 3.4]) in the special case of $\mathrm{RCD}(K, \infty)$ spaces. As we are going to see in the ensuing result, to obtain the total variation of a BV function it is sufficient to restrict the attention only to those competitors that are Sobolev regular. The proof is based on a parabolic approximation argument that builds upon the technical results developed in Section 1.2.1.

**Theorem 5.14** (Representation formula for $|Df|$ on RCD spaces)**.** *Let $(X, \mathsf{d}, \mathfrak{m})$ be an $\mathrm{RCD}(K, \infty)$ space and $f \in \mathrm{BV}(X)$. Then it holds that*

$$|Df|(X) = \sup \left\{ \int f \, \mathrm{div}(v) \, \mathrm{d}\mathfrak{m} \;\middle|\; v \in H^{1,2}_C(TX) \cap D(\mathrm{div}),\ |v| \leq 1 \ \mathfrak{m}\textit{-a.e.},\ \mathrm{div}(v) \in L^\infty(\mathfrak{m}) \right\}. \tag{5.10}$$

*Proof.* Building upon [**74**, Theorem 3.4] and Proposition 1.56 (recall that we have $b \in \mathrm{Der}^{2,2}(X)$ for every $b \in \mathrm{Der}^{\infty,\infty}(X)$ such that $\mathrm{supp}(b)$ is bounded, thanks to Remark 1.54) we deduce the following identity

(5.11)
$$|Df|(U) = \sup\left\{ \int_U f \mathrm{div}(v)\,\mathrm{d}\mathfrak{m} \;\middle|\; v \in D(\mathrm{div}), |v| \le 1 \ \mathfrak{m}\text{-a.e.},\ \mathrm{div}(v) \in L^\infty(\mathfrak{m}),\ \mathrm{supp}(v) \Subset U \right\}$$

for any $f \in \mathrm{BV}(X, \mathsf{d}, \mathfrak{m})$ and any open set $U \subset X$.

We want to prove (5.10) starting from (5.11) by means of a regularization argument. Call $S$ the right hand side of (5.10). We know by (5.11) that $|Df|(X) \ge S$. In order to prove the converse inequality, fix any $\varepsilon > 0$. The identity (5.11) guarantees the existence of a vector field $v \in D(\mathrm{div})$ – with $|v| \le 1$ in the $\mathfrak{m}$-a.e. sense and $\mathrm{div}(v) \in L^\infty(\mathfrak{m})$ – such that $\int f \,\mathrm{div}(v)\,\mathrm{d}\mathfrak{m} > |Df|(X) - \varepsilon/2$. Now define $v_t := e^{Kt}\,\mathsf{h}_{\mathrm{H},t}(v)$ for every $t > 0$. Notice that $v_t \in H^{1,2}_C(TX) \cap D(\mathrm{div})$ by Proposition 5.13. Since $\mathrm{div}(v) \in L^\infty(\mathfrak{m})$ and $\mathrm{div}(v_t) = e^{Kt} P_t(\mathrm{div}(v))$, we deduce from the weak maximum principle that $\mathrm{div}(v_t) \in L^\infty(\mathfrak{m})$ as well. More precisely, one has $\|\mathrm{div}(v_t)\|_{L^\infty(\mathfrak{m})} \le e^{Kt}\,\|\mathrm{div}(v)\|_{L^\infty(\mathfrak{m})}$ for all $t > 0$. Moreover, the weak maximum principle also guarantees that

$$|v_t| = e^{Kt}\,|\mathsf{h}_{\mathrm{H},t}(v)| \overset{(5.8)}{\le} \sqrt{P_t(|v|^2)} \le 1 \quad \text{in the } \mathfrak{m}\text{-a.e. sense.}$$

Given that $\lim_{t\searrow 0} \mathrm{div}(v_t) = \mathrm{div}(v)$ in $L^2(\mathfrak{m})$, we can find $t_n \searrow 0$ such that $\mathrm{div}(v_{t_n})(x) \to \mathrm{div}(v)(x)$ holds for $\mathfrak{m}$-a.e. $x \in X$. Being $\big(\mathrm{div}(v_{t_n})\big)_n$ a bounded sequence in $L^\infty(\mathfrak{m})$, we can finally conclude that $\lim_n \int f \,\mathrm{div}(v_{t_n})\,\mathrm{d}\mathfrak{m} = \int f \,\mathrm{div}(v)\,\mathrm{d}\mathfrak{m}$ by dominated convergence theorem. Therefore, there exists $n \in \mathbb{N}$ such that $w := v_{t_n}$ satisfies

$$\int f \,\mathrm{div}(w)\,\mathrm{d}\mathfrak{m} > \int f \,\mathrm{div}(v)\,\mathrm{d}\mathfrak{m} - \frac{\varepsilon}{2} > |Df|(X) - \varepsilon.$$

This shows that $|Df|(X) < S + \varepsilon$, whence $|Df|(X) \le S$ by arbitrariness of $\varepsilon$, as desired. □

*Proof of Theorem 5.6.* First of all, let us define $\mu_t := P_t^*|D\chi_E|\mathfrak{m}$ for every $t > 0$. Recall that $\mu_t \rightharpoonup |D\chi_E|$ in duality with $C_b(X)$ as $t \searrow 0$. Let us also set

$$\nu_t := \chi_{\{P_t^*|D\chi_E|>0\}} \frac{\nabla P_t \chi_E}{P_t^*|D\chi_E|} \in L^0(TX) \quad \text{for every } t > 0.$$

It follows from the 1-Bakry-Émery estimate (1.23) that $|DP_t\chi_E| \le e^{-Kt} P_t^*|D\chi_E|$ holds $\mathfrak{m}$-a.e., thus accordingly $\nu_t \in L^\infty(TX)$ and $|\nu_t| \le e^{-Kt}$ is satisfied in the $\mathfrak{m}$-a.e. sense. Call

$$\mathcal{V} := \left\{ v \in H^{1,2}_C(TX) \cap D(\mathrm{div}) \;\middle|\; |v| \in L^\infty(\mathfrak{m}) \right\}$$

and fix $v \in \mathcal{V}$. The Leibniz rule for the divergence ensures that $\phi v \in D(\mathrm{div})$ for any $\phi \in \mathrm{Lip}_b(X)$, so the usual integration-by-parts formula yields

(5.12)
$$\int P_t\chi_E \,\mathrm{div}(\phi v)\,\mathrm{d}\mathfrak{m} = -\int \phi\,\langle \nabla P_t \chi_E, v\rangle\,\mathrm{d}\mathfrak{m} = -\int \phi\,\langle v, \nu_t\rangle\,\mathrm{d}\mu_t \quad \text{for all } \phi \in \mathrm{Lip}_b(X).$$

Moreover, observe that $\langle v, \nu_t\rangle \in L^\infty(\mu_t)$ and $\big\|\langle v, \nu_t\rangle\big\|_{L^\infty(\mu_t)} \le e^{-Kt}\,\||v|\|_{L^\infty(\mathfrak{m})}$ for every $t > 0$. Let us call $\sigma_t := \langle v, \nu_t\rangle \mu_t$ for all $t > 0$. Fix any sequence $t_n \searrow 0$. Since $\mu_{t_n} \rightharpoonup |D\chi_E|$ in duality with $C_b(X)$, we know that $(\mu_{t_n})_n$ is tight by Prohkorov theorem. Given that $\sup_n \|\langle v, \nu_{t_n}\rangle\|_{L^\infty(\mu_{t_n})}$ is finite, we deduce that $(\sigma_{t_n})_n$ is tight as well. By using Prohkorov theorem again, we can thus take a subsequence $(t_{n_i})_i$ such that $\sigma_{t_{n_i}} \rightharpoonup \sigma$ in duality with $C_b(X)$ for some finite (signed) Borel measure $\sigma$ on $X$. Since $\mathrm{Lip}_b(X)$ is dense in $C_b(X)$ and the identity in (5.12) gives

$$\int \phi \, \mathrm{d}\sigma = \lim_{i\to\infty} \int \phi \, \mathrm{d}\sigma_{t_{n_i}} = -\int_E \mathrm{div}(\phi v) \, \mathrm{d}\mathfrak{m} \quad \text{for every } \phi \in \mathrm{Lip}_b(X),$$

we see that $\sigma$ is independent of the chosen sequence $(t_{n_i})_i$. Hence, $\sigma_t \rightharpoonup \sigma$ in duality with $C_b(X)$ as $t \searrow 0$. Given any non-negative function $\phi \in C_b(X)$, it thus holds that

$$\left| \int \phi \, \mathrm{d}\sigma \right| \leq \lim_{t\searrow 0} \int \phi \, |\langle v, \nu_t \rangle| \, \mathrm{d}\mu_t \leq e^{|K|} \, \| |v| \|_{L^\infty(\mathfrak{m})} \lim_{t\searrow 0} \int \phi \, \mathrm{d}\mu_t = e^{|K|} \, \| |v| \|_{L^\infty(\mathfrak{m})} \int \phi \, \mathrm{d}|D\chi_E|,$$

whence $\sigma \ll |D\chi_E|$ and its Radon-Nikodým derivative $L(v) := \frac{\mathrm{d}\sigma}{\mathrm{d}|D\chi_E|}$ belongs to $L^\infty(|D\chi_E|)$. Consequently, taking into account (5.12) we deduce that

$$\int_E \mathrm{div}(\phi v) \, \mathrm{d}\mathfrak{m} = -\int \phi \, L(v) \, \mathrm{d}|D\chi_E| \quad \text{for every } v \in \mathcal{V} \text{ and } \phi \in \mathrm{Lip}_b(X). \tag{5.13}$$

Furthermore, one also has that

$$\lim_{t\searrow 0} \int \phi \, \langle v, \nu_t \rangle \, \mathrm{d}\mu_t = \int \phi \, L(v) \, \mathrm{d}|D\chi_E| \quad \text{for every } v \in \mathcal{V} \text{ and } \phi \in \mathrm{Lip}_b(X). \tag{5.14}$$

Observe that for any $v \in \mathcal{V}$ and $\phi \in \mathrm{Lip}_b(X)$, $\phi \geq 0$ it holds that

$$\begin{aligned}
\left| \int \phi \, L(v) \, \mathrm{d}|D\chi_E| \right| &\overset{(5.14)}{=} \lim_{t\searrow 0} \left| e^{Kt} \int \phi \, \langle v, \nu_t \rangle \, \mathrm{d}\mu_t \right| \\
&\leq \lim_{t\searrow 0} \left( \|\phi\|_{L^\infty(\mathfrak{m})} \, \| |v| \|_{L^\infty(\mathfrak{m})} \int \left| 1 - e^{Kt} |\nu_t| \right| \mathrm{d}\mu_t + \int \phi \left\langle v, \frac{\nu_t}{|\nu_t|} \right\rangle \mathrm{d}\mu_t \right) \\
&\overset{(5.2)}{\leq} \lim_{t\searrow 0} \int \phi \, |v| \, \mathrm{d}\mu_t \overset{(5.3)}{=} \lim_{t\searrow 0} \int \mathrm{tr}_E \left( P_t(\phi|v|) \right) \mathrm{d}|D\chi_E| \\
&\overset{(5.4)}{=} \int \phi \, \mathrm{tr}_E(|v|) \, \mathrm{d}|D\chi_E|.
\end{aligned}$$

In the last two equalities we used the fact that $|v| \in H^{1,2}(X)$. By arbitrariness of $\phi$, we obtain that $|L(v)| \leq \mathrm{tr}_E(|v|)$ holds $|D\chi_E|$-a.e. for all $v \in \mathcal{V}$. Let us now define $\omega : \mathrm{tr}_E(\mathcal{V}) \to L^1(|D\chi_E|)$ as

$$\omega\Big( \mathrm{tr}_E(v) \Big) := L(v) \quad \text{for every } v \in \mathcal{V}. \tag{5.15}$$

The operator $L : \mathcal{V} \to L^\infty(|D\chi_E|)$ is linear by its very construction, whence by exploiting the inequality $|L(v)| \leq \mathrm{tr}_E(|v|)$ we can conclude that $\omega$ is well-posed, linear and satisfying

$$\Big| \omega(v) \Big| \leq |v| \quad |D\chi_E|\text{-a.e.} \quad \text{for every } v \in \mathrm{tr}_E(\mathcal{V}).$$

Since $\mathrm{TestV}(X) \subset \mathcal{V}$ and $\mathrm{TestV}_E(X)$ is dense in $L^2_E(TX)$, we infer from Lemma 5.11 that $\bar{\mathrm{tr}}_E(\mathcal{V})$ is a dense linear subspace of $L^2_E(TX)$. Therefore, we know from [**87**, Proposition 1.4.8] that $\omega$ can be uniquely extended to an element $\omega \in L^2_E(T^*X) := L^2_E(TX)^*$ satisfying $|\omega| \leq 1$ in the $|D\chi_E|$-a.e. sense. We denote by $\nu_E \in L^2_E(TX)$ the vector field corresponding to $\omega$ via the Riesz isomorphism. By combining (5.13) (with $\phi \equiv 1$) and (5.15), we conclude that (5.1) is satisfied. It only remains to show that $|\nu_E| \geq 1$ holds $|D\chi_E|$-a.e.. In order to do it, just observe that Theorem 5.14 yields

$$\begin{aligned}|D\chi_E|(X) &\leq \sup_{\substack{v\in\mathcal{V},\\ |v|\leq 1\ \mathfrak{m}\text{-a.e.}}} \int_E \operatorname{div}(v)\, \mathrm{d}\mathfrak{m} \overset{(5.1)}{=} \sup_{\substack{v\in\mathcal{V},\\ |v|\leq 1\ \mathfrak{m}\text{-a.e.}}} -\int \Big\langle \operatorname{tr}_E(v), \nu_E \Big\rangle\, \mathrm{d}|D\chi_E| \leq \int |\nu_E|\, \mathrm{d}|D\chi_E| \\ &\leq |D\chi_E|(X),\end{aligned}$$

whence each inequality must be an equality. This clearly forces the $|D\chi_E|$-a.e. equality $|\nu_E| = 1$. The element $\nu_E$ is uniquely determined by (5.1) as the space $\operatorname{tr}_E(\mathcal{V})$ is dense in $L^2_E(TX)$. Finally, the last part of the statement is an immediate consequence of Lemma 5.11. □

## 2. Uniqueness of tangents for sets of finite perimeter

In this section we prove a uniqueness theorem (up to negligible sets) for blow-ups of sets with finite perimeter over $\mathrm{RCD}(K, N)$ metric measure spaces. We refer to Definition 4.30 for the notion of tangent to a set of finite perimeter.

**Theorem 5.15.** *Let $(X, \mathsf{d}, \mathfrak{m})$ be an $\mathrm{RCD}(K, N)$ m.m.s. with essential dimension $1 \leq n \leq N$, $E \subset X$ be a set of finite perimeter. Then, for $|D\chi_E|$-a.e. $x \in X$, there exists $k = 1, \dots, n$ such that*

$$\operatorname{Tan}_x(X, \mathsf{d}, \mathfrak{m}, E) = \Big\{(\mathbb{R}^k, \mathsf{d}_{Eucl}, c_k\mathscr{L}^k, 0^k, \{x_k > 0\})\Big\}.$$

Let us explain the strategy of its proof. The starting point of our analysis is Theorem 4.32 that we restate below for the reader's convenience.

**Theorem 5.16.** *Let $(X, \mathsf{d}, \mathfrak{m})$ be an $\mathrm{RCD}(K, N)$ m.m.s. and $E \subset X$ be a set of locally finite perimeter. Then $E$ admits a Euclidean half-space as tangent at $x$ for $|D\chi_E|$-a.e. $x \in X$, that is to say*

$$\Big(\mathbb{R}^k, \mathsf{d}_{Eucl}, c_k\mathscr{L}^k, 0^k, \{x_k > 0\}\Big) \in \operatorname{Tan}_x(X, \mathsf{d}, \mathfrak{m}, E), \qquad \textit{for some } k \in [1, N].$$

The existence of a Euclidean half space along a fixed scale is a regularity information which can be propagated at any location and scale up to a set which is small with respect to the perimeter measure, yielding uniqueness of tangents. From a technical point of view, our construction heavily relies on the use of *harmonic $\delta$-splitting maps* already introduced in Section 7.1. The *propagation of regularity step* is a consequence of a weighted maximal argument which was suggested in [**57**]. Let us point out that, in order for the whole procedure to work, the fact that perimeter measures have codimension-1 (see Lemma 5.1) and the fact that harmonic functions satisfy $L^2$ Hessian bounds play a key role. The strategy would completely fail if perimeter measures had codimension bigger than or equal to 2.

**2.1. Propagation of the $\delta$-splitting property.** In the next result we are concerned with the propagation of the property of being a $\delta$-splitting map. We are going to prove that, if $\alpha \in (0, 2)$, outside a set of small codimension-$\alpha$ content any $\delta$-splitting map at a given scale is a $C_{N,\alpha}\delta^{1/4}$ splitting map at any scale. The proof is based on a weighted maximal function argument.

**Proposition 5.17.** *Let $\alpha \in (0, 2)$ and $N > 1$. There exist constants $C_N > 0$ and $C_{N,\alpha} > 0$ such that, for any $0 < \delta < 1$, any $\mathrm{RCD}(-1, N)$ m.m.s. $(X, \mathsf{d}, \mathfrak{m})$, any $p \in X$ and for any*

*$\delta$-splitting map $u := (u_1, \dots, u_k) : B_2(p) \to \mathbb{R}^k$, there exists a Borel set $G \subset B_1(p)$ with $\mathscr{H}_5^{h_\alpha}(B_1(p) \setminus G) < C_N \sqrt{\delta}\mathfrak{m}(B_2(p))$ such that for any $x \in G$ it holds*

$$\sup_{0<r<1} r^\alpha \fint_{B_r(x)} |\operatorname{Hess} u_a|^2 \, d\mathfrak{m} \le \sqrt{\delta} \qquad \text{for any} \quad a = 1, \dots, k, \tag{5.16}$$

*and*

$$u : B_r(x) \to \mathbb{R}^k \qquad \text{is a } C_{N,\alpha}\delta^{1/4}\text{-splitting map for any } 0 < r < 1/2. \tag{5.17}$$

*Proof.* Let us start proving (5.16). To this aim fix any $a = 1, \dots, k$ and denote by $C_P$ and $C_D$ the Poincaré and the doubling constants over balls of radius 10 of $(X, \mathsf{d}, \mathfrak{m})$. To be more precise $C_P$ is a constant in the $(1,2)$-Poincaré inequality with $\lambda = 2$ as in (1.41). This inequality is available on $\mathrm{RCD}(K,N)$ m.m.s. with constant depending only on $K$ and $N$. In particular, since $(X, \mathsf{d}, \mathfrak{m})$ is an $\mathrm{RCD}(-1,N)$, $C_P$ depends only on $N$. The same conclusion holds for $C_D$ thanks to the Bishop-Gromov inequality (1.13).

Set

$$G := \left\{ x \in B_1(p) : \sup_{0<r<1} r^\alpha \fint_{B_r(x)} |\operatorname{Hess} u_a|^2 \, d\mathfrak{m} \le \sqrt{\delta} \right\}.$$

We claim that $\mathscr{H}_5^{h_\alpha}(B_1(p) \setminus G) < C_N \sqrt{\delta}\mathfrak{m}(B_2(p))$. For any $x \in B_1(p) \setminus G$ we choose $\rho_x \in (0,1)$ satisfying

$$\rho_x^\alpha \fint_{B_{\rho_x}(x)} |\operatorname{Hess} u_a|^2 \, d\mathfrak{m} > \sqrt{\delta}. \tag{5.18}$$

Observe that the family $\{B_{\rho_x}(x)\}_{x \in B_1(p) \setminus G}$ covers $B_1(p) \setminus G$. Using Vitali's covering lemma we can find a subfamily of disjoint balls $\{B_{\rho_i}(x_i)\}_{i \in \mathbb{N}}$ such that $B_1(p) \setminus G \subset \cup_{i \in \mathbb{N}} B_{5\rho_i}(x_i)$. This gives the sought conclusion

$$\begin{aligned}
\mathscr{H}_5^{h_\alpha}(B_1(p) \setminus G) \le & \sum_{i \in \mathbb{N}} h_\alpha(B_{5\rho_i}(x_i)) = \sum_{i \in \mathbb{N}} \frac{\mathfrak{m}(B_{5\rho_i}(x_i))}{(5\rho_i)^\alpha} \\
\le & C_N \sum_{i \in \mathbb{N}} \frac{\mathfrak{m}(B_{\rho_i}(x_i))}{\rho_i^\alpha} \le C_N \sum_{i \in \mathbb{N}} \frac{1}{\sqrt{\delta}} \int_{B_{\rho_i}(x_i)} |\operatorname{Hess} u_a|^2 \, d\mathfrak{m} \\
\le & C_N \frac{1}{\sqrt{\delta}} \int_{B_2(p)} |\operatorname{Hess} u_a|^2 \, d\mathfrak{m} \le C_N \sqrt{\delta}\mathfrak{m}(B_2(p)),
\end{aligned}$$

where we used the definition of $\mathscr{H}_5^{h_\alpha}$, the Bishop-Gromov inequality, (5.18) and the fact that $u$ is a $\delta$-splitting map.

In order to verify (5.17) we just need to check that, for $a, b = 1, \dots, k$,

$$\fint_{B_r(x)} |\nabla u_a \cdot \nabla u_b - \delta_{a,b}| \, d\mathfrak{m} < C_{N,\alpha}\delta^{1/4} \quad \text{for any } x \in G, \ 0 < r < 1.$$

To this aim let us set $f_{a,b} := |\nabla u_a \cdot \nabla u_b - \delta_{a,b}|$ and note that $|\nabla f_{a,b}| \le C_N(|\operatorname{Hess} u_a| + |\operatorname{Hess} u_b|)$ as a consequence of Definition 1.80(i) and (1.37). Whence, the Poincaré inequality and (5.16) yield

$$\begin{aligned}
\left| \fint_{B_r(x)} f_{a,b} \, d\mathfrak{m} - \fint_{B_{r/2}(x)} f_{a,b} \, d\mathfrak{m} \right| \le & C_P r \left( \fint_{B_{2r}(x)} |\nabla f_{a,b}|^2 \, d\mathfrak{m} \right)^{1/2} \\
\le & C_N C_P \left( r^2 \fint_{B_{2r}(x)} |\operatorname{Hess} u_a|^2 \, d\mathfrak{m} + r^2 \fint_{B_{2r}(x)} |\operatorname{Hess} u_b|^2 \, d\mathfrak{m} \right)^{1/2}
\end{aligned}$$

$$\leq C_N C_P \delta^{1/4} r^{1-\alpha/2}$$

for any $0 < r < 1/2$. Applying a telescopic argument it is simple to see that

$$\left| \fint_{B_{2^{-1}}(x)} f_{a,b} \, \mathrm{d}\mathfrak{m} - \fint_{B_{2^{-k}}(x)} f_{a,b} \, \mathrm{d}\mathfrak{m} \right| \leq C_\alpha C_N C_P \delta^{1/4}, \qquad \text{for any } k > 1.$$

Therefore, for any $0 < r < 1/2$ we take $k \in \mathbb{N}$ such that $2^{-k-1} < r \leq 2^{-k}$ and using that $u : B_2(p) \to \mathbb{R}^k$ is a $\delta$-splitting map we get

$$\begin{aligned}
\fint_{B_r(x)} f_{a,b} \, \mathrm{d}\mathfrak{m} \leq & C_D 2^N \fint_{B_{2^{-k}}(x)} f_{a,b} \, \mathrm{d}\mathfrak{m} \\
\leq & C_D 2^N \left| \fint_{B_{1/2}(x)} f_{a,b} \, \mathrm{d}\mathfrak{m} - \fint_{B_r(x)} f_{a,b} \, \mathrm{d}\mathfrak{m} \right| + C_D 2^N \fint_{B_{1/2}(x)} f_{a,b} \, \mathrm{d}\mathfrak{m} \\
\leq & 2^N C_D C_\alpha C_N C_P \delta^{1/4} + 8^N C_D^2 \fint_{B_2(p)} f_{a,b} \, \mathrm{d}\mathfrak{m} \\
\leq & C_{N,\alpha} \delta^{1/4}.
\end{aligned}$$

□

For the purposes of this thesis we just need to consider the case $\alpha = 1$ in Proposition 5.17. This is related to the fact that boundaries of sets with finite perimeter are codimension-1 objects. In order to ease notation in the sequel we will write $h$ in place of $h_1$.

We are going to use several times the following scale invariant version of Proposition 5.17.

**Corollary 5.18.** *Let $(X, \mathsf{d}, \mathfrak{m}, p)$ be an $\mathrm{RCD}(K, N)$ p.m.m.s. and $u : B_{4r}(p) \to \mathbb{R}^k$ a $\delta$-splitting map for some $r > 0$ such that $|K| r^2 \leq 4$ and $r < 1/2$. Then there exists $G \subset B_{2r}(p)$ with*

$$\mathscr{H}^h_5(B_{2r}(p) \setminus G) \leq \mathscr{H}^h_{10r}(B_{2r}(p) \setminus G) \leq C_N \sqrt{\delta} \frac{\mathfrak{m}(B_{2r}(p))}{2r}$$

*such that $u : B_s(x) \to \mathbb{R}^k$ is a $C_N \delta^{1/4}$-splitting map for any $x \in G$ and any $0 < s < r$.*

*Proof.* Apply Proposition 5.17 to the rescaled space $(X, (2r)^{-1}\mathsf{d}, \mathfrak{m}(B_{2r}(p))^{-1}\mathfrak{m}, p)$. □

**2.2. Uniqueness of tangents and consequences.** Let $(X, \mathsf{d}, \mathfrak{m})$ be an $\mathrm{RCD}(K, N)$ metric measure space with essential dimension $n \leq N$ (Cf. Theorem 3.1) and let $E \subset X$ be a set of locally finite perimeter. For any $k = 1, \dots, n$ we define $A_k \subset X$ as

$$\Big\{ x \in X \ : \ \Big( \mathbb{R}^k, \mathsf{d}_{\mathrm{Eucl}}, c_k \mathscr{L}^k, 0^k, \{x_k > 0\} \Big) \in \mathrm{Tan}_x(X, \mathsf{d}, \mathfrak{m}, E), \text{ but for no } (Y, \varrho, \mu, y) \text{ with}$$
$$\mathrm{diam}(Y) > 0 \text{ it holds } (Y \times \mathbb{R}^k, \varrho \times \mathsf{d}_{\mathrm{Eucl}}, \mu \times \mathscr{L}^k, (y, 0^k), \{x_k > 0\}) \in \mathrm{Tan}_x(X, \mathsf{d}, \mathfrak{m}, E) \Big\}.$$

With arguments analogous to those in [**121**, Lemma 6.1] one can show that $A_k$ is a $|D\chi_E|$-measurable set for any $k = 1, \dots, n$.

**Lemma 5.19.** *Under the assumptions above*

$$|D\chi_E| \left( X \setminus \bigcup_{k=1}^n A_k \right) = 0.$$

*Proof.* As a consequence of Theorem 4.32 we have

$$|D\chi_E| \left( X \setminus \bigcup_{k=1}^n A'_k \right) = 0,$$

where

$$A'_k := \Big\{x \in X : \Big(\mathbb{R}^k, \mathsf{d}_{\rm Eucl}, c_k\mathscr{L}^k, 0^k, \{x_k > 0\}\Big) \in \mathrm{Tan}_x(X, \mathsf{d}, \mathfrak{m}, E) \text{ but } (\mathbb{R}^m, \mathsf{d}_{\rm Eucl}, c_m\mathscr{L}^m, 0^m, \{x_m > 0\}) \notin \mathrm{Tan}_x(X, \mathsf{d}, \mathfrak{m}, E) \text{ for any } m > k\Big\}.$$

The measurability of the $A'_k$'s can be verified as in the case of the $A_k$'s.

It is clear that $A_k \subset A'_k$, let us prove $|D\chi_E|(A'_k \setminus A_k) = 0$. We argue by contradiction. If the claim is false we can find $x \in A'_k \setminus A_k$ such that the iterated tangent property of Theorem 4.38 holds true. Since $x \in A'_k \setminus A_k$ we can find $(Y, \varrho, \mu, y) \in \mathrm{RCD}(0, N-k)$ with $\mathrm{diam}(Y) > 0$ such that

$$(Y \times \mathbb{R}^k, \varrho \times \mathsf{d}_{\rm Eucl}, \mu \times \mathscr{L}^k, (y, 0^k), \{x_k > 0\}) \in \mathrm{Tan}_x(X, \mathsf{d}, \mathfrak{m}, E).$$

Moreover $\mathrm{Tan}_{(y',x,0)}(Y \times \mathbb{R}^k, \varrho \times \mathsf{d}_{\rm Eucl}, \mu \times \mathscr{L}^k, \{x_k > 0\}) \subset \mathrm{Tan}(E, x)$ for any $(y', x) \in Y \times \mathbb{R}^{k-1}$, thanks to Theorem 4.38. Thus, choosing $(y', x, 0) \in Y \times \mathbb{R}^k$ such that Theorem 4.32 holds and $y'$ is regular in $Y$ we get the sought contradiction, since the essential dimension of $Y$ is bigger or equal than one (otherwise $\mathrm{diam}(Y) = 0$). $\square$

We are now in a position to conclude the proof of Theorem 5.15.

*Proof of Theorem 5.15.* In light of Lemma 5.19 it is enough to prove that $A_k$ coincides up to a $|D\chi_E|$-negligible set with

$$\Big\{x \in X : \ \mathrm{Tan}_x(X, \mathsf{d}, \mathfrak{m}, E) = \Big\{(\mathbb{R}^k, \mathsf{d}_{\rm Eucl}, c_k\mathscr{L}^k, 0^k, \{x_k > 0\})\Big\}\Big\}.$$

Let us assume without loss of generality that $A_k \subset B_2(p)$ for some $p \in X$. We claim that, for any $\eta > 0$, there exists $G^\eta \subset A_k$ with

$$\mathscr{H}^h_5(A_k \setminus G^\eta) \le C_N \eta \,\mathrm{Per}(E, B_2(p)) \tag{5.19}$$

such that, for any $x \in G^\eta$ and for any $(Y, \varrho, \mu, y) \in \mathrm{Tan}_x(X, \mathsf{d}, \mathfrak{m})$, there exists a pointed $\mathrm{RCD}(0, N-k)$ m.m.s. $(Z, \mathsf{d}_Z, \mathfrak{m}_Z, z)$ satisfying

$$\mathsf{d}_{pmGH}((Y, \varrho, \mu, y), (\mathbb{R}^k \times Z, (0, z)) \le \eta. \tag{5.20}$$

Observe that the claim implies our conclusion. Indeed if we fix $\eta > 0$ and set $\eta_i := \eta 2^{-i}$ then $G_\eta := \cup_{i\in\mathbb{N}} G^{\eta_i}$ satisfies $\mathscr{H}^h_5(A_k \setminus G_\eta) = 0$ and thus $\mathrm{Per}(E, A_k \setminus G_\eta) = 0$ thanks to Lemma 5.1. Moreover, for any $x \in G_\eta$, (5.20) holds. We conclude observing that $G := \cap_{k\in\mathbb{N}} G_{2^{-k}}$ still satisfies $\mathrm{Per}(E, A_k \setminus G) = 0$ and any tangent cone at $x \in G$ splits off a factor $\mathbb{R}^k$. By definition of $A_k$ we deduce that the only tangent at $x \in G$ is the Euclidean space of dimension $k$.

Let us pass to the verification of the claim. Fix $\delta \in (0, 1/2)$ and take $\varepsilon > 0$ as in Proposition 1.85. Of course we can assume $\varepsilon \le \delta$. We wish to prove that there exists a disjoint family of balls $\{B_{r_i}(x_i)\}_{i\in\mathbb{N}}$ such that $r_i^2|K| \le \varepsilon$ for any $i \in \mathbb{N}$ and

(i) $A_k \cap B_1(p) \subset \cup_{i\in\mathbb{N}} B_{5r_i}(x_i)$;
(ii) $\mathsf{d}_{pmGH}\Big((X, r_i^{-1}\mathsf{d}, \mathfrak{m}_x^{r_i}, x_i), (\mathbb{R}^k, \mathsf{d}_{\rm Eucl}, c_k\mathscr{L}^k, 0^k)\Big) \le \varepsilon$;
(iii) $\frac{\omega_{k-1}}{\omega_k}(1-\varepsilon)\frac{\mathfrak{m}(B_{r_i}(x_i))}{r_i} \le \mathrm{Per}(E, B_{r_i}(x_i)) \le \frac{\omega_{k-1}}{\omega_k}(1+\varepsilon)\frac{\mathfrak{m}(B_{r_i}(x_i))}{r_i}$.

Indeed, for any $x \in A_k$ there exists a sequence of radii $r_i \to 0$ such that

$$\lim_{i\to\infty} \mathsf{d}_{pmGH}\Big((X, r_i^{-1}\mathsf{d}, \mathfrak{m}_x^{r_i}, x), (\mathbb{R}^k, \mathsf{d}_{\rm Eucl}, \mathscr{L}^k, 0^k)\Big) = 0 \quad \text{and} \quad \lim_{i\to\infty} \frac{r_i \,\mathrm{Per}(E, B_{r_i}(x))}{\mathfrak{m}(B_{r_i}(x))} = \frac{\omega_{k-1}}{\omega_k},$$

as a consequence of Theorem 4.32, see also (5.25). Therefore, for any $x \in A_k$ we can choose $r_x^2|K| \le \varepsilon$ such that the pair $(x, r_x)$ satisfies (ii) and (iii). In order to get a disjoint family of balls satisfying (i) we have just to apply Vitali's Lemma to $\{B_{r_x}(x)\}_{x\in A_k\cap B_1(p)}$.

Let us now focus the attention on a single ball $B_{20r_i}(x_i) \subset X$. Corollary 1.86 yields the existence of a $\delta$-splitting map

$$u^i : B_{5r_i}(x_i) \to \mathbb{R}^k.$$

Thanks to Corollary 5.18 we can find $G_i \subset B_{5r_i}(x_i)$ with

$$\mathscr{H}_5^h(B_{5r_i}(x_i) \setminus G_i) \le C_N\sqrt{\delta}\frac{\mathfrak{m}(B_{5r_i}(x_i))}{5r_i} \tag{5.21}$$

and such that $u^i : B_s(x) \to \mathbb{R}^k$ is a $C_N\delta^{1/4}$-splitting map for any $x \in G_i$ and any $0 < s < 5r_i$. Applying Corollary 1.84, up to assuming $\delta$ small enough, we deduce that at any $x \in G_i$ (5.20) holds true.

To conclude let us verify that $G := \cup_{i\in\mathbb{N}} G_i$ satisfies (5.19). Using (iii), (5.21) and the Bishop-Gromov inequality (1.15) we get

$$\begin{aligned}
\mathscr{H}_5^h(A_k \setminus G) \le & \sum_{i\in\mathbb{N}} \mathscr{H}_5^{h_1}(B_{5r_i}(x_i) \setminus G_i) \le \sum_{i\in\mathbb{N}} C_N\sqrt{\delta}\frac{\mathfrak{m}(B_{5r_i}(x_i))}{5r_i} \\
\le & C_N\sqrt{\delta}\sum_{i\in\mathbb{N}}\frac{\mathfrak{m}(B_{r_i}(x_i))}{r_i} \le C_N\sqrt{\delta}\sum_{i\in\mathbb{N}}\mathrm{Per}(E, B_{r_i}(x_i)) \\
\le & C_N\sqrt{\delta}\,\mathrm{Per}(E, B_2(p)).
\end{aligned}$$

Since we can assume $\delta < \eta^2$ we get the sought estimate. ☐

Let $(X, \mathsf{d}, \mathfrak{m})$ be an $\mathrm{RCD}(K, N)$ metric measure space and $E \subset X$ a set of locally finite perimeter. For any $k = 1, \dots, n$, where $n$ is the essential dimension of $(X, \mathsf{d}, \mathfrak{m})$, we set

$$\mathcal{F}_k E := \left\{x \in X \ : \ \mathrm{Tan}_x(X, \mathsf{d}, \mathfrak{m}, E) = \left\{(\mathbb{R}^k, \mathsf{d}_{\mathrm{Eucl}}, c_k\mathscr{L}^k, 0^k, \{x_k > 0\})\right\}\right\}.$$

We know thanks to Theorem 5.15 that $\mathrm{Per}(E,\cdot)$ is concentrated on $\mathcal{F}E := \cup_{k=1}^n \mathcal{F}_k E$ and, from now on, we shall call $\mathcal{F}E$ the reduced boundary of $E$.

The result about uniqueness of tangents that we just proved allows to obtain a representation formula for the perimeter measure in terms of the codimension-1 Hausdorff measure.

**Corollary 5.20.** *Let $(X, \mathsf{d}, \mathfrak{m})$ be an $\mathrm{RCD}(K, N)$ m.m.s. with essential dimension $n$. Let $E \subset X$ be a set of locally finite perimeter. Then*

$$|D\chi_E| = \sum_{k=1}^n \frac{\omega_{k-1}}{\omega_k}\mathscr{H}^h \llcorner \mathcal{F}_k E. \tag{5.22}$$

*Proof.* Let us begin by proving the identity

$$\lim_{r\to 0}\sup_{x\in B_s(y),\ s\le r}\frac{s\,|D\chi_E|\,(B_s(y))}{\mathfrak{m}(B_s(y))} = \frac{\omega_{k-1}}{\omega_k} \quad \text{for any } x \in \mathcal{F}_k E. \tag{5.23}$$

First notice that, for $x \in \mathcal{F}_k E$, we have

$$\begin{aligned}
\lim_{r\to 0}\frac{r\,|D\chi_E|\,(B_r(x))}{\mathfrak{m}(B_r(x))} = & \lim_{r\to 0}\frac{r\,|D\chi_E|\,(B_r(x))}{C(x,r)}\cdot\frac{C(x,r)}{\mathfrak{m}(B_r(x))} = \lim_{r\to 0}\frac{|D^r\chi_E|\,(B_1(x))}{\mathfrak{m}_x^r(B_1(x))} \\
= & \frac{\mathscr{H}^{k-1}(B_1(0))}{\mathscr{H}^k(B_1(0))} = \frac{\omega_{k-1}}{\omega_k},
\end{aligned}$$

where $C(x,r)$ is introduced in Definition 2.1, and the weak convergence of the rescaled perimeter measures to the perimeter measure of a half-space play a role. This yields the inequality

$$\lim_{r\to 0}\sup_{x\in B_s(y),\ s\le r}\frac{s\,|D\chi_E|\,(B_s(y))}{\mathfrak{m}(B_s(y))}\ge\frac{\omega_{k-1}}{\omega_k}\quad\text{for any } x\in\mathcal{F}_kE.$$

Let us now prove the converse inequality. It suffices to show that, for any sequence of radii $r_i\to 0$ and points $x_i\in B_{r_i}(x)$, it holds

$$\limsup_{i\to\infty}\frac{r_i|D\chi_E|(B_{r_i}(x_i))}{\mathfrak{m}(B_{r_i}(x_i))}\le\frac{\omega_{k-1}}{\omega_k}.$$

We consider the sequence $(X,\mathsf{d}r_i^{-1},C(x,r_i)^{-1}\mathfrak{m},x)$, that converges to $(\mathbb{R}^k,\mathsf{d}_{\mathrm{Eucl}},c_k\mathscr{L}^k,0^k)$, and we assume $x_i\to z\in\overline{B}_1(0^k)$. Arguing as above we write

$$\begin{aligned}\limsup_{i\to\infty}\frac{r_i\,|D\chi_E|\,(B_{r_i}(x_i))}{\mathfrak{m}(B_{r_i}(x_i))}&=\limsup_{i\to\infty}\frac{r_i\,|D\chi_E|\,(B_{r_i}(x_i))}{C(x_i,r_i)}\cdot\frac{C(x_i,r_i)}{\mathfrak{m}(B_{r_i}(x_i))}\\&=\lim_{i\to\infty}\frac{|D^{r_i}\chi_E|\,(B_1(x_i))}{\mathfrak{m}^{r_i}_{x_i}(B_1(x_i))}\le\frac{\left|D\chi_{\{x_n>0\}}\right|(\overline{B}_1(z))}{\omega_k}\le\frac{\omega_{k-1}}{\omega_k}.\end{aligned}$$

Having (5.23) at hand the stated conclusion follows from [**118**, Theorem 3].

□

## 3. Rectifiability of the reduced boundary

The main achievement of this section is the rectifiability result for the reduced boundary of sets with finite perimeter. With this theorem we complete the picture about the generalization of De Giorgi's theorem to the framework of $\mathrm{RCD}(K,N)$ spaces.

**Theorem 5.21.** *Let $(X,\mathsf{d},\mathfrak{m})$ be an $\mathrm{RCD}(K,N)$ m.m.s. and $E\subset X$ be a set of locally finite perimeter. Then, for any $k=1,\dots,n$, $\mathcal{F}_kE$ is $\Big(|D\chi_E|\,,(k-1)\Big)$-rectifiable.*

Let us recall that a set is $\Big(|D\chi_E|\,,\ell\Big)$-rectifiable if up to a $|D\chi_E|$-negligible set it can be covered by $\cup_{i\in\mathbb{N}}A_i$ where any $A_i$ is bi-Lipschitz equivalent to a Borel subset of $\mathbb{R}^\ell$.

**Remark 5.22.** We point out that, given any $\varepsilon>0$, the maps providing rectifiability of the reduced boundary in Theorem 5.21 can be taken $(1+\varepsilon)$-bi-Lipschitz.

**Remark 5.23.** It is worth mentioning that Theorem 5.21 is stronger than Theorem 2.11. Indeed, given an $\mathrm{RCD}(K,N)$ m.m.s. $(Z,\mathsf{d}_Z,\mathfrak{m}_Z)$ we can consider $X:=Z\times\mathbb{R}$ endowed with the product structure, and the set of finite perimeter $E:=\{(z,t)\in Z\times\mathbb{R}\,:\,t>0\}$. Applying Theorem 5.21 to $E\subset X$ we get the rectifiability result for $Z$.

Let us outline the strategy of proof of Theorem 5.21. First of all, up to intersecting with a ball and thanks to the locality of perimeter and tangents, we can assume that $E$ has finite measure and perimeter. The bi-Lipschitz maps from subsets of $\mathcal{F}_kE$ to $\mathbb{R}^{k-1}$ providing rectifiability are going to be *suitable approximations* of the $(k-1)$ coordinate maps over the hyperplane where the perimeter concentrates after the blow-up. Better said, they will be the first $(k-1)$ components of a $(k,\delta)$-splitting map "$\delta$-orthogonal to the exterior normal $\nu_E$ to the boundary of $E$". Proving existence of these maps requires some technical work which builds upon the Gauss–Green formula Theorem 5.6. The rigorous statement is as follows.

**Proposition 5.24.** *Let $(X,\mathsf{d},\mathfrak{m})$ be an $\mathrm{RCD}(K,N)$ m.m. space and $E\subset X$ a set of finite perimeter and measure. For any $\delta>0$, $r_0>0$ and $|D\chi_E|$-a.e. $x\in\mathcal{F}_kE$ there exist $r=r_{x,\delta}<r_0$ and a $\delta$-splitting map $u=(u_1,\dots,u_{k-1}):B_r(x)\to\mathbb{R}^{k-1}$ such that*

$$\frac{r}{\mathfrak{m}(B_r(x))}\int_{B_r(x)}|\nu_E\cdot\nabla u_\alpha|\,\mathrm{d}|D\chi_E|<\delta,\quad \textit{for } \alpha=1,\dots,k-1.$$

The second step in the proof of Theorem 5.21 is showing that the map built in Proposition 5.24 is indeed bi-Lipschitz with its image if restricted to suitable subsets of $\mathcal{F}_kE$ (see Proposition 5.27 below for the rigorous statement). These subsets are obtained by collecting points $x\in\mathcal{F}_kE$ such that $B_s(x)\cap E$ is $\varepsilon$-close, in a suitable sense, to $B_s(0^k)\cap\{x_k>0\}$ for any $s\le r_0$, where $r_0>0$ is a fixed radius.

**Definition 5.25.** Given $\varepsilon>0$ and $r_0>0$, we define $(\mathcal{F}_kE)_{r_0,\varepsilon}$ as the set of points $x\in\mathcal{F}_kE$ satisfying

(i) $\mathsf{d}_{pmGH}\left(\left(X,s^{-1}\mathsf{d},\frac{\mathfrak{m}}{\mathfrak{m}(B_s(x))},x\right),\left(\mathbb{R}^k,\mathsf{d}_{\mathrm{Eucl}},\frac{1}{\omega_k}\mathscr{L}^k,0^k\right)\right)<\varepsilon$ for any $s\le r_0$;

(ii)

$$\left|\frac{\mathfrak{m}(B_s(x)\cap E)}{\mathfrak{m}(B_s(x))}-\frac{1}{2}\right|+\left|\frac{s\,|D\chi_E|\,(B_s(x))}{\mathfrak{m}(B_s(x))}-\frac{\omega_{k-1}}{\omega_k}\right|<\varepsilon\quad\text{for any } s\le r_0. \tag{5.24}$$

**Remark 5.26.** Observe that, if $x\in\mathcal{R}_k$, then one has

$$\lim_{r\to0}\frac{\int_{B_r(x)}\left(1-\frac{\mathsf{d}(x,y)}{r}\right)\,\mathrm{d}\mathfrak{m}(y)}{\mathfrak{m}(B_r(x))}=\frac{1}{k+1}. \tag{5.25}$$

Moreover it can be easily checked that $x\in\mathcal{R}_k$ if and only if

$$\lim_{r\to0}\mathsf{d}_{pmGH}\left(\left(X,r^{-1}\mathsf{d},\frac{\mathfrak{m}}{\mathfrak{m}(B_r(x))},x\right),\left(\mathbb{R}^k,\mathsf{d}_{\mathrm{Eucl}},\frac{1}{\omega_k}\mathscr{L}^k,0^k\right)\right)=0.$$

As a consequence of Theorem 5.15 and Remark 5.26, for any $\varepsilon>0$, we have

$$\mathcal{F}_kE=\bigcup_{0<r<1}(\mathcal{F}_kE)_{r,\varepsilon}\quad\text{and}\quad(\mathcal{F}_kE)_{r,\varepsilon}\subset(\mathcal{F}_kE)_{r',\varepsilon}\ \text{for } r'<r.$$

Hence for any $\eta>0$ there exists $r=r(\eta)>0$ such that

$$|D\chi_E|\left(\mathcal{F}_kE\setminus(\mathcal{F}_kE)_{s,\varepsilon}\right)<\eta,\quad\text{for any }\ 0<s<r. \tag{5.26}$$

**Proposition 5.27.** *Let $N>1$, $K\in\mathbb{R}$ and $k\in[1,N]$ be fixed. For any $\eta>0$ there exists $\varepsilon=\varepsilon(\eta,N)<\eta$ such that, if $(X,\mathsf{d},\mathfrak{m})$ is an $\mathrm{RCD}(K,N)$ m.m.s., $E\subset X$ is a set of finite perimeter and finite measure, $p\in(\mathcal{F}_kE)_{2s,\varepsilon}$ for some $s\in(0,|K|^{-1/2})$ and there exists an $\varepsilon$-splitting map $u:B_{2s}(p)\to\mathbb{R}^{k-1}$ such that*

$$\frac{s}{\mathfrak{m}(B_{2s}(x))}\int_{B_{2s}(x)}|\nu_E\cdot\nabla u_a|\,\mathrm{d}|D\chi_E|<\varepsilon,\quad\textit{for any } a=1,\dots,k-1, \tag{5.27}$$

*then there exists $G\subset B_s(p)$ that satisfies:*

(i) *$G\cap(\mathcal{F}_kE)_{2s,\varepsilon}$ is bi-Lipschitz to a Borel subset of $\mathbb{R}^{k-1}$. More precisely,*

$$\Big||u(x)-u(y)|-\mathsf{d}(x,y)\Big|\le C_N\eta\,\mathsf{d}(x,y),\quad\forall x,y\in(\mathcal{F}_kE)_{2s,\varepsilon}\cap G; \tag{5.28}$$

(ii) $\mathscr{H}^h_5(B_s(p)\setminus G)<C_N\eta\frac{\mathfrak{m}(B_s(p))}{s}$.

Let us now prove Theorem 5.21 assuming Proposition 5.24 and Proposition 5.27.

*Proof of Theorem 5.21.* Assume without loss of generality that $E$ has finite perimeter and measure, and that $\mathcal{F}_k E \subset B_2(p)$ for some $p \in X$. We claim that, for any $\eta > 0$, we can decompose $\mathcal{F}_k E = G^\eta \cup B^\eta \cup R^\eta$, where $G^\eta$ is $(k-1)$-rectifiable and

$$\mathscr{H}_5^h(B^\eta) + |D\chi_E|(R^\eta) \le C_{N,K} |D\chi_E|(B_2(p))\eta + \eta. \tag{5.29}$$

Observe that the claim easily gives the sought conclusion. Indeed, setting $\eta_i := \eta 2^{-i}$, $G_\eta := \cup_i G^{\eta_i}$ and $R_\eta := \cup_{i\in\mathbb{N}} R^{\eta_i}$, $G_\eta$ is still $(k-1)$-rectifiable and it holds

$$\mathscr{H}_5^h((\mathcal{F}_k E \setminus G_\eta) \setminus R_\eta) = 0,$$

hence, as a consequence of Lemma 5.1, $|D\chi_E|(\mathcal{F}_k E \setminus G_\eta) \setminus R_\eta) = 0$. Therefore

$$|D\chi_E|(\mathcal{F}_k E \setminus G_\eta) \le |D\chi_E|(R_\eta) \le C_N |D\chi_E|(B_2(p))\eta + \eta.$$

Setting $G := \cup_{i\in\mathbb{N}} G_{2^{-i}}$, we get that $G$ is still $(k-1)$-rectifiable and coincides with $\mathcal{F}_k E$ up to a $|D\chi_E|$-negligible set.

Let us now prove the claim. To this aim fix $r > 0$ and $\varepsilon > 0$. We cover $(\mathcal{F}_k E)_{r,\varepsilon}$ with balls of radius smaller than $r/5$ with center in $(\mathcal{F}_k E)_{r,\varepsilon}$ such that the assumptions of Proposition 5.27 are satisfied. The possibility of building such a covering is a consequence of Theorem 5.15 and of Proposition 5.24. By Vitali's lemma, we can extract a disjoint family $\left\{B_{r_i/5}(x_i)\right\}_{i\in\mathbb{N}}$ such that $(\mathcal{F}_k E)_{r,\varepsilon} \subset \cup_i B_{r_i}(x_i)$. Applying Proposition 5.27 above, for any $i \in \mathbb{N}$ we can find $G_i \subset B_{r_i}(x_i)$ such that $G_i \cap (\mathcal{F}_k E)_{r,\varepsilon}$ is $(k-1)$-rectifiable and $\mathscr{H}_5^h(B_{r_i}(x_i) \setminus G_i) < C_N \eta \frac{\mathfrak{m}(B_{r_i}(x_i))}{r_i}$. Set $G_r^\eta := (\mathcal{F}_k E)_{r,\varepsilon} \cap (\cup_i G_i)$ and observe that

$$\begin{aligned}
\mathscr{H}_5^h((\mathcal{F}_k E)_{r,\varepsilon} \setminus G_r^\eta) \le& \sum_{i\in\mathbb{N}} \mathscr{H}_5^h(B_{r_i}(x_i) \setminus G_i) \le \sum_{i\in\mathbb{N}} C_N \eta \frac{\mathfrak{m}(B_{r_i}(x_i))}{r_i} \\
\le& C_N \eta \sum_{i\in\mathbb{N}} \frac{\mathfrak{m}(B_{r_i/5}(x_i))}{r_i/5} \le C_{N,K}\eta \sum_{i\in\mathbb{N}} |D\chi_E|(B_{r_i/5}(x_i)) \\
\le& C_{N,K}\eta\, |D\chi_E|(B_2(p)),
\end{aligned}$$

where we used the Bishop-Gromov inequality (1.13) and

$$\frac{\mathfrak{m}(B_{r_i/5}(x_i))}{r_i/5} \le C(k) |D\chi_E|(B_{r_i/5}(x_i)),$$

that holds true provided $\varepsilon$ is small enough.
Setting $B_r^\eta := (\mathcal{F}_k E)_{r,\varepsilon} \setminus G_r^\eta$, the argument above gives the decomposition

$$(\mathcal{F}_k E)_{r,\varepsilon} = G_r^\eta \cup B_r^\eta,$$

where $G_r^\eta$ is $(k-1)$-rectifiable and $\mathscr{H}_5^h(B_r^\eta) \le C_{N,K}\eta\, |D\chi_E|(B_2(p))$. Let us now choose $r > 0$ small enough to have (5.26). This allows us to write

$$\mathcal{F}_k E = G_r^\eta \cup B_r^\eta \cup (\mathcal{F}_k E \setminus (\mathcal{F}_k E)_{r,\varepsilon}) =: G^\eta \cup B^\eta \cup R^\eta$$

and to conclude the proof. $\square$

**3.1. Proof of Proposition 5.24.** Let us start by recalling that one of the main results of Section 1 was proving that the exterior normal is indeed an element of $L^2_E(TX)$ (see Theorem 5.6). In the following, to simplify the notation, we shall write $v$ in place of $\mathrm{tr}_E(v)$ for any $v \in H^{1,2}_C(TX) \cap D(\mathrm{div})$.

**Definition 5.28.** Let $(X, \mathsf{d}, \mathfrak{m})$ be an $\mathrm{RCD}(K, N)$ m.m.s. and $E \subset X$ of finite perimeter. Given $x \in X$ and a sequence $r_i \downarrow 0$ we say that $\left\{u^{r_i} := (u_1^{r_i}, \dots, u_{k-1}^{r_i}) : B_{r_i}(x) \to \mathbb{R}^{k-1}\right\}_{i\in\mathbb{N}}$ is a good approximation of the boundary of $E$ at $x$ if the following conditions hold true:

(i) there exists a sequence $\delta_i \to 0$ such that $u^{r_i} : B_{r_i}(x) \to \mathbb{R}^{k-1}$ is a $\delta_i$-splitting map with $u^{r_i}(x) = 0$;

(ii) there exists $(Z, \mathsf{d}_Z)$ that realizes the convergences

$$(X, r_i^{-1}\mathsf{d}, \mathfrak{m}_x^{r_i}, x) \to (\mathbb{R}^k, \mathsf{d}_{\mathrm{Eucl}}, c_k\mathscr{L}^k, 0^k) \quad \text{and} \quad E_{r_i} \to \{x_k > 0\} \text{ locally strongly in } BV$$

and $r_i^{-1}u_\alpha^{r_i} \to x_\alpha$ in $H^{1,2}$-strong on $B_1(0^k)$ along the sequence

$$(X, r_i^{-1}\mathsf{d}, \mathfrak{m}_x^{r_i}, x) \to (\mathbb{R}^k, \mathsf{d}_{\mathrm{Eucl}}, c_k\mathscr{L}^k, 0^k),$$

for any $\alpha = 1, \dots, k-1$.

**Lemma 5.29.** *Let $(X, \mathsf{d}, \mathfrak{m})$ be an $\mathrm{RCD}(K, N)$ m.m. space and $E \subset X$ a set of finite perimeter and finite measure. Then for any $p \in X$ and for any $\varepsilon > 0$ there exists $V \in \mathrm{TestV}(X)$ such that*

$$\int_{B_2(p)} |\nu_E - V|^2 \,\mathrm{d}|D\chi_E| \le \varepsilon,$$

*where $\nu_E$ is the exterior normal of $E$.*
*Moreover, there exists $G \subset B_1(p)$ with $\mathscr{H}^h(B_1(p) \setminus G) \le C_{K,N}\sqrt{\varepsilon}$ and such that, for any $x \in G$, it holds*

$$\limsup_{r\to 0} \frac{r}{\mathfrak{m}(B_r(x))} \int_{B_r(x)} |\nu_E - V|^2 \,\mathrm{d}|D\chi_E| \le \sqrt{\varepsilon}.$$

*Proof.* The first conclusion follows from Theorem 5.6, where we proved that the normal is an element of $L^2_E(TX)$, and Lemma 5.11, yielding density of $\mathrm{tr}_E\left(\mathrm{TestV}(X)\right)$ in $L^2_E(TX)$.

To prove the second part of the statement we set

$$G := \left\{ x \in B_1(p) \;:\; \limsup_{r\downarrow 0} \frac{r}{\mathfrak{m}(B_r(x))} \int_{B_r(x)} |\nu_E - V|^2 \,\mathrm{d}|D\chi_E| \le \sqrt{\varepsilon} \right\}.$$

Then, for any $r_0 > 0$ and for any $x \in B_1(p) \setminus G$, there exists $r_x < r_0$ such that

$$\frac{r_x}{\mathfrak{m}(B_{r_x}(x))} \int_{B_{r_x}(x)} |\nu - V|^2 \,\mathrm{d}|D\chi_E| > \sqrt{\varepsilon}.$$

Hence, applying Vitali's covering theorem we can find a disjoint set of balls $\{B_{r_i}(x_i)\}$ such that $\{B_{5r_i}(x_i)\}$ is a covering of $B_1(p) \setminus G$. Now we can estimate, for any $r_0 > 0$,

$$\begin{aligned}
\mathscr{H}^h_{5r_0}(B_1(p) \setminus G) &\le \sum_{i=0}^{\infty} \frac{\mathfrak{m}(B_{5r_i}(x_i))}{5r_i} \le C_{K,N} \sum_{i=0}^{\infty} \frac{\mathfrak{m}(B_{r_i}(x_i))}{r_i} \\
&\le \frac{C_{K,N}}{\sqrt{\varepsilon}} \sum_{i=0}^{\infty} \int_{B_{r_i}(x_i)} |\nu - V|^2 \,\mathrm{d}|D\chi_E| \le \frac{C_{K,N}}{\sqrt{\varepsilon}} \int_{B_2(p)} |\nu - V|^2 \,\mathrm{d}|D\chi_E| \\
&\le C_{K,N}\sqrt{\varepsilon}.
\end{aligned}$$

The conclusion follows letting $r_0 \downarrow 0$. $\square$

*Proof of Proposition 5.24.* The proof is divided in 3 steps. Aim of the first one is to prove that good approximations of the boundary are regular enough to guarantee that the scalar product between their gradient and the gradient of any given test function leaves a well-defined trace over the reduced boundary of $E$. In the second step we combine the outcome of the first one, the approximation result of Lemma 5.29 and the *orthogonality in weak sense* between the normal vector and the coordinates of its orthogonal hyperplane guaranteed by the Gauss–Green formula, to get that gradients of good approximations of the boundary leave a trace even when coupled with the normal to the boundary and that this trace is 0. In the last step we prove existence of good approximations of the boundary and combine it with steps 1 and 2 to get the sought conclusion.

**Step 1.** Observe that it suffices to restrict the attention to the ball $B_1(p) \subset X$, for any $p \in X$.

We claim that for any function $\varphi \in \mathrm{Test}(X, \mathsf{d}, \mathfrak{m})$ there exists a $|D\chi_E|$-negligible set $\mathrm{N} \subset \mathcal{F}_k E \cap B_1(p)$ such that, for any $x \in \mathcal{F}_k E \cap B_1(p) \setminus \mathrm{N}$ and any good approximation of the boundary of $E$ at $x$ with radii $r_i \downarrow 0$ and maps $\left\{u^{r_i} := (u_1^{r_i}, \dots, u_{k-1}^{r_i}) \,:\, B_{r_i}(x) \to \mathbb{R}^{k-1}\right\}_{i\in\mathbb{N}}$, there exist a subsequence $r_{i_j}$ and $c(x) = (c_1(x), \dots, c_{k-1}(x)) \in \mathbb{R}^{k-1}$ such that

(5.30)
$$\lim_{j\to\infty} \frac{r_{i_j}}{\mathfrak{m}(B_{r_{i_j}}(x))} \int_{B_{r_{i_j}}(x)} \left|\nabla u_\alpha^{r_{i_j}} \cdot \nabla\varphi - c_\alpha(x)\right|^2 \mathrm{d}|D\chi_E| = 0, \quad \text{for any } \alpha = 1, \dots, k-1.$$

Assume without loss of generality that $|\nabla\varphi| \le 1$. Let us fix also $\alpha \in \{1 \dots, k-1\}$ and set $g_i := \nabla u_\alpha^{r_i} \cdot \nabla\varphi$. We have

(i) $\|g_i\|_{L^\infty(B_{r_i}(x))} \le C_N$;

(ii) $r_i^2 ⨍_{B_{r_i}(x)} |\nabla g_i|^2 \,\mathrm{d}\mathfrak{m} \le 2\delta_i + C_N r_i^2 ⨍_{B_{r_i}(x)} |\operatorname{Hess}\varphi|^2 \,\mathrm{d}\mathfrak{m}$, where $\delta_i$ is as in Definition 5.28.

Since $\operatorname{Hess}\varphi \in L^2(B_2(p), \mathfrak{m})$, by Lemma 1.69 and Lemma 5.1, we deduce that

$$\lim_{r\to 0} r^2 ⨍_{B_r(x)} |\operatorname{Hess}\varphi|^2 \,\mathrm{d}\mathfrak{m} = 0$$

for any $x \in X$ outside a $|D\chi_E|$-negligible set depending only on $\varphi$. Therefore we can assume that $x$ does not belong to this set obtaining

$$\lim_{i\to\infty} r_i^2 ⨍_{B_{r_i}(x)} |\nabla g_i|^2 \,\mathrm{d}\mathfrak{m} = 0.$$

This gives that, up to subsequence, $g_i \to c_\alpha(x)$ in $H^{1,2}$-strong on $B_1(0^k)$ along the sequence in Definition 5.28(ii). Here we have used (5.25). Taking into account Proposition 1.40, it follows that $(g_i - c_\alpha(x)) \to 0$ in $H^{1,2}$-strong on $B_1(0^k)$ and thus, reading the convergence in the starting space,

$$⨍_{B_{r_{i_j}}(x)} |g_{i_j} - c_\alpha(x)|^2 \,\mathrm{d}\mathfrak{m} + r_{i_j}^2 ⨍_{B_{r_{i_j}}(x)} |\nabla g_{i_j}|^2 \,\mathrm{d}\mathfrak{m} =: \varepsilon_j \to 0 \quad \text{as } j \to \infty. \tag{5.31}$$

We wish to prove that, up to excluding another $|D\chi_E|$-negligible set depending only on $E$, (5.31) gives (5.30). More precisely we are going to prove that (5.31) implies (5.30) at any $x \in X$ such that $x \in E_{r_0,C}$ for some $r_0 > 0$ and $C > 1$, where

$$E_{r_0,C} := \left\{ y \in X : \; C^{-1} \le \frac{r|D\chi_E|(B_r(y))}{\mathfrak{m}(B_r(y))} \le C \;\forall r < r_0 \right\}, \tag{5.32}$$

and

$$\lim_{r\to 0}\frac{|D\chi_E|(E_{r_0,C}\cap B_r(x))}{|D\chi_E|(B_r(x))}=1. \tag{5.33}$$

Observe that (5.32) and (5.33) are satisfied at $|D\chi_E|$-a.e. point in $\mathcal{F}E$ thanks to Theorem 5.15, the asymptotic doubling property of $|D\chi_E|$ and elementary considerations. In order to keep notations short, from now on we set $r_j := r_{i_j}$ and $g_j := g_{i_j}$.
We claim that, for any $j$ such that $r_j \le r_0/5$, it holds

$$|D\chi_E|\Big(E_{r_0,C}\cap B_{r_j}(x)\cap\Big\{|g_j-c_\alpha(x)|^2\ge\sqrt{\varepsilon_j}\Big\}\Big)\le CC_{N,K}\sqrt{\varepsilon_j}\frac{\mathfrak{m}(B_{r_j}(x))}{r_j}, \tag{5.34}$$

where $\varepsilon_j$ is as in (5.31) and $r_0$ and $C$ are as in (5.32).
Notice that (5.34), together with the Chebyshev inequality, (i) and (5.33), give (5.30).
Let us see how to establish (5.34). Fix any $j$ such that $r_j\le r_0/5$ and let us set

$$(X_j,\mathsf{d}_j,\mathfrak{m}_j,x):=\left(X,r_j^{-1}\mathsf{d},\frac{\mathfrak{m}}{\mathfrak{m}(B_{r_j}(x))},x\right).$$

With a slight abuse of notation we use the notations $E_{r_0,C}$ and $g_j$ also in $X_j$. Let us observe that, when read in $X_j$, (5.31) turns into

$$\fint_{B_1^j(x)}|g_j-c_\alpha(x)|^2\,\mathrm{d}\mathfrak{m}_j+\fint_{B_1^j(x)}|\nabla g_j|^2\,\mathrm{d}\mathfrak{m}_j\le\varepsilon_j.$$

Moreover, a telescopic argument as in the proof of Proposition 5.17 gives

$$\begin{aligned}&B_1^j(x)\cap E_{r_0,C}\cap\Big\{|g_j-c_\alpha(x)|^2\ge C_{N,K}\sqrt{\varepsilon_j}\Big\}\\&\qquad\subset B_1^j(x)\cap E_{r_0,C}\cap\left\{z\ :\ \sup_{0<s<1}s\fint_{B_s^j(z)}|\nabla g_j|^2\,\mathrm{d}\mathfrak{m}_j>\sqrt{\varepsilon_j}\right\}.\end{aligned}$$

Using Vitali's lemma we can find a disjoint family $\Big\{B^j_{s_i}(z_i)\Big\}_{i\in\mathbb{N}}$ with $s_i\le 1$ and $z_i\in B_1^j(x)\cap E_{r_0,C}\cap\Big\{z\ :\ \sup_{0<s<1}s\fint_{B_s^j(z)}|\nabla g_j|^2\,\mathrm{d}\mathfrak{m}_j>\sqrt{\varepsilon_j}\Big\}$ for any $i\in\mathbb{N}$ such that

$$B_1^j(x)\cap E_{r_0,C}\cap\left\{z\ :\ \sup_{0<s<1}s\fint_{B_s^j(z)}|\nabla g_j|^2\,\mathrm{d}\mathfrak{m}_j>\sqrt{\varepsilon_j}\right\}\subset\bigcup_{i\in\mathbb{N}}B^j_{5s_i}(z_i).$$

Taking into account (5.32) and the defining identities

$$B^j_{s_i}(z_i)=B_{r_js_i}(z_i),\quad \mathfrak{m}_j=\frac{\mathfrak{m}}{\mathfrak{m}(B_{r_j}(x))},$$

we get

$$\begin{aligned}&\frac{r_j}{\mathfrak{m}(B_{r_j}(x))}|D\chi_E|\Big(E_{r_0,C}\cap B_{r_j}(x)\cap\Big\{|g_j-c_\alpha(x)|^2\ge\sqrt{\varepsilon_j}\Big\}\Big)\\&\qquad\le\frac{r_j}{\mathfrak{m}(B_{r_j}(x))}\sum_{i\in\mathbb{N}}|D\chi_E|(B_{5r_js_i}(z_i))\le\frac{CC_{N,K}r_j}{\mathfrak{m}(B_{r_j}(x))}\sum_{i\in\mathbb{N}}\frac{\mathfrak{m}(B_{r_js_i}(z_i))}{r_js_i}\\&\qquad=CC_{N,K}\sum_{i\in\mathbb{N}}\frac{\mathfrak{m}_j(B^j_{s_i}(z_i))}{s_i}\le\frac{CC_{N,K}}{\sqrt{\varepsilon_j}}\int_{B_1^j(x)}|\nabla g_j|^2\,\mathrm{d}\mathfrak{m}_j\le CC_{N,K}\sqrt{\varepsilon_j}.\end{aligned}$$

**Step 2.** We prove that, for $|D\chi_E|$-a.e. $x \in \mathcal{F}_k E$ and any good approximation of the boundary of $E$ at $x$ with radii $r_i \downarrow 0$ and maps $\left\{u^{r_i} := (u_1^{r_i}, \dots, u_{k-1}^{r_i}) \,:\, B_{r_i}(x) \to \mathbb{R}^{k-1}\right\}_{i\in\mathbb{N}}$, there exists a subsequence $r_{i_j} \to 0$ such that

$$\lim_{j\to\infty} \frac{r_{i_j}}{\mathfrak{m}(B_{r_{i_j}}(x))} \int_{B_{r_{i_j}}(x)} |\nu_E \cdot \nabla u_\alpha^{r_{i_j}}| \,\mathrm{d}|D\chi_E| = 0 \quad \text{for any } \alpha = 1, \dots, k-1. \tag{5.35}$$

Let us restrict our attention as above to $\mathcal{F}_k E \cap B_1(p)$.

We claim that, for any $\varepsilon > 0$, there exists $G_\varepsilon \subset B_1(p) \cap \mathcal{F}_k E$ with

$$\mathscr{H}^h(B_1(p) \cap \mathcal{F}_k E \setminus G_\varepsilon) \le C_{N,K}\sqrt{\varepsilon}$$

and such that, for any $x \in G_\varepsilon$, and any $\left\{u^{r_i} := (u_1^{r_i}, \dots, u_{k-1}^{r_i}) \,:\, B_{r_i}(x) \to \mathbb{R}^{k-1}\right\}_{i\in\mathbb{N}}$ good approximation of the boundary of $E$ at $x$, there exists a subsequence $r_{i_j} \to 0$ satisfying

$$\limsup_{j\to\infty} \frac{r_{i_j}}{\mathfrak{m}(B_{r_{i_j}}(x))} \int_{B_{r_{i_j}}(x)} |\nu_E \cdot \nabla u_\alpha^{r_{i_j}}| \,\mathrm{d}|D\chi_E| \le C_{N,K}\varepsilon^{1/4} \quad \text{for any } \alpha = 1, \dots, k-1. \tag{5.36}$$

Before proving the claim let us see how it implies (5.35).

Fix $\varepsilon > 0$, set $\varepsilon_i := \varepsilon 2^{-i}$ and take $G^\varepsilon := \cup_{i\in\mathbb{N}} G_{\varepsilon_i}$. Then we have $|D\chi_E|(B_1(p) \cap \mathcal{F}_k E \setminus G^\varepsilon) = 0$, thanks to Lemma 5.1, and (5.36) holds for any $x \in G^\varepsilon$. Therefore the set $\cap_{i\in\mathbb{N}} G^{\varepsilon_i}$ has full $|D\chi_E|$-measure in $B_1(p) \cap \mathcal{F}_k E$ and has the sought property.

The remaining part of this step is devoted to the proof of (5.36). Let $\varepsilon > 0$ be fixed, take $G$ and $V$ as in Lemma 5.29. Recalling that any test vector field can be represented as $\sum_{i=1}^m \eta_i \nabla\varphi_i$ with $\eta_i, \varphi_i \in \mathrm{Test}(X, \mathsf{d}, \mathfrak{m})$ for some $m \in \mathbb{N}$ and using Step 1, we conclude that there exists $G_\varepsilon \subset G \cap \mathcal{F}_k E$ with $|D\chi_E|(G \cap \mathcal{F}_k E \setminus G_\varepsilon) = 0$ and the property that, for any $x \in G_\varepsilon$ and $\left\{u^{r_i} := (u_1^{r_i}, \dots, u_{k-1}^{r_i}) \,:\, B_{r_i}(x) \to \mathbb{R}^{k-1}\right\}_{i\in\mathbb{N}}$ good approximation of the boundary of $E$ at $x$, there exists $c(x) := (c_1(x), \dots, c_{k-1}(x)) \in \mathbb{R}^{k-1}$ and a subsequence $r_{i_j} \to 0$ such that

$$\lim_{j\to\infty} \frac{r_{i_j}}{\mathfrak{m}(B_{r_{i_j}}(x))} \int_{B_{r_{i_j}}(x)} |\nabla u_\alpha^{r_{i_j}} \cdot V - c_\alpha(x)|^2 \,\mathrm{d}|D\chi_E| = 0 \quad \text{for } \alpha = 1, \dots, k-1. \tag{5.37}$$

In order to conclude the proof it suffices to show that

$$|c(x)| \le C_{K,N}\varepsilon^{1/4}. \tag{5.38}$$

Indeed, in that case, one has

$$\begin{aligned}
&\limsup_{j\to\infty} \frac{r_{i_j}}{\mathfrak{m}(B_{r_{i_j}}(x))} \int_{B_{r_{i_j}}(x)} |\nu_E \cdot \nabla u_\alpha^{r_{i_j}}| \,\mathrm{d}|D\chi_E| \\
&\quad\le C_N \limsup_{j\to\infty} \left( \frac{r_{i_j}}{\mathfrak{m}(B_{r_{i_j}}(x))} \int_{B_{r_{i_j}}(x)} |\nu_E - V|^2 \,\mathrm{d}|D\chi_E| \right)^{1/2} \\
&\qquad + \limsup_{j\to\infty} \frac{C_N r_{i_j}}{\mathfrak{m}(B_{r_{i_j}}(x))} \int_{B_{r_{i_j}}(x)} |\nabla u_\alpha^{r_{i_j}} \cdot V| \,\mathrm{d}|D\chi_E| \\
&\quad\le C_N\varepsilon^{1/4} + \lim_{j\to\infty} C_N \left( \frac{r_{i_j}}{\mathfrak{m}(B_{r_{i_j}}(x))} \int_{B_{r_{i_j}}(x)} |\nabla u_\alpha^{r_{i_j}} \cdot V - c_\alpha(x)|^2 \,\mathrm{d}|D\chi_E| \right)^{1/2}
\end{aligned}$$

$$
\begin{aligned}
&+|c_\alpha(x)|\frac{r_{i_j}|D\chi_E|(B_{r_{i_j}}(x))}{\mathfrak{m}(B_{r_{i_j}}(x))}\\
&\le C_{K,N}\varepsilon^{1/4},
\end{aligned}
$$

where we used (5.37), (5.38) and the fact that $x\in\mathcal{F}_kE$.

In order to prove (5.38) we simplify the notation setting $r_{i_j}=:r_j$. Choose a smooth function $\psi_\infty:\mathbb{R}^k\to\mathbb{R}$ with compact support in $B_1(0^k)$ and such that $\int_{\{x_k=0\}}\psi_\infty\,\mathrm{d}\mathscr{L}^{k-1}=:C_k>0$. Then we consider a sequence $\psi_j\in\mathrm{Lip}(X,\mathsf{d})$ with $\mathrm{supp}(\psi_j)\subset B_{r_j}(x)$, $\|\psi_j\|_{L^\infty}\le 2$ and $\psi_j\to\psi_\infty$ strongly in $H^{1,2}$ along the sequence in Definition 5.28(ii), whose existence is proved in Lemma 1.46. Observe now that

(5.39)
$$\lim_{j\to\infty}\frac{r_j}{\mathfrak{m}(B_{r_j}(x))}\int_E\nabla\psi_j\cdot\nabla u_\alpha^{r_j}\,\mathrm{d}\mathfrak{m}=c_k\int_{\{x_k>0\}}\nabla\psi_\infty\cdot e_\alpha\,\mathrm{d}\mathscr{L}^k=0,\quad\text{for }\alpha=1,\dots,k-1,$$

and

$$\lim_{j\to\infty}\frac{r_j}{\mathfrak{m}(B_{r_j}(x))}\int\psi_jV\cdot\nabla u_\alpha^{r_j}\,\mathrm{d}|D\chi_E|=C_kc_\alpha(x),\quad\text{for }\alpha=1,\dots,k-1,\tag{5.40}$$

where the last equality in (5.39) is obtained integrating by parts and to prove (5.40) we used (5.37). Building upon (5.39), (5.40), Theorem 5.6 and Lemma 5.29, we get (5.37):

$$
\begin{aligned}
C_k|c_\alpha(x)|&=\left|\lim_{j\to\infty}\frac{r_j}{\mathfrak{m}(B_{r_j}(x))}\left(\int_E\nabla\psi_j\cdot\nabla u_\alpha^{r_j}\,\mathrm{d}\mathfrak{m}+\int\psi_jV\cdot\nabla u_\alpha^{r_j}\,\mathrm{d}|D\chi_E|\right)\right|\\
&=\left|\lim_{j\to\infty}\frac{r_j}{\mathfrak{m}(B_{r_j}(x))}\left(-\int\psi_j\nu\cdot\nabla u_\alpha^{r_j}\,\mathrm{d}|D\chi_E|+\int\psi_jV\cdot\nabla u_\alpha^{r_j}\,\mathrm{d}|D\chi_E|\right)\right|\\
&\le\limsup_{j\to\infty}\frac{C_Nr_j}{\mathfrak{m}(B_{r_j}(x))}\int_{B_{r_j}(x)}|\nu_E-V|\,\mathrm{d}|D\chi_E|\\
&\le C_{N,K}\varepsilon^{1/4}.
\end{aligned}
$$

Note that in order to apply the Gauss–Green formula in the previous estimate the fact that $u_\alpha^{r_j}$ is locally the restriction of a $H^{2,2}(X,\mathsf{d},\mathfrak{m})$ function (see Remark 1.81) plays a role.

**Step 3.** In order to conclude the proof we just observe that, since

$$\mathsf{d}_{pmGH}\Big((X,r^{-1}\mathsf{d},\mathfrak{m}_x^r,x),(\mathbb{R}^k,\mathsf{d}_{\mathrm{Eucl}},c_k\mathscr{L}^k,0^k)\Big)\to 0$$

as $r\downarrow 0$ and the blow-up of the set of finite perimeter is a half-space (in the sense of $BV_{\mathrm{loc}}$ convergence, as we pointed out after Definition 4.30), a slight modification of Proposition 1.85[1] provides, for any sequence $r_i\downarrow 0$, existence of a good approximation of the boundary of $E$ at $x$ with maps $\left\{u^{r_i}:=(u_1^{r_i},\dots,u_{k-1}^{r_i}):B_{r_i}(x)\to\mathbb{R}^{k-1}\right\}_{i\in\mathbb{N}}$ (observe that Proposition 1.85 gives $\delta_i$-splitting maps defined on the balls of radius 1 of the rescaled spaces for a sequence $\delta_i\downarrow 0$ and then rescale these functions). The sought conclusion follows now from what we obtained in the previous step. $\square$

[1]With the splitting functions defined on balls of radius 1 in place of 5.

**3.2. Proof of Proposition 5.27.** The proof is divided in three steps. Aim of the first one is to provide a bridge between analysis and geometry suitable for this context. We prove that, whenever at a certain location and scale the set of finite perimeter is quantitatively close to a half-space in a Euclidean space and there is a $(k-1,\delta)$-splitting map which is also $\delta$-orthogonal to the normal vector in the sense of (5.27), then the $(k-1,\delta)$-splitting map is an $\eta$-isometry (in the scale invariant sense) when restricted to the support of the perimeter. The second step is analytic and dedicated to the propagation of the $\delta$-orthogonality condition. In the last one we get the bi-Lipschitz property relying on the observation that a map which is an $\eta$-isometry (in the scale invariant sense) at any location and scale is bi-Lipschitz.

**Step 1.** Let $N>0$, $K\in\mathbb{R}$ and $k\in[1,N]$ be fixed. We claim that, for any $\eta>0$, there exists $\delta=\delta_{\eta,N}\le\eta$ such that, for any pointed $\mathrm{RCD}(K,N)$ m.m.s. $(X,\mathsf{d},\mathfrak{m},x)$ and for any set of finite perimeter and finite measure $E\subset X$ such that, for some $0<r<|K|^{-1/2}$,

(i) $\mathsf{d}_{pmGH}\left(\left(X,(2r)^{-1}\mathsf{d},\frac{\mathfrak{m}}{\mathfrak{m}(B_{2r}(x))},x\right),\left(\mathbb{R}^k,\mathsf{d}_{\mathrm{Eucl}},\frac{1}{\omega_k}\mathscr{L}^k,0^k\right)\right)<\delta$;

(ii)

$$\left|\frac{\mathfrak{m}(B_t(x)\cap E)}{\mathfrak{m}(B_t(x))}-\frac{1}{2}\right|+\left|\frac{t\,|D\chi_E|\,(B_t(x))}{\mathfrak{m}(B_t(x))}-\frac{\omega_{k-1}}{\omega_k}\right|<\delta\quad\text{for any } t\le 2r;\tag{5.41}$$

(iii) there exists $u:=(u_1,\dots,u_{k-1}):B_{2r}(x)\to\mathbb{R}^{k-1}$ a $\delta$-splitting map satisfying

$$\frac{r}{\mathfrak{m}(B_{2r}(x))}\int_{B_{2r}(x)}|\nu_E\cdot\nabla u_a|\,\mathrm{d}|D\chi_E|<\delta,\quad\text{for any } a=1,\dots,k-1,\tag{5.42}$$

then $u:\operatorname{supp}|D\chi_E|\cap B_r(x)\to B_r^{\mathbb{R}^{k-1}}(u(x))$ is an $\eta r$-GH isometry.

By scaling it is enough to prove the claim when $r=1/2$ and $|K|\le 4$. Let us argue by contradiction. Then we could find $\eta>0$, a sequence $(X_n,\mathsf{d}_n,\mathfrak{m}_n,E_n,x_n)$, points $z_1^n,z_2^n\in\operatorname{supp}|D\chi_{E_n}|\cap B_{1/2}(x_n)$, and $1/n$-splitting maps $u^n:B_1(x_n)\to\mathbb{R}^{k-1}$ satisfying (i), (ii) and (iii) with $\delta=1/n$, $u^n(x_n)=0$ and

$$\Big||u^n(z_1^n)-u^n(z_2^n)|-\mathsf{d}_n(z_1^n,z_2^n)\Big|\ge\eta,\quad\forall n\in\mathbb{N}.\tag{5.43}$$

Notice that $\mathsf{d}_n(z_1^n,z_2^n)\ge\min\{\eta/(C_N-1),\eta\}$ since $u^n$ is $C_N$-Lipschitz.

Observe that, by (i)

$$(X_n,\mathsf{d}_n,\mathfrak{m}_n,x_n)\to\left(\mathbb{R}^k,\mathsf{d}_{\mathrm{Eucl}},\frac{1}{\omega_k}\mathscr{L}^k,0^k\right)\quad\text{in the pmGH topology.}$$

We can assume the existence of a metric space $(Z,\mathsf{d}_Z)$ realizing this convergence (cf. Section 4.2). Since $E_n$ satisfies the bound

$$\left|\frac{\mathfrak{m}_n(E_n\cap B_t(x_n))}{\mathfrak{m}_n(B_t(x_n))}-\frac{1}{2}\right|+\left|\frac{t|D\chi_{E_n}|(B_t(x_n))}{\mathfrak{m}_n(B_t(x_n))}-\frac{\omega_{k-1}}{\omega_k}\right|<1/n\quad\text{for any } t\le 1,\tag{5.44}$$

up to extracting a subsequence, $E_n\cap B_1(x_n)\to F\cap B_1(0^k)$ in $L^1$-strong, where $F$ is of locally finite perimeter in $B_1(0^k)$ thanks to Proposition 4.29.

Up to extracting again a subsequence we can assume $u^n\to u^\infty$ strongly in $H^{1,2}$ on $B_1(0^k)$, where $u^\infty:B_1^{\mathbb{R}^k}(0)\to\mathbb{R}^{k-1}$ is the restriction of an orthogonal projection, as a consequence of Proposition 1.34 and Theorem 1.47. We assume, without loss of generality, that $u^\infty(x)=(x_1,\dots,x_{k-1})$ for any $x\in B_1(0^k)$.

We claim that $\mathscr{L}^k\left(\left(F\cap B_1(0^k)\right)\Delta\left(\{x_k>0\}\cap B_1(0^k)\right)\right)=0$ and

$$\int g\ \mathrm{d}|D\chi_{E_n}|\to\int g\ \mathrm{d}|D\chi_{\{x_k>0\}}|\quad\text{for any } g\in C(Z) \text{ with } \mathrm{supp}(g)\subset B_{1/2}(0^k). \tag{5.45}$$

This would imply that $z_1^\infty, z_2^\infty\in\{x_k=0\}$, therefore $|u^\infty(z_1^\infty)-u^\infty(z_2^\infty)|=\mathsf{d}_{\mathrm{Eucl}}(z_1^\infty,z_2^\infty)$ that contradicts (5.43).
In order to verify the claim we argue as in the proof of the second step of Proposition 5.24. We choose a smooth function $\psi_\infty:\mathbb{R}^k\to\mathbb{R}$ with compact support in $B_1(0^k)$. Then we consider a sequence $\psi_n\in\mathrm{Lip}(X_n,\mathsf{d}_n)$ with $\mathrm{supp}(\psi_n)\subset B_1(x_n)$, $\|\psi_n\|_{L^\infty}+\||\nabla\psi_n|\|_{L^\infty}\le 4$ and $\psi_n\to\psi_\infty$ strongly in $H^{1,2}$ along the sequence $(X_n,\mathsf{d}_n,\mathfrak{m}_n,x_n)$, whose existence is proved in Lemma 1.46. Observe now that

$$\nabla\psi_n\cdot\nabla u_a^n\to\nabla\psi_\infty\cdot e_a=\frac{\partial\psi_\infty}{\partial x_a}\quad\text{in } L^2\text{-strong, for any } a=1,\dots,k-1,$$

by Proposition 1.40(i) and Proposition 1.40(iii). This observation, along with Proposition 1.40(ii) and Remark 4.19, gives

$$\int_F\frac{\partial\psi_\infty}{\partial x_a}\ \mathrm{d}\frac{\mathscr{L}^k}{\omega_k}=\lim_{n\to\infty}\int_{E_n}\nabla\psi_n\cdot\nabla u_a^n\ \mathrm{d}\mathfrak{m}_n. \tag{5.46}$$

We can now use (5.46), Theorem 5.6 and (iii) to conclude that

$$\begin{aligned}\left|\int_F\frac{\partial\psi_\infty}{\partial x_a}\ \mathrm{d}\frac{\mathscr{L}^k}{\omega_k}\right|&=\lim_{n\to\infty}\left|\int_{E_n}\nabla\psi_n\cdot\nabla u_a^n\ \mathrm{d}\mathfrak{m}_n\right|\\&=\lim_{n\to\infty}\left|\int\psi_n\nabla u_a^n\cdot\nu_{E_n}\ \mathrm{d}|D\chi_{E_n}|\right|\\&\le\lim_{n\to\infty}\int|\psi_n|\,|\nabla u_a^n\cdot\nu_{E_n}|\ \mathrm{d}|D\chi_{E_n}|=0,\end{aligned}$$

for $a=1,\dots,k-1$. Since $\psi_\infty\in C_c^\infty(B_1(0^k))$ is arbitrary we obtain that

$$\mathscr{L}^k\left(\left(F\cap B_1(0^k)\right)\Delta\left(\{x_k>\lambda\}\cap B_1(0^k)\right)\right)=0\quad\text{for some }\lambda\in\mathbb{R}.$$

Using again (5.44) we get $\mathscr{L}^k(F\cap B_1(0^k))=\omega_k/2$ that forces $\lambda=0$.
Let us finally prove (5.45). To this end we use again (5.44) with $t=1/2$ obtaining that

$$\lim_{n\to\infty}|D\chi_{E_n}|\,(B_{1/2}(x_n))=\frac{\omega_{k-1}}{2^{k-1}}=\left|D\chi_{\{x_k>0\}}\right|(B_{1/2}(0^k)).$$

We can now apply the third conclusion of Proposition 4.29 and conclude.

**Step 2.** By assumption there exists an $\varepsilon$-splitting map $u:B_{2s}(p)\to\mathbb{R}^{k-1}$ such that (5.27) holds true. We wish to propagate now both the $\varepsilon$-splitting condition and the orthogonality condition (5.27) at any scale and point outside a set of small $\mathscr{H}_5^h$-measure. More precisely we are going to prove that there exists a set $G\subset B_s(p)$ with $\mathscr{H}_5^h(B_s(p)\setminus G)\le C_N\sqrt{\varepsilon}\frac{\mathfrak{m}(B_s(p))}{s}$ such that

(i) for any $x\in G$, $0<r<s$, $u:B_r(x)\to\mathbb{R}^{k-1}$ is a $C_N\varepsilon^{1/4}$-splitting map;
(ii) for any $x\in G$, $0<r<s$, it holds

$$\frac{r}{\mathfrak{m}(B_r(x))}\int_{B_r(x)}|\nu_E\cdot\nabla u_a|\ \mathrm{d}|D\chi_E|<\sqrt{\varepsilon},\quad\text{for } a=1,\dots,k-1. \tag{5.47}$$

We can find a set $G'$ satisfying the measure estimate and (i) applying Corollary 5.18. Hence it is enough to find a set $G''$ satisfying the measure estimate and (ii) and to take $G := G' \cap G''$. To do so we apply a standard maximal argument. Let us fix $a = 1, \dots, k-1$ and set

$$M(x) := \sup_{0<r<s} \frac{r}{\mathfrak{m}(B_r(x))} \int_{B_r(x)} |\nu_E \cdot \nabla u_a| \, \mathrm{d}|D\chi_E|.$$

We claim that $G'' := \{x \in B_s(p) \, : \, M(x) < \sqrt{\varepsilon}\}$ has the sought properties.
Indeed, for any $x \in B_s(p) \setminus G''$, there exists $\rho_x \in (0, s)$ such that

$$\frac{\rho_x}{\mathfrak{m}(B_{\rho_x}(x))} \int_{B_{\rho_x}(x)} |\nu_E \cdot \nabla u_a| \, \mathrm{d}|D\chi_E| \geq \sqrt{\varepsilon}. \tag{5.48}$$

Applying Vitali lemma to the family of balls $\{B_{\rho_x}(x)\}_{x \in B_s(p)\setminus G''}$ we find a disjoint subfamily $\{B_{r_i}(x_i)\}_{i\in\mathbb{N}}$ such that $B_s(p) \setminus G'' \subset \cup_i B_{5r_i}(x_i)$. Taking into account the disjointedness of the covering, (5.48), (5.27) and the Bishop-Gromov inequality, we can compute

$$\begin{aligned}
\mathscr{H}_5^h(B_s(p) \setminus G'') \leq & \sum_{i\in\mathbb{N}} h(B_{5r_i}(x_i)) = \sum_{i\in\mathbb{N}} \frac{\mathfrak{m}(B_{5r_i}(x_i))}{5r_i} \\
\leq & C_N \sum_{i\in\mathbb{N}} \frac{\mathfrak{m}(B_{r_i}(x_i))}{r_i} \leq C_N \sum_{i\in\mathbb{N}} \varepsilon^{-1/2} \int_{B_{r_i}(x_i)} |\nu_E \cdot \nabla u_a| \, \mathrm{d}|D\chi_E| \\
\leq & C_N \varepsilon^{-1/2} \int_{B_{2s}(p)} |\nu_E \cdot \nabla u_a| \, \mathrm{d}|D\chi_E| \leq C_N \sqrt{\varepsilon} \, \frac{\mathfrak{m}(B_{2s}(p))}{s}.
\end{aligned}$$

**Step 3.** We claim now that for any $\eta > 0$ there exists $\varepsilon = \varepsilon_{\eta,N} > 0$ small enough such that for any $0 < r < s$ and $x \in G \cap (\mathcal{F}_k E)_{2s,\varepsilon}$ the map

$$u = (u_1, \dots, u_{k-1}) : \operatorname{supp} |D\chi_E| \cap B_r(x) \to \mathbb{R}^{k-1} \qquad \text{is an } r\eta\text{-GH isometry.} \tag{5.49}$$

The claim is a consequence of Step 1. Indeed, for any $x \in G \cap (\mathcal{F}_k E)_{2s,\varepsilon}$ and any $r \in (0, s)$, the conditions (i) and (ii) of Step 1 are satisfied by definition of $(\mathcal{F}_k E)_{2s,\varepsilon}$. Moreover $u$ is a $C_N \varepsilon^{1/4}$-splitting map on $B_r(x)$ satisfying (5.47), hence also the assumption (iii) of Step 1 is satisfied for $\varepsilon$ small enough.

In order to conclude the proof we have just to check the conclusion (i) in the statement of Proposition 5.27, since the conclusion (ii) follows from Step 2 choosing $\varepsilon$ small enough so that $\sqrt{\varepsilon} < \eta$. To this aim, take $x, y \in G \cap (\mathcal{F}_k E)_{2s,\varepsilon}$ and choose $r := \mathsf{d}(x, y)$. Our claim (5.49) ensures that

$$\Big| |u(x) - u(z)| - \mathsf{d}(x, z) \Big| \leq r\eta \qquad \text{for any } z \in \operatorname{supp} |D\chi_E| \cap B_r(x),$$

therefore we can take $z = y$ and conclude.

CHAPTER 6

# Approximation in Lusin's sense of Sobolev by Lipschitz functions on $\mathrm{RCD}(K,\infty)$ spaces

We say that a function $f : X \to \mathbb{R}$ on a metric measure space $(X, \mathsf{d}, \mathfrak{m})$ is approximable in Lusin's sense by Lipschitz functions if, given any $\varepsilon > 0$, there exist a Lipschitz function $g : X \to \mathbb{R}$ and a Borel set $A \subset X$ such that $\mathfrak{m}(X \setminus A) < \varepsilon$ and $f \equiv g$ on $A$. In Euclidean metric measure structures it is well known that this property is equivalent to an almost everywhere differentiability, in an approximate sense. A quantitative version of this Lusin-Lipschitz property, namely

$$|f(x) - f(y)| \le \mathsf{d}(x, y)\Big(g(x) + g(y)\Big) \qquad \text{for some nonnegative } g \in L^p(\mathfrak{m}) \tag{6.1}$$

holds for $W^{1,p}$ functions, $p \in (1,\infty)$, in Euclidean spaces (see [**114**]). It is also well-known that in the class of metric measure spaces $(X, \mathsf{d}, \mathfrak{m})$ satisfying the doubling and 1-Poincaré inequality, the property (6.1) *characterizes* $W^{1,p}$ functions, while for general metric measure structures it is the basis of the definition of the so-called Hajlasz Sobolev functions (see e.g. [**101**]). The quantitative Lusin-Lipchitz property has seen many applications in several fields of analysis and geometry. Just to mention one, in Chapter 3, following the original idea in [**23, 63**], we have heavily used this property to obtain a regularity result for flows of Sobolev velocity fields.

Following closely [**8**] in this chapter we prove the Lipschitz approximation result for Sobolev functions in the class of $\mathrm{RCD}(K,\infty)$ metric measure structures, introducing a new strategy which does not rely on the doubling property of the reference measure.

**Theorem 6.1.** *Let $(X, \mathsf{d}, \mathfrak{m})$ be an $\mathrm{RCD}(K,\infty)$ space. For any $\alpha \in (1, 2)$, $f \in W^{1,2}(X, \mathsf{d}, \mathfrak{m})$, there exists a $\mathfrak{m}$-negligible set $N \subset X$ such that, for every $x, y \in X \setminus N$ with $\mathsf{d}(x, y) \le 1/(K^-)^2$ one has*

$$|f(x) - f(y)| \le C_\alpha \mathsf{d}(x, y)(g(x) + g(y)), \tag{6.2}$$

$$g := \left(\sup_{t>0} P_t|\nabla f|^\alpha\right)^{1/\alpha} + \sup_{t>0} |P_t\sqrt{-\Delta} f| \in L^2(X, \mathfrak{m}),$$

*where $K^-$ denotes the negative part of the curvature bound $K$. Here $P_t$ denotes the heat semigroup on $(X, \mathsf{d}, \mathfrak{m})$ whose infinitesimal generator is $\Delta$.*

The main tool in the proof of Theorem 6.1 is the heat semigroup $P_t$ associated to the Sobolev class $W^{1,2}(X, \mathsf{d}, \mathfrak{m})$. More precisely, given $x, y \in X$ we consider $P_t f$ when $t \sim \mathsf{d}(x, y)^2$ and we estimate

$$|f(x) - f(y)| \le |f(x) - P_t f(x)| + |P_t f(x) - P_t f(y)| + |P_t f(y) - f(y)|.$$

Roughly speaking the estimates of all terms involve $|\nabla f|$, but while the estimate of the oscillation $|P_t(x) - P_t(y)|$ involves mostly the curvature properties of the metric measure space, the estimate of $f-P_tf$ builds upon a general result for symmetric Markov semigroup. More precisely, we introduce the representation formula (see (6.3))

$$P_tv-v=\int_0^\infty K(s,t)P_s\sqrt{-L}v\ \mathrm{d}s\qquad \forall v\in D(\sqrt{-L}),\quad \forall t\ge 0$$

where $K$ is a suitable kernel and $L$ the infinitesimal generator of a Markov semigroup $R$. This formula provides the correct integrability estimates, at the price of working with the nonlocal operator $\sqrt{-L}$.

## 1. Abstract semigroup tools

Aim of this section is to provide a pointwise representation formula for symmetric Markov semigroups which will play a role in the sequele. Throughout this section $G$ denotes a separable Hilbert space.

**Proposition 6.2.** *Let $P_t=e^{tL}$ be a continuous semigroup acting on $G$, with infinitesimal generator $L:D(L)\subset G\to G$. If $-L$ is a positive selfadjoint operator, one has the representation formula*

$$P_tv-v=\int_0^\infty K(s,t)P_s\sqrt{-L}v\ \mathrm{d}s\qquad \forall v\in D(\sqrt{-L}),\quad \forall t\ge 0, \tag{6.3}$$

*(understanding the integral in Bochner's sense) for a suitable kernel $K:\mathbb{R}_+\times\mathbb{R}_+\to\mathbb{R}$ independent of $P_t$ and satisfying*

$$\int_0^\infty |K(s,t)|\,\mathrm{d}s=\frac{4}{\sqrt{\pi}}\sqrt{t}\qquad \forall t\ge 0. \tag{6.4}$$

*Proof.* Let us begin by proving that

$$e^{-bt}-1=\int_0^\infty K(s,t)\sqrt{b}e^{-bs}\,\mathrm{d}s,\qquad \forall t\ge 0,\quad \forall b\ge 0 \tag{6.5}$$

for some kernel $K$ satisfying (6.4). For $b>0$ (if $b=0$ (6.5) is obvious), we use the identity

$$e^{-bt}=\frac{1}{\sqrt{\pi}}\int_t^\infty \frac{1}{(s-t)^{1/2}}\sqrt{b}e^{-bs}\,\mathrm{d}s,\qquad \forall t\in\mathbb{R}, \tag{6.6}$$

which follows immediately from

$$\int_t^\infty \frac{e^{-bs}}{(s-t)^{1/2}}\,\mathrm{d}s=e^{-bt}\int_t^\infty \frac{e^{-b(s-t)}}{(s-t)^{1/2}}\,\mathrm{d}s=\frac{e^{-bt}}{\sqrt{b}}\int_0^\infty \frac{e^{-s}}{\sqrt{s}}\,\mathrm{d}s=\frac{e^{-bt}}{\sqrt{b}}\sqrt{\pi},$$

where we have used the well known identity $\int_0^\infty \frac{e^{-s}}{\sqrt{s}}ds=\sqrt{\pi}$.

Using (6.6) we find

$$e^{-bt}-1=e^{-bt}-e^{-b0}=\int_{\mathbb{R}}\frac{1}{\sqrt{\pi}}\left(\frac{\chi_{s>t}}{(s-t)^{1/2}}-\frac{\chi_{s>0}}{s^{1/2}}\right)\sqrt{b}e^{-bs}\,\mathrm{d}s,$$

so that, setting

$$K(s,t):=\frac{1}{\sqrt{\pi}}\left(\frac{\chi_{s>t}}{(s-t)^{1/2}}-\frac{\chi_{s>0}}{s^{1/2}}\right)$$

we obtain (6.5). Let us now check that

$$\int_0^\infty |K(s,t)|\,\mathrm{d}s=\frac{4}{\sqrt{\pi}}\sqrt{t}\quad \text{for any } t\ge 0.$$

For $t > 0$ we have

$$\begin{aligned}\int_0^\infty |K(s,t)|\,\mathrm{d}s =& \frac{1}{\sqrt{\pi}}\int_0^t \left(\frac{1}{s^{1/2}}ds + \frac{1}{\sqrt{\pi}}\int_t^\infty \frac{1}{(s-t)^{1/2}} - \frac{1}{s^{1/2}}\right)\mathrm{d}s\\ =&\frac{2}{\sqrt{\pi}}\sqrt{t} + \frac{2}{\sqrt{\pi}}((s-t)^{1/2} - s^{1/2})|_{s=t}^{s=\infty}\\ =&\frac{4}{\sqrt{\pi}}\sqrt{t}.\end{aligned}$$

Using standard notions of functional calculus we can write

$$P_t = \int_0^\infty e^{-\lambda t}\,\mathrm{d}E(\lambda) \qquad \forall t > 0,$$

where $E$ is the spectral measure associated to $L$. For $v \in D(\sqrt{-L})$, from (6.5) we obtain

$$\begin{aligned}P_t v - v =& \int_0^\infty (e^{-\lambda t} - 1)\,\mathrm{d}E(\lambda)v\\ =& \int_0^\infty \int_0^\infty K(s,t)\sqrt{\lambda}e^{-\lambda s}\,\mathrm{d}s\,\mathrm{d}E(\lambda)v\\ =& \int_0^\infty K(s,t)\int_0^\infty \sqrt{\lambda}e^{-\lambda s}\,\mathrm{d}E(\lambda)v\,\mathrm{d}s,\end{aligned}$$

where all integrals are well defined since $v \in D(\sqrt{-L})$ implies $\int_0^\infty \sqrt{\lambda}\,\mathrm{d}E(\lambda)v < \infty$. We finally observe that

$$\int_0^\infty \sqrt{\lambda}e^{-\lambda s}\,\mathrm{d}E(\lambda)v = P_s\sqrt{-L}v = \sqrt{-L}P_s v \qquad \forall v \in D(\sqrt{-L}),$$

that concludes the proof. □

We now particularize the previous result to the case of Markov symmetric semigroups (see e.g. [**133**, page 65]). Let $(X, \mathcal{F}, \mathfrak{m})$ be an abstract measure space, with $\mathfrak{m}$ $\sigma$-finite, and let $P_t$ be a symmetric Markov semigroup acting on $G = L^2(X, \mathcal{F}, \mathfrak{m})$. In this class of semigroups, which have a canonical extension to a contraction semigroup in all $L^p(X, \mathcal{F}, \mathfrak{m})$ spaces, $1 < p < \infty$, one can always find, for all $f \in G$, versions of $P_t f$, $t > 0$, with the property that $t \mapsto P_t f(x)$ is continuous (in fact, analytic) in $(0, \infty)$ for $\mathfrak{m}$-a.e. $x \in X$ (cf. [**133**, page 72] for a proof). For such continuous version, besides the Littlewood-Paley inequality (cf. [**133**, page 74])

$$\int_X \int_0^\infty t\Big|\frac{\mathrm{d}}{\mathrm{d}t}P_t f(x)\Big|^2\,\mathrm{d}t\,\mathrm{d}\mathfrak{m}(x) \le \frac{1}{4}\int_X |f|^2\,\mathrm{d}\mathfrak{m}, \tag{6.7}$$

we shall also use the following powerful result from the theory of Markov semigroups (see for instance [**133**, page 73]).

**Theorem 6.3** (Maximal inequality)**.** *For $p \in (1, \infty]$ one has, for some $C_p < \infty$,*

$$\|\sup_{t>0} P_t f\|_p \le C_p\|f\|_p \qquad \forall f \in L^p(X, \mathcal{F}, \mathfrak{m}).$$

*In addition, for all $f \in L^p(X, \mathcal{F}, \mathfrak{m})$, one has $P_t f \to f$ $\mathfrak{m}$-a.e. as $t \to 0^+$.*

**Proposition 6.4.** *For every $f \in D(\sqrt{-L})$, one has $\mathfrak{m}$-a.e. continuous version of $P_t$ satisfying*

$$|P_t f(x) - f(x)| \le \frac{4\sqrt{t}}{\sqrt{\pi}} \sup_{s>0} |P_s \sqrt{-L} f|(x) \qquad \text{for } \mathfrak{m}\text{-a.e. } x \in X. \tag{6.8}$$

*Proof.* By Proposition 6.2 we have

$$P_t f(x) - f(x) = \int_0^\infty K(r,t) P_s \sqrt{-L} f(x)\,\mathrm{d}r, \qquad \text{for } \mathfrak{m}\text{-a.e. } x \in X, \tag{6.9}$$

so that

$$|P_t f(x) - f(x)| \le \int_0^\infty |K(r,t)|\,\mathrm{d}r \cdot \sup_{s>0} |P_s \sqrt{-L} f|(x) = \frac{4\sqrt{t}}{\sqrt{\pi}} \sup_{s>0} |P_s \sqrt{-L} f|(x),$$

where we have tacitly used that $t \to P_t f(x)$ is continuous for $\mathfrak{m}$-a.e. $x \in X$. □

## 2. Proof of the Theorem 6.1

As we have already mentioned in the introduction the starting point of the proof of Theorem 6.1 is the inequality

$$|f(x) - f(y)| \le |P_t f(x) - f(x)| + |P_t f(y) - P_t f(x)| + |P_t f(y) - f(y)|, \tag{6.10}$$

which is relevant for our purpose when $t \sim \mathsf{d}(x,y)^2$.

Let us define

$$I_t(x_0, x_1) := |P_t f(x_0) - P_t f(x_1)|, \qquad J_t(x) := |f(x) - P_t f(x)|,$$

and study those functions separately.

**2.1. Estimates on $I_t(x_0, x_1)$.** The main ingredient is *Wang's infinite-dimensional Harnack inequality* for nonnegative functions $g \in L^2(X, \mathsf{d}, \mathfrak{m})$, namely

$$(P_t g)^\alpha(x) \le P_t g^\alpha(y) \exp\left(\frac{\alpha \mathsf{d}(x,y)^2}{2\sigma_K(t)(\alpha - 1)}\right) \qquad \forall x,\, y \in X \tag{6.11}$$

with $\alpha > 1$ and $\sigma_K(t) = K^{-1}(e^{2Kt} - 1)$ if $K \neq 0$, $\sigma_0(t) = 2t$.

This inequality is a consequence of the synthetic lower bound on the Ricci curvature and can be established along the lines of the proof of Wang's log-Harnack inequality given in [**18**].

**Proposition 6.5.** *For every $\alpha \in (1,2]$, $t > 0$ and $f \in W^{1,2}(X, \mathsf{d}, \mathfrak{m})$ one has*

$$|P_t f(x_0) - P_t f(x_1)| \le \mathsf{d}(x_0, x_1) e^{-Kt} \exp\left\{\frac{\mathsf{d}(x_0,x_1)^2}{2\sigma_K(t)(\alpha - 1)}\right\} (P_t |\nabla f|^\alpha(x_0))^{1/\alpha} \qquad \forall x_1 \in X. \tag{6.12}$$

*for $\mathfrak{m}$-a.e. $x_0 \in X$.*

*Proof.* By a simple truncation argument, it is not restrictive to assume that $f$ is bounded, so that all functions $g_r := P_r f$, $r > 0$ are bounded and Lipschitz. If we establish the pointwise inequality

$$|P_t g_r(x_0) - P_t g_r(x_1)| \le \mathsf{d}(x_0, x_1) e^{-Kt} \exp\left\{\frac{\mathsf{d}(x_0,x_1)^2}{2\sigma_K(t)(\alpha - 1)}\right\} (H_t |\nabla g_r|^\alpha(x_0))^{1/\alpha}$$

for all $x_0$, $x_1 \in X$ we can then pass to the limit as $r \to 0$ and use the pointwise continuity of the semigroup on bounded functions to achieve (6.12).

Let $(x_s)_{s\in[0,1]}$ be a unit speed geodesics connecting $x_0$ and $x_1$, we have

$$|P_t g_r(x_0) - P_t g_r(x_1)| \le \int_0^1 \left|\frac{\mathrm{d}}{\mathrm{d}s} P_t g_r(x_s)\right| \mathrm{d}s \le \mathsf{d}(x_0, x_1) \int_0^1 |\nabla P_t g_r|(x_s)\,\mathrm{d}s.$$

We can now use the Bakry-Emery estimate $|\nabla P_t g_r| \le e^{-Kt} P_t |\nabla g_r|$, to get

$$|P_t g_r(x_0) - P_t g_r(x_1)| \le \int_0^1 \left|\frac{\mathrm{d}}{\mathrm{d}s} P_t g_r(x_s)\right| \mathrm{d}s \le \mathsf{d}(x_0, x_1) \int_0^1 P_t|\nabla g_r|(x_s)\,\mathrm{d}s.$$

By using Wang's Harnack inequality (6.11) we get

$$\begin{aligned}
\int_0^1 P_t|\nabla g_r|(x_s)ds \le& \left(\int_0^1 \Big(H_t|\nabla g_r|(x_s)\Big)^\alpha \mathrm{d}s\right)^{1/\alpha} \\
\le& \left(\int_0^1 P_t|\nabla g_r|^\alpha(x_0) \exp\left\{\frac{\alpha\mathsf{d}(x_0,x_s)^2}{2\sigma_K(t)(\alpha-1)}\right\} \mathrm{d}s\right)^{1/\alpha} \\
\le& (P_t|\nabla g_r|^\alpha(x_0))^{1/\alpha} \exp\left\{\frac{\mathsf{d}(x_0,x_1)^2}{2\sigma_K(t)(\alpha-1)}\right\}.
\end{aligned}$$

The proof is complete. □

**2.2. Estimate on $J_t(x)$.** We look for a pointwise estimate of the form

$$|P_t f(x) - f(x)| \le \sqrt{t}\, g(x),$$

where $g$ is a nonnegative function satisfying

$$\|g\|_{L^2} \le C\,\|\nabla f\|_{L^2}.$$

Natural candidates are $g(x) := \sup_{t>0} P_t|\nabla f|(x)$, as in the finite-dimensional theory, or $g(x) = \sup_{t>0} |P_t\sqrt{-\Delta} f|(x)$. Here we focus on the latter.

Let us recall that for every $g \in D(\Delta)$ (that is a dense subset of $W^{1,2}$) we have

$$\left\|\sqrt{-\Delta} g\right\|_{L^2}^2 = \int_X \sqrt{-\Delta} g\,\sqrt{-\Delta} g\,\mathrm{d}\mathfrak{m} = -\int_X g\Delta g\,\mathrm{d}\mathfrak{m} = \int_X |\nabla g|^2\,\mathrm{d}\mathfrak{m}, \tag{6.13}$$

so that $W^{1,2}(X, \mathsf{d}, \mathfrak{m}) \subset D(\sqrt{-\Delta})$. Hence, from Proposition 6.4 we get

$$|P_t f(x) - f(x)| \le \frac{4\sqrt{t}}{\sqrt{\pi}} \sup_{s>0} |P_s\sqrt{-\Delta} f|(x) \qquad \text{for } \mathfrak{m}\text{-a.e. } x \in X. \tag{6.14}$$

and

$$\left\|\sup_{s>0} |P_s\sqrt{-\Delta} f|\right\|_{L^2} \le C\,\|\nabla f\|_{L^2}.$$

**2.3. Conclusion.** We are now in position to conclude the proof of Theorem 6.1. Recalling the decomposition (6.10) and and applying Proposition 6.5, (6.14) with $t := \mathsf{d}(x,y)^2$ we find an $\mathfrak{m}$-negligible set $N$ such that

$$|f(x)-f(y)| \leq \mathsf{d}(x,y)\frac{4}{\pi}(\sup_{t>0}|P_t\sqrt{-\Delta}f|(x)+\sup_{t>0}|P_t\sqrt{-\Delta}f|(y)) \\ +\mathsf{d}(x,y)\exp\left\{-K\mathsf{d}(x,y)^2+\frac{\mathsf{d}(x,y)^2}{2\sigma_K(\mathsf{d}(x,y)^2)(\alpha-1)}\right\}\sup_{t>0}P_t|\nabla f|^\alpha(x),$$

for every $x,y\in X\setminus N$. Where $\sigma_K(t)=K^{-1}(e^{2Kt}-1)$ if $K\neq 0$, $\sigma_0(t)=2t$. In order to conclude the proof we show that

$$\exp\left\{-K\mathsf{d}(x,y)^2+\frac{\mathsf{d}(x,y)^2}{2\sigma_K(\mathsf{d}(x,y)^2)(\alpha-1)}\right\}\leq C_\alpha,$$

for every $x,y\in X$ satisfying $\mathsf{d}(x,y)\leq 1/(K^-)^2$. When $K\geq 0$, using that $\sigma_K(t)\geq 2t$, we obtain

$$-K\mathsf{d}(x,y)^2+\frac{\mathsf{d}(x,y)^2}{2\sigma_K(\mathsf{d}(x,y)^2)(\alpha-1)}\leq\frac{1}{4(1-\alpha)}\quad\text{for every } x,y\in X.$$

When $K<0$ it is easily seen that $\sigma_K(t)\geq 2te^{2Kt}$, thus we deduce
(6.15)

$$-K\mathsf{d}(x,y)^2+\frac{\mathsf{d}(x,y)^2}{2\sigma_K(\mathsf{d}(x,y)^2)(\alpha-1)}\leq -K\mathsf{d}(x,y)^2+\frac{1}{4(1-\alpha)}e^{-2K\mathsf{d}(x,y)^2}\quad\text{for every } x,y\in X.$$

The conclusion easily follows using that $\mathsf{d}(x,y)\leq 1/(K^-)^2$.

# Bibliography


[1] G. Alberti, G. Crippa, and A. L. Mazzucato, *Loss of regularity for the continuity equation with non-Lipschitz velocity field*, Ann. PDE **5** (2019), no. 1, Paper No. 9, 19.

[2] L. Ambrosio, *Corso introduttivo alla teoria geometrica della misura ed alle superfici minime*, Appunti dei Corsi Tenuti da Docenti della Scuola. [Notes of Courses Given by Teachers at the School], Scuola Normale Superiore, Pisa, 1997 (Italian).

[3] ———, *Some fine properties of sets of finite perimeter in Ahlfors regular metric measure spaces*, Adv. Math. **159** (2001), no. 1, 51–67.

[4] ———, *Fine properties of sets of finite perimeter in doubling metric measure spaces*, Set-Valued Anal. **10** (2002), no. 2-3, 111–128. Calculus of variations, nonsmooth analysis and related topics.

[5] ———, *Transport equation and Cauchy problem for $BV$ vector fields*, Invent. Math. **158** (2004), no. 2, 227–260.

[6] ———, *Calculus, heat flow and curvature-dimension bounds in metric measure spaces*, Proceedings of the International Congress of Mathematicians—Rio de Janeiro 2018. Vol. I. Plenary lectures, World Sci. Publ., Hackensack, NJ, 2018, pp. 301–340.

[7] L. Ambrosio, E. Brué, and D. Semola, *Rigidity of the 1-Bakry-Émery inequality and sets of finite perimeter in RCD spaces*, Geom. Funct. Anal. **29** (2019), no. 4, 949–1001.

[8] L. Ambrosio, E. Bruè, and D. Trevisan, *Lusin-type approximation of Sobolev by Lipschitz functions, in Gaussian and* $\mathrm{RCD}(K,\infty)$ *spaces*, Adv. Math. **339** (2018), 426–452.

[9] L. Ambrosio, M. Colombo, and S. Di Marino, *Sobolev spaces in metric measure spaces: reflexivity and lower semicontinuity of slope*, Variational methods for evolving objects, Adv. Stud. Pure Math., vol. 67, Math. Soc. Japan, [Tokyo], 2015, pp. 1–58.

[10] L. Ambrosio and S. Di Marino, *Equivalent definitions of $BV$ space and of total variation on metric measure spaces*, J. Funct. Anal. **266** (2014), no. 7, 4150–4188.

[11] L. Ambrosio, N. Fusco, and D. Pallara, *Functions of bounded variation and free discontinuity problems*, Oxford Mathematical Monographs, The Clarendon Press, Oxford University Press, New York, 2000.

[12] L. Ambrosio and N. Gigli, *A user's guide to optimal transport*, Modelling and optimisation of flows on networks, Lecture Notes in Math., vol. 2062, Springer, Heidelberg, 2013, pp. 1–155.

[13] L. Ambrosio, N. Gigli, A. Mondino, and T. Rajala, *Riemannian Ricci curvature lower bounds in metric measure spaces with $\sigma$-finite measure*, Trans. Amer. Math. Soc. **367** (2015), no. 7, 4661–4701.

[14] L. Ambrosio, N. Gigli, and G. Savaré, *Gradient flows in metric spaces and in the space of probability measures*, 2nd ed., Lectures in Mathematics ETH Zürich, Birkhäuser Verlag, Basel, 2008.

[15] ———, *Density of Lipschitz functions and equivalence of weak gradients in metric measure spaces*, Rev. Mat. Iberoam. **29** (2013), no. 3, 969–996.

[16] ———, *Calculus and heat flow in metric measure spaces and applications to spaces with Ricci bounds from below*, Invent. Math. **195** (2014), no. 2, 289–391.

[17] ———, *Metric measure spaces with Riemannian Ricci curvature bounded from below*, Duke Math. J. **163** (2014), no. 7, 1405–1490.

[18] ———, *Bakry-Émery curvature-dimension condition and Riemannian Ricci curvature bounds*, Ann. Probab. **43** (2015), no. 1, 339–404.

[19] L. Ambrosio and S. Honda, *New stability results for sequences of metric measure spaces with uniform Ricci bounds from below*, Measure theory in non-smooth spaces, Partial Differ. Equ. Meas. Theory, De Gruyter Open, Warsaw, 2017, pp. 1–51.

[20] ———, *Local spectral convergence in* $\mathrm{RCD}^*(K,N)$ *spaces.* part A, Nonlinear Anal. **177** (2018), no. part A, 1–23.

[21] L. Ambrosio, S. Honda, and D. Tewodrose, *Short-time behavior of the heat kernel and Weyl's law on* $\mathrm{RCD}^*(K,N)$ *spaces*, Ann. Global Anal. Geom. **53** (2018), no. 1, 97–119.

[22] L. Ambrosio, B. Kleiner, and E. Le Donne, *Rectifiability of sets of finite perimeter in Carnot groups: existence of a tangent hyperplane*, J. Geom. Anal. **19** (2009), no. 3, 509–540.

[23] L. Ambrosio, M. Lecumberry, and S. Maniglia, *Lipschitz regularity and approximate differentiability of the DiPerna-Lions flow*, Rend. Sem. Mat. Univ. Padova **114** (2005), 29–50 (2006).

[24] L. Ambrosio, A. Mondino, and G. Savaré, *On the Bakry-Émery condition, the gradient estimates and the local-to-global property of* $\mathrm{RCD}^*(K,N)$ *metric measure spaces*, J. Geom. Anal. **26** (2016), no. 1, 24–56.

[25] ———, *Nonlinear diffusion equations and curvature conditions in metric measure spaces*, Mem. Amer. Math. Soc. **262** (2019), no. 1270, v+121.

[26] L. Ambrosio, F. Stra, and D. Trevisan, *Weak and strong convergence of derivations and stability of flows with respect to MGH convergence*, J. Funct. Anal. **272** (2017), no. 3, 1182–1229.

[27] L. Ambrosio and D. Trevisan, *Well-posedness of Lagrangian flows and continuity equations in metric measure spaces*, Anal. PDE **7** (2014), no. 5, 1179–1234.

[28] ———, *Lecture notes on the DiPerna-Lions theory in abstract measure spaces*, Ann. Fac. Sci. Toulouse Math. (6) **26** (2017), no. 4, 729–766 (English, with English and French summaries).

[29] G. Antonelli, E. Brué, and D. Semola, *Volume bounds for the quantitative singular strata of non collapsed RCD metric measure spaces*, Anal. Geom. Metr. Spaces **7** (2019), no. 1, 158–178.

[30] T. Aubin, *Some nonlinear problems in Riemannian geometry*, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 1998. MR1636569

[31] K. Bacher and K.-T. Sturm, *Localization and tensorization properties of the curvature-dimension condition for metric measure spaces*, J. Funct. Anal. **259** (2010), no. 1, 28–56.

[32] D. Bakry and M. Émery, *Diffusions hypercontractives*, Séminaire de probabilités, XIX, 1983/84, Lecture Notes in Math., vol. 1123, Springer, Berlin, 1985, pp. 177–206 (French).

[33] E. Brué, M. Colombo, and C. De Lellis, *Positive solutions of transport equations and classical nonuniqueness of characteristic curves*, arXiv:2003.00539.

[34] E. Brué, S. Di Marino, and F. Strá, *Linear Lipschitz and* $C^1$ *extension operators through random projection*, arXiv:1801.07533.

[35] E. Brué and Q.-H. Nguyen, *Sharp regularity estimates for solutions of the continuity equation drifted by Sobolev vector fields*, arXiv:1806.03466.

[36] ———, *Sobolev estimates for solutions of the transport equation and ODE flows associated to non-Lipschitz drifts*, Accepted at Mathematische Annalen, arXiv:1905.02995.

[37] ———, *On the Sobolev space of functions with derivative of logarithmic order*, Adv. Nonlinear Anal. **9** (2020), no. 1, 836–849.

[38] ———, *Advection diffusion equations with Sobolev velocity field*, arXiv:2003.08198.

[39] E. Bruè, Q.-H. Nguyen, and G. Stefani, *A maximal function characterisation of absolutely continuous measures and Sobolev functions*, Atti Accad. Naz. Lincei Rend. Lincei Mat. Appl. **30** (2019), no. 3, 599–614.

[40] E. Bruè, E. Pasqualetto, and D. Semola, *Rectifiability of the reduced boundary of sets with finite perimeter over* $\mathrm{RCD}(K,N)$ *spaces*, arXiv:1909.00381.

[41] ———, *Rectifiability of* $\mathrm{RCD}(K,N)$ *spaces via* $\delta$*-splitting maps*, arXiv:2001.07911.

[42] E. Bruè and D. Semola, *Regularity of Lagrangian flows over* $\mathrm{RCD}^*(K,N)$ *spaces*, Accepted at J. Reine Angew. Math., arXiv:1803.04387.

[43] ———, *Constancy of the dimension for* $\mathrm{RCD}(K,N)$ *spaces via regularity of Lagrangian flows*, Accepted at Comm. Pure and Applied Math., arXiv:1804.07128.

[44] P. Buser, *A note on the isoperimetric constant*, Ann. Sci. École Norm. Sup. (4) **15** (1982), no. 2, 213–230.

[45] Yu. Burago, M. Gromov, and G. Perelman, *A. D. Aleksandrov spaces with curvatures bounded below*, Uspekhi Mat. Nauk **47** (1992), no. 2(284), 3–51, 222 (Russian, with Russian summary); English transl., Russian Math. Surveys **47** (1992), no. 2, 1–58.

[46] F. Cavalletti and E. Milman, *The globalization theorem for the curvature dimension condition*, arXiv:1612.07623.

[47] F. Cavalletti and A. Mondino, *Sharp and rigid isoperimetric inequalities in metric-measure spaces with lower Ricci curvature bounds*, Invent. Math. **208** (2017), no. 3, 803–849.

[48] J. Cheeger, *Differentiability of Lipschitz functions on metric measure spaces*, Geom. Funct. Anal. **9** (1999), no. 3, 428–517.

[49] J. Cheeger and T. H. Colding, *Almost rigidity of warped products and the structure of spaces with Ricci curvature bounded below*, C. R. Acad. Sci. Paris Sér. I Math. **320** (1995), no. 3, 353–357 (English, with English and French summaries).

[50] ———, *Lower bounds on Ricci curvature and the almost rigidity of warped products*, Ann. of Math. (2) **144** (1996), no. 1, 189–237.

[51] ———, *On the structure of spaces with Ricci curvature bounded below. I*, J. Differential Geom. **46** (1997), no. 3, 406–480.

[52] ———, *On the structure of spaces with Ricci curvature bounded below. II*, J. Differential Geom. **54** (2000), no. 1, 13–35.

[53] ———, *On the structure of spaces with Ricci curvature bounded below. III*, J. Differential Geom. **54** (2000), no. 1, 37–74.

[54] J. Cheeger and D. Gromoll, *The splitting theorem for manifolds of nonnegative Ricci curvature*, J. Differential Geometry **6** (1971/72), 119–128.

[55] J. Cheeger, W. Jiang, and A. Naber, *Rectifiability of singular sets in noncollapsed spaces with Ricci curvature bounded below*, arXiv:1805.07988.

[56] J. Cheeger and A. Naber, *Lower bounds on Ricci curvature and quantitative behavior of singular sets*, Invent. Math. **191** (2013), no. 2, 321–339.

[57] ———, *Regularity of Einstein manifolds and the codimension 4 conjecture*, Ann. of Math. (2) **182** (2015), no. 3, 1093–1165.

[58] T. H. Colding and A. Naber, *Sharp Hölder continuity of tangent cones for spaces with a lower Ricci curvature bound and applications*, Ann. of Math. (2) **176** (2012), no. 2, 1173–1229.

[59] P. Constantin, A. Kiselev, L. Ryzhik, and A. Zlatoš, *Diffusion and mixing in fluid flow*, Ann. of Math. (2) **168** (2008), no. 2, 643–674.

[60] D. Cordero-Erausquin, R. J. McCann, and M. Schmuckenschläger, *A Riemannian interpolation inequality à la Borell, Brascamp and Lieb*, Invent. Math. **146** (2001), no. 2, 219–257.

[61] M. Coti Zelati, G. M. DelGadino, and M. E. Tarek, *On the relation between enhanced dissipation time-scales and mixing rates*, arXiv:1806.03258.

[62] L. Caravenna and G. Crippa, *Uniqueness and Lagrangianity for solutions with lack of integrability of the continuity equation*, C. R. Math. Acad. Sci. Paris **354** (2016), no. 12, 1168–1173 (English, with English and French summaries).

[63] G. Crippa and C. De Lellis, *Estimates and regularity results for the DiPerna-Lions flow*, J. Reine Angew. Math. **616** (2008), 15–46.

[64] S. Daneri and G. Savaré, *Eulerian calculus for the displacement convexity in the Wasserstein distance*, SIAM J. Math. Anal. **40** (2008), no. 3, 1104–1122.

[65] E. De Giorgi, *Su una teoria generale della misura $(r-1)$-dimensionale in uno spazio ad $r$ dimensioni*, Ann. Mat. Pura Appl. (4) **36** (1954), 191–213 (Italian).

[66] ———, *Nuovi teoremi relativi alle misure $(r-1)$-dimensionali in uno spazio ad $r$ dimensioni*, Ricerche Mat. **4** (1955), 95–113 (Italian).

[67] D. Clement, G. Nicola, and P. Enrico, *Quasi-continuous vector fields on* RCD *spaces*, arXiv:1903.04302.

[68] C. De Lellis and L. Székelyhidi Jr., *The Euler equations as a differential inclusion*, Ann. of Math. (2) **170** (2009), no. 3, 1417–1436.

[69] ———, *Dissipative continuous Euler flows*, Invent. Math. **193** (2013), no. 2, 377–407.

[70] G. De Philippis and N. Gigli, *From volume cone to metric cone in the nonsmooth setting*, Geom. Funct. Anal. **26** (2016), no. 6, 1526–1587, DOI 10.1007/s00039-016-0391-6. MR3579705

[71] ———, *Non-collapsed spaces with Ricci curvature bounded from below*, J. Éc. polytech. Math. **5** (2018), 613–650 (English, with English and French summaries).

[72] G. De Philippis, A. Marchese, and F. Rindler, *On a conjecture of Cheeger*, Measure theory in non-smooth spaces, Partial Differ. Equ. Meas. Theory, De Gruyter Open, Warsaw, 2017, pp. 145–155.

[73] G. De Philippis and F. Rindler, *On the structure of $\mathcal{A}$-free measures and applications*, Ann. of Math. (2) **184** (2016), no. 3, 1017–1039.

[74] D. M. Simone, *Sobolev and* BV *spaces on metric measure spaces via derivations and integration by parts*, arXiv:1409.5620.

[75] D. M. Simone, P. Enrico, and S. Elefterios, *Infinitesimal Hilbertianity of locally CAT($\kappa$) spaces*, arXiv:1812.02086.

[76] R. J. DiPerna and P.-L. Lions, *Ordinary differential equations, transport theory and Sobolev spaces*, Invent. Math. **98** (1989), no. 3, 511–547.

[77] M. Erbar, K. Kuwada, and K.-T. Sturm, *On the equivalence of the entropic curvature-dimension condition and Bochner's inequality on metric measure spaces*, Invent. Math. **201** (2015), no. 3, 993–1071.

[78] L. C. Evans and R. F. Gariepy, *Measure theory and fine properties of functions*, Studies in Advanced Mathematics, CRC Press, Boca Raton, FL, 1992.

[79] W. H. Fleming, *Flat chains over a finite coefficient group*, Trans. Amer. Math. Soc. **121** (1966), 160–186.

[80] H. Federer and W. H. Fleming, *Normal and integral currents*, Ann. of Math. (2) **72** (1960), 458–520.

[81] K. Fukaya, *Collapsing of Riemannian manifolds and eigenvalues of Laplace operator*, Invent. Math. **87** (1987), no. 3, 517–547.

[82] M. Fukushima, Y. Oshima, and M. Takeda, *Dirichlet forms and symmetric Markov processes*, De Gruyter Studies in Mathematics, vol. 19, Walter de Gruyter & Co., Berlin, 1994.

[83] N. Gigli, *An overview of the proof of the splitting theorem in spaces with non-negative Ricci curvature*, Anal. Geom. Metr. Spaces **2** (2014), no. 1, 169–213, DOI 10.2478/agms-2014-0006. MR3210895

[84] ______, *The splitting theorem in non-smooth context*, arXiv:1302.5555.

[85] ______, *Optimal maps in non branching spaces with Ricci curvature bounded from below*, Geom. Funct. Anal. **22** (2012), no. 4, 990–999.

[86] ______, *On the differential structure of metric measure spaces and applications*, Mem. Amer. Math. Soc. **236** (2015), no. 1113, vi+91.

[87] ______, *Nonsmooth differential geometry—an approach tailored for spaces with Ricci curvature bounded from below*, Mem. Amer. Math. Soc. **251** (2018), no. 1196, v+161.

[88] ______, *Lecture notes on differential calculus on* RCD *spaces*, Publ. Res. Inst. Math. Sci. **54** (2018), no. 4, 855–918.

[89] N. Gigli and B.-X. Han, *The continuity equation on metric measure spaces*, Calc. Var. Partial Differential Equations **53** (2015), no. 1-2, 149–177.

[90] ______, *Independence on $p$ of weak upper gradients on* RCD *spaces*, J. Funct. Anal. **271** (2016), no. 1, 1–11.

[91] N. Gigli, A. Mondino, and T. Rajala, *Euclidean spaces as weak tangents of infinitesimally Hilbertian metric measure spaces with Ricci curvature bounded below*, J. Reine Angew. Math. **705** (2015), 233–244.

[92] N. Gigli, A. Mondino, and G. Savaré, *Convergence of pointed non-compact metric measure spaces and stability of Ricci curvature bounds and heat flows*, Proc. Lond. Math. Soc. (3) **111** (2015), no. 5, 1071–1129.

[93] N. Gigli and S. Mosconi, *The abstract Lewy-Stampacchia inequality and applications*, J. Math. Pures Appl. (9) **104** (2015), no. 2, 258–275.

[94] N. Gigli and E. Pasqualetto, *Equivalence of two different notions of tangent bundle on rectifiable metric measure spaces*, Accepted at Communications in Analysis and Geometry, arXiv: 1611.09645.

[95] ______, *Behaviour of the reference measure on* RCD *spaces under charts*, Accepted at Communications in Analysis and Geometry, arXiv: 1607.05188.

[96] M. Gromov, *Metric structures for Riemannian and non-Riemannian spaces*, Reprint of the 2001 English edition, Modern Birkhäuser Classics, Birkhäuser Boston, Inc., Boston, MA, 2007. Based on the 1981 French original; With appendices by M. Katz, P. Pansu and S. Semmes; Translated from the French by Sean Michael Bates.

[97] A. Grigor′yan, *Heat kernels on weighted manifolds and applications*, The ubiquitous heat kernel, Contemp. Math., vol. 398, Amer. Math. Soc., Providence, RI, 2006, pp. 93–191.

[98] P. Hajłasz and P. Koskela, *Sobolev met Poincaré*, Mem. Amer. Math. Soc. **145** (2000), no. 688, x+101.

[99] B.-X. Han, *Ricci tensor on* $\mathrm{RCD}^*(K,N)$ *spaces*, J. Geom. Anal. **28** (2018), no. 2, 1295–1314.
[100] J. Heinonen, *Lectures on analysis on metric spaces*, Universitext, Springer-Verlag, New York, 2001.
[101] ———, *Nonsmooth calculus*, Bull. Amer. Math. Soc. (N.S.) **44** (2007), no. 2, 163–232.
[102] J. Heinonen and P. Koskela, *Quasiconformal maps in metric spaces with controlled geometry*, Acta Math. **181** (1998), no. 1, 1–61.
[103] S. Honda, *Bakry-Émery conditions on almost smooth metric measure spaces*, Anal. Geom. Metr. Spaces **6** (2018), no. 1, 129–145.
[104] P.-E. Jabin, *Critical non-Sobolev regularity for continuity equations with rough velocity fields*, J. Differential Equations **260** (2016), no. 5, 4739–4757, DOI 10.1016/j.jde.2015.11.028. MR3437603
[105] R. Jiang, *Cheeger-harmonic functions in metric measure spaces revisited*, J. Funct. Anal. **266** (2014), no. 3, 1373–1394.
[106] R. Jiang, H. Li, and H. Zhang, *Heat kernel bounds on metric measure spaces and some applications*, Potential Anal. **44** (2016), no. 3, 601–627.
[107] V. Kapovitch and A. Mondino, arXiv:1907.02614.
[108] V. Kapovitch, A. Lytchak, and A. Petrunin, *Metric-measure boundary and geodesic flow on Alexandrov spaces*, arXiv:1705.04767.
[109] M. Kell and A. Mondino, *On the volume measure of non-smooth spaces with Ricci curvature bounded below*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **18** (2018), no. 2, 593–610.
[110] Y. Kitabeppu, *A sufficient condition to a regular set being of positive measure on* RCD *spaces*, Potential Anal. **51** (2019), no. 2, 179–196.
[111] E. Le Donne, *Metric spaces with unique tangents*, Ann. Acad. Sci. Fenn. Math. **36** (2011), no. 2, 683–694.
[112] J. R. Lee and A. Naor, *Extending Lipschitz functions via random metric partitions*, Invent. Math. **160** (2005), no. 1, 59–95.
[113] F. Léger, *A new approach to bounds on mixing*, Math. Models Methods Appl. Sci. **28** (2018), no. 5, 829–849.
[114] F. C. Liu, *A Luzin type property of Sobolev functions*, Indiana Univ. Math. J. **26** (1977), no. 4, 645–651.
[115] P. Li and S.-T. Yau, *On the parabolic kernel of the Schrödinger operator*, Acta Math. **156** (1986), no. 3-4, 153–201.
[116] J. Lott and C. Villani, *Ricci curvature for metric-measure spaces via optimal transport*, Ann. of Math. (2) **169** (2009), no. 3, 903–991, DOI 10.4007/annals.2009.169.903. MR2480619
[117] Z. M. Ma and M. Röckner, *Introduction to the theory of (nonsymmetric) Dirichlet forms*, Universitext, Springer-Verlag, Berlin, 1992.
[118] V. Magnani, *On a measure-theoretic area formula*, Proc. Roy. Soc. Edinburgh Sect. A **145** (2015), no. 4, 885–891, DOI 10.1017/S030821051500013X. MR3415533
[119] M. Miranda Jr., *Functions of bounded variation on "good" metric spaces*, J. Math. Pures Appl. (9) **82** (2003), no. 8, 975–1004 (English, with English and French summaries).
[120] S. Modena and L. Székelyhidi Jr., *Non-uniqueness for the transport equation with Sobolev vector fields*, Ann. PDE **4** (2018), no. 2, Paper No. 18, 38.
[121] A. Mondino and A. Naber, *Structure theory of metric measure spaces with lower Ricci curvature bounds*, J. Eur. Math. Soc. (JEMS) **21** (2019), no. 6, 1809–1854.
[122] A. Nakayasu, *Metric viscosity solutions for Hamilton-Jacobi equations of evolution type*, Adv. Math. Sci. Appl. **24** (2014), no. 2, 333–351.
[123] M. Obata, *Certain conditions for a Riemannian manifold to be isometric with a sphere*, J. Math. Soc. Japan **14** (1962), 333–340.
[124] F. Otto and C. Villani, *Generalization of an inequality by Talagrand and links with the logarithmic Sobolev inequality*, J. Funct. Anal. **173** (2000), no. 2, 361–400.
[125] P. Petersen, *Riemannian geometry*, Graduate Texts in Mathematics, vol. 171, Springer-Verlag, New York, 1998.
[126] A. Petrunin, *Semiconcave functions in Alexandrov's geometry*, Surveys in differential geometry. Vol. XI, Surv. Differ. Geom., vol. 11, Int. Press, Somerville, MA, 2007, pp. 137–201.
[127] ———, *Alexandrov meets Lott-Villani-Sturm*, Münster J. Math. **4** (2011), 53–64.

[128] D. Preiss, *Geometry of measures in* $\mathbf{R}^n$*: distribution, rectifiability, and densities*, Ann. of Math. (2) **125** (1987), no. 3, 537–643, DOI 10.2307/1971410. MR890162

[129] T. Rajala, *Local Poincaré inequalities from stable curvature conditions on metric spaces*, Calc. Var. Partial Differential Equations **44** (2012), no. 3-4, 477–494.

[130] ———, *Interpolated measures with bounded density in metric spaces satisfying the curvature-dimension conditions of Sturm*, J. Funct. Anal. **263** (2012), no. 4, 896–924.

[131] T. Rajala and K.-T. Sturm, *Non-branching geodesics and optimal maps in strong* $\mathrm{CD}(K,\infty)$*-spaces*, Calc. Var. Partial Differential Equations **50** (2014), no. 3-4, 831–846.

[132] M.-K. von Renesse and K.-T. Sturm, *Transport inequalities, gradient estimates, entropy, and Ricci curvature*, Comm. Pure Appl. Math. **58** (2005), no. 7, 923–940.

[133] E. M. Stein, *Topics in harmonic analysis related to the Littlewood-Paley theory.*, Annals of Mathematics Studies, No. 63, Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, 1970.

[134] K. T. Sturm, *Analysis on local Dirichlet spaces. III. The parabolic Harnack inequality*, J. Math. Pures Appl. (9) **75** (1996), no. 3, 273–297.

[135] K.-T. Sturm, *On the geometry of metric measure spaces. I*, Acta Math. **196** (2006), no. 1, 65–131.

[136] ———, *On the geometry of metric measure spaces. II*, Acta Math. **196** (2006), no. 1, 133–177.

[137] C. Villani, *Topics in optimal transportation*, Graduate Studies in Mathematics, vol. 58, American Mathematical Society, Providence, RI, 2003.

[138] ———, *Optimal transport*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 338, Springer-Verlag, Berlin, 2009. Old and new.

[139] N. Weaver, *Lipschitz algebras and derivations. II. Exterior differentiation*, J. Funct. Anal. **178** (2000), no. 1, 64–112.

[140] B. White, *A new proof of the compactness theorem for integral currents*, Comment. Math. Helv. **64** (1989), no. 2, 207–220.