LINEAR LOGIC IN A REFUTATIONAL SETTING

MARIO PIAZZA, GABRIELE PULCINI, AND MATTEO TESI

Abstract. Sequent-style refutation calculi with non-invertible rules are challenging to design because multiple proof-search strategies need to be simultaneously verified. In this paper, we present a refutation calculus for the multiplicative-additive fragment of linear logic (MALL) whose binary rule for the multiplicative conjunction ($\otimes$) and the unary rule for the additive disjunction ($\oplus$) fail invertibility. Specifically, we design a cut-free hypersequent calculus HMALL, which is equivalent to MALL, and obtained by transforming the usual tree-like shape of derivations into a parallel and linear structure. Next, we develop a refutation calculus $\overline{\mathsf{HMALL}}$ based on the calculus HMALL. As far as we know, this is also the first refutation calculus for a substructural logic. Finally, we offer a *fractional* semantics for MALL — whereby its formulas are interpreted by a rational number in the closed interval [0, 1] of rational numbers — thus extending to the substructural landscape the project of fractional semantics already pursued for classical and modal logics.

Keywords. Multiplicative-additive linear logic · Refutation calculi · Hypersequents · Fractional semantics.

1. Introduction

To paraphrase the opening lines of Tolstoy's *Anna Karenina*, it can be said that all logical systems are similar in accepting validities, but each system has its own unique way of refuting invalidities. Proof-theory reveals that the art of refuting is often more challenging than that of proving. Specifically, in the context of sequent calculus, proving involves presenting a derivation for a particular sequent, whereas refuting requires verifying that no such derivation is possible [22, 23, 12, 20]. This computational asymmetry between refuting and proving is not surprising given some fundamental characteristics of logic. For example, the set of theorems in first-order logic can be recursively enumerated, unlike the set of refutations, which cannot be decided according to Post's theorem [6].

Abstractly speaking, the design of a sequent-style refutation system may be regarded as a shift from a *universal* requirement:

$$\text{There is not a proof of the sequent } \vdash \Gamma$$

to an *existential* one:

$$\text{There is a refutation of } \vdash \Gamma.$$

In standard sequent calculi, refutation requires checking whether every possible derivation fails to produce a proof of a given sequent. This way of proceeding by exclusions can be tediously long and usually need to heavily rely on the sub-formula property. Conversely, a refutation system allows us to directly produce a certification of the underivability of a sequent by 'proving its unprovability'.

Needless to say, only decidable logics can be characterized by means of a sound and complete refutation system. In a way, developing a sound and complete refutation system helps us to bridge the gap between syntax and semantics by providing a comprehensive and, so to speak, self-contained description of the logical system under analysis. Furthermore, the challenge of refutation increases when dealing with calculi that include non-invertible rules, as they require surveying multiple proof-search strategies simultaneously. This is the case when dealing with linear logic, where the binary rule for the multiplicative conjunction $\otimes$ and the unary rule for the additive disjunction $\oplus$ are both non-invertible. In the $\otimes$-rule, the conclusion's context is split among the premises, while the $\oplus$-rule requires selecting one disjunct while discarding the other. From a more general standpoint, non-invertibility can be viewed as a loss of information: whenever a rule is not invertible, its premises taken together contain more information than the conclusion.

In this paper, we address the issue of developing a refutation calculus for linear logic. As a preliminary step, we introduce an alternative proof-theoretic platform for the multiplicative and additive fragment of linear logic (MALL) [11]. Specifically, we reformulate the sequent system MALL into a proof-system HMALL in which derivations come as chains of hypersequents. Unlike traditional approaches to hypersequents, where the bar '$|$' is typically interpreted disjunctively, we work with a metalogical interpretation of hypersequents in which the separation symbol is interpreted conjunctively. This alternative proof representation has the advantage that every topmost hypersequent displays the topmost sequents of the corresponding derivation in MALL as its components. This feature is crucial because the parallel structure of hypersequents allows for more information to be stored than the standard sequential approach. In fact, a linear and non-branching proof representation makes it possible to conceive of rules with multiple premises as displaying different possible derivations.

Intuitively, the reading of linear connectives in a refutation setting is dual to the one of linear logic. Indeed, refuting an expression whose principal connective is the additive unary connective $\oplus$ amounts to producing a refutation of both disjuncts. Refuting the additive conjunction & consists in producing a refutation of one of the conjuncts. The meaning of the multiplicative disjunction, the $⅋$, is unaltered due to the invertibility, whereas the refutation of the multiplicative conjunction (also called *tensor*), $\otimes$, is obtained by exhibiting a refutation of one of the conjuncts of the tensor with a suitable context.

The construction of a refutation system for a substructural logic offers insights into the nature of substructural derivations. While a proof in linear logic can be seen as a process that satisfies certain constraints on the use of resources, a refutation can be seen as a process that violates these constraints. For instance, the sequent $\vdash \overline{p}⅋(p \otimes p)$ still encodes a kind of process due to the presence of the linear connectives $⅋$ and $\otimes$, but it is unprovable precisely because it involves duplicating the resource p. More in general, the rules of the refutation calculus are formulated to describe and govern the properties of anti-theorems in MALL.

To illustrate this, consider the example of the (anti-)sequent $\vdash p \oplus q, r \& s$. In a refutation setting, the hypersequent bar regains its usual disjunctive reading. Applying the rule & backward, we get the (anti-)hypersequent $\vdash p \oplus q, r \mid \vdash p \oplus q, s$. Now, to apply the rule $\oplus$, we need to make a choice between p and q, which may lead to different derivations. The hypersequential structure below stores all the relevant information:

$$\dfrac{\dfrac{\dfrac{\dashv p, r \mid \dashv p, s \qquad \dashv p, r \mid \dashv q, s}{\dashv p, r \mid \dashv p \oplus q, s}\ \hat{\oplus} \qquad \dfrac{\dashv q, r \mid \dashv p, s \qquad \dashv q, r \mid \dashv q, s}{\dashv q, r \mid \dashv p \oplus q, s}\ \hat{\oplus}}{\dashv p \oplus q, r \mid \dashv p \oplus q, s}\ \hat{\oplus}}{\dashv p \oplus q, r \& s}\ \hat{\&}$$

It is easy to observe that the components of each top-hypersequent correspond exactly to the topmost sequents of a derivation in HMALL of $\dashv p \oplus q, r \mid \dashv p \oplus q, s$. Therefore, the tree-like structure of hypersequents derivations enables us to recover the invertibility of the rules for the connective $\oplus$. Similar considerations lead to an analogous result for the connective $\otimes$.

In our previous work [19], a hypersequent calculus $\overline{\mathbf{HK}}$ for the modal logic K was presented which combined rules both for proving and refuting. These systems are labelled as *hybrid* calculi. To the best of our knowledge, this is the first refutation calculus to employ hypersequents. In our opinion, the approach seems to be general as it allows to handle non invertible connectives by employing the additional structure of hypersequents.

We begin by employing this approach to design a refutation calculus for the additive part of MALL, namely ALL. We then shift the focus to the multiplicative fragment of MALL, MLL [7, 14], and demonstrate that combining the methods used for ALL and MLL results in a refutation system for MALL. The soundness and completeness of the refutation calculi are established using purely proof-theoretic methods. Specifically, soundness is ensured by designing rules that guarantee that the premises' unprovability entails the conclusion's unprovability. Completeness follows from the fact that the branches of a refutation encode all possible derivations of a sequent.

Finally, we continue our work on the project of fractional semantics with respect to MALL, which was initiated with classical logic [18] and continued with modal logics [19, 1]. Fractional semantics is a multi-valued semantics that is driven by pure proof-theoretic considerations, and its truth-values are the rational numbers in the closed interval $[0, 1]$. However, dealing with substructural calculi requires special attention to accommodate non-invertible connectives. In the context of fractional semantics, this naturally leads to *two* different possible interpretations of a given sequent, which is in contrast to the cases of classical and modal logics.

The development of a fractional interpretation for a logic is closely linked to its decidability and the concept of refutation. A value is assigned to each formula of the language and this assignment is functional. Such value is determined via the set of refutations of the given formula and gives a quantitative evaluation of the degree of provability of a certain sequent.

The plan of our paper is as follows. Section 2 sets out the hypersequent framework for MALL and describes the structural properties of the resulting system. Section 3 is devoted to display a refutation system for ALL. Sections 4 and 5 deal with the extension to systems containing the tensor connective, namely MLL and MALL. Section 6 describes the fractional interpretation of linear logic, while Section 7 contains some concluding remarks.

2. A hypersequent presentation for MALL

Given a set of atomic sentences $\mathbf{AT} = \{p, q, \ldots\} \cup \{\overline{p}, \overline{q}, \ldots\}$, the set $\mathcal{F}$ of formulas is defined by the following grammar:

$$\mathcal{F} ::= \mathbf{AT} \,|\, \mathcal{F} \parr \mathcal{F} \,|\, \mathcal{F} \otimes \mathcal{F} \,|\, \mathcal{F} \& \mathcal{F} \,|\, \mathcal{F} \oplus \mathcal{F}.$$

The connectives come in two pairs: *additive* conjunction (with) and disjunction (plus) and *multiplicative* conjunction (tensor) and disjunction (par). We use capital Greek letters $\Gamma, \Delta, \ldots$ to indicate finite multisets of formulas from $\mathcal{F}$. As usual, we write Γ, Δ and Γ, A to indicate the multiset union $\Gamma \uplus \Delta$ and $\Gamma \uplus [A]$, respectively. In other words, by writing $\Gamma' \uplus \Gamma'' = \Gamma$, we mean that Γ', Γ'' is a bipartition of Γ. If $\Gamma = [A_1, A_2, \ldots, A_n]$ and $\circ \in \{\parr, \otimes, \oplus, \&\}$, we write $\circ\Gamma$ to indicate the formula $A_1 \circ A_2 \circ \cdots \circ A_n$. Linear negation $A^\perp$ is inductively defined as usual exploiting the well-known De Morgan dualities [9]. The expression $A \multimap B$ abridges $A^\perp \parr B$. The rules of MALL are recalled in the left column of Figure 1 [11].

We fix some terminological convention which will hold for all the systems discussed in the paper:

- A *derivation* of a sequent is any configuration obtained by an exhaustive backward application of the rules of the calculus.
- A *proof* is a derivation in which every topmost sequent is an initial sequent. The endsequent is said to be *provable*.
- A sequent is *refutable* or *not provable* if no derivation of it is a proof.

Remark 1. The notion of derivation can be considered as a type of decomposition. Essentially, it involves applying the rules of a logical system exhaustively in a bottom-up manner until every formula is reduced to a set of atomic sentences. The concept is well-defined, as the sequent calculus for MALL has a terminating proof-search process, with backtracking necessary to handle non-invertible rules for the connectives $\oplus$ and $\otimes$.

As usual, the *height* of a derivation is taken to be the number of nodes in a branch of maximal length.

We use the notation MLL and HMLL to indicate the *multiplicative* fragments $(\otimes, \parr)$ of MALL and HMALL, respectively. Similarly, ALL and HALL denote the *additive* fragments $(\oplus, \&)$. The two calculi will be shown to be equivalent with respect to their usual sequent-style presentations. However, we immediately observe that in HMALL (and its

MALL	HMALL
AXIOM	
$\dfrac{}{\vdash p, \overline{p}}\, ax$	$\dfrac{}{\vdash p_1, \overline{p}_1 \mid \cdots \mid \vdash p_n, \overline{p}_n}\, ax$
LOGICAL RULES	
$\dfrac{\vdash \Gamma, A \qquad \vdash \Delta, B}{\vdash \Gamma, \Delta, A \otimes B}\, \otimes$	$\dfrac{\mathcal{G} \mid \vdash \Gamma, A \mid \vdash \Delta, B}{\mathcal{G} \mid \vdash \Gamma, \Delta, A \otimes B}\, \otimes$
$\dfrac{\vdash \Gamma, A, B}{\vdash \Gamma, A \parr B}\, \parr$	$\dfrac{\mathcal{G} \mid \vdash \Gamma, A, B}{\mathcal{G} \mid \vdash \Gamma, A \parr B}\, \parr$
$\dfrac{\vdash \Gamma, A \qquad \vdash \Gamma, B}{\vdash \Gamma, A \& B}\, \&$	$\dfrac{\mathcal{G} \mid \vdash \Gamma, A \mid \vdash \Gamma, B}{\mathcal{G} \mid \vdash \Gamma, A \& B}\, \&$
$\dfrac{\vdash \Gamma, A}{\vdash \Gamma, A \oplus B}\, \oplus_{\mathcal{R}}$ $\dfrac{\vdash \Gamma, B}{\vdash \Gamma, A \oplus B}\, \oplus_{\mathcal{R}}$	$\dfrac{\mathcal{G} \mid \vdash \Gamma, A}{\mathcal{G} \mid \vdash \Gamma, A \oplus B}\, \oplus_{\mathcal{L}}$ $\dfrac{\mathcal{G} \mid \vdash \Gamma, B}{\mathcal{G} \mid \vdash \Gamma, A \oplus B}\, \oplus_{\mathcal{L}}$

FIGURE 1. The MALL calculus and its hypersequent version HMALL.

subsystems), derivations are non-branching structures. Consequently, we gain as a payoff an additional logical space which will be exploited to define the refutation systems.

Hypersequents are denoted by capital Latin letters $\mathcal{G}, \mathcal{H}, \mathcal{I}, \ldots$. Following Avron [3, 4], a (one-sided) hypersequent is a list of ordinary (one-sided) sequents separated by the bar symbol '$|$'; that is, $\mathcal{G} =, \vdash \Gamma_1 \mid \vdash \Gamma_2 \mid \cdots \mid \vdash \Gamma_n$. Hypersequents are considered up to the order in which their components are displayed. Specifically, $\mathcal{G} \mid \mathcal{H} \mid \mathcal{I}$ is equivalent to $\mathcal{G} \mid \mathcal{I} \mid \mathcal{H}$. We write $\vdash \Gamma \in \mathcal{G}$ to indicate that the sequent $\vdash \Gamma$ is one of those displayed by $\mathcal{G}$. If a certain hypersequent consists of a unique component $\vdash \Gamma$, we simply indicate it with $\vdash \Gamma$ instead of the longer expression $\varnothing \mid \vdash \Gamma$. An *hyperclause* is an hypersequent $\mathcal{G}$ in which only atomic sentences occur. Specifically, if $\vdash \Gamma \in \mathcal{G}$, then $\Gamma \subset \mathbf{AT}$.

The intended interpretation of a hypersequent $\vdash \Gamma_1 \mid \vdash \Gamma_2 \mid \cdots \mid \vdash \Gamma_n$ in the calculus HMALL is:

$$\vdash \Gamma_i \textit{ is provable for every } i \in \{1, ..., n\}.$$

Therefore, the hypersequents admit here a metalogical interpretation in terms of conjunctions and not of disjunction as usual.

$$
\begin{array}{cl}
\hline \mathcal{G}_1 & r_{\langle 1,1\rangle} \\
\hline & r_{\langle 1,2\rangle} \\
\vdots & \\
\hline \mathcal{G}_n & r_{\langle 1,n\rangle} \\
\hline \mathcal{G} & r_{\langle 1,n+1\rangle}
\end{array}
\quad \circ \quad
\begin{array}{cl}
\hline \mathcal{H}_1 & r_{\langle 2,1\rangle} \\
\hline & r_{\langle 2,2\rangle} \\
\vdots & \\
\hline \mathcal{H}_m & r_{\langle 2,m\rangle} \\
\hline \mathcal{H} & r_{\langle 2,m+1\rangle}
\end{array}
\quad = \quad
\begin{array}{cl}
\hline \mathcal{G}_1 \,|\, \mathcal{H}_1 & r_{\langle 1,1\rangle} \\
\hline & r_{\langle 1,2\rangle} \\
\vdots & \\
\hline \mathcal{G}_n \,|\, \mathcal{H}_1 & r_{\langle 1,n\rangle} \\
\hline \mathcal{G} \,|\, \mathcal{H}_1 & r_{\langle 1,n+1\rangle} \\
\hline & r_{\langle 2,1\rangle} \\
\vdots & \\
\hline \mathcal{G} \,|\, \mathcal{H}_m & r_{\langle 2,m\rangle} \\
\hline \mathcal{G} \,|\, \mathcal{H} & r_{\langle 2,m+1\rangle}
\end{array}
$$

FIGURE 2. Serial composition of two HMALL-proofs.

$$
\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash p,\overline{p}}\,ax}{\vdash p,\overline{p}\oplus q}\,\oplus_{\mathcal{L}} \qquad \dfrac{\dfrac{}{\vdash q,\overline{q}}\,ax}{\vdash \overline{p}\oplus q,\overline{q}}\,\oplus_{\mathcal{L}}}{\vdash p\&\overline{q},\overline{p}\oplus q}\,\& \qquad \dfrac{}{\vdash p,\overline{p}}\,ax}{\dfrac{\vdash (p\&\overline{q})\otimes p,\overline{p}\oplus q,\overline{p}}{\vdash (p\&\overline{q})\otimes p,(\overline{p}\oplus q)\parr\overline{p}}\,\parr}\,\otimes
\qquad\qquad
\begin{array}{cl}
\hline \vdash \overline{q},q \,|\vdash \overline{p},p \,|\vdash p,\overline{p} & ax \\
\hline \vdash \overline{q},q \,|\vdash \overline{p}\oplus q,p \,|\vdash p,\overline{p} & \oplus_{\mathcal{L}} \\
\hline \vdash \overline{q},\overline{p}\oplus q \,|\vdash \overline{p}\oplus q,p \,|\vdash p,\overline{p} & \oplus_{\mathcal{R}} \\
\hline \overline{p}\oplus q,p\&\overline{q} \,|\vdash p,\overline{p} & \& \\
\hline \vdash (p\&\overline{q})\otimes p,\overline{p}\oplus q,\overline{p} & \otimes \\
\hline \vdash (p\&\overline{q})\otimes p,(\overline{p}\oplus q)\parr\overline{p} & \parr
\end{array}
$$

FIGURE 3. A proof in MALL together with its HMALL version.

Lemma 2.1. *If* HMALL *proves the two hypersequents* $\mathcal{G}$ *and* $\mathcal{H}$*, then it also proves* $\mathcal{G}\,|\mathcal{H}$*.*

Proof. Given two HMALL-proofs π and ρ ending in $\mathcal{G}$ and $\mathcal{H}$, respectively, we can produce their 'serial composition' as illustrated in Figure 2 so as to obtain a proof $\pi \circ \rho$ ending in $\mathcal{G}\,|\mathcal{H}$. □

Lemma 2.2. *If* MALL *proves* $\vdash \Gamma$*, then* HMALL *proves the hypersequent* $\vdash \Gamma$*.*

Proof. The proof is a simple routine induction on the height $h(\pi)$ of π, we just need the previous lemma to set things up when binary rules are applied. □

Example 2.1. Figure 3 reports, side by side, a MALL proof together with its hypersequent version in HMALL.

Theorem 2.3. HMALL *proves the hypersequent* $\mathcal{G}$ *if, and only if, each* $\vdash \Gamma \in \mathcal{G}$ *is provable in* MALL*.*

Proof. ($\Rightarrow$) We can prove this easily by emphasizing the soundness property for the MALL-rules and using induction on the height $h(\pi)$ of the HMALL-proof π ending in $\mathcal{G}$.

($\Leftarrow$) We proceed by induction on the number of sequents displayed by $\mathcal{G}$.

Base: The base case is immediately provided by Lemma 2.2.

Step: Assume that the sequents $\vdash \Gamma_1, \vdash \Gamma_2, \ldots, \vdash \Gamma_n, \vdash \Gamma_{n+1}$ are all provable in MALL. By the inductive hypothesis, there exist two HMALL-proofs π and ρ ending in $\vdash \Gamma_1 \,|\vdash \Gamma_2 \,|\cdots|\vdash \Gamma_n$ and $\vdash \Gamma_{n+1}$, respectively. By following the steps indicated in Figure 2, we can easily obtain a third HMALL-proof $\pi \circ \rho$ ending in $\vdash \Gamma_1 \,|\vdash \Gamma_2 \,|\cdots|\vdash \Gamma_n \,|\vdash \Gamma_{n+1}$.

$\square$

Corollary 2.4. *If* HMALL *proves the hypersequent* $\mathcal{G} \mid \vdash \Gamma$, *with* $\mathcal{G} \neq \varnothing$, *then it also proves* $\mathcal{G}$.

Proof. Let $\mathcal{G} \equiv \vdash \Gamma_1 \mid \vdash \Gamma_2 \mid \cdots \mid \vdash \Gamma_n \mid \vdash \Gamma$. According to Theorem 2.3, each sequent $\vdash \Gamma_1, \vdash \Gamma_2, \ldots, \vdash \Gamma_n, \vdash \Gamma$ turns out to be provable in MALL. From this latter fact, again by Theorem 2.3, we conclude that HMALL proves $\vdash \Gamma_1 \mid \vdash \Gamma_2 \mid \cdots \mid \vdash \Gamma_n$. $\square$

Proposition 2.5. *If* HMALL *proves the sequent* $\vdash A \oplus B$, *then it also proves at least one sequent of* $\vdash A$ *or* $\vdash B$. *Moreover, if* HMALL *proves the sequent* $\vdash A \oplus B, C \oplus D$, *then at least one of the following sequents is provable too:*

$$\vdash A, C \oplus D; \qquad \vdash B, C \oplus D; \qquad \vdash A \oplus B, C; \qquad \vdash A \oplus B, D.$$

Proof. In absence of the structural rules, the last inference of any HMALL proof ending in $\vdash A \oplus B$ must be the $\oplus$-application forming the formula $A \oplus B$. The second part of the claim can be proved likewise. $\square$

We call HMALL$^+$ the system obtained from HMALL by adding the following unary version of the cut-rule:

$$\frac{\mathcal{G} \mid \vdash \Gamma, A \mid \vdash \Delta, \overline{A}}{\mathcal{G} \mid \vdash \Gamma, \Delta}\ \mathit{cut}$$

The cut-elimination theorem can be thus reformulated.

Theorem 2.6. *Any* HMALL$^+$*-proof* π *can be turned into a* HMALL*-proof ending in the same hypersequent.*

Proof. The same procedure designed to eliminate cut-applications from MALL-proofs can be applied *mutatis mutandis* to its hypersequent version HMALL. We report below the hyersequent versions of some key reductions.

- Immediate reduction ax/ax:

$$\frac{\dfrac{}{\mathcal{G} \mid \vdash \Gamma, p \mid \vdash \Delta, \overline{p}}\ ax}{\mathcal{G} \mid \vdash \Gamma, \Delta}\ \mathit{cut} \quad \longrightarrow \quad \frac{}{\mathcal{G} \mid \vdash \Gamma, \Delta}\ ax$$

- Parallel reduction $\otimes/⅋$.

$$\frac{\dfrac{\dfrac{\begin{matrix}\pi_1 \\ \vdots \\ \mathcal{G} \mid \vdash \Gamma, A \mid \vdash \Delta, B \mid \vdash \Lambda, \overline{A}, \overline{B}\end{matrix}}{\mathcal{G} \mid \vdash \Gamma, \Delta, A \otimes B \mid \vdash \Lambda, \overline{A}, \overline{B}}\ \otimes}{\mathcal{G} \mid \vdash \Gamma, \Delta, A \otimes B \mid \vdash \Lambda, \overline{A} ⅋ \overline{B}}\ ⅋}{\mathcal{G} \mid \vdash \Gamma, \Delta, \Lambda}\ \mathit{cut} \quad \longrightarrow \quad \frac{\dfrac{\begin{matrix}\pi_1 \\ \vdots \\ \mathcal{G} \mid \vdash \Gamma, A \mid \vdash \Delta, B \mid \vdash \Lambda, \overline{A}, \overline{B}\end{matrix}}{\mathcal{G} \mid \vdash \Gamma, \Lambda, \overline{B} \mid \vdash \Delta, B}\ \mathit{cut}}{\mathcal{G} \mid \vdash \Gamma, \Delta, \Lambda}\ \mathit{cut}$$

- Parallel reduction &/⊕. Here below, $\pi_1 \upharpoonleft \vdash \Gamma, \overline{B}$ indicates the HMALL proof of $\mathcal{G} \,|\, \vdash \Gamma, A \,|\, \vdash \Delta, \overline{A}$ obtained from π_1 according to the algorithm indicated in the demonstration of Corollary 2.4.

$$
\dfrac{\dfrac{\dfrac{\begin{array}{c}\pi_1\\ \vdots\\ \mathcal{G} \,|\, \vdash \Gamma, A \,|\, \vdash \Gamma, B \,|\, \vdash \Delta, \overline{A}\end{array}}{\mathcal{G} \,|\, \vdash \Gamma, A \& B \,|\, \vdash \Delta, \overline{A}}\,\&}{\mathcal{G} \,|\, \vdash \Gamma, A \& B \,|\, \vdash \Delta, \overline{A} \oplus \overline{B}}\,\oplus_{\mathcal{R}}}{\mathcal{G} \,|\, \vdash \Gamma, \Delta}\,\mathit{cut}
\quad\longrightarrow\quad
\dfrac{\begin{array}{c}\pi_1 \upharpoonleft \vdash \Gamma, \overline{B}\\ \vdots\\ \mathcal{G} \,|\, \vdash \Gamma, A \,|\, \vdash \Delta, \overline{A}\end{array}}{\mathcal{G} \,|\, \vdash \Gamma, \Delta}\,\mathit{cut}
$$

□

Corollary 2.7. *Both the ⅋ and the &-connectives are invertible in* HMALL, *that is:*

(1) If HMALL *proves* $\mathcal{G} \,|\, \vdash \Gamma, A \parr B$, *then it also proves the hypersequent* $\mathcal{G} \,|\, \vdash \Gamma, A, B$.

(2) If HMALL *proves* $\mathcal{G} \,|\, \vdash \Gamma, A \& B$, *then it also proves both the hypersequents* $\mathcal{G} \,|\, \vdash \Gamma, A$ *and* $\mathcal{G} \,|\, \vdash \Gamma, B$.

Proof. We handle point (1) and (2) separately. Let us begin with item (1) and assume the provability of $\mathcal{G} \,|\, \vdash \Gamma, A \parr B$. The hypersequents $\vdash A, \overline{A}$ and $\vdash B, \overline{B}$ are both provable in HMALL. By Theorem 2.3, there is a proof π_1 ending in the compound hypersequent $\mathcal{G} \,|\, \vdash \Gamma, A \parr B \,|\, \vdash A, \overline{A} \,|\, \vdash B, \overline{B}$. The proof π_1 can be extended into an HMALL$^+$-proof π in the way indicated below so as to end in $\mathcal{G} \,|\, \vdash \Gamma, A, B$. By Theorem 2.6, the proof π admits a cut-free version π'.

$$
\dfrac{\dfrac{\begin{array}{c}\pi_1\\ \vdots\\ \mathcal{G} \,|\, \vdash \Gamma, A \parr B \,|\, \vdash A, \overline{A} \,|\, \vdash B, \overline{B}\end{array}}{\mathcal{G} \,|\, \vdash \Gamma, A \parr B \,|\, \vdash A, B, \overline{A} \otimes \overline{B}}\,\otimes}{\mathcal{G} \,|\, \vdash \Gamma, A, B}\,\mathit{cut}
\quad\longrightarrow\quad
\begin{array}{c}\pi'_1\\ \vdots\\ \mathcal{G} \,|\, \vdash \Gamma, A, B\end{array}
$$

As for item (2), we can proceed in a similar way. In particular, we consider the HMALL$^+$-proof ending in $\mathcal{G} \,|\, \vdash \Gamma, A$. Again, Theorem 2.6 guarantees the existence of a HMALL-proof ending in the same hypersequent.

$$
\dfrac{\dfrac{\begin{array}{c}\pi_1\\ \vdots\\ \mathcal{G} \,|\, \vdash \Gamma, A \& B \,|\, \vdash A, \overline{A}\end{array}}{\mathcal{G} \,|\, \vdash \Gamma, A \& B \,|\, \vdash A, \overline{A} \oplus \overline{B}}\,\oplus_{\mathcal{R}}}{\mathcal{G} \,|\, \vdash \Gamma, A}\,\mathit{cut}
\quad\longrightarrow\quad
\begin{array}{c}\pi'_1\\ \vdots\\ \mathcal{G} \,|\, \vdash \Gamma, A\end{array}
$$

That is, in virtue of Lemma 2.1, it proves $\mathcal{G} \,|\, \vdash \Gamma, A \,|\, \vdash \Gamma, B$. □

Remark 2. The results presented in this section can also be understood as showing that classical propositional logic can be expressed in a hypersequent calculus HGS4 that does not contain any binary rules. This calculus can be obtained simply by replacing the HMALL-axiom rule with its *weakened* version:

$$\frac{}{\vdash \Gamma_1, p_1, \overline{p}_1 \mid \cdots \mid \vdash \Gamma_n, p_n, \overline{p}_n}\, ax$$

and by renaming the multiplicative unary rule and the additive binary rule with their classical counterpart:

$$\frac{\mathcal{G} \mid \vdash \Gamma, A, B}{\mathcal{G} \mid \vdash \Gamma, A \vee B} \vee/\parr \qquad \frac{\mathcal{G} \mid \vdash \Gamma, A \mid \vdash \Gamma, B}{\mathcal{G} \mid \vdash \Gamma, A \wedge B} \wedge/\&.$$

3. A hypersequent refutation system for ALL

We will present the refutation systems for MALL in a step-by-step manner. First, we will discuss the case of the additive fragment of MALL, followed by the multiplicative fragment, and finally the complete system. By treating each system separately, we can emphasize both the differences and the similarities between them.

We first introduce the notion of an *anti-hypersequent*, which is based on the concept of an *anti-sequent* [5, 12]. One-sided anti-sequents are denoted by reversing the turnstile symbol, as follows: $\dashv \Gamma$. Unlike ordinary sequents, anti-sequents are explicitly introduced to handle logical invalidity. In the present case, we use the anti-sequent $\dashv \Gamma$ to express the *in*validity of the linear formula $\parr\Gamma$. Anti-hypersequents are indicated by placing a hat over the capital letters used to denote ordinary hypersequents, that is: $\hat{\mathcal{G}}, \hat{\mathcal{H}}, \hat{\mathcal{I}}, \ldots$.

As hypersequents allow us to generalize the structure of sequents, anti-hypersequents are designed to add an additional dimension to the structure of anti-sequents. In particular, an anti-hypersequent consists of finite multisets of anti-sequents, i.e., $\hat{\mathcal{G}} =, \dashv \Gamma_1 \mid \dashv \Gamma_2 \mid \cdots \mid \dashv \Gamma_n$. The intended interpretation of an anti-hypersequent $\hat{\mathcal{G}} =, \dashv \Gamma_1 \mid \dashv \Gamma_2 \mid \cdots \mid \dashv \Gamma_n$ is: $\vdash \Gamma_i$ *is refutable for some* $i \in 1, ..., n$. Therefore, in a refutational setting, hypersequents regain their usual metalogical interpretation in terms of disjunction.

Once we restrict ourselves to ALL, each sequent occurring in a proof must display exactly two formulas. This observation leads us to the following remark:

Remark 3. If HALL proves the hypersequent $\mathcal{G}$ and $\vdash \Gamma \in \mathcal{G}$, then $\#\Gamma = 2$, where $\#\Gamma$ denotes the cardinality of the multiset Γ.

Remark 3 permits us to recall the following deductive properties of the plus-connective ($\oplus$) and, consequently, to streamline the development of a refutation system for ALL.

Lemma 3.1. *The following two facts hold true:*

(1) If $\mathcal{G} \mid \vdash p, A \oplus B$ is provable in HALL*, then at least one of the following two hypersequents is also provable*

$$\mathcal{G} \mid \vdash p, A \qquad \mathcal{G} \mid \vdash p, B$$

(2) If $\mathcal{G} \mid \vdash A \oplus B, C \oplus D$ is provable in HALL*, then at least one of the following four hypersequents is also provable.*

$$\mathcal{G} \mid \vdash A, C \oplus D \quad \mathcal{G} \mid \vdash B, C \oplus D \quad \mathcal{G} \mid \vdash A \oplus B, C \quad \mathcal{G} \mid \vdash A \oplus B, D$$

IDENTITY GROUP

$$\dfrac{}{\dashv \Gamma_1 \,|\, \dashv \Gamma_2 \,|\, \cdots \,|\, \dashv \Gamma_n}\ \overline{ax} \quad \#\Gamma_i \neq 2 \text{ or } \Gamma_i \subset \mathbf{AT} \text{ and } \Gamma_i \neq [p, \overline{p}], \text{ for some } 1 \leqslant i \leqslant n.$$

LOGICAL RULES

$$\dfrac{\hat{\mathcal{G}} \,|\, \dashv \Gamma, A \,|\, \dashv \Gamma, B}{\hat{\mathcal{G}} \,|\, \dashv \Gamma, A \& B}\ \hat{\&}$$

$$\dfrac{\hat{\mathcal{G}} \,|\, \dashv A, p \qquad \hat{\mathcal{G}} \,|\, \dashv B, p}{\hat{\mathcal{G}} \,|\, \dashv A \oplus B, p}\ \hat{\oplus}_1$$

$$\dfrac{\hat{\mathcal{G}} \,|\, \dashv A, C \oplus D \qquad \hat{\mathcal{G}} \,|\, \dashv B, C \oplus D \qquad \hat{\mathcal{G}} \,|\, \dashv A \oplus B, C \qquad \hat{\mathcal{G}} \,|\, \dashv A \oplus B, D}{\hat{\mathcal{G}} \,|\, \dashv A \oplus B, C \oplus D}\ \hat{\oplus}_2$$

FIGURE 4. The sequent calculus of $\overline{\mathsf{HALL}}$

Proof. Analogous to the proof of Proposition 2.5. □

The rules of $\overline{\mathsf{HALL}}$, shown in Figure 4, are obtained by combining the information provided by Remark 3 and Lemma 3.1. One notable feature of the calculus $\overline{\mathsf{HALL}}$ is its incorporation of a form of proof-theoretic focusing [2]. Specifically, the rules are designed to impose an order on their applications, such that the connective $\oplus$ can only be analyzed once there are no formulas with & as their principal connective. This relies on the fact that the rule & is invertible, unlike the one for $\oplus$. We conclude this section by showing that $\overline{\mathsf{HALL}}$ actually qualifies as a refutation system for ALL.

We can now prove that $\overline{\mathsf{HALL}}$ turns out to be sound and complete with respect to the set of ALL invalidities.

Theorem 3.2 (Soundness). *If* $\overline{\mathsf{HALL}}$ *proves the anti-hypersequent* $\hat{\mathcal{G}}$*, then there is a* $\dashv \Gamma \in \hat{\mathcal{G}}$ *such that the sequent* $\vdash \Gamma$ *is not provable in* ALL.

Proof. The proof can be easily led by induction on the height of $\overline{\mathsf{HALL}}$ derivations.

Here we just consider the specific case regarding the application of the $\hat{\oplus}_2$-rule. By inductive hypothesis we get the unprovability of one of the components of the premises of $\hat{\oplus}_2$. The crucial case is the one in which each of the following components turns out to be not provable in ALL:

$$\vdash A, C \oplus D; \qquad \vdash B, C \oplus D; \qquad \vdash A \oplus B, C; \qquad \vdash A \oplus B, D.$$

We claim that $\vdash A \oplus B, C \oplus D$ is not provable in ALL. Assume, by contradiction, that it is. By Lemma 3.1 we should conclude the provability of at least one of the sequents displayed above.

□

IDENTITY GROUP

$$\frac{}{\dashv \Gamma}\ \overline{ax} \quad \Gamma \subset \mathbf{AT} \text{ and } \Gamma \neq [p, \overline{p}]$$

LOGICAL RULES

$$\frac{\dashv \Gamma, A}{\dashv \Gamma, A \& B}\ \&_{\mathcal{L}} \qquad\qquad \frac{\dashv \Gamma, B}{\dashv \Gamma, A \& B}\ \&_{\mathcal{R}}$$

$$\frac{\dashv \Gamma, B \qquad \dashv \Gamma, A}{\dashv \Gamma, A \oplus B}\ \oplus$$

FIGURE 5. The dual system $\overline{\mathsf{ALL}}$.

Remark 4. The cut rule cannot be included in the purely refutational setting provided by $\overline{\mathsf{HALL}}$, as it would violate the soundness property, i.e., the preservation of unprovability. To illustrate this, consider the following counterexample:

$$\frac{\dashv p \& r, \overline{q} \oplus \overline{r}, s \qquad \dashv \overline{s}}{\vdash p \& r, \overline{q} \oplus \overline{r}}\ cut$$

The following theorem furnishes a form of completeness regarding refutation: every sequent that cannot be proven in ALL can be proved by a derivation in $\overline{\mathsf{HALL}}$. This result relies on a more general result that we will prove in the next section, which demonstrates that our hypersequent refutation calculi encode information about every possible derivation of an endsequent in the corresponding sequent calculus, see Lemma 4.7.

Theorem 3.3 (Completeness). *If* ALL *does not prove the sequent* $\vdash \Gamma$*, then* $\overline{\mathsf{HALL}}$ *proves the anti-hypersequent* $\dashv \Gamma$*.*

Proof. We can assume that $\#\Gamma = 2$. Since $\vdash \Gamma$ is not provable in ALL, any possible ALL-derivation π ending in $\vdash \Gamma$ must contain at least one non-axiomatic topmost sequent. Since the components of every top hypersequent of every branch $\mathcal{B}$ of a derivation of Γ in $\overline{\mathsf{HALL}}$ are all the initial sequents of a derivation in ALL, we can conclude that such top hypersequents are instances of $\overline{ax}$. Therefore we get the desired conclusion. □

Let us conclude this section by observing that, when refutation is on focus, soundness may be a property much more difficult to prove than completeness.

Example 3.1. To provide a concrete example, let us consider the dual system $\overline{\mathsf{ALL}}$ displayed in Figure 5, which is obtained by inverting the rules for $\oplus$ with the one for &. This system is indeed complete in the sense that if the sequent $\vdash \Gamma$ is refutable in ALL, then $\overline{\mathsf{ALL}}$ proves the anti-sequent $\dashv \Gamma$. In other words, any unprovable sequent Γ in ALL can be formally proved in $\overline{\mathsf{ALL}}$. However, the converse does not hold, i.e., validity is not guaranteed. Specifically, there exist sequents that can be refuted in $\overline{\mathsf{ALL}}$, but are provable in ALL. For instance, consider the sequent $\vdash p \& q, \overline{p} \oplus \overline{q}$:

$$\dfrac{\dfrac{\overline{\vdash q, \overline{p}}\ \overline{ax}}{\vdash p\&q, \overline{p}}\ \& \qquad \dfrac{\overline{\vdash p, \overline{q}}\ \overline{ax}}{\vdash p\&q, \overline{q}}\ \&}{\vdash p\&q, \overline{p} \oplus \overline{q}}\ \oplus$$

4. A hypersequent refutation system for MLL

In this section, we present a hypersequent refutation system for MLL. However, defining a refutation system for MLL is a more complex task for two reasons. Firstly, unlike ALL, MLL does not allow for a specific bound on the cardinality of formulas in provable sequents. Secondly, the connective $\otimes$ is more challenging to handle. This is because of the multiplicative nature of the rule governing it, which necessitates considering all the possible partitions of the multiset of formulas in the conclusion.

We begin with proving some preliminary results for the calculus MLL. All the results hold also for the system HMLL with inessential modifications in the proofs. Let us start with the following lemma.

Lemma 4.1 (**Height-preserving invertibility of the $\invamp$**). *Let π be an* MLL*-derivation for $\vdash \Gamma, A \invamp B$. Then there is a derivation δ ending in $\vdash \Gamma, A, B$ such that $h(\delta) \leqslant h(\pi)$.*

Proof. We proceed with an induction on the height $h(\pi)$ of π.

Base: If $h(\pi) = 1$, then π cannot end in the sequent $\vdash \Gamma, A \invamp B$, and hence the theorem claim is trivially verified.

Step: For $h(\pi) > 1$, we need to consider two cases depending on the last rule applied in π. Here, we discuss the case where the last rule applied is $\invamp$.

($\invamp$-rule) Assume the derivation π comes shaped as follows:

$$\dfrac{\begin{array}{c}\pi_1\\ \vdots\\ \vdash \Gamma', C, D, A \invamp B\end{array}}{\vdash \Gamma', C \invamp D, A \invamp B}\ \invamp$$

By inductive hypothesis, there is a derivation π_1' ending in $\vdash \Gamma', C, D, A, B$. Then, the desired derivation π' can be easily obtained from π_1' by adding a final application of the $\invamp$-rule as indicated below:

$$\dfrac{\begin{array}{c}\pi_1'\\ \vdots\\ \vdash \Gamma', C, D, A, B\end{array}}{\vdash \Gamma', C \invamp D, A, B}\ \invamp$$

□

We need now to distinguish the subset $\mathcal{F}^{\otimes} \subset \mathcal{F}$, whose formulas do not have the *par* ($\invamp$) as principal connective; more formally:

$$\mathcal{F}^{\otimes} = \big\{ A \in \mathcal{F} \,|\, (A \in \mathbf{AT}) \text{ or } (A \equiv B \otimes C, \text{ with } B, C \in \mathcal{F}\,) \big\}$$

A sequent $\vdash \Gamma$ is said to be *reduced* whenever $\Gamma \subset \mathcal{F}^{\otimes}$.

We start with the following observation.

Definition 1 (Strict equivalence of sequents). Two sequents are said to be *strictly equivalent* when one can be derived from the other by means of a series of $\parr$-applications.

Example 4.1. According to the previous definition, the sequents $\vdash\ p, q, u \parr v, r \otimes t$ and $\vdash\ p \parr q, u \parr v, r \otimes t$ are strictly equivalent.

Lemma 4.2. *Given an* MLL*-derivation of* $\vdash \Gamma$*, there is an* MLL*-derivation* δ *of* $\vdash \Gamma'$ *such that:*

- (i) Γ' *is reduced and strictly equivalent to* Γ*,*
- (ii) δ *has at most the same height as* π*.*

Proof. Consider a derivation of $\vdash \Gamma$ and apply height-preserving invertibility of the $\parr$-rule to obtain a derivation of the sequent $\vdash \Gamma'$, where $\vdash \Gamma'$ is reduced and strictly equivalent to $\vdash \Gamma$. □

In what follows, reduced sequents $\vdash \Gamma$ will be written $\vdash \Gamma^{at}, \Gamma^{\otimes}$, so as to make explicit the partition of Γ into atomic sentences Γ^{at} and other formulas whose principal connective is $\otimes$.

To obtain a refutation calculus for MLL, the key is to identify a weakened criterion for the connective $\otimes$ to be invertible. This can be achieved by utilizing the analyticity of the calculus MLL and determining the last rule applied when a reduced sequent is given.

Lemma 4.3 (Splitting tensor). *For any provable reduced sequent* $\vdash \Gamma^{at}, \Gamma^{\otimes}$ *in* MLL *with* $\Gamma^{\otimes} \neq \varnothing$*, there is a formula* $A \otimes B \in \Gamma^{\otimes}$ *such that:*

- (i) *the sequents* $\vdash \Gamma'^{at}, \Gamma'^{\otimes}, A$ *and* $\vdash \Gamma''^{at}, \Gamma''^{\otimes}, B$ *are both provable,*
- (ii) $\Gamma'^{at} \uplus \Gamma''^{at} = \Gamma^{at}$ *and* $\Gamma'^{\otimes} \uplus \Gamma''^{\otimes} = \Gamma^{\otimes} - [A \otimes B]$*.*

Proof. This is a simple consequence of the cut-elimination theorem. □

These preliminary results set the stage for the following characterization of refutability for reduced sequents.

Theorem 4.4. *A reduced sequent* $\vdash \Gamma^{at}, \Gamma^{\otimes}$ *is unprovable in* MLL *if, and only if, for every formula* $A \otimes B \in \Gamma^{\otimes}$ *and every bipartition* $\Gamma' \uplus \Gamma''$ *of* $\Gamma = \Gamma^{at}, \Gamma^{\otimes}$*, at least one of the two sequents* $\vdash \Gamma', A$ *and* $\vdash \Gamma'', B$ *turns out to be unprovable in* MLL*.*

Proof. We will prove the two directions of the biconditional separately.

($\Rightarrow$) Assume that there exists a sequent $\vdash \Gamma^{at}, \Gamma^{\otimes}$ that is refutable in MLL. We further assume that there exists an $A \otimes B \in \Gamma^{\otimes}$ and a partition $\Gamma' \uplus \Gamma'' = \Gamma$ such that $\vdash \Gamma', A$ and $\vdash \Gamma'', B$ are both provable in MLL. By a single application of the $\otimes$-rule, we obtain a contradiction, and hence $\vdash \Gamma^{at}, \Gamma^{\otimes}$ is provable in MLL.

($\Leftarrow$) Assume that $\vdash \Gamma^{at}, \Gamma^{\otimes}$ is provable in MLL. By Lemma 4.3, we obtain a contradiction, and hence the assumption must be false. Therefore, there exists a sequent $\vdash \Gamma^{at}, \Gamma^{\otimes}$ that is refutable in MLL. □

IDENTITY GROUP

$$\dfrac{}{\dashv \Gamma_1 \mid \dashv \Gamma_2 \mid \cdots \mid \dashv \Gamma_n}\ \overline{ax} \quad \Gamma_i \subset \mathbf{AT} \text{ for all } i \text{ and } \Gamma_i \neq [\,p, \overline{p}\,]$$

LOGICAL RULES

$$\dfrac{\{\hat{\mathcal{G}} \mid \dashv \Pi'_{1j_1}, A_1 \mid \dashv \Pi''_{1j_1}, B_1 : j_1 \in J_1\} \qquad \ldots \qquad \{\hat{\mathcal{G}} \mid \dashv \Pi'_{nj_n}, A_n \mid \dashv \Pi''_{nj_n}, B_n : j_n \in J_n\}}{\hat{\mathcal{G}} \mid \dashv \Gamma, A_1 \otimes B_1, \ldots, A_n \otimes B_n}\ \hat{\otimes}^*$$

$$\dfrac{\hat{\mathcal{G}} \mid \dashv \Gamma, A, B}{\hat{\mathcal{G}} \mid \dashv \Gamma, A \parr B}\ \hat{\parr}$$

FIGURE 6. The sequent calculus of $\overline{\mathsf{HMLL}}$

The hypersequent calculus $\overline{\mathsf{HMLL}}$, shown in Figure 6, is designed to capture the notion of MLL-refutability. Unlike ALL, when dealing with multiplicative connectives, we cannot assume any context restriction regarding the number or type of formulas occurring in the sequent. The rules of $\overline{\mathsf{HMLL}}$ are derived directly from the refutability conditions established in the previous section. As before, the calculus is obtained by combining these conditions with a form of proof-theoretic focusing. The same approach is used for the refutation system for MALL, which we will discuss next.

($*$) The following conditions need to be fulfilled for the rule $\hat{\otimes}$:

(1) $\dashv \Gamma, A_1 \otimes B_1, ..., A_n \otimes B_n$ is a reduced sequent, where Γ is a multiset of atomic formulas.
(2) $\Pi'_{ij_i} \uplus \Pi''_{ij_i} = [\Gamma, A_1 \otimes B_1, ..., A_n \otimes B_n] - [A_i \otimes B_i]$.
(3) J_i is the index set of all the possible pairs of contexts $\Pi'_{ij_i}, \Pi''_{ij_i}$ which are partitions of $[\Gamma, A_1 \otimes B_1, ..., A_n \otimes B_n] - [A_i \otimes B_i]$ and satisfy condition 2.

In other words, the rule $\hat{\otimes}$ has as many premises as the submultisets arising from the application of rule $\otimes$ in MLL for every formula $A_i \otimes B_i$ (essentially, the rule is obtained from Theorem 2.4).

Example 4.2. Here below a formal HMLL-refutation for the linearly invalid sequent $\vdash (p \parr p^\perp) \parr (q \otimes q^\perp)$.

$$\dfrac{\dfrac{\dfrac{}{\dashv p, p^\perp, q \mid \dashv q^\perp}\ \overline{ax} \quad \dfrac{}{\dashv p, q \mid \dashv p^\perp, q^\perp}\ \overline{ax} \quad \dfrac{}{\dashv p^\perp, q \mid \dashv p, q^\perp}\ \overline{ax} \quad \dfrac{}{\dashv q \mid \dashv p^\perp, p, q^\perp}\ \overline{ax}}{\dfrac{\dashv p, p^\perp, q \otimes q^\perp}{\dashv p \parr p^\perp, q \otimes q^\perp}\ \hat{\parr}}\ \hat{\otimes}}{\dashv (p \parr p^\perp) \parr (q \otimes q^\perp)}\ \hat{\parr}$$

It is easy to check that each branch of the previous derivation encodes an unsuccessful HMLL proof-search strategy:

$$\dfrac{\dfrac{\dfrac{\vdash p, p^\perp, q \mid \vdash q}{\vdash p, p^\perp, q \otimes q^\perp}\otimes}{\vdash p \parr p^\perp, q \otimes q^\perp}\parr}{\vdash (p \parr p^\perp) \parr (q \otimes q^\perp)}\parr \qquad \dfrac{\dfrac{\dfrac{\vdash p, q \mid \vdash p^\perp, q}{\vdash p, p^\perp, q \otimes q^\perp}\otimes}{\vdash p \parr p^\perp, q \otimes q^\perp}\parr}{\vdash (p \parr p^\perp) \parr (q \otimes q^\perp)}\parr$$

$$\dfrac{\dfrac{\dfrac{\vdash p^\perp, q \mid \vdash p, q}{\vdash p, p^\perp, q \otimes q^\perp}\otimes}{\vdash p \parr p^\perp, q \otimes q^\perp}\parr}{\vdash (p \parr p^\perp) \parr (q \otimes q^\perp)}\parr \qquad \dfrac{\dfrac{\dfrac{\vdash q \mid \vdash p, p^\perp, q}{\vdash p, p^\perp, q \otimes q^\perp}\otimes}{\vdash p \parr p^\perp, q \otimes q^\perp}\parr}{\vdash (p \parr p^\perp) \parr (q \otimes q^\perp)}\parr$$

The following theorem establishes the soundness of HMLL with respect to the set of linearly (multiplicative) invalid sequents. The proof of soundness does not rely on an external mathematical structure designed to interpret linear logic, such as phase semantics [11, 9] or quantale semantics [17], but is purely syntactic and carried out through proof-theoretic considerations.

Theorem 4.5. *If* $\overline{\mathsf{HMLL}}$ *proves* $\hat{\mathcal{G}}$, *then there is an antisequent* $\dashv \Gamma \in \hat{\mathcal{G}}$ *such that* $\vdash \Gamma$ *is refutable in* MLL.

Proof. By induction on the height of the $\overline{\mathsf{HMLL}}$-derivation ending in $\hat{\mathcal{G}}$. If $n = 0$, immediate. If $n > 0$ we distinguish two subcases, if the last rule applied is $\hat{\parr}$, the proof follows straightforwardly from an application of the induction hypothesis and, if needed, an application of $\parr$. If the last rule applied is $\hat{\otimes}$, we have:

$$\dfrac{\{\hat{\mathcal{G}} \mid \dashv \Pi'_{1j_1}, A_1 \mid \vdash \Pi''_{1j_1}, B_1 : j_1 \in J_1\} \qquad \dots \qquad \{\hat{\mathcal{G}} \mid \dashv \Pi'_{nj_n}, A_n \mid \vdash \Pi''_{nj_n}, B_n : j_n \in J_n\}}{\hat{\mathcal{G}} \mid \dashv \Gamma, A_1 \otimes B_1, \dots, A_n \otimes B_n}\hat{\otimes}$$

We apply the induction hypothesis to each of the premises of the rule. Now, if one application of the induction hypothesis yields that there is a sequent in $\hat{\mathcal{G}}$ which is refutable in MLL, we have obtained the desired conclusion. The interesting case is the one in which for every $i \in \{1, \dots, n\}$ there is Π_{ij_i}, C_i, where C_i is either A_i or B_i and Π_{ij_i} is either Π'_{ij_i} or Π''_{ij_i}. In this case we exploit the Lemma 4.3 to conclude that $\Gamma, A_1 \otimes B_1, \dots, A_n \otimes B_n$ is not provable. □

Corollary 4.6. *If* $\dashv A$ *is provable in* $\overline{\mathsf{HMLL}}$, *then* $\vdash A$ *is not provable in* MLL.

Proof. This is simply a specific case of the soundness property. □

We would like to obtain the converse result, i.e., if A is not provable in MLL, then there is a proof of $\dashv A$ in $\overline{\mathsf{HMLL}}$. This amounts to an internal form of completeness with respect to the set of MLL invalid formulas.

We start by proving the following proposition which ensures that the search of a derivation in $\overline{\mathsf{HMLL}}$ is exhaustive. In particular, we show that $\overline{\mathsf{HMLL}}$ encodes every derivation of a sequent in MLL. To prove this crucial result we work with HMLL — the multiplicative fragment of HMALL — which is equivalent to MLL, but closer to the refutation system $\overline{\mathsf{HMLL}}$.

From now on, we will indicate with $\mathsf{top}(\pi)$ the multiset of top-hypersequents displayed by the derivation π.

Lemma 4.7. *Given an* HMLL*-derivation* π *having* $\mathcal{H}$ *as top-hypersequent, there is an* $\overline{\mathsf{HMLL}}$*-derivation* ρ *ending in the same hypersequent and such that* $\mathcal{H} \in \mathsf{top}(\rho)$*.*

Proof. The proof runs by induction on the height of π. For $n = 1$, the proof is trivial. If $n > 0$, we distinguish cases according to the last rule applied.

- If the last rule applied is a $\parr$-application, we have:

$$\frac{\hat{\mathcal{G}} \mid \vdash \Gamma, A, B}{\hat{\mathcal{G}} \mid \vdash \Gamma, A \parr B} \parr$$

  We apply the induction hypothesis to the direct subproof delivering the premise and so we obtain a derivation ρ' of $\hat{\mathcal{G}} \mid \dashv \Gamma, A, B$ with a top anti-hypersequent $\hat{\mathcal{H}}$ such that $\mathsf{top}(\pi') = \hat{\mathcal{H}}$. An application of the rule $\hat{\parr}$ yields the desired result.
- If the last rule applied is $\otimes$, we have:

$$\frac{\hat{\mathcal{G}} \mid \vdash \Gamma, A \mid \vdash \Delta, B}{\hat{\mathcal{G}} \mid \vdash \Gamma, \Delta, A \otimes B} \otimes$$

  To start with, we apply height-preserving invertibility of the $\parr$-rule in HMLL to obtain the reduced sequent $\hat{\mathcal{G}} \mid \vdash \Gamma^{\otimes}, \Gamma^{at}, A \mid \vdash \Delta^{\otimes}, \Delta^{at}, B$. Hence we apply the induction hypothesis to get a derivation ρ' of $\hat{\mathcal{G}} \mid \dashv \Gamma^{\otimes}, \Gamma^{at}, A \mid \dashv \Delta^{\otimes}, \Delta^{at}, B$ in $\overline{\mathsf{HMLL}}$ with a top hypersequent $\hat{\mathcal{H}}$ in ρ' such that $\mathsf{top}(\pi') = \hat{\mathcal{H}}$. We construct the following derivation:

$$\cfrac{\cfrac{\ldots \qquad \cfrac{\overline{\hat{\mathcal{H}}}}{\hat{\mathcal{G}} \mid \dashv \Gamma^{\otimes}, \Gamma^{at}, A \mid \dashv \Delta^{\otimes}, \Delta^{at}, B}\,\vdots\,\rho' \qquad \ldots}{\hat{\mathcal{G}} \mid \dashv \Gamma^{\otimes}, \Gamma^{at}, \Delta^{\otimes}, \Delta^{at}, A \otimes B}\,\hat{\otimes}}{\hat{\mathcal{G}} \mid \dashv \Gamma, \Delta, A \otimes B}\,\hat{\parr}$$

  where the double line indicates a finite cluster of $\hat{\parr}$-applications. The premises not displayed are the ones obtained by the bottom-up application of the rule $\hat{\otimes}$ in system $\overline{\mathsf{HMLL}}$. □

Theorem 4.8. *If* $\vdash A$ *is not provable in* MLL*, then* $\dashv A$ *is provable in* $\overline{\mathsf{HMLL}}$*.*

Proof. If $\vdash A$ is not provable in MLL, then all the possible derivations of $\vdash A$ end with at least one non-axiomatic sequent. By Lemma 4.7, the components of a top-hypersequent in a derivation of a formula A in $\overline{\mathsf{HMLL}}$ correspond to the topmost sequents of a derivation of A in MLL. Since by hypothesis every derivation of $\vdash A$ contains at least one non-axiomatic sequent, then every topmost hypersequent in every branch $\mathcal{B}$ of a derivation of $\dashv A$ in $\overline{\mathsf{MLL}}$ is an instance of $\overline{ax}$. Therefore we get the desired conclusion. □

5. $\overline{\mathsf{HMALL}}$: a refutation calculus for MALL

To develop a unified refutational framework for MALL, we must take into account all the requirements to handle ALL and MLL. In particular, we still rely on the invertibility of the & and ⅋ rules, and we must also consider the potential interaction between the remaining two linear connectives, $\otimes$ and $\oplus$. To do so, we need to expand the concept of a *reduced sequent* as follows:

Definition 2. A reduced sequent in MALL is a syntactic object of the form $\vdash \Gamma, \Delta, \Sigma$ where Γ, Δ and Σ contain exclusively literals, formulas whose main connective is $\otimes$ and formulas whose main connective is $\oplus$, respectively.

As a consequence of the analyticity of HMALL, we obtain another criterion of partial invertibility for the connectives $\otimes$ and $\oplus$.

Lemma 5.1. *A reduced sequent $\vdash \Gamma^{at}, A_1 \otimes B_n, ..., A_n \otimes B_n, C_1 \oplus D_1, ..., C_m \oplus D_m$ is provable if and only if one of the following two conditions is fulfilled:*

- *the sequents $\vdash \Delta_1, A_i$ and $\vdash \Delta_2, B_i$ are both provable for some $i \in \{1, ..., n\}$ and $\Delta_1 \uplus \Delta_2 = [\Gamma^{at}, A_1 \otimes B_n, ..., A_n \otimes B_n, C_1 \oplus D_1, ..., C_m \oplus D_m] - [A_i \otimes B_i]$;*
- *at least one of the following two sequents $\vdash \Delta, C_j$ and $\vdash \Delta, D_j$ turns out to be provable for some $j \in \{1, ..., m\}$ and $\Delta = [\Gamma^{at}, A_1 \otimes B_n, ..., A_n \otimes B_n, C_1 \oplus D_1, ..., C_m \oplus D_m] - [C_j \oplus D_j]$.*

Therefore, the refutation rule which we obtain combines the features of the systems for ALL and MLL. It is a mixed rule which acts both on $\otimes$ and $\oplus$, i.e., on the non-invertible connectives. In brief, it adds to the premises of the rule for MLL all the premises resulting from the criterion for $\oplus$.

$$\frac{\{\hat{\mathcal{G}} \mid \dashv \Pi'_{ij_i}, A_i \mid \dashv \Pi''_{ij_i}, B_i \;:\; j_1 \in J_i,\; 1 \leqslant i \leqslant n\} \qquad \{\hat{\mathcal{G}} \mid \dashv \Delta_k, E_k \;:\; 1 \leqslant k \leqslant m\}}{\hat{\mathcal{G}} \mid \dashv \Gamma, A_1 \otimes B_1, ..., A_n \otimes B_n, C_1 \oplus D_1, ..., C_n \oplus D_n}\;\hat{\otimes}/\hat{\oplus}$$

where:

- Γ is a multiset of atomic formulas.
- $\Pi'_{ij_i} \uplus \Pi''_{ij_i} = [\Gamma, A_1 \otimes B_1, ..., A_n \otimes B_n, C_1 \oplus D_1, ..., C_n \oplus D_n] - [A_i \otimes B_i]$ and J_i is the index of the set of the multisets of possible partitions of the conclusion.
- E_k is C_k or D_k and Δ_k is $[\Gamma, A_1 \otimes B_1, ..., A_n \otimes B_n, C_1 \oplus D_1, ..., C_n \oplus D_n] - [C_k \otimes D_k]$.

The complete system refutation system $\overline{\mathsf{HMALL}}$ for MALL is displayed in Figure 7. This shows how the task of finding a refutation system for MALL needs to combine features of ALL and MLL. We now give a concrete example of the functioning of the system.

Example 5.1. Consider the MALL-invalid formula $\vdash p \,\&(p \multimap q) \multimap q$. Then, we can produce a proof of $\dashv (\overline{p} \oplus (\overline{p} \otimes \overline{q}))⅋q$ in $\overline{\mathsf{HMALL}}$.

IDENTITY GROUP

$$\frac{}{\dashv \Gamma_1 \mid \dashv \Gamma_2 \mid \cdots \mid \dashv \Gamma_n}\ \overline{ax} \qquad \Gamma_i \subset \mathbf{AT} \text{ for all } i \text{ and } \Gamma_i \neq [p, \overline{p}]$$

LOGICAL RULES

$$\frac{\hat{\mathcal{G}} \mid \dashv \Gamma, A \mid \dashv \Gamma, B}{\dashv \Gamma, A \& B}\ \hat{\&}$$

$$\frac{\{\hat{\mathcal{G}} \mid \dashv \Pi'_{ij_i}, A_i \mid \dashv \Pi''_{ij_i}, B_i : j_1 \in J_i, 1 \leq i \leq n\} \qquad \{\hat{\mathcal{G}} \mid \dashv \Delta_k, E_k : 1 \leq k \leq m\}}{\hat{\mathcal{G}} \mid \dashv \Gamma, A_1 \otimes B_1, ..., A_n \otimes B_n, C_1 \oplus D_1, ..., C_n \oplus D_n}\ \hat{\otimes}/\hat{\oplus}$$

$$\frac{\hat{\mathcal{G}} \mid \dashv \Gamma, A, B}{\hat{\mathcal{G}} \mid \dashv \Gamma, A \parr B}\ \hat{\parr}$$

FIGURE 7. The sequent calculus of $\overline{\mathsf{HMALL}}$

$$\dfrac{\dfrac{\dfrac{}{\dashv \overline{p}, q}\ \overline{ax} \qquad \dfrac{\dfrac{}{\dashv \overline{p} \mid \dashv q, \overline{q}}\ \overline{ax} \qquad \dfrac{}{\dashv \overline{p}, q \mid \dashv \overline{q}}\ \overline{ax}}{\dashv \overline{p} \otimes \overline{q}, q}\ \hat{\otimes}/\hat{\oplus}}{\dashv \overline{p} \oplus (\overline{p} \otimes \overline{q}), q}\ \hat{\otimes}/\hat{\oplus}}{\dashv (\overline{p} \oplus (\overline{p} \otimes \overline{q})) \parr q}\ \hat{\parr}$$

Again, our system proves sound and complete, in the sense that $\vdash A$ is not provable in MALL if and only if $\overline{\mathsf{HMALL}}$ proves $\dashv A$.

Theorem 5.2 (**Soundness**). *If* $\overline{\mathsf{HMALL}}$ *proves* $\hat{\mathcal{G}}$*, then there is an antisequent* $\dashv \Gamma \in \hat{\mathcal{G}}$ *such that* $\vdash \Gamma$ *is not provable in* MALL*.*

Proof. As in the case of MLL, see Lemma 5.1. □

Corollary 5.3. *If* HMALL *proves* $\dashv \Gamma$*, then* $\vdash \Gamma$ *is not provable in* MALL*.*

Lemma 5.4. *Given an* HMALL*-derivation* π *having* $\mathcal{H}$ *as top-hypersequent, there is an* $\overline{\mathsf{HMALL}}$*-derivation* ρ *ending in the same hypersequent and such that* $\mathcal{H} \in \mathsf{top}(\rho)$*.*

Proof. The proof follows the same structure as Lemma 4.7, we argue by induction on $h(\pi)$. We detail the case in which the last rule applied is the rule $\oplus$.

$$\frac{\hat{\mathcal{G}} \mid \vdash \Gamma, A}{\hat{\mathcal{G}} \mid \vdash \Gamma, A \oplus B}\ \oplus$$

We apply the induction hypothesis to the premise of the rule and we obtain the following derivation:

$$\begin{array}{c} \overline{\hat{\mathcal{H}}} \\ \vdots \rho' \\ \hat{\mathcal{G}} \mid \dashv \Gamma, A \end{array}$$

Next, by design of the rules, we now that there is a (possibly empty) segment ρ'', such that:

$$\dfrac{}{\hat{\mathcal{H}}}$$
$$\vdots \rho'$$
$$\hat{\mathcal{G}} \mid \dashv \Gamma_1^{at}, \Gamma_1^{\otimes}, \Gamma_1^{\oplus}, A \mid \ldots \mid \dashv \Gamma_n^{at}, \Gamma_n^{\otimes}, \Gamma_n^{\oplus}, A$$
$$\vdots \rho''$$
$$\hat{\mathcal{G}} \mid \dashv \Gamma, A$$

where ρ'' contains only applications of the rules $\hat{\parr}$ and $\hat{\&}$. The desired conclusion follows rearranging the derivation as follows:

$$\dfrac{\ldots \quad \begin{array}{c}\dfrac{}{\hat{\mathcal{H}}} \\ \vdots \rho' \\ \hat{\mathcal{G}} \mid \dashv \Gamma_1^{at}, \Gamma_1^{\otimes}, \Gamma_1^{\oplus}, A \mid \ldots \mid \dashv \Gamma_n^{at}, \Gamma_n^{\otimes}, \Gamma_n^{\oplus}, A\end{array} \quad \hat{\mathcal{G}} \mid \dashv \Gamma_1^{at}, \Gamma_1^{\otimes}, \Gamma_1^{\oplus}, B \mid \ldots \mid \dashv \Gamma_n^{at}, \Gamma_n^{\otimes}, \Gamma_n^{\oplus}, A \quad \ldots}{\hat{\mathcal{G}} \mid \dashv \Gamma_1^{at}, \Gamma_1^{\otimes}, \Gamma_1^{\oplus}, A \oplus B \mid \ldots \mid \dashv \Gamma_n^{at}, \Gamma_n^{\otimes}, \Gamma_n^{\oplus}, A} \hat{\oplus}/\hat{\otimes}$$
$$\vdots \delta$$
$$\hat{\mathcal{G}} \mid \dashv \Gamma_1^{at}, \Gamma_1^{\otimes}, \Gamma_1^{\oplus}, A \oplus B \mid \ldots \mid \dashv \Gamma_n^{at}, \Gamma_n^{\otimes}, \Gamma_n^{\oplus}, A \oplus B$$
$$\vdots \rho''$$
$$\hat{\mathcal{G}} \mid \dashv \Gamma, A \oplus B$$

where δ contains applications of the rule $\hat{\oplus}/\hat{\otimes}$.

The other cases are analogous (e.g. the case in which the last rule applied is $\otimes$) or simpler (whenever the last rule applied is $\parr$ or $\&$). ☐

Theorem 5.5 (Completeness). *If $\vdash A$ is not provable in* MALL*, then $\dashv A$ is provable in* $\overline{\text{HMALL}}$*.*

Proof. See the proof of Theorem 4.8. ☐

The soundness and completeness of our calculus follow the same pattern as the one detailed for $\overline{\text{HMLL}}$. Additionally, our calculus has some advantages over the standard presentation of MALL. Specifically, the lack of invertibility of the rules in MALL means that it does not enjoy the property of *stability*. Stability refers to the property that each derivation of the same (hyper)sequent has the same multiset of top (hyper)sequents. In other words, in MALL it is possible to have two different derivations of the same endsequent with different multisets of topmost sequents. This is a fundamental aspect of derivability in MALL.

Proposition 5.6. *Stability fails in* MALL*.*

Proof. Let us consider the sequent $\vdash (p \oplus q) \parr (\overline{p} \oplus \overline{q})$. We can have the two following derivations:

$$\dfrac{\dfrac{\dfrac{\vdash p, \overline{p}}{\vdash p, \overline{p} \oplus \overline{q}}\oplus}{\vdash p \oplus q, \overline{p} \oplus \overline{q}}\oplus}{\vdash (p \oplus q) \parr (\overline{p} \oplus \overline{q})}\parr \qquad \dfrac{\dfrac{\dfrac{\vdash q, \overline{q}}{\vdash q, \overline{p} \oplus \overline{q}}\oplus}{\vdash p \oplus q, \overline{p} \oplus \overline{q}}\oplus}{\vdash (p \oplus q) \parr (\overline{p} \oplus \overline{q})}\parr$$

Clearly, the multisets of topmost sequents differ. □

This property is recovered within the system $\overline{\mathsf{HMALL}}$. Indeed, two derivations of the same endsequent will share the same multiset of top hypersequents. Essentially, two derivations of the same endsequent in $\overline{\mathsf{HMALL}}$ may differ only with respect to the order of application of the rules and, possibly, to the height.

We observe that, in contrast to HMALL, the rules of $\overline{\mathsf{HMALL}}$ are all invertible, albeit without preserving the height. As we will see, this property essentially depends on the hypersequential structure. From a conceptual standpoint, the full invertibility of the rules ensures that there is no loss of information in the process of building a derivation by the backward application of the rules.

Theorem 5.7. *The following statements hold for the system* $\overline{\mathsf{HMALL}}$*:*

(1) If π_1 is a derivation of $\dashv \hat{\mathcal{G}} \mid \Gamma, A \parr B$, then there is a derivation π_2 of $\dashv \hat{\mathcal{G}} \mid \Gamma, A, B$ and $h(\pi_2) \leqslant h(\pi_1)$ and $\mathsf{top}(\pi_1) = \mathsf{top}(\pi_2)$.

(2) If π_1 is a derivation of $\dashv \hat{\mathcal{G}} \mid \Gamma_1, A_1 \& B_1 \mid ... \mid \Gamma_m, A_m \& B_m$, then there is a derivation π_2 of $\dashv \hat{\mathcal{G}} \mid \Gamma_1, A_1 \mid \Gamma_1, B_1 \mid ... \mid \Gamma_m, A_m \mid \Gamma_m, B_m$ for any m and $h(\pi_2) \leqslant 2 \cdot h(\pi_1)$ and $\mathsf{top}(\pi_1) = \mathsf{top}(\pi_2)$.

(3) If π is a derivation of $\dashv \hat{\mathcal{G}} \mid \Gamma^{at}, A_1 \otimes B_1, ..., A_m \otimes B_m, C_1 \oplus D_1, ..., C_n \oplus D_n$, then there are derivations $\pi_1, ..., \pi_k$ for any premise of the mixed rule $\hat{\otimes}/\hat{\oplus}$ and $h(\pi_i) \leqslant h(\pi)$ and top$(\pi) = \bigcup_{1 \leqslant i \leqslant k} \mathsf{top}(\pi_i)$.

Proof. We discuss the items separately. Item (3) is immediate due to the context restriction imposed on the mixed rule as no other rule can be applied apart from the one yielding the desired premise. As for item (2), we argue by induction on $h(\pi_1)$. If $h(\pi_1) = 0$, the proof is trivial by the design of the initial sequents. If $h(\pi_1) > 0$, then we distinguish two subcases. If the last rule acted on another component, then the proof follows from the application of the induction hypothesis to the premises and then of the rule again. If the last rule acted in $\Gamma_j, A_j \& B_j$ for some j with $1 \leqslant j \leqslant m$, then we distinguish cases according to the last rule applied. If $A_j \& B_j$ is principal, it is enough to apply the induction hypothesis to the premise to get the desired result. If the last rule applied is $\parr$, we have:

$$\dfrac{\overset{n-1}{\dashv} \hat{\mathcal{G}} \mid \dashv \Gamma_1, A_1 \& B_1, C, D \mid ... \mid \dashv \Gamma_m, A_m \& B_m}{\overset{n}{\dashv} \hat{\mathcal{G}} \mid \dashv \Gamma_1, A_1 \& B_1, C \parr D \mid ... \mid \dashv \Gamma_m, A_m \& B_m}\hat{\parr}$$

We proceed as follows:

$$\dfrac{\dfrac{\begin{array}{c}\vdots\ \mathrm{IH}\\ \overset{2n-2}{\dashv}\ \hat{\mathcal{G}} \mid\ \dashv \Gamma_1, A_1, C, D \mid\ \dashv \Gamma_1, B_1, C, D \mid \ldots \mid\ \dashv \Gamma_m, A_m \mid\ \dashv \Gamma_m, B_m\end{array}}{\overset{2n-1}{\dashv}\ \hat{\mathcal{G}} \mid\ \dashv \Gamma_1, A_1, C \parr D \mid\ \dashv \Gamma_1, B_1, C, D \mid \ldots \mid\ \dashv \Gamma_m, A_m \mid\ \dashv \Gamma_m, B_m}\ \hat{\parr}}{\overset{2n}{\dashv}\ \hat{\mathcal{G}} \mid\ \dashv \Gamma_1, A_1, C \parr D \mid\ \dashv \Gamma_1, B_1, C \parr D \mid \ldots \mid\ \dashv \Gamma_m, A_m \mid\ \dashv \Gamma_m, B_m}\ \hat{\parr}$$

If the last rule applied is &, we proceed analogously:

$$\dfrac{\overset{n-1}{\dashv}\ \hat{\mathcal{G}} \mid\ \dashv \Gamma_1, A_1 \& B_1, C \mid\ \dashv \Gamma_1, A_1 \& B_1, D \mid \ldots \mid\ \dashv \Gamma_m, A_m \& B_m}{\overset{n}{\dashv}\ \hat{\mathcal{G}} \mid\ \dashv \Gamma_1, A_1 \& B_1, C \& D \mid \ldots \mid\ \dashv \Gamma_m, A_m \& B_m}\ \hat{\&}$$

We proceed as follows:

$$\dfrac{\dfrac{\begin{array}{c}\vdots\ \mathrm{IH}\\ \overset{2n-2}{\dashv}\ \hat{\mathcal{G}} \mid\ \dashv \Gamma_1, A_1, C \mid\ \dashv \Gamma_1, B_1, C \mid\ \dashv \Gamma_1, A_1, D \mid\ \dashv \Gamma_1, B_1, D \mid \ldots \mid\ \dashv \Gamma_m, A_m \mid\ \dashv \Gamma_m, B_m\end{array}}{\overset{2n-1}{\dashv}\ \hat{\mathcal{G}} \mid\ \dashv\dashv \Gamma_1, A_1, C \& D \mid\ \dashv \Gamma_1, B_1, C \mid\ \dashv \Gamma_1, B_1, D \mid \ldots \mid\ \dashv \Gamma_m, A_m \mid\ \dashv \Gamma_m, B_m}\ \hat{\&}}{\overset{2n}{\dashv}\ \hat{\mathcal{G}} \mid\ \dashv \Gamma_1, A_1, C \& D \mid\ \dashv \Gamma_1, B_1, C \& D \mid \ldots \mid\ \dashv \Gamma_m, A_m \mid\ \dashv \Gamma_m, B_m}\ \hat{\&}$$

As the reader can see, the preservation of the height fails in the case of permuting $\hat{\&}$ downwards with respect to $\hat{\parr}$ and $\hat{\&}$, as it leads to a duplication in the application of the rules $\hat{\parr}$ and $\hat{\&}$ in both the components of the hypersequential structure. This imposes to prove the preservation of the double value of the height. We can omit the specific details about item (1) since this can handled similarly to item (2). ☐

Theorem 5.8 (Stability). *The calculus* $\overline{\mathsf{HMALL}}$ *satisfies stability.*

Proof. Given two proofs π_1 and π_2 of the same anti-hypersequent $\hat{\mathcal{G}}$ we want to show that $\mathsf{top}(\pi_1) = \mathsf{top}(\pi_2)$. We argue by induction on the height $h(\pi_1)$ of π_1. If $h(\pi_1) = 0$, then the claim is trivial. If $h(\pi_1) > 0$, we distinguish cases according to the last rule applied in π_1. Hence we apply Theorem 5.7 and then the induction hypothesis to the premises to get the desired conclusion.

Let us give a concrete example of this qualitative analysis by supposing that the last rule applied in π_1 is a $\hat{\&}$-application:

$$\dfrac{\begin{array}{c}\vdots\ \pi_1'\\ \hat{\mathcal{G}}' \mid\ \dashv \Gamma, A \mid\ \dashv \Gamma, B\end{array}}{\hat{\mathcal{G}}' \mid\ \dashv \Gamma, A \& B}\ \hat{\&}$$

We apply Theorem 5.7 to the derivation π_2 and we get a derivation π_2' with $\mathsf{top}(\pi_2) = \mathsf{top}(\pi_2')$ and $h(\pi_2') \leqslant 2 \cdot h(\pi_2)$. We apply the induction hypothesis on the height of the derivation since $h(\pi_1') < h(\pi_1)$ and we get the following chain of equalities:

$$\mathsf{top}(\pi_1) = \mathsf{top}(\pi_1') =^{IH} \mathsf{top}(\pi_2') = \mathsf{top}(\pi_2)$$

which gives the desired conclusion. Other cases can be treated likewise. □

This result paves the way for a fractional reading of linear logic.

6. Fractional-valued MALL

In this section, we aim to establish a fractional semantics for MALL, which is the broadest decidable fragment of linear logic. As we have already noted in the cases of classical and modal logics, the significance of adopting such a semantic approach to linear logic is primarily due to its inherently proof-theoretic nature. More specifically, the interpretations assigned to the formulas are solely determined by the structure of the proofs of the sequent associated with them [18, 19]. From a conceptual perspective, this is desirable because it is in line with a traditional goal of linear logic, which is to reject a Tarskian-like interpretation of logic in which expressions are interpreted within a corresponding metalanguage [10]. In fact, each formula is assigned a set of refutations that determines its interpretation.

To define a fractional interpretation of a given (decidable) logic an adequate proof-theoretic platform is required. In particular, a sequent calculus should satisfy three conditions: (*i*) termination of the proof search; (*ii*) invertibility of the rules, and (*iii*) stability. To the extent that the (hyper)sequent calculus $\overline{\mathsf{HMALL}}$ meets all these properties, it offers the possibility to fractionally evaluate MALL formulas. Since the first condition requires decidability of the underlying logic, it is necessary to have a decidable logic before a fractional interpretation can be expressed. A sequent calculus that is fully invertible and terminates for a decidable logic can be seen as a means of bridging the gap between semantics and syntax. To demonstrate that a sequent is invalid, it is sufficient to construct a proof of it within a refutation system.

However, when designing a fractional interpretation for MALL formulas, we are immediately facing a specific difficulty. In fact, while in the cases of classical and modal logics the interpretation of a formula can be determined by the decomposition resulting from the bottom-up applications of the rules, in $\overline{\mathsf{HMALL}}$ — although the decomposition is unique — each branch corresponds to a possible MALL-derivation of the endsequent. More specifically, in an HMALL-derivation tree of $\vdash A$, each path from one of the top hypersequents $\mathcal{G}$ to the end-hypersequent $\vdash A$ encodes a MALL-derivation whose top-sequents are exactly those occurring in $\mathcal{G}$.

Two possible interpretations arise from this fact. The first interpretation involves assigning a fractional value equal to the maximal value of the topmost hypersequents. The second interpretation involves determining a value based on the ratio between tautological top-hypersequents and the total number of top-hypersequents. Both solutions are interesting. The first solution expresses the idea that the fractional value corresponds to the derivation that is closest to proving the endsequent. The second solution is analogous to the assignment used in classical and modal logic, although it is not conservative with respect to theorems of linear logic.

Formula	Value
$(p \multimap (q \multimap r)) \multimap ((p \& q) \multimap r)$	$0.\overline{6}$
$(p \multimap (p \multimap q)) \multimap (p \multimap q)$	$0.\overline{6}$
$p \,\&(p \multimap q) \multimap q$	0.5
$p \multimap p \otimes p$	0.5
$(p \otimes \overline{p}) \otimes (q \parr \overline{q})$	$0.\overline{3}$
$p \multimap (q \multimap p)$	0

FIGURE 8. Some linear formulas together with their fractional value.

Let us start by exploring the first interpretation. Recall that $\mathsf{top}(\pi)$ denotes the multiset of top-hypersequents displayed by an $\overline{\mathsf{HMALL}}$-derivation π. We need now to refine our notation by writing $\mathsf{top}_1(\pi)$ and $\mathsf{top}_0(\pi)$ to denote the multiset of valid and *in*valid top-hypersequents in π, respectively. Clearly, $\mathsf{top}_1(\pi) \uplus \mathsf{top}_0(\pi) = \mathsf{top}(\pi)$.

Thanks to the property of stability (cf. Theorem 5.8), we can drop the reference to the derivation π ending in $\vdash \Gamma$ and use directly $\mathsf{top}(\Gamma)$ to refer the unique multiset of top-hypersequents resulting from any decomposition of $\vdash \Gamma$. Since the calculus $\overline{\mathsf{HMALL}}$ encodes every possible derivation of MALL, the completeness with respect to the set of refutations guarantees that the resulting fractional value is determined by the derivations which have the maximal number of tautological initial sequents.

Definition 3. Given a formula A and a derivation π of $\dashv A$, we assign to every top-hypersequent $\vdash \Pi_1 \mid \ldots \mid \vdash \Pi_n$ in π a fractional value $\frac{m}{n}$, where $m = \#[\Pi_i \,|\, \Pi_i = [p, \overline{p}]]$. We denote with $\mathsf{top}^{\mathsf{m}}(\Gamma)$ the multiset of top-hypersequents with maximal fractional value. The fractional value $[\![\Gamma]\!]$ of Γ is defined as $\mathtt{max}\{\frac{m}{n} \mid \frac{m}{n}$ is the value of $\mathcal{G} \in \mathsf{top}^{\mathsf{m}}(\Gamma)\}$.

We start discussing a couple of examples; a list of linear formulas together with their fractional value is displayed in Figure 8.

Example 6.1. Consider the formula $\dashv p \multimap (p \otimes p)$ and consider the following decomposition:

$$\dfrac{\dfrac{\dfrac{}{\dashv \overline{p}, p \mid \dashv p}\,\overline{ax} \qquad \dfrac{}{\dashv p \mid \dashv \overline{p}, p}\,\overline{ax}}{\dashv \overline{p}, p \otimes p}\,\hat{\otimes}}{\dashv \overline{p} \parr (p \otimes p)}\,\hat{\parr}$$

In this case, $\mathsf{top}^{\mathsf{m}}(\dashv \overline{p} \parr (p \otimes p)) = [\dashv \overline{p}, p \mid \dashv p; \dashv \overline{p}, p \mid \dashv p]$ so that $[\![\overline{p} \parr (p \otimes p)]\!] = \frac{1}{2}$.

Example 6.2. Consider now the formula $p \,\&(p \multimap q) \multimap q$ together with the corresponding decomposion (which actually qualifies, in this case, as an $\overline{\mathsf{HMALL}}$-proof):

$$\dfrac{\dfrac{\dfrac{}{\dashv \overline{p}, q}\,\overline{ax} \qquad \dfrac{\dfrac{}{\dashv \overline{p} \mid \dashv q, \overline{q}}\,\overline{ax} \qquad \dfrac{}{\dashv \overline{p}, q \mid \dashv \overline{q}}\,\overline{ax}}{\dashv \overline{p} \otimes \overline{q}, q}\,\hat{\otimes}}{\dashv \overline{p} \oplus (\overline{p} \otimes \overline{q}), q}\,\hat{\otimes}}{\dashv (\overline{p} \oplus (\overline{p} \otimes \overline{q})) \parr q}\,\hat{\parr}$$

In this case, $\mathsf{top}^{\mathsf{m}}((\overline{p} \oplus (\overline{p} \otimes \overline{q})) \parr q) = [\dashv \overline{p} \mid \dashv q, \overline{q}]$, therefore we get $[\![(\overline{p} \oplus (\overline{p} \otimes \overline{q})) \parr q]\!] = \frac{1}{2}$.

Degrees of provability. As mentioned in the introductory remarks to this section, there is an alternative method for measuring the fractional value of a given sequent. This involves considering the ratio between the number of tautological initial (hyper)sequents and the total number of topmost (hyper)sequents, analogously to the fractional semantics used in classical and modal logics. However, in the context of linear logic, this approach is not adequate, as it would result in a loss of conservativity over the class of theorems of linear logic.

Example 6.3. Let us consider an example of this phenomenon. The sequent $\vdash p \oplus (q \parr \overline{q})$ is indeed provable in MALL. The refutation system $\overline{\mathsf{HMALL}}$ produces the following decomposition:

$$\dfrac{\dashv p \qquad \dfrac{\dashv q, \overline{q}}{\dashv q \parr \overline{q}}\,\hat{\parr}}{\dashv p \oplus (q \parr \overline{q})}\,\hat{\oplus}$$

As usual, each branch corresponds to a derivation in MALL. The total number of derivations of the sequent in MALL is 2. One of the two is a proof of $\vdash p \oplus (q \parr \overline{q})$ in MALL, the other one is not. Since the sequent is provable, we would expect the fractional value to be 1. However, if we took into account the ratio between the number of tautological initial (hyper)sequents and the total number of initial (hyper)sequents, we would obtain 0.5, thereby losing conservativity.

Nevertheless, when restricted to provable sequents, such valuation introduces *degrees of provability*. In fact, for a sequent to be provable it is enough to find a derivation in MALL. Compare the sequent of the previous example with $\vdash (p \parr \overline{p}) \oplus (q \parr \overline{q})$. In this case the sequent admits two different derivations in MALL which are both proofs.

$$\dfrac{\dfrac{\dfrac{}{\vdash p, \overline{p}}\,ax}{\vdash p \parr \overline{p}}\,\hat{\parr} \qquad \dfrac{\dfrac{}{\vdash q, \overline{q}}\,ax}{\vdash q \parr \overline{q}}\,\hat{\parr}}{\vdash (p \parr \overline{p}) \oplus (q \parr \overline{q})}\,\hat{\oplus}$$

As a consequence, the assigned value is 1.

In this case the fractional value expresses the ratio between the number of proofs and the total number of derivations of the endsequent. Therefore, such second approach may be regarded as a mean to assess the degree of provability of a given sequent, i.e., if a sequent which receives the value 1, then all its derivations are proofs.

7. Concluding remarks

In conclusion, we would like to outline some themes for further research. Perhaps not surprisingly, it would be intriguing to explore a refutational setting for more expressive fragments of linear logic that involve *exponentials*. Another natural task could be to increase the efficiency of the refutation system $\overline{\mathsf{HMALL}}$. In this direction, it could be worthwhile to design calculi that incorporate the interaction of both proofs and refutations, as well as proof systems that deal with "hybrid" or "bilateral" hyper-sequents of the form $\vdash \Gamma_1 \mid \cdots \mid \vdash \Gamma_n \mid \dashv \Delta_1 \cdots \mid \dashv \Delta_m$. We believe that a better understanding of the refutational side of MALL could also aid in improving the syntax of proof-nets, which still exhibit unsatisfactory aspects when additive links are involved [13]. Similar to how the results in [19] provide an algorithm for computing the (deductively minimal) abductive hypothesis in the context of modal logic K [8, 16], we suspect that an analogous result could be smoothly achieved for $\overline{\mathsf{MALL}}$ by using the refutation system $\overline{\mathsf{HMALL}}$. This could be of interest in light of the well-known applications of some fragments of linear logic to the field of computational linguistics [21, 15].

Regarding fractional semantics, it is worth exploring whether a similar approach could be applied to linear logic with exponentials. These exponentials can be seen as S4 modalities in disguise, and since fractional semantics offers an alternative, purely syntactic interpretation of Dugundji's theorem in the case of modal logic, it could be intriguing to investigate if a similar approach can be used for linear logic with exponentials.

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Scuola Normale Superiore di Pisa
Email address: mario.piazza@sns.it

Dipartimento di studi letterari, filosofici e di storia dell'arte
Università di Roma "Tor Vergata"
Email address, G. Pulcini: gabriele.pulcini@uniroma2.it

Scuola Normale Superiore di Pisa
Email address: matteo.tesi@sns.it