CLASSIFICATION OF RATIONAL ANGLES IN PLANE LATTICES II

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Abstract. This paper is a continuation of an earlier one, and completes a classification of the configurations of points in a plane lattice that determine angles that are rational multiples of π. We give a complete and explicit description of lattices according to which of these configurations can be found among their points.

In the recent paper [DVZ22] we considered the problem of classifying lattices (of rank 2) in the plane $\mathbb{R}^2$ and sets of ordered triples of points of the lattice defining angles with amplitudes which are *rational* multiples of π. We will call them 'rational angles'. (We notice at once that the middle point in any such triple can be assumed to be the origin.)

This kind of problem appeared to us rather spontaneously, although it is treated only in a scattered form in the existing literature, and is actually almost absent from it (we found this fact somewhat surprising). But at least the problem is a natural generalisation of some old investigations, some of which are recalled and summarised in [DVZ22]; for example, those on regular polygons with vertices on a plane lattice. At the end of the introduction we shall give a further motivation: the description of closed polygons with vertices on a given lattice and all angles rational.

In fact, we are prompted to point out at once that the problem itself has quite a number of different aspects: for instance, we can vary both the lattice and the points; and we can ask both for all possible configurations of points that determine rational angles in a given lattice, and also for an exact description of the lattices for which a given configuration is possible (and is 'maximal' for that lattice). Furthermore, when both the lattice and the configuration are given, we can ask for a description of all the systems of points in that lattice that give copies of the given configuration.

All of this is explained in detail in the first paper [DVZ22], where we give a sort of 'list' of different questions that can be asked, and where we outline a programme for obtaining complete answers to them. In that paper we also carried out many of the steps in this programme; let us now recall the strategy and the main issues that arise.

0.1. **Trigonometric diophantine equations.** It turns out that any kind of 'complete' classification, whatever that may mean, leads in the first place to certain (rather laborious) arithmetical questions involving *algebraic equations to be solved in unknown roots of unity*.

This is a rather well-known topic in itself, closely related to (and in a sense originating from) the so-called *Manin-Mumford conjecture* and certain analogous problems formulated by Lang, which led to extensive research in several directions (see for example the book by the fourth author [Zan12] and the survey paper [Zan23]). But in fact equations to be solved in roots of unity had also appeared independently, already in Gordan's 1877 investigations on finite subgroups of $\mathrm{PGL}_2(\mathbb{C})$. They were then studied systematically by other authors, in particular by Mann [Man65] and a decade later by Conway and Jones [CJ76]. Further results and methods were then developed by other authors (we cite, for example, the paper [DZ00] by the first and last author of the present work).

In particular, the paper [CJ76] by Conway and Jones improved on some of Mann's (crucial) estimates for the maximum order of the roots of unity appearing in a suitably normalised equation. More generally, their analysis also covered what they called *trigonometric diophantine equations*, meaning, roughly speaking, algebraic equations (or systems of algebraic equations) in variables $x_1, \dots, x_m, y_1, \dots, y_n$, to be solved with rational (or integral) x_i and roots of unity y_j. The terminology is, of course, motivated by the fact that these latter constraints amount to the *rationality* of the *angles* $\theta_j \in \mathbb{R}/\mathbb{Z}$ when we write $y_j = \exp(2\pi\sqrt{-1}\theta_j)$.

2010 *Mathematics Subject Classification.* 11H06, 14G05, 11D61, 51M05.
Key words and phrases. plane lattices, trigonometric diophantine equations, rational points on curves.

In fact, the work of Conway and Jones provided a formal theory and a kind of algorithm for reducing any trigonometric diophantine equation (or system) to an ordinary diophantine system, namely by getting rid of the variables y_i in roots of unity. In particular, this procedure allows one to 'solve' any given algebraic equation (e.g., defined over $\mathbb{Q}$) purely in roots of unity, but is actually more general than this.[1]

0.2. **About our previous paper and the current one.** In fact, the problem we are considering here turned out to be a *mixed problem* of exactly this sort, namely falling within the scope of Conway and Jones's theory. After reducing the problem to a trigonometric diophantine equation, the two main steps of our previous work on this subject were essentially as follows:

(1) to bound the orders of the roots of unity that can appear corresponding to 'non-trivial' configurations, and
(2) to classify the possible configurations according to these orders.

We fully described the configurations where the orders are unbounded, and showed that they can be suitably parametrised. Let us comment on these steps.

First goal. The bounds mentioned above were fully explicit. Nevertheless, the number of cases to be analysed individually was so large that part of the resulting classification, though possible in principle, was not carried out in full detail. Our first objective in the present paper is to address this aspect: we refine some of the previous estimates and complete the computational part of the proof.

Second goal. More interestingly, already in the earlier paper it turned out that a few such configurations, corresponding to certain specific orders of the roots of unity involved, could be parametrised by rational points on certain curves of genus up to 5. This fact truly gives our basic problem a doubly diophantine aspect.

It is well known that the rational points on (smooth) curves can be parametrised by $\mathbb{P}_1(\mathbb{Q})$ (genus 0), or form a group of finite rank (genus 1), or form a finite set (genus > 1). In principle, however, only the first of these cases can be solved effectively with today's knowledge. Hence, even though after the results of the paper [DVZ22] the curves in question were known (and thus the classification was in a sense complete), it was not entirely clear whether all solutions could be written down 'explicitly'.

Our second purpose in this paper is to carry out this step and complete the classification in a fully explicit way. In fact, the particular structure of the curves in question allows us to (a) compute the Mordell-Weil groups of the genus-1 curves and (b) compute the finitely many remaining finite sets of rational points. This is done by a combination of methods, the last step using the Chabauty-Skolem methods, see § 8. We remark that the chain of reductions which allowed us to reduce the problem from curves of genus up to 5 to curves of genus 2 for which known algorithmic methods apply was not guaranteed, *a priori*, to exist.

0.3. **Final comments.** Our discussion of the situation fits well within the scope of the theory of Conway and Jones mentioned above.

As far as we know, this is the first 'practical' example where some naturally given trigonometric diophantine equations lead to *non-trivial* ordinary diophantine equations. Indeed, to the best of our knowledge, in all the previous applications of the theory of Mann and/or Conway and Jones, the main difficulties lay, so to speak, in the 'trigonometric part' of the problem, while the purely diophantine questions turned out to be of a milder nature, usually yielding an overdetermined system even in complex numbers.[2] See for instance the paper [KKPR20] for a remarkable recent example of a geometric problem (already raised by Conway and Jones) leading to equations in roots of unity. In [KKPR20] ordinary diophantine equations do not appear to a large extent, while the computational complexity in dealing with the trigonometric part is more substantial. The authors use several tools, also of a theoretical nature, to reduce this complexity, and many more subtle ideas and devices are needed to achieve the desired goal.

[1] The possibility of such a reduction was already implicit in Mann's inequalities, but the paper by Conway and Jones emphasises this aspect and provides a formal treatment of it.

[2] In fact, this also happens in some of the cases analysed here, but other cases lead to the high-genus curves already mentioned.

0.4. **A further curious motivation.** As we hinted above, we recently realised that the subject of these two papers of ours is closely related to the subject of *polygons with rational angles* (which are relevant in other areas, such as the theory of billiards).

We formally state this link, which is of a very elementary nature, in the following proposition, where by *rational n-tuple* we mean a set of n elements of a lattice of rank 2 in $\mathbb{R}^2$ (which we identify with $\mathbb{C}$) such that any two of them form a rational angle with the origin. In fact, as in [DVZ22], it is easy to see that one can equivalently work with the vector space V generated by the lattice over $\mathbb{Q}$. We call such a vector space V simply a *space*. We have:

Proposition 0.1. A space V as above contains a rational n-tuple if and only if it contains an n-gon with rational angles and no pair of parallel sides.

Proof. If $p_1, \dots, p_n \in V$ are the vertices of such an n-gon, then setting $v_i = p_{i+1} - p_i$ for $i = 1, \dots, n$ (interpreting p_{n+1} as p_1) gives a rational n-tuple. Indeed, each v_i lies in V, the angle between v_{i-1} and v_i is equal to $p_{i-1}\widehat{p_i}p_{i+1}$, and the fact that there is no pair of parallel sides implies that each pair of v_i is $\mathbb{R}$-linearly independent.

Conversely, assume that $v_0, \dots, v_{n-1}$ is a rational n-tuple. Without loss of generality we can assume that the v_i are contained in the same half-plane and are ordered by increasing argument. Set now $p_1 = 0$ and $p_{i+1} = p_i + v_i$ for $i = 1, \dots, n-2$, so that $p_{n-1} = v_1 + v_2 + \cdots + v_{n-2}$. The point $p_{n-1} \in V$ lies in the angle generated by v_0, v_n, so that $p_{n-1} = av_0 + bv_n$ for some positive rational numbers a, b. Set $p_n = bv_n$. Then $p_1p_2 \cdots p_n$ is a rational n-gon. The angle between two consecutive sides is the same as that between two of the v_i, and one can see that the polygon does not cross itself by checking that the coordinates of the sequence p_i $i = 1, \dots, n-1$ with respect to the basis v_0, v_n form two increasing sequences. □

0.5. **Proofs and computations.** Once the theoretical method has been established, the proofs of the main results are carried out with the help of a computer. We describe our approach in Section 1, after recalling some necessary preliminaries from [DVZ22] and setting up our notation, and give details in the remaining sections.

We immediately point out that we develop a rather general strategy to handle trigonometric-diophantine equations which greatly reduces the combinatorial difficulty of analysing all possible 'vanishing subsums' of a given equation. In particular, Proposition 2.2 allows us to check certain properties not for every decomposition of the given equation into vanishing subsums, but rather for every subset of its monomials of a fixed size. For a sense of how the proposition can be applied, and of the enormous improvement that results from this approach, we refer the reader to Proposition 5.2: the computation that proves this result involves checking the 2,002 subsets of cardinality 9 of a set of 14 elements. If we had to consider all possible decompositions into vanishing subsums, we would instead have to loop over all possible 190,899,322 partitions of a set of 14 elements. Even more strikingly, Proposition 7.1 is proved by considering 8,436,285 subsets, rather than the approximately $5.5 \cdot 10^{20}$ partitions of a set of 27 elements. In future work, we plan to apply this strategy to other trigonometric-diophantine equations: for example, the main conjecture of [Wan23] should be amenable to our methods.

The main computations were carried out in MAGMA V2.28-4 [BCP97]; the corresponding source code and output files are available at

https://github.com/DavideLombardoMath/rational-angles-lattices.

All the computer calculations required for this paper took less than 1.5 hours in total on a 10-core Intel Core i9-10900KF CPU @ 3.70GHz with 32GiB of RAM.

Acknowledgements

We thank Alessandra Caraceni for her help with the euclidean proof presented in the appendix. We thank Evan O'Dorney for his comments and his thorough Math Review of the first part of this work. The second and third authors are members of the GNSAGA group of the Istituto Nazionale di Alta Matematica.

1. Notation and conventions

We recall here the main definitions and results from [DVZ22].

For each $n \geq 1$ we write ζ_n for a primitive n-th root of unity, and we denote by U the set of all complex roots of unity.

We call *space* the $\mathbb{Q}$-span of two $\mathbb{R}$-linearly independent elements of $\mathbb{C}$.

We will say that two spaces V_1, V_2 are *homothetic* if one is sent to the other by a homothety of the form $z \mapsto \lambda z$ for some fixed $\lambda \in \mathbb{C}^*$. We will say that they are *equivalent* if V_1 is homothetic to V_2 or its complex conjugate $\overline{V_2}$. The configurations of rational angles appearing in a lattice Λ depend only on the corresponding space $V = \Lambda \otimes \mathbb{Q}$, and in fact only on its homothety class. For this reason, we can – whenever desired – assume that the lattice (or the corresponding space) is of the form $\langle 1, \tau \rangle$ for a suitable $\tau \in \mathbb{C}$.

If V is a space, we call *rational angle* in V a pair (v_1, v_2) of $\mathbb{R}$-linearly independent elements of V such that the argument of v_1/v_2 is a rational multiple of π. Swapping v_1 and v_2 or multiplying them by non-zero rational numbers defines the same angle. More generally, we call *rational n-tuple* an n-tuple of elements of V such that any two of them form a rational angle.

We say that a space is of type $\textcircled{n}$ if it contains a rational n-tuple, and we extend this notation additively by saying that it is of type $\textcircled{n}+\textcircled{m}$ if it contains a rational n-tuple and a disjoint rational m-tuple, and so on. There is an obvious partial order on the possible types of a given space, and we say that V is of a certain *exact* type if that type is maximal for V.

In [DVZ22] it was proved that the existence of two independent rational angles leads to algebraic equations for the generator τ of the normalised lattice $\langle 1, \tau \rangle$. The shape of these equations depends on whether the two angles are adjacent. Having three independent rational angles then leads to two equations for the generator of the lattice, from which one can recover a diophantine-trigonometric equation whose exact form depends on the relative position of the three angles. In particular, spaces of types $\textcircled{4}$, $\textcircled{3}+\textcircled{2}$, and $3\textcircled{2}$ lead to different equations, which have been worked out in [DVZ22, § 3]; we will treat them separately, although using the same techniques.

For a rational angle (v_1, v_2) the root of unity involved in the diophantine-trigonometric equation has an argument which is double to the geometric amplitude of the angle between v_1 and v_2; that is, if μ is a root of unity such that $\mu v_1/v_2 \in \mathbb{R}$, then the equation only involves μ^2. We will sometimes refer to the root of unity μ^2 as the *squared amplitude* of the angle.

Spaces of type $\textcircled{4}$ have already been fully classified in [DVZ22], spaces of type $\textcircled{3}+\textcircled{2}$ will be classified in Section 5, and type $3\textcircled{2}$, the most general, will take up most of the effort of this paper. The diophantine-trigonometric equation arising from a space of type $3\textcircled{2}$ is represented by a polynomial $P \in \mathbb{Z}[a, b, c, d, x, y, z]$ which is of degree 2 in each of the variables x, y, z and homogeneous of degree 4 in the variables a, b, c, d.

The 27 terms of the polynomial P are listed in the following table, where each term is described as a monomial in x, y, z multiplied by a coefficient in a, b, c, d.

(1)

Monomials	Coefficient
$x^2y^2z^2, 1$	$-b(a-c)(b-d)d$
x^2y^2z, z	$b(abc+abd-2acd-2bcd+c^2d+cd^2)$
x^2yz^2, y	$d(a^2b+ab^2-2abc-2abd+acd+bcd)$
xy^2z^2, x	$(a-c)(b-d)(ab+cd)$
x^2y^2, z^2	$-b(b-c)c(a-d)$
x^2yz, yz	$-a^2bc-ab^2c-a^2bd-ab^2d+8abcd-ac^2d-bc^2d-acd^2-bcd^2$
x^2z^2, y^2	$-a(b-c)(a-d)d$
xy^2z, xz	$-2a^2b^2+a^2bc+ab^2c-2abc^2+a^2bd+ab^2d+$ $+ac^2d+bc^2d-2abd^2+acd^2+bcd^2-2c^2d^2$
xyz^2, xy	$-2a^2b^2+a^2bc+ab^2c+a^2bd+ab^2d-2a^2cd+$ $-2b^2cd+ac^2d+bc^2d+acd^2+bcd^2-2c^2d^2$
y^2z^2, x^2	$-a(a-c)c(b-d)$
x^2y, yz^2	$c(a^2b+ab^2-2abc-2abd+acd+bcd)$
x^2z, y^2z	$a(abc+abd-2acd-2bcd+c^2d+cd^2)$
xy^2, xz^2	$(b-c)(a-d)(ab+cd)$
xyz	$2(2a^2b^2-a^2bc-ab^2c+2abc^2-a^2bd-ab^2d+2a^2cd-4abcd+$ $+2b^2cd-ac^2d-bc^2d+2abd^2-acd^2-bcd^2+2c^2d^2)$

Classifying spaces of type $3\textcircled{2}$ amounts to

- determining all solutions to the equation

$$P(a,b,c,d,x,y,z)=0 \tag{2}$$

with $x, y, z \in U$ and $(a : b : c : d) \in \mathbb{P}_3(\mathbb{Q})$ and satisfying the non-degeneracy conditions

$$x, y, z \neq 1, \qquad a, b, c, d \text{ distinct and non-zero;} \tag{3}$$

- understanding if and how these solutions correspond to actual geometric configurations.

In the rest of the paper, whenever we speak of solutions to (2), we implicitly mean that conditions (3) are also satisfied.

1.1. **Correspondence between algebraic solutions and geometric configurations.** Given a solution to (2) such that $ab \neq cd$, we can set

$$\tau = \frac{(cd-ab)x(y-1)(z-1)}{a(y-x)(z-1)+b(xy-1)(z-1)-c(z-x)(y-1)-d(xz-1)(y-1)}; \tag{4}$$

then $V = \langle 1, \tau \rangle_{\mathbb{Q}}$ gives a space with three rational angles $(1, \tau), (\tau + a, \tau + b), (\tau + c, \tau + d)$.

Solutions to (2) such that $ab = cd$ do not necessarily determine a space; this case occurs when the two quadratic equations for τ [DVZ22, 3.11] arising from the two rational angles $(\tau + a, \tau + b), (\tau + c, \tau + d)$ are proportional. This case will be treated separately in Section 6.

1.2. **Symmetries of the equation.** If (a,b,c,d,x,y,z) is a solution of (2) so are

$$\begin{aligned}
&(b,a,c,d,x,y^{-1},z)\\
&(a,b,d,c,x,y,z^{-1})\\
&\left(\frac{1}{a},\frac{1}{b},\frac{1}{c},\frac{1}{d},x^{-1},y,z\right)\\
&(c,d,a,b,x,z,y)\\
&(b(b-c)(b-d),a(b-c)(b-d),b(a-c)(d-b),b(a-d)(c-b),y,x,z)\,.
\end{aligned} \tag{5}$$

The condition $ab \neq cd$ is preserved by these substitutions. Moreover, if $ab \neq cd$ holds, the values of τ given by (4) before and after the substitution produce two homothetic spaces.

Let G be the group of signed 3×3 permutation matrices. The formulas (5) clearly induce an action of G on the set of solutions of (2) by considering the action on x,y,z.

We will call the least common multiple of the multiplicative orders of x,y,z their *common order*. Notice in particular that x,y,z all belong to the cyclotomic field $\mathbb{Q}(\zeta_n)$, where n is their common order.

Let now (a,b,c,d,x,y,z) be a solution to Equation (2) for which x,y,z have common order n. For any $\sigma \in \mathrm{Gal}\left(\mathbb{Q}(\zeta_n)/\mathbb{Q}\right)$, the 7-uple $(a,b,c,d,\sigma(x),\sigma(y),\sigma(z))$ is again a solution to Equation (2). More generally, if σ is an automorphism in $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, then $(a,b,c,d,\sigma(x),\sigma(y),\sigma(z))$ is a solution to (2) whenever (a,b,c,d,x,y,z) is. This action of the Galois group on the set of solutions is non-trivial on the spaces $\langle 1,\tau\rangle_{\mathbb{Q}}$, even up to homothety.

Finally, the actions of G and of $\mathrm{Gal}\left(\overline{\mathbb{Q}}/\mathbb{Q}\right)$ on the set of solutions of (2) commute, hence we get an action of $G\times \mathrm{Gal}\left(\overline{\mathbb{Q}}/\mathbb{Q}\right)$. We will mostly study solutions of Equation (2) up to the action of this group, as made precise in the following definition.

Definition 1.1. We say that two solutions (a,b,c,d,x,y,z) and (a',b',c',d',x',y',z') of Equation (2) are *symmetry-equivalent* if they lie in the same orbit under the action of $G\times \mathrm{Gal}\left(\overline{\mathbb{Q}}/\mathbb{Q}\right)$.

1.3. **Outline of the steps leading to the full classification.** We summarise here how the classification will be broken down:

- Part 1: Solve Equation (2) in the cases when one of a,b,c,d is zero or when two of them coincide. These solutions correspond to spaces of type (3)+(2) (or of type (4), which have already been classified in [DVZ22]); this is done is Section 5.
- Part 2: Find all solutions with $ab=cd$. These correspond to spaces of type (2)+(2)+(2) in which the two quadratic equations [DVZ22, 3.11] coming from the angles $(\tau+a,\tau+b)$ and $(\tau+c,\tau+d)$ are proportional. These spaces are easier to study because, under this assumption, the polynomial P factors. After discarding trivial factors and using the symmetries mentioned above, the equation reduces to a sum of four cosines of rational angles, which was already studied by Conway and Jones. It is important to note, however, that in this case a solution to the equation does not necessarily lead to a valid geometric configuration. We will work out precisely when this happens; this is done in Section 6.

  Taking the results of [DVZ22] and of Parts 1 and 2 into account, we can now assume that a,b,c,d are all distinct, different from zero, and satisfy $ab\neq cd$.
- Part 3: Find all solutions to (2). This is worked out in Section 7.

Although Parts 1 and 2 above are much easier than Part 3, all three of them involve solving a diophantine-trigonometric equation of the same nature. This is achieved by applying the same strategy, which we break down as follows:

- Step 1: Improve the bound given in [DVZ22, Theorem 1.1] on the common order of x,y,z. This is achieved by systematically studying which subsets of the monomials in the equation $P(a,b,c,d,x,y,z)=0$ can add to zero (in the language of [CJ76], these are called *vanishing subsums*). Our approach is fairly general and is described in Section 2, see in particular Proposition 2.2. This already leads to substantial improvements over the bound found in [DVZ22, § 5].
- Step 2: Further limit which prime powers appear in the common order of x,y,z. We achieve this by considering a certain subgroup of the full Galois group of $\mathbb{Q}(x,y,z)/\mathbb{Q}$ and applying the results of Parts 1 and 2 to bypass several subcases. The details are given in Section 4. This is a key step in making the computation feasible and leads to a substantial improvement over [DVZ22].

- Step 3: Once the bound on the common order is small enough, one can exhaustively check all possibilities for x, y, z, leading to finitely many algebraic varieties (most of them empty). For each of them, the set of rational points has to be computed; of course, this is not guaranteed to be possible *a priori*, but by a piece of good luck this step can be done for all the varieties involved in our computations (for example, we find curves of genus 2, 3, 5 whose rational points can be determined either because they admit maps to elliptic curves of rank 0 or because they are amenable to the Chabauty-Coleman approach).

Putting together the results of [DVZ22] and of this paper, the following table summarises all the non-trivial exact types that can occur and where they are described.

Exact type	Spaces	Notes
$\infty\,(6)$	$\mathbb{Q}(\sqrt{-3})$.	CM and Superrectangular [DVZ22, § 4.2]
$\infty\,(4)$	$\mathbb{Q}(\sqrt{-1})$.	CM and Superrectangular [DVZ22, § 4.2]
$\infty\,(2)$	Imaginary quadratic fields different from $\mathbb{Q}(\sqrt{-1})$ and $\mathbb{Q}(\sqrt{-3})$	CM and Rectangular [DVZ22, § 4.2]
(4)	All non-CM Superrectangular spaces and the two dodecagonal spaces	[DVZ22, § 7]
$2\,(3)$	Four spaces	Theorem 5.6
$6\,(2)$	Four elliptic families	Section 7
$5\,(2)$	A rational family	Section 6
$4\,(2)$	A rational family and four elliptic families	Section 7
$3\,(2)$	Three spaces	Section 6

We call these types "non-trivial" since the corresponding spaces possess three independent rational angles. For the sake of completeness, we observe that the "trivial" exact types are attained as well.

Proposition 1.2. There are spaces of exact types 0, (2), $2\,(2)$ and (3).

Proof. Consider, in the space $\langle 1, \tau\rangle$, the points $a_0\tau + a_1$ and $b_0\tau + b_1$. The angle $(a_0\tau + a_1, b_0\tau + b_1)$ is rational if and only if there exists a root of unity $\mu \neq \pm 1$ such that $\frac{b_0\tau+b_1}{a_0\tau+a_1}\mu \in \mathbb{R}$. We recall from [DVZ22, § 3] that this last condition is equivalent to the equation

$$a_0b_0\tau\overline{\tau} + a_0b_1\frac{\mu^2\tau - \overline{\tau}}{\mu^2 - 1} + a_1b_0\frac{\mu^2\overline{\tau} - \tau}{\mu^2 - 1} + a_1b_1 = 0. \tag{6}$$

For fixed a_0, b_0, a_1, b_1, μ, the set of $\tau = x + iy$ which satisfy this equation is a conic in the (real) plane (x, y). It is easily checked that if (a_0, a_1) and (b_0, b_1) are not proportional the equation is not identically satisfied in τ.

Since $\mathbb{C} \cong \mathbb{R}^2$ is not the union of countably many conics, there are infinitely many τ such that $\langle 1, \tau\rangle$ has no rational angles at all, i.e. is of exact type 0. In fact, the set of τ that generate a space with at least one rational angle has Lebesgue measure zero.

Similarly, if we fix $\theta \in U$ and set $\tau = r\theta$, we see that for angles $(a_0\tau + a_1, b_0\tau + b_1)$ different from $(1, \tau)$ Equation (6) has at most 2 solutions in r. Varying (a_0, a_1, b_0, b_1) over the countably many possible choices of angles $(a_0\tau + a_1, b_0\tau + b_1)$, we see that there are only countably many r such that $\langle 1, r\theta\rangle$ has a rational angle different from $(1, \tau)$. Therefore, for any given amplitude θ, there are uncountably many spaces with exactly one rational angle of amplitude θ and no other rational angle, i.e. spaces of exact type (2).

Turning to spaces of exact type $2\,(2)$, we begin by fixing a root of unity θ with large order and consider the lattices generated by complex numbers τ of the form $\tau = r\theta$ with $r \in \mathbb{R}$. According to [DVZ22, § 3.3], the angle $(\tau + a, \tau + b)$ is a rational angle with squared amplitude y if and only if

$$\tau^2(y - 1) + \tau\,[a(y - x) + b(xy - 1)] + abx(y - 1) = 0; \tag{7}$$

it is also useful to consider the corresponding equation for r, namely

$$r^2 + r\frac{a(y - x) + b(xy - 1)}{\theta(y - 1)} + ab = 0. \tag{8}$$

It was proved in [DVZ22, Theorem 1.1] that any space with three independent angles with large enough common order must contain a right angle, so, in order to find a space of exact type $2\,(2)$, we will fix the

second angle in a way that forbids the existence of a right angle in the same space. We can select $y \in U$ and $a, b \in \mathbb{Q}$ so that the degree of y over $\mathbb{Q}(x)$ is large and the left-hand side of (8) is irreducible (as a polynomial in r) over $\mathbb{Q}(x, y)$. This implies that the degree of τ over $\mathbb{Q}(x)$ is large. Suppose that the space $\langle 1, \tau \rangle$ contains a right angle. We can then apply (7) with $y = -1$ (the squared amplitude of a right angle) to obtain that τ has degree at most 2 over $\mathbb{Q}(x)$, contradiction.

A similar argument applies, using [DVZ22, § 3.2] instead of [DVZ22, § 3.3], to show the existence of spaces of exact type (3). □

2. General strategy for subsums

Building upon the strategy in [DVZ22] we explain here how to bound the common order of x, y, z occurring in a solution of (2).

We will find it useful, here and after, to employ the following notation:

(1) we consider elements of $\mathbb{Z}^3$ as vectors with three coordinates, and a vector $\underline{\alpha}$ of $\mathbb{Z}^3$ will be understood to have coordinates $(\alpha_1, \alpha_2, \alpha_3)$;
(2) given two vectors $\underline{\alpha}$ and $\underline{\beta}$ of $\mathbb{Z}^3$ we denote by $\underline{\alpha} \cdot \underline{\beta} = \alpha_1\beta_1 + \alpha_2\beta_2 + \alpha_3\beta_3$ their scalar product;
(3) given nonzero complex numbers z_1, z_2, z_3 and $\underline{\alpha} \in \mathbb{Z}^3$, we denote by $(z_1, z_2, z_3)^{\underline{\alpha}}$ the complex number $z_1^{\alpha_1} z_2^{\alpha_2} z_3^{\alpha_3}$; similarly, if x, y, z are three unknowns, we will write $(x, y, z)^{\underline{\alpha}}$ to denote the rational function $x^{\alpha_1} y^{\alpha_2} z^{\alpha_3}$.

Consider an equation of the form

$$f(x, y, z) := \sum_{\underline{i} \in \{0,1,2\}^3} p_{\underline{i}} \cdot (x, y, z)^{\underline{i}} = 0 \tag{9}$$

where each $p_{\underline{i}}$ is a rational number (in our applications, $p_{\underline{i}}$ will depend polynomially on the rational variables a, b, c, d, but this is not important for the present discussion). Let $(x, y, z) \in U^3$ be a solution of Equation (9). We fix a decomposition of $\{0, 1, 2\}^3$ as a disjoint union $\coprod_{j=1}^{r} I_j$ with the following properties:

(1) for each $j = 1, \dots, r$ we have $\sum_{\underline{i} \in I_j} p_{\underline{i}} \cdot (x, y, z)^{\underline{i}} = 0$;
(2) if I is a proper nonempty subset of I_j some $j = 1, \dots, r$ we have $\sum_{\underline{i} \in I} p_{\underline{i}} \cdot (x, y, z)^{\underline{i}} \neq 0$.

In other words, we fixed a maximally refined decomposition of (9) into vanishing subsums.

Remark 2.1. This decomposition is not necessarily unique, as shown for example by $1+1+(-1)+(-1)$, where we can group together the first and third summands (resp. second and fourth), or the first and the fourth (resp. second and third).

For each $j = 1, \dots, r$ we also fix an arbitrary element $\underline{i}_j$ of I_j and define

$$M_j := \langle \underline{i} - \underline{i}_j \;\big|\; \underline{i} \in I_j \rangle_{\mathbb{Z}} \subseteq \mathbb{Z}^3, \tag{10}$$

where we interpret elements of $\{0, 1, 2\}^3$ as elements of the free $\mathbb{Z}$-module $\mathbb{Z}^3$. Notice that the free $\mathbb{Z}$-module M_j is independent of the choice of $\underline{i}_j \in I_j$. Finally we set $M := M_1 + \cdots + M_r$.

Proposition 2.2. The following hold:

(1) If the quotient $\mathbb{Z}^3/M$ is infinite, there exist $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{Z}$, not all equal to zero, such that for all $t \in \mathbb{C}^\times$ the triple $(xt^{\alpha_1}, yt^{\alpha_2}, zt^{\alpha_3})$ is a solution to Equation (9).
(2) If the quotient $\mathbb{Z}^3/M$ is finite (so that it is a finite abelian group), let e denote its exponent and let $\ell := \max\{|I_j| : j = 1, \dots, r\}$ be the length of the longest vanishing subsum. The common order of x, y, z divides en, where $n = \prod_{\substack{p \text{ prime} \\ p \leq \ell}} p$.

Proof. Suppose first that $\mathbb{Z}^3/M$ is infinite. In this case, the vector subspace $M \otimes_{\mathbb{Z}} \mathbb{Q}$ of $\mathbb{Q}^3$ has dimension strictly less than 3. In particular, the orthogonal subspace $(M \otimes \mathbb{Q})^\perp$ is non-trivial, and we conclude that there exists a non-zero $\underline{\alpha} \in \mathbb{Z}^3$ that is orthogonal to all the elements of M, hence in particular to all the

elements of every M_j, for $j = 1, \dots, r$. We can now write

$$\begin{aligned} f(xt^{\alpha_1}, yt^{\alpha_2}, zt^{\alpha_3}) &= \sum_{j=1}^{r} \sum_{\underline{i} \in I_j} p_{\underline{i}} \cdot (xt^{\alpha_1}, yt^{\alpha_2}, zt^{\alpha_3})^{\underline{i}} \\ &= \sum_{j=1}^{r} t^{\underline{\alpha} \cdot \underline{i}_j} \sum_{\underline{i} \in I_j} p_{\underline{i}} \cdot (x, y, z)^{\underline{i}} t^{\underline{\alpha} \cdot (\underline{i} - \underline{i}_j)}. \end{aligned}$$

By construction of $\underline{\alpha}$, every scalar product $\underline{\alpha} \cdot (\underline{i} - \underline{i}_j)$ vanishes (notice that $\underline{i} - \underline{i}_j$ lies in M_j), so the previous expression evaluates to

$$f(xt^{\alpha_1}, yt^{\alpha_2}, zt^{\alpha_3}) = \sum_{j=1}^{r} t^{\underline{\alpha} \cdot \underline{i}_j} \sum_{\underline{i} \in I_j} p_{\underline{i}} \cdot (x, y, z)^{\underline{i}} = 0,$$

where we have used the fact that (again by construction) $\sum_{\underline{i} \in I_j} p_{\underline{i}} \cdot (x, y, z)^{\underline{i}}$ vanishes for every j.

Suppose on the other hand that $\mathbb{Z}^3/M$ is finite. Multiplying the relation $\sum_{\underline{i} \in I_j} p_{\underline{i}} \cdot (x, y, z)^{\underline{i}} = 0$ by the invertible element $(x, y, z)^{-\underline{i}_j}$ we obtain, for every j, the equality

$$\sum_{\underline{i} \in I_j} p_{\underline{i}} \cdot (x, y, z)^{\underline{i} - \underline{i}_j} = 0.$$

This is a vanishing sum of rational multiples of roots of unity, satisfying the following two properties:

(1) No proper subsum vanishes: this follows from property (2) in the choice of the decomposition $\{0, 1, 2\}^3 = \coprod I_j$;
(2) it is *normalised*, in the sense that the root of unity 1 does appear in the sum (in the summand corresponding to $\underline{i} = \underline{i}_j$).

By [CJ76, Theorem 5], we obtain that the common order of the roots of unity $(x, y, z)^{\underline{i} - \underline{i}_j}$ is a squarefree number n_j such that $\sum_{p | n_j} (p - 2) \leq |I_j| - 2 \leq \ell - 2$. Notice in particular that n_j divides the n in the statement.

By definition of exponent of an abelian group we know that the vector $(e, 0, 0)$ belongs to M, that is, we may write $(e, 0, 0)$ as a linear combination (with integer coefficients) of elements of the form $\underline{i} - \underline{i}_j$ for $\underline{i} \in I_j$. Raising (x, y, z) to the power $(e, 0, 0)$, we obtain that x^e may be written as a product of terms of the form $(x, y, z)^{\underline{i} - \underline{i}_j}$, each of which – by what we already showed – is a root of unity of order dividing n. Clearly, this implies that x is a root of unity of order dividing en. The same argument, applied to the vectors $(0, e, 0)$ e $(0, 0, e)$, shows that the orders of y and z also divide en, which concludes the proof. □

Definition 2.3. We will say that a solution (a, b, c, d, x, y, z) to Equation (2) *belongs to an unbounded family* if it satisfies the conclusion of case (1) of the previous proposition.

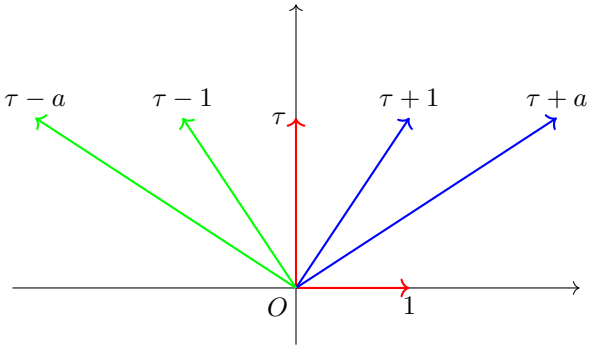
For such solutions there is no absolute bound on the degree of the field $\mathbb{Q}(x, y, z)$. These solutions are classified in [DVZ22, § 6], where it is proved that they can only arise from *rectangular* and *superrectangolar* spaces (cf. the definitions in [DVZ22, § 7.1]). We review these spaces in Section 3 for the reader's convenience. These spaces will appear repeatedly as exceptional cases in the classifications we obtain in the following sections.

3. Rectangular spaces and CM spaces

We recall here some classes of spaces which play a special role in the classification.

3.1. **Rectangular spaces.** We call *rectangular* any space containing a rational angle of $\pi/2$. Up to homothety, these can be characterized as those generated by $1, \tau$ with a purely imaginary τ, or as those generated by $1, \tau$ with $|\tau| = 1$ ([DVZ22, Lemma 4.1]).

Solutions to (2) belonging to unbounded families come from rectangular spaces, and indeed the presence of a right angle provides a geometric explanation (by reflection) for the existence of a third rational angle when a second one is given.

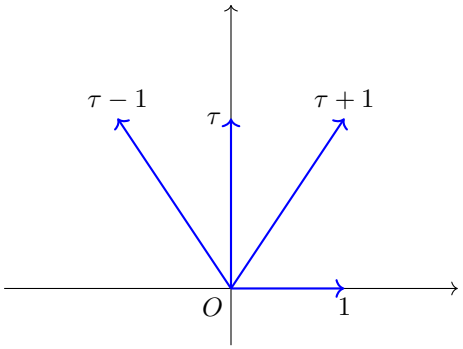


Up to homothety, we can describe the configuration as follows: fix a rational number $a \neq 0, \pm 1$ and a root of unity $y \neq 1$; if the equation

$$\tau^2 + (a-1)\frac{y+1}{y-1}\tau - a = 0$$

has a purely imaginary solution τ, then $1, \tau$ generate a rectangular space having a rational angle $(\tau + 1, \tau + a)$ with squared amplitude y.

3.2. **Superrectangular spaces.** We call a rectangular space whose right angle belongs to a rational triple *superrectangular*. It is easy to see that superrectangular spaces are homothetic to a space of the form $\langle 1, \zeta \rangle_{\mathbb{Q}}$ for some $\zeta \in U$, and that their rational triple always extends to a rational quadruple.



Up to homothety, we can describe the configuration as follows: fix a root of unity $y \neq 1$ and set

$$\tau = \frac{y+1}{y-1};$$

then $1, \tau$ generate a superrectangular space with a rational quadruple $(1, \tau, \tau + 1, \tau - 1)$ such that the rational angle $(1, \tau + 1)$ has squared amplitude equal to y.

It was proved in [DVZ22, Theorem 7.2] that every superrectangular space has exact type (4) with only two exceptions (up to homothety): the fields $\mathbb{Q}(\sqrt{-1})$ and $\mathbb{Q}(\sqrt{-3})$. These two cases belong to another class of special rectangular spaces with additional structure.

3.3. **CM spaces.** Every imaginary quadratic field is clearly a space in our sense; we call *CM space* any space homothetic to an imaginary quadratic field. It is easy to see that distinct imaginary quadratic fields are not homothetic, and that each of them contains infinitely many rational angles. Furthermore, any space containing infinitely many rational angles must be a CM space [DVZ22, Theorem 4.3].

One can compute the exact type of CM spaces and find that:

- $\mathbb{Q}(\sqrt{-3})$ contains infinitely many rational sextuples;
- $\mathbb{Q}(\sqrt{-1})$ contains infinitely many rational quadruples;
- any other CM space contains infinitely many rational right angles.

See [DVZ22, Theorem 4.2] for details.

We state here a simple lemma that cuts down the amount of trivial cases we will have to consider in our subsequent computations:

Lemma 3.1. Any space which contains two distinct right angles is a CM space.

Proof. Two distinct right angles cannot be adjacent (recall that, with our conventions, (v_1, v_2) and $(v_1, -v_2)$ define the same angle). Therefore we can fix one of the angles as $(1, \tau)$, the second one as $(\tau + b_0, \tau + b_1)$, and [DVZ22, Equation (3.11)] holds with $x = y = -1$. We thus have

$$\tau^2 = b_0 b_1,$$

which is a (negative) rational number, and therefore $\langle 1, \tau \rangle_{\mathbb{Q}}$ is a CM space. □

4. The Galois conjugates argument

We discuss here an argument involving cyclotomic extensions that is used to speed up the computation.

Let (a, b, c, d, x, y, z) be a solution to Equation (2), and let $n = n(x, y, z)$ be the common order of the roots of unity x, y, z. Let q be an integer dividing n, write $N := n/q$, and assume $(N, q) = 1$ (in this case we will say that q divides n *exactly*). We can write

$$x = \xi_1 x', \quad y = \xi_2 y', \quad z = \xi_3 z',$$

where $\xi_i = \zeta_q^{e_i}$ (with $e_1, e_2, e_3 \in \{0, 1, \dots, q-1\}$) and $x', y', z' \in U$ have order prime to q. Notice in particular that there is only a finite number of possibilities for the triple (ξ_1, ξ_2, ξ_3).

The extensions $\mathbb{Q}(\zeta_q)/\mathbb{Q}$ and $\mathbb{Q}(\zeta_N)/\mathbb{Q}$ are linearly disjoint, hence we have a canonical identification $\mathrm{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q}) \cong \mathrm{Gal}(\mathbb{Q}(\zeta_N)/\mathbb{Q}) \times \mathrm{Gal}(\mathbb{Q}(\zeta_q)/\mathbb{Q})$. We will use this identification to consider $\mathrm{Gal}(\mathbb{Q}(\zeta_q)/\mathbb{Q})$ as the subgroup of $\mathrm{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q})$ fixing $\mathbb{Q}(\zeta_N)$. Observe now that by assumption we have

$$P(a, b, c, d, \xi_1 x', \xi_2 y', \xi_3 z') = P(a, b, c, d, x, y, z) = 0,$$

and that the coefficients of P, the quantities a, b, c, d, and the roots of unity x', y', z' are all contained in $\mathbb{Q}(\zeta_N)$. As a consequence, for every $\sigma \in \mathrm{Gal}(\mathbb{Q}(\zeta_q)/\mathbb{Q})$ we have

$$P(a, b, c, d, \sigma(\xi_1) x', \sigma(\xi_2) y', \sigma(\xi_3) z') = 0.$$

For each fixed choice of $\xi_1 = \zeta_q^{e_1}, \xi_2 = \zeta_q^{e_2}, \xi_3 = \zeta_q^{e_3}$ such that the least common multiple of the orders of ξ_1, ξ_2, ξ_3 is q (equivalently, the greatest common divisor (e_1, e_2, e_3, q) is 1), we let V_{ξ_1,ξ_2,ξ_3} be the variety over $\mathbb{Q}(\zeta_q)$ defined in $\mathbb{A}^7$ by

$$P(a, b, c, d, \sigma(\xi_1) x', \sigma(\xi_2) y', \sigma(\xi_3) z') = 0 \quad \forall \sigma \in \mathrm{Gal}(\mathbb{Q}(\zeta_q)/\mathbb{Q}); \tag{11}$$

notice that here the 7 variables are a, b, c, d, x', y', z'. By construction, V_{ξ_1,ξ_2,ξ_3} is defined over $\mathbb{Q}(\zeta_q)$. The ideal defining it is stable under $\mathrm{Gal}(\mathbb{Q}(\zeta_q)/\mathbb{Q})$, so it defines in a natural way a corresponding variety over $\mathbb{Q}$, which by a slight abuse of notation we still denote by V_{ξ_1,ξ_2,ξ_3} (or by $V_{\xi_1,\xi_2,\xi_3,\mathbb{Q}}$ when we want to stress the field of definition). Finally, we let W_{ξ_1,ξ_2,ξ_3} be the projection of V_{ξ_1,ξ_2,ξ_3} to $\mathbb{A}^4_{\mathbb{Q}}(a, b, c, d)$.

It is often convenient in our computations to consider the varieties V_{ξ_1,ξ_2,ξ_3} and W_{ξ_1,ξ_2,ξ_3}. If W_{ξ_1,ξ_2,ξ_3} does not have any rational points, or more generally if none of its rational points satisfies the non-degeneracy conditions (3), then we conclude that for no solution to (2) the common order of x, y, z is exactly divisible by q. This technique allows us to further bound the possible common orders in a computationally efficient way. Note that Equation (11) shows that V_{ξ_1,ξ_2,ξ_3} is defined by $\varphi(q)$ equations, so, as q grows, it becomes increasingly unlikely for V_{ξ_1,ξ_2,ξ_3} to have (non-trivial) rational points. This heuristic expectation is fully confirmed by our computations.

5. Spaces with a triple of adjacent rational angles

In this section we classify the spaces of type $\textcircled{3}+\textcircled{2}$. This is a much simpler task than the classification of spaces of type $3\textcircled{2}$, but of the same nature. Therefore, this section will also provide a blueprint for how the main part of the proof will be organised and what computations we need to perform.

Spaces of type $\textcircled{3}+\textcircled{2}$ are described by the following equation, computed in [DVZ22, § 3.4]:

$$\begin{aligned} P_{\textcircled{3}+\textcircled{2}}(a, b, c, x, y, z) = {} & (2a^2 - a(b+c) + 2bc)xy - abx^2y - acy \\ & - (2a^2 - a(b+c) + 2bc)xyz + abyz + acx^2yz \\ & + b(a-c)x^2 + c(a-b)y^2 - a(a-b)xy^2 - a(a-c)x \\ & - b(a-c)y^2z - c(a-b)x^2z + a(a-b)xz + a(a-c)xy^2z = 0. \end{aligned} \tag{12}$$

We seek solutions with $x, y, z \in U$ and $(a : b : c) \in \mathbb{P}_2(\mathbb{Q})$ satisfying the following non-degeneracy conditions:

$$x, y, z \neq 1, \qquad x \neq y, \qquad a, b, c \text{ distinct and non-zero.} \tag{13}$$

Given such a solution, setting

$$\tau = ax\frac{y-1}{x-y}$$

and $V = \langle 1, \tau \rangle_{\mathbb{Q}}$ gives a space with a rational triple $(1, \tau, \tau + a)$ and a rational angle $(\tau + b, \tau + c)$.

Remark 5.1.

(1) In principle, there would be no need of treating this case separately, as it can be recovered from the general equation (2) when $|\{0, a, b, c, d\}| < 5$. Specifically, Equation (12) can be obtained from (2) by setting $b = 0$ and renaming the variables through the substitution $(a, c, d, x, y, z) \mapsto (a, b, c, x, x/y, z)$. By working with Equation (12), however, we are able to avoid a great amount of spurious degenerate solutions without geometric meaning, leading to cleaner and faster computations.

(2) Not every symmetry taken into account by Definition 1.1 is also valid for Equation (12); in particular, the roles of the three variables x, y, z are now genuinely different, so that we cannot freely permute them. We still have that, for every solution (a, b, c, x, y, z) to (12), the following are also solutions:

$$\begin{gathered}(a, c, b, x, y, z^{-1}) \\ (-a, -a + b, -a + c, y, x, z) \\ \left(\frac{1}{a}, \frac{1}{b}, \frac{1}{c}, x^{-1}, yx^{-1}, z\right)\end{gathered} \tag{14}$$

and they lead to values of τ generating spaces homothetic to the original one. As far as the Galois action is concerned, on the other hand, we still have that for every solution (a, b, c, x, y, z) to (12) and every $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, the 6-uple $(a, b, c, \sigma(x), \sigma(y), \sigma(z))$ is again a solution to (12).

(3) We observe that the space $V = \langle 1, \tau \rangle_{\mathbb{Q}}$ only depends on x and y (and not on a, b, c). As a consequence, we know *a priori* that—outside of the unbounded families already mentioned—there is only a *finite* list of spaces of type (3) + (2), because the orders of x and y are bounded by a universal constant.

We employ the strategy outlined in Section 2, fixing a maximal decomposition of the 14 summands in (12) into vanishing subsums.

Proposition 5.2. With notation as in Proposition 2.2,

(1) if $\ell \geq 9$ then $M = \mathbb{Z}^3$, $e = 1$, and the common order n of x, y, z is squarefree and satisfies $\sum_{p|n}(p-2) \leq 14 - 2$;

(2) in all cases, if $\mathbb{Z}^3/M$ is finite, then its exponent e satisfies $e \leq 10$.

Proof. The proof is computational. For part (1) we proceed as follows: we list all 9-elements subsets of the terms appearing in (12), we think of such a subset as being the I_j realising the maximal length, and we check that the corresponding module $M_j \subseteq \mathbb{Z}^3$ is all of $\mathbb{Z}^3$. Notice that it suffices to perform this operation on subsets with exactly nine elements: if I_j, I_j' are two subsets with $I_j \subseteq I_j'$, the corresponding modules M_j and M_j' satisfy $M_j \subseteq M_j'$, hence the fact that the equality $M_j = \mathbb{Z}^3$ holds for all subsets I_j with $|I_j| = 9$ implies that the same statement holds for all I_j with $|I_j| \geq 9$. Finally, given a solution of (12), the statement about n follows upon applying [CJ76, Theorem 5] to the vanishing sum indexed by the subset of maximal cardinality in the partition $\{I_j\}$.

For part (2), let V be the set consisting of the 14 vectors giving the exponents of (x, y, z) in the 14 summands of (12), and let $W = \{v_1 - v_2 : v_1, v_2 \in V\}$. By construction, given a solution of (12) and a maximal decomposition into vanishing subsums $\{I_j\}$, the module M is generated by vectors in W. We loop over all 3-elements subsets $\{w_1, w_2, w_3\}$ of W, and for each we check whether $\mathbb{Z}^3/(\mathbb{Z}w_1 + \mathbb{Z}w_2 + \mathbb{Z}w_3)$ is finite. If it is, we compute its exponent, and find that it always bounded above by 10.

Notice that, when M has finite index, we can always find three vectors of the form $\underline{i} - \underline{i}_j$, with $\underline{i} \in I_j$, that generate a finite-index subgroup of M: this follows from basic linear algebra upon tensoring M with $\mathbb{Q}$. In particular, given a solution to (12) that does not belong to an unbounded family, the corresponding

module M contains one of the modules with three generators that we considered above. This shows that the exponent of $\mathbb{Z}^3/M$ is bounded above by 10 in all cases when $\mathbb{Z}^3/M$ is finite. □

Corollary 5.3. Let (a,b,c,x,y,z) be a solution of (12) not belonging to an unbounded family. The common order n of x,y,z divides $2^4\cdot 3^3\cdot 5^2\cdot 7^2\cdot 11\cdot 13$ and is not divisible by $2^3\cdot 3^2$.

Proof. Let (a,b,c,x,y,z) be a solution to (12). We continue with the notation of Proposition 2.2. One of the following holds:

(1) $\mathbb{Z}^3/M$ is infinite: then (a,b,c,x,y,z) belongs to an unbounded family, which is not the case by assumption;
(2) $\ell \geq 9$: then by Proposition 5.2 (1) we have $n \mid \prod_{p\leq 13} p$;
(3) $\ell < 9$ and $\mathbb{Z}^3/M$ is finite: by Proposition 2.2 we obtain $n \mid e\prod_{p\leq \ell} p \mid e\prod_{p\leq 7} p$, where (by part (2) of Proposition 5.2) e is an integer not exceeding 10. For any such integer e the conclusion in the statement holds.

□

Consider now a solution (a,b,c,x,y,z) to Equation (12), where x,y,z have common order n. As in Section 4, we let q be an integer such that $q \mid n$ and $(n, n/q) = 1$. We also write $x = \xi_1 x', y = \xi_2 y', z = \xi_3 z'$ with $\xi_i = \zeta_q^{e_i}$ and $e_i \in [0, q-1]$. Notice that since q divides the common order of x,y,z we have $(q, e_1, e_2, e_3) = 1$.

We see then that (a,b,c,x,y,z) gives rise to a $\overline{\mathbb{Q}}$-point on the variety V_{ξ_1,ξ_2,ξ_3} defined in $\mathbb{A}^6$ by

$$P_{\textcircled{3}+\textcircled{2}}(a,b,c,\sigma(\xi_1)x',\sigma(\xi_2)y',\sigma(\xi_3)z') = 0 \quad \forall \sigma \in \mathrm{Gal}(\mathbb{Q}(\zeta_q)/\mathbb{Q}).$$

We denote by $V_{\xi_1,\xi_2,\xi_3,\mathbb{Q}}$ the $\mathbb{Q}$-variety obtained from V_{ξ_1,ξ_2,ξ_3} (see Section 4). The solution (a,b,c,x,y,z) also gives rise to a $\overline{\mathbb{Q}}$-point of $V_{\xi_1,\xi_2,\xi_3,\mathbb{Q}}$. The following is proved by a direct computation.

Proposition 5.4. Let $q \in \{3^3, 5^2, 7^2, 7, 11, 13, 3^2\cdot 5, 3\cdot 5\}$. Let $e_1,e_2,e_3 \in [0, q-1]$ be integers such that $(e_1,e_2,e_3,q) = 1$ and let $\xi_i = \zeta_q^{e_i}$. Define a set $S \subseteq \{x'-1, y'-1, z'-1, x'-y'\}$ as follows:

(1) $x'-1 \in S$ if and only if $e_1 = 0$ (respectively, $y'-1 \in S$ if and only if $e_2 = 0$, and $z'-1 \in S$ if and only if $e_3 = 0$);
(2) $x'-y' \in S$ if and only if $e_1 = e_2$.

The intersection of $V_{\xi_1,\xi_2,\xi_3,\mathbb{Q}}$ with $\{abc(a-b)(b-c)(c-a)\prod_{s\in S} s \neq 0\}$ is empty.

Corollary 5.5. With the same hypotheses as in Corollary 5.3, the common order n of x,y,z divides $2^4\cdot 3^2\cdot 5$ and is not divisible by $3\cdot 5$.

Proof. By Corollary 5.3 we already know that $n \mid 2^4\cdot 3^3\cdot 5^2\cdot 7^2\cdot 11\cdot 13$ and that $2^3\cdot 3^2 \nmid n$. Suppose now that $13 \mid n$: the previous divisibility implies in particular $(13, n/13) = 1$, and therefore (a,b,c,x,y,z) gives rise to a $\overline{\mathbb{Q}}$-point on $V_{\xi_1,\xi_2,\xi_3,\mathbb{Q}}$ for suitable 13-th roots of unity ξ_1,ξ_2,ξ_3. Proposition 5.4 shows that one of the following holds:

(1) $abc(a-b)(b-c)(c-a) = 0$, which contradicts the non-degeneracy conditions (13);
(2) $e_1 = 0$ and $x' = 1$, which implies $x = \zeta_q^{e_1}x' = 1$, again contradicting (13). An identical argument applies in the cases $y' = 1$ and $z' = 1$.
(3) $e_1 = e_2$ and $x' = y'$, which implies $x = \xi_1 x' = \xi_2 y' = y$, once again contradicting (13).

In all cases we obtain a contradiction, which shows that $13 \nmid n$. A similar argument also shows that $7^2 \nmid n, 11 \nmid n, 5^2 \nmid n$, and $3^3 \nmid n$. Furthermore, once we know that $3^3 \nmid n$, we can repeat the same argument again with $3^2\cdot 5$ (since at this point we know that $3^2\cdot 5 \mid n \Rightarrow (3^2\cdot 5, n/(3^2\cdot 5)) = 1$), and obtain that $45 \nmid n$. In a similar way we prove that 7 and 15 do not divide n, which concludes the proof. □

The bound on the common order of (x,y,z) is now small enough that we can easily enumerate all solutions of Equation (12) outside of the unbounded families (recall that these correspond to rectangular spaces), and for each of them describe the rational angles contained in the corresponding space:

Theorem 5.6. Let V be a space of type $\textcircled{3}+\textcircled{2}$. Then one of the following holds:

(1) V is a rectangular space;

(2) V is homothetic to $\langle 1, \zeta_8^2 + \zeta_8 + 1\rangle_{\mathbb{Q}}$; it is of exact type $2\textcircled{3}$. Setting $\tau = \zeta_8^2 + \zeta_8 + 1$, the two rational triples are $(1, \tau, \tau - 1)$ and $(\tau + 1, \tau - 1/2, \tau - 2)$; we have

$$\arg(\tau) = \frac{\pi}{4}, \qquad \arg(\tau - 1) = \frac{3}{8}\pi, \qquad \arg\left(\frac{\tau - 1/2}{\tau + 1}\right) = \frac{\pi}{8}, \qquad \arg\left(\frac{\tau - 2}{\tau + 1}\right) = \frac{3}{8}\pi.$$

(3) V is homothetic to $\langle 1, \zeta_5^2 + 2\zeta_5 + 2\rangle_{\mathbb{Q}}$; it is of exact type $2\textcircled{3}$. Setting $\tau = \zeta_5^2 + 2\zeta_5 + 2$, the two rational triples are $(1, \tau, \tau - 1)$ and $(\tau - 1/3, \tau - 4, \tau - 5/4)$; we have

$$\arg(\tau) = \frac{3}{10}\pi, \qquad \arg(\tau - 1) = \frac{2}{5}\pi, \qquad \arg\left(\frac{\tau - 4}{\tau - 1/3}\right) = \frac{2}{5}\pi, \qquad \arg\left(\frac{\tau - 5/4}{\tau - 1/3}\right) = \frac{\pi}{10}.$$

(4) V is homothetic to $\langle 1, 2\zeta_5^3 + \zeta_5 + 2\rangle_{\mathbb{Q}}$; it is of exact type $2\textcircled{3}$. Setting $\tau = 2\zeta_5^3 + \zeta_5 + 2$, the two rational triples are $(1, \tau, \tau - 1)$ and $(\tau - 1/3, \tau - 4, \tau - 5/4)$; we have

$$\arg(\tau) = -\frac{\pi}{10}, \qquad \arg(\tau - 1) = -\frac{4}{5}\pi, \qquad \arg\left(\frac{\tau - 4}{\tau - 1/3}\right) = -\frac{4}{5}\pi, \qquad \arg\left(\frac{\tau - 5/4}{\tau - 1/3}\right) = -\frac{7}{10}\pi.$$

(5) V is homothetic to $\langle 1, 2\zeta_{12}^3 + 2\zeta_{12}^2 - \zeta_{12} - 1\rangle_{\mathbb{Q}}$; it is of exact type $2\textcircled{3}$. Setting $\tau = 2\zeta_{12}^3 + 2\zeta_{12}^2 - \zeta_{12} - 1$, the two rational triples are $(1, \tau, \tau - 1)$ and $(\tau + 3, \tau - 2, \tau - 3/4)$; we have

$$\arg(\tau) = \frac{7}{12}\pi, \qquad \arg(\tau - 1) = \frac{2}{3}\pi, \qquad \arg\left(\frac{\tau - 2}{\tau + 3}\right) = \frac{5}{12}\pi, \qquad \arg\left(\frac{\tau - 3/4}{\tau + 3}\right) = \frac{\pi}{3}.$$

The spaces in parts (2)–(5) are illustrated in Section 10.

6. Spaces with $ab = cd$

Under the assumption that $ab = cd$, equation (2) splits, and after removing multiple factors and trivial solutions one is left with

$$f_{ab=cd} := (d-b)xyz + (b-c)xy + (a-d)xz + (c-a)yz + (c-a)x + (a-d)y + (b-c)z + (d-b) = 0. \tag{15}$$

As explained in Section 1.1, in this case it is not possible to recover a single τ that generates the corresponding space. A solution to (15) need not correspond to a space, and when it does, τ has (in general) degree two over $\mathbb{Q}(x, y, z)$. One finds two (in principle non-homothetic) spaces corresponding to the same algebraic solution. This peculiarity justifies a separate study of this case.

As in Proposition 5.2 one shows:

Proposition 6.1. With notation as in Proposition 2.2,

(1) if $\ell \geq 5$ then $M = \mathbb{Z}^3$, $e = 1$, and the (squarefree) common order n of x, y, z satisfies $\sum_{p|n}(p-2) \leq 8 - 2$;
(2) in all cases, if $\mathbb{Z}^3/M$ is finite, then its exponent e satisfies $e \leq 4$.

In this case, the argument of Section 4 is not necessary: we immediately obtain the following corollary, which will be enough for our purposes.

Corollary 6.2. Let (a, b, c, d, x, y, z) be a solution to Equation (15) not belonging to an unbounded family and let n be the least common multiple of the orders of x, y, z. Then n divides one among $\{2 \cdot 3 \cdot 5, 2 \cdot 3 \cdot 7, 2^3 \cdot 3, 2 \cdot 3^2\}$.

Proof. In case (1) of Proposition 6.1 one checks immediately that m divides $2 \cdot 3 \cdot 5$ or $2 \cdot 3 \cdot 7$. In case (2), we can assume $\ell \leq 4$, and Proposition 2.2 (2) yields that the common order of x, y, z divides $6e$, where $e \in \{1, 2, 3, 4\}$. □

The bound provided by the previous corollary is sharp enough that we can test all possible cases.

Theorem 6.3. Let (a, b, c, d, x, y, z) be a solution to (15). Then one of the following holds:

(1) the solution belongs to an unbounded family;
(2) two of the variables a, b, c, d are equal to zero;
(3) $x = y = z = -1$;
(4) $n = 10$, and the solution is symmetry-equivalent to $\left(3, \frac{1}{2}, \frac{3}{2}, 1, \zeta_{10}, \zeta_{10}^2, \zeta_{10}^3\right)$;
(5) $n = 12$, and the solution is symmetry-equivalent to one of the two solutions

$$\left(2, \frac{2}{3}, \frac{4}{3}, 1, \zeta_{12}, \zeta_{12}^3, \zeta_{12}^5\right), \quad \left(\frac{4}{3}, \frac{2}{3}, \frac{8}{9}, 1, \zeta_{12}, \zeta_{12}^4, \zeta_{12}^5\right);$$

(6) $n = 6$, and the solution is symmetry-equivalent to one of the following:
   (a) $\left(\frac{3cd}{c+2d}, \frac{c+2d}{3}, c, d, \zeta_6, \zeta_6, \zeta_6^2\right)$ for some $(c,d) \in \mathbb{Q}^2$ with $c+2d \neq 0$;
   (b) $\left(2d, \frac{1}{2}c, c, d, \zeta_6, \zeta_6^2, \zeta_6^2\right)$ for some $(c,d) \in \mathbb{Q}^2$;
   (c) $\left(\frac{3d^2-4bd}{2d-3b}, b, \frac{3bd-4b^2}{2d-3b}, d, \zeta_6, \zeta_6^2, \zeta_6^3\right)$ for some $(b,d) \in \mathbb{Q}^2$ with $2d-3b \neq 0$.

As mentioned, it is not necessarily the case that every solution to the algebraic equations actually leads to a valid geometric configuration. Concerning the geometric interpretation of the previous theorem, we have the following result:

Theorem 6.4.

(1) If the solution corresponds to a space, it is a rectangular space;
(2) if the solution corresponds to a space, it is a superrectangular space;
(3) the solution $(a,b,c,d,-1,-1,-1)$ with $ab = cd$ is geometrically acceptable if and only if $ab < 0$; in this case, the corresponding spaces are CM (Lemma 3.1);
(4) these solution corresponds to 2 geometric configurations, up to homothety. Setting $(a,b,c,d) = (3, \frac{1}{2}, \frac{3}{2}, 1)$ and

$$\tau_1 = -\frac{1}{2}\sqrt{\frac{1}{2}\left(2\sqrt{5}+\sqrt{52\sqrt{5}+45}+13\right)}\cdot e^{\frac{3\pi i}{10}}, \quad \tau_2 = -\frac{1}{2}\sqrt{\frac{1}{2}\left(2\sqrt{5}-\sqrt{52\sqrt{5}+45}+13\right)}\cdot e^{\frac{3\pi i}{10}}$$

gives the two non-equivalent configurations. For both values of τ one has

$$\arg(\tau/\overline{\tau}) = \frac{3}{10}\cdot 2\pi, \quad \arg\left(\frac{\tau+a}{\tau+b} \,/\, \frac{\overline{\tau}+a}{\overline{\tau}+b}\right) = \frac{2}{5}\cdot 2\pi, \quad \arg\left(\frac{\tau+c}{\tau+d} \,/\, \frac{\overline{\tau}+c}{\overline{\tau}+d}\right) = \frac{1}{10}\cdot 2\pi,$$

and these are the only rational angles in $\langle 1, \tau\rangle$, which is in particular a space of exact type $3\,(2)$;
(5) none of these solutions gives rise to a geometric configuration;
(6) the families (a) and (b) parametrise the same spaces. The parametrisation in (b) gives a geometrically valid solution if and only if $c^2 - 8cd + 4d^2 > 0$. For such c, d there are two corresponding values of τ, and they generate homothetic spaces. Denote by $V_{[c:d]}$ the resulting space (up to homothety). The space $V_{[c:d]}$ generically has 4 rational angles, of squared amplitudes $\zeta_6^{\pm 1}, \zeta_6^{\pm 2}$. It has fewer than 4 rational angles if and only if $c+2d = 0$, in which case $V_{[c:d]}$ is of exact type $3\,(2)$. The space $V_{[c:d]}$ has more than 4 rational angles if it is either the CM space $\mathbb{Q}(\zeta_3)$, or if it is not CM but has a right angle, in which case it is of exact type $5\,(2)$. The latter possibility happens if and only if the space $V_{[c:d]}$ corresponds to a point on family (c), which is in fact a rational sub-family of (b).

Proof. Parts (1)-(5) are straightforward computations, so we only prove (6).

The solutions in families (a) and (b) for the same pair (c,d) and the same angle $z = \zeta_6^2$ lead to the same equations (7), (8), hence parametrise the same spaces, which we describe in greater detail below.

We show that the family (c) is a sub-family of (b). Clearly, the family in (b) is equivalent to $\left(c, d, 2d, \frac{1}{2}c, \zeta_6, \zeta_6^2, \zeta_6^2\right)$. Given a point

$$\left(\frac{3d_0^2-4b_0d_0}{2d_0-3b_0}, b_0, \frac{3b_0d_0-4b_0^2}{2d_0-3b_0}, d_0, \zeta_6, \zeta_6^2, \zeta_6^3\right)$$

in family (c), we take

$$c = d_0\frac{3d_0-4b_0}{2d_0-3b_0}, \quad d = b_0 :$$

this gives a point on family (b), and the corresponding values of τ satisfy the same quadratic equation (specifically, that coming from eliminating $\overline{\tau}$ from the equations $\tau/\overline{\tau} = \zeta_6$ and $\frac{\tau+c}{\tau+d}\Big/\overline{\left(\frac{\tau+c}{\tau+d}\right)} = \zeta_6^2$). Thus, (c) is a sub-family of (b).

Let us show that it is precisely the subfamily such that the quantity $(c-d)(c-4d)$ is a square. Indeed, on the one hand we have

$$\begin{aligned}(c-d)(c-4d) &= \left(d_0\frac{3d_0-4b_0}{2d_0-3b_0}-b_0\right)\left(d_0\frac{3d_0-4b_0}{2d_0-3b_0}-4b_0\right)\\ &= 3\frac{d_0^2-2b_0d_0+b_0^2}{2d_0-3b_0}\cdot 3\frac{d_0^2-4b_0d_0+4b_0^2}{2d_0-3b_0}\\ &= \left(3\frac{(d_0-b_0)(d_0-2b_0)}{2d_0-3b_0}\right)^2;\end{aligned}$$

conversely, let $P_{(b)} = (c_0, d_0, 2d_0, \frac{1}{2}c_0, \zeta_6, \zeta_6^2, \zeta_6^2)$ be a point on family (b) and suppose that $(c_0-d_0)(c_0-4d_0)$ is a square. We take $b = d_0$ and d such that

$$c_0 = \frac{3d^2-4bd}{2d-3b} \Leftrightarrow c_0(2d-3d_0) = 3d^2-4d_0d \Leftrightarrow 3d^2-2d(2d_0+c_0)+2c_0d_0 = 0.$$

The (reduced) discriminant of this quadratic equation in d is $(2d_0+c_0)^2-9c_0d_0 = 4d_0^2-5c_0d_0+c_0^2 = (c_0-d_0)(c_0-4d_0)$, which is a square by assumption. Thus, a point of family (b) for which $(c_0-d_0)(c_0-4d_0)$ is a square gives rise to the point $P_{(c)} = \left(\frac{3d^2-4bd}{2d-3b}, b, \frac{3bd-4b^2}{2d-3b}, d, \zeta_6, \zeta_6^2, \zeta_6^3\right)$ on family (c). Note that $P_{(c)}$ gives the same quadratic equation for τ as the original point $P_{(b)}$. Finally, we should check that the point $P_{(c)}$ that we obtained does in fact satisfy $2d-3b \neq 0$. Assume the contrary. Replacing the values found above for b, d we obtain

$$\begin{aligned}2d = 3b &\Leftrightarrow 4d_0+2c_0 \pm 2\sqrt{(c_0-d_0)(c_0-4d_0)} = 9d_0\\ &\Leftrightarrow 4(c_0^2-5c_0d_0+4d_0^2) = (5d_0-2c_0)^2\\ &\Leftrightarrow 9d_0^2 = 0,\end{aligned}$$

which corresponds to a CM space.

Finally, we describe which solutions correspond to geometric configurations. This happens when (8) has a real solution, which happens if and only if $c^2-8cd+4d^2 > 0$. Let us show that in this case to a point $[c:d] \in \mathbb{P}_1(\mathbb{Q})$ corresponds a unique lattice up to homotethy.

Indeed, on the one hand it is clear that rescaling (c,d) by a non-zero rational number gives a homothetic space. On the other hand, for fixed (c,d), the relevant values of τ are obtained as the roots of a quadratic equation (so there are in principle two τ for any given (c,d)), but in fact the two lattices obtained in this way are always homothetic. To see this, notice that τ_1, τ_2 are the roots of the equation

$$(-\zeta_6^2+2)\tau^2+(2c+d)\tau+(\zeta_6+1)cd = 0;$$

it follows easily that $\tau_1\tau_2$ and $\tau_1+\tau_2$ both belong to the 2-dimensional $\mathbb{Q}$-vector space $\mathbb{Q}(\zeta_6)$, hence the three numbers $1, \tau_1\tau_2$ and $\tau_1+\tau_2$ are $\mathbb{Q}$-linearly dependent. It is easy to see that—in complete generality—the lattices $\langle 1, \tau_1\rangle_{\mathbb{Q}}$ and $\langle 1, \tau_2\rangle_{\mathbb{Q}}$ are homothetic if and only if the four numbers $1, \tau_1, \tau_2, \tau_1\tau_2$ are linearly dependent over $\mathbb{Q}$ (see [DVZ22, Remark 2.1]), which proves the claim.

Let $[c,d] \in \mathbb{P}_1(\mathbb{Q})$ be such that $c^2-8cd+4d^2 > 0$. Fix

$$\tau = -\zeta_{12}\frac{\sqrt{3}}{6}(c+2d+\sqrt{c^2-8cd+4d^2})$$

and write $V_{[c:d]} = \langle 1, \tau\rangle$.

Let $S = \{[c:d] \in \mathbb{P}_1(\mathbb{Q}) \mid c^2-8cd+4d^2 \text{ is a rational square}\} \cup \{[2:-1]\}$.

The map

$$\begin{array}{rccc} f: & \mathbb{P}_1(\mathbb{Q}) & \to & \{\text{homothety classes of spaces}\}\\ & [c:d] & \mapsto & \text{class of } V_{[c:d]}\end{array}$$

is 2-to-1 on $\mathbb{P}_1(\mathbb{Q}) \setminus S$. The involution $\iota : [c:d] \mapsto [4d:c]$ preserves the fibres of f.

If $c^2-8cd+4d^2$ is a rational square, then $V_{[c:d]} = \mathbb{Q}(\zeta_3)$ is a CM space; this happens for infinitely many $[c:d]$.

If $c+2d = 0$ we have that $V_{[2:-1]}$ has exact type $3\textcircled{2}$.

In all other cases $V_{[c:d]}$ contains 4 rational angles $(1,\tau), (\tau+\frac{3cd}{c+2d}, \tau+\frac{c+2d}{3}), (\tau+2d, \tau+\frac{1}{2}c), (\tau+c, \tau+d)$, all distinct and not adjacent, with squared amplitudes $\zeta_6, \zeta_6, \zeta_6^2, \zeta_6^2$ respectively.

The space $V_{[c:d]}$ contains additional rational angles if and only if $(c-d)(c-4d)$ is a square. In this case, $(\tau+e, \tau+f)$, with e, f given by $\frac{(c+2d)\pm\sqrt{(c-d)(c-4d)}}{3}$ respectively, is a right angle and there are no other rational angles in this space. This rational sub-family is preserved by the involution ι.

To see that no other angles can appear, observe that such a square amplitude must lie in the field $\mathbb{Q}(\tau, \zeta_6) = \mathbb{Q}(\zeta_6, \sqrt{c^2-8cd+4d^2})$, which is an extension of $\mathbb{Q}(\zeta_6)$ of degree at most 2. The only possibilities are $\zeta_6^1, \zeta_6^2, \zeta_6^3$ or a primitive 12th root of unity; in this latter case however we must have that $\sqrt{c^2-8cd+4d^2} \in \mathbb{Q}(\zeta_{12}) \cap \mathbb{R} = \mathbb{Q}(\sqrt{3})$, and this can only happen if $c^2-8cd+4d^2$ is a rational square because 3 is not a square modulo 4.

We finally remark that all four Galois conjugates of $V_{[c:d]}$ are homothetic to $V_{[c:d]}$. We have already showed it for the Galois morphism that acts trivially on ζ_6. We now notice that $-\overline{\zeta_{12}}\frac{\sqrt{3}}{6}(c+2d-\sqrt{c^2-8cd+4d^2})$ times τ is a rational number, so this conjugate too generates a space homothetic to $V_{[c:d]}$.

As for $\overline{\tau}$, one can check directly that $\sqrt{c^2-8cd+4d^2}$ is a $\mathbb{Q}$-linear combination of 1 and $\tau+\overline{\tau}$, and from there that $\overline{V_{[c:d]}}$ is also homothetic to $V_{[c:d]}$. □

7. The 2+2+2 case

7.1. **Bounding the degree.** Again in analogy with Proposition 5.2, one proves:

Proposition 7.1. With notation as in Proposition 2.2,

(1) if $\ell \geq 10$ then $e \leq 2$, and the common order n of x, y, z divides $2\prod_{p\leq\ell} p$ (hence divides $2\prod_{p\leq 23} p$);
(2) in all cases, if $\mathbb{Z}^3/M$ is finite, then its exponent e satisfies $e \in \{1, \dots, 22\} \cup \{24, 26, 30\}$.

Remark 7.2. A coarser, but still useful, bound as in (2) can be obtained by noticing that the exponent of $\mathbb{Z}^3/M$ is bounded by the non-zero determinant of a 3×3 matrix A with columns given by three vectors of the form $\underline{i}-\underline{i}_j$ (notation as in the proof of Proposition 5.2). Since each coefficient of $\underline{i}-\underline{i}_j$ is bounded in absolute value by 2, one immediately obtains that $\det(A) \leq 3!\cdot 2^3 = 48$.

The argument of Corollary 5.3 now gives the following.

Corollary 7.3. Let (a, b, c, d, x, y, z) be a solution to Equation (2). If the solution does not belong to an unbounded family, the common order n of x, y, z satisfies at least one of the following:

(1) n is squarefree or twice a squarefree, and divisible only by primes ≤ 23;
(2) n is of the form em, with m squarefree and divisible only by primes ≤ 7, and $e \in \{1, ..., 22\} \cup \{24, 26, 30\}$.

In particular, n divides $2^5\cdot 3^3\cdot 5^2\cdot 7^2\cdot 11\cdot 13\cdot 17\cdot 19\cdot 23$.

Reasoning as in Corollary 5.5, one can then check the following.

Proposition 7.4. Let (a, b, c, d, x, y, z) be a solution to Equation (2), and let n be the common order of x, y, z. The following implication holds. Suppose that q divides n exactly, where q is one of the following:

- $11, 13, 17, 19, 23$;
- $2^5, 3^3, 5^2, 7^2$;
- $5\cdot 7, 3^2\cdot 7, 2^4\cdot 7, 2^3\cdot 7$;
- $3^2\cdot 5, 2^4\cdot 5$.

Then $[a : b : c : d] \in \mathbb{P}_3(\mathbb{Q})$ satisfies one of the following:

- $ab = cd$;
- $abcd(a-b)(a-c)(a-d)(b-c)(b-d)(c-d) = 0$.

Corollary 7.5. In the setting of Proposition 7.4, at least one of the following holds:

(1) $ab = cd$;
(2) $abcd(a-b)(a-c)(a-d)(b-c)(b-d)(c-d) = 0$;
(3) (a, b, c, d, x, y, z) belongs to an unbounded family;
(4) $n \in \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 14, 15, 16, 18, 20, 21, 24, 28, 30, 36, 40, 42, 48, 60, 72, 84, 120, 144\}$.

Solutions as in (1) are classified in Section 6, while solutions falling in case (2) are either geometrically meaningless (if, for example, $a = b$) or correspond to spaces having at least a triple of adjacent rational angles. These have been classified in Section 5. The unbounded families are described in Section 3. Thus,

we just need to study Equation (2) for the finitely many values of n listed in (4), and we may do so under the additional constraints $abcd(a-b)(a-c)(a-d)(b-c)(b-d)(c-d) \neq 0$, $ab \neq cd$.

Remark 7.6. For a given value of $n \geq 3$ and fixed n-th roots of unity x, y, z, equation (2) becomes the equation of a surface S which is in general defined over $\mathbb{Q}(\zeta_n)^+$ and not defined over $\mathbb{Q}$ unless x, y, z are fourth or sixth roots of unity (see the appendix of [DVZ22] for discussion of the reducibility and field of definition of S). Rational points $(a : b : c : d)$ on S must therefore lie in the intersection of $\frac{1}{2}\varphi(n)$ conjugates of S. Heuristically, we should expect this intersection to consist of curves when $\varphi(n) = 4$ (which happens precisely for $n = 5, 8, 10, 12$), and at most of isolated points otherwise. This is precisely what happens, and the isolated rational points are typically highly singular points of S. These points, such as $(1 : 1 : 1 : 1)$ or $(1 : 0 : 0 : 0)$, often show up for many different values of n and many choices of the roots of unity x, y, z. In this respect, it is maybe worth mentioning that, for $n = 30$ and $(x, y, z) = (\zeta_{30}^5, \zeta_{30}^6, \zeta_{30}^{10})$, we find the 'exceptional' rational point $(a : b : c : d) = (2 : 2 : 4 : 1)$, which is a solution only for this choice of n and roots of unity (up to symmetry equivalence); however it does not satisfy the nondegeneracy conditions and does not correspond to any geometric configuration.

The computation to be performed now amounts to finding the rational points on a certain number of algebraic varieties, most of which are empty or consist only of finitely many points; however curves of genus up to 5 are also encountered. We discuss in Proposition 7.9 which curves we encounter, and we present in Section 8 the techniques used to study the set of their rational points. The outcome of the computation is the following theorem:

Theorem 7.7. Let (a, b, c, d, x, y, z) be a solution to Equation (2) satisfying the non-degeneracy conditions (3) and the additional constraints $abcd(a-b)(a-c)(a-d)(b-c)(b-d)(c-d) \neq 0$, $ab \neq cd$. Let n be the common order of x, y, z. Then $n \in \{5, 8, 10, 12\}$.

Building on Remark 7.6, we explicitly note that for $n \notin \{5, 8, 10, 12\}$ we only need to study the rational points of a zero-dimensional scheme, which of course does not pose any great difficulties. It would be possible to study directly the 1-dimensional intersections that arise for $n \in \{5, 8, 10, 12\}$, but we find it easier to analyse these cases more directly and from a more geometric perspective, which we do in the following subsections. We note that the direct analysis, with much more tedious computations, leads to the same conclusions that we report in the rest of this section.

7.2. **General structure.** In listing all solutions for a given degree we will present them as a rationally parametrised family of spaces having a certain angle, with additional angles appearing when certain expressions are squares in $\mathbb{Q}$. We give here the general structure of these theorems. Note that, if the space V contains an angle of squared amplitude ζ_n^k, then (choosing generators that form an angle of that squared amplitude and rescaling suitably) we can write $V = \langle 1, \tau \rangle_{\mathbb{Q}}$, where τ is in $\mathbb{R}\zeta_{2n}^k$.

Theorem 7.8. Fix (n, k) in $\{(8, 1), (10, 1), (10, 2), (12, 1), (12, 3)\}$. The $\mathbb{Q}$-vector subspace of $\mathbb{Q}(\zeta_n)$ given by $\mathbb{Q}(\zeta_n) \cap \mathbb{R}\zeta_{2n}^k$ has dimension 2. Let $\tau_{1,0}, \tau_{0,1} \in \mathbb{Q}(\zeta_n)$ be the basis of this space given by the following table

(n, k)	$\tau_{1,0}$	$\tau_{0,1}$
$(8, 1)$	$\zeta_8^2 - \zeta_8^3$	$1 + \zeta_8$
$(10, 1)$	$-2 - 2\zeta_{10}^2 + \zeta_{10}^3$	$1 + \zeta_{10}$
$(10, 2)$	$1 + \zeta_{10}^2$	ζ_{10}
$(12, 1)$	$\zeta_{12} + \zeta_{12}^2 - \zeta_{12}^3$	$1 + \zeta_{12}$
$(12, 3)$	$\zeta_{12} + \zeta_{12}^2$	$1 + \zeta_{12}^3$

For $(\lambda, \mu) \in \mathbb{Q}^2 \setminus \{(0, 0)\}$, set

$$\tau_{\lambda,\mu} = \lambda\tau_{1,0} + \mu\tau_{0,1}.$$

The space $\langle 1, \tau_{\lambda,\mu} \rangle_{\mathbb{Q}}$ only depends on $[\lambda : \mu] \in \mathbb{P}_1(\mathbb{Q})$. We denote it by $V_{[\lambda:\mu]}$.

(1) Fix $e \in \mathbb{N}$. There exists a homogeneous quartic form $f_{n,k,e}(\lambda, \mu) \in \mathbb{Q}[\lambda, \mu]$ such that the following holds: the space $V_{[\lambda:\mu]}$ contains an angle with squared amplitude ζ_n^e if and only if $f_{n,k,e}(\lambda, \mu)$ is the square of a rational number u.

(2) There exist four rational functions $a_1, a_2, b_1, b_2 \in \mathbb{Q}(\lambda, \mu)$ such that, setting

$$a_\pm(\lambda, \mu, u) = a_1(\lambda, \mu) \pm a_2(\lambda, \mu)u, \quad b_\pm(\lambda, \mu, u) = b_1(\lambda, \mu) \pm b_2(\lambda, \mu)u,$$

the following holds. Suppose that $(\lambda, \mu, u) \in \mathbb{Q}^3$, with $(\lambda, \mu) \neq (0, 0)$, is a solution of the equation $u^2 = f_{n,k,e}(\lambda, \mu)$: then, the rational angles in the space $V_{[\lambda:\mu]}$ with squared amplitude ζ_n^e are given by

$$(\tau_{\lambda,\mu} + a_+(\lambda, \mu, u), \tau_{\lambda,\mu} + b_+(\lambda, \mu, u)), \quad (\tau_{\lambda,\mu} + a_-(\lambda, \mu, u), \tau_{\lambda,\mu} + b_-(\lambda, \mu, u)).$$

These two angles may coincide or have a side in common.

(3) The equation $u^2 = f_{n,k,e}(\lambda, \mu)$ defines a curve in the weighted projective space $\mathbb{P}_{\mathbb{Q}}(1, 1, 2)$ (the variables are taken in the order λ, μ, u). This curve is either the union of two irreducible components of genus 0 or a curve of genus 1.

The theorem simply describes what happens carrying out the computation. For the angles considered, $\mathbb{Q}(\zeta_n) \cap \mathbb{R}\zeta_{2n}^k$ has dimension 2, and cannot have dimension greater than 2, as $\mathbb{Q}(\zeta_n) \cap \mathbb{R}$ has also dimension 2, and the whole $\mathbb{Q}(\zeta_n)$ has dimension 4. The existence of angles of a fixed amplitude correspond to the existence of rational points on the conic given by (6) or (7), which in this case is defined over a real quadratic field. A rational point must therefore lie in the intersection with the conjugate conic, and expressing $\tau_{[\lambda,\mu]}$ it terms of the basis leads to the forms $f_{n,k,e}$.

We now prove in general certain geometric properties of a class of curves that we encounter in the computations.

Proposition 7.9. Let $f_1(\lambda, \mu)$ and $f_2(\lambda, \mu)$ be two non-proportional homogeneous forms of degree 4. Let C be the curve defined in $\mathbb{P}_{\mathbb{Q}}(1, 1, 2, 2)$ (weighted projective space in the variables λ, μ, u, v, of respective weights $1, 1, 2, 2$) by

$$C : \begin{cases} u^2 = f_1(\lambda, \mu) \\ v^2 = f_2(\lambda, \mu) \end{cases}$$

The following hold:

(1) C is a Galois cover of $\mathbb{P}_{1,\mathbb{Q}}$ with Galois group $(\mathbb{Z}/2\mathbb{Z})^2$;
(2) if f_1, f_2 have no common factors, C is smooth, has genus 5, and is an étale double cover of the hyperelliptic, genus 3 curve $C' : w^2 = f_1(\lambda, \mu) f_2(\lambda, \mu)$ (sitting in $\mathbb{P}_{\mathbb{Q}}(1, 1, 4)$);
(3) if the greatest common divisor of f_1, f_2 is a quadratic polynomial $q_1(\lambda_1, \lambda)$, then C is not smooth, and its desingularisation $\tilde{C}$ has genus 3. Write $f_1 = q_1(\lambda, \mu)q_2(\lambda, \mu)$, $f_2 = q_1(\lambda, \mu)q_3(\lambda, \mu)$, where each $q_i(\lambda, \mu)$ has degree 2. The curve C (hence also $\tilde{C}$) admits degree-2 maps to the three genus-1 curves $w^2 = q_i(\lambda, \mu)q_j(\lambda, \mu)$ (sitting in $\mathbb{P}_{\mathbb{Q}}(1, 1, 2)$) for $i, j = 1, 2, 3$ with $i \neq j$.

Proof.

(1) The automorphisms $\alpha : [\lambda : \mu : u : v] \mapsto [\lambda : \mu : -u : v]$ and $\beta : [\lambda : \mu : u : v] \mapsto [\lambda : \mu : u : -v]$ generate a subgroup G of $\mathrm{Aut}_{\mathbb{Q}}(C)$ isomorphic to $(\mathbb{Z}/2\mathbb{Z})^2$. The quotient map $C \to C/G$ can be identified with

$$\begin{array}{rccc} f : & C & \to & \mathbb{P}_{1,\mathbb{Q}} \\ & [\lambda : \mu : u : v] & \mapsto & [\lambda : \mu] : \end{array}$$

indeed, it is clear that the fibers of f are invariant under the action of G, and that the degree of f is $4 = |G|$.

(2) Smoothness of C follows immediately from the fact that all derivatives should vanish at a non-smooth point, which would force f_1, f_2 to have a common factor. To compute the genus of C we study the ramification of f. As C is smooth, f is the quotient map by a group action, and the group action is tame since we are working in characteristic 0, it suffices to understand the fixed points of each non-identity element in G. The fixed points of α (respectively β) can be identified with the points $[\lambda : \mu : v]$ that solve $f_1(\lambda, \mu) = 0$ and $v^2 = f_2(\lambda, \mu)$ (respectively, $f_2(\lambda, \mu) = 0$ and $u^2 = f_1(\lambda, \mu)$), hence – using the assumption that f_1, f_2 have no common factors – both α and β have 8 fixed points. The automorphism $\alpha\beta$ has no fixed points, again because of the hypothesis that f_1, f_2 have no common factors. The Riemann-Hurwitz formula then yields

$$2g(C) - 2 = 4(2g(\mathbb{P}_1) - 2) + \sum_{h \in G} (\mathrm{ord}(h) - 1) \# \mathrm{Fix}(h) = -8 + 8 + 8 \Longrightarrow g(C) = 5.$$

To prove the last statement of (2), notice that

$$\begin{array}{ccc} C & \to & C' \\ [\lambda : \mu : u : v] & \mapsto & [\lambda : \mu : uv] \end{array}$$

is well-defined, of degree 2, and étale (it is the quotient map for the action of $\alpha\beta$, which as already observed has no fixed points). Furthermore, C' is hyperelliptic and defined by a separable polynomial of degree 8, hence has genus $3 = \frac{8-2}{2}$.

(3) Without loss of generality we may work with the base-change of C to an algebraic closure of $\mathbb{Q}$. Write D_i for the divisor cut out on $\mathbb{P}_1$ by $q_i(\lambda,\mu) = 0$: it consists of 2 distinct points of multiplicity 1, and the supports of D_1, D_2, D_3 are pairwise disjoint. To see that C is not smooth, one may check that the points $[\lambda : \mu : 0 : 0]$, where $[\lambda : \mu]$ is in the support of D_1, are singular. The maps to the three genus-1 curves described in the statement can be taken to be

$$\begin{array}{ccc} C & \to & w^2 = q_1(\lambda,\mu)q_2(\lambda,\mu) \\ [\lambda:\mu:u:v] & \mapsto & [\lambda:\mu:u] \end{array}$$

$$\begin{array}{ccc} C & \to & w^2 = q_1(\lambda,\mu)q_3(\lambda,\mu) \\ [\lambda:\mu:u:v] & \mapsto & [\lambda:\mu:v] \end{array}$$

and

$$\begin{array}{ccc} C & \to & w^2 = q_2(\lambda,\mu)q_3(\lambda,\mu) \\ [\lambda:\mu:u:v] & \mapsto & [\lambda:\mu:uv/q_1(\lambda,\mu)]. \end{array}$$

To compute the genus of $\tilde{C}$ we work at the level of function fields and analyse the ramification of the map $\tilde{f} : \tilde{C} \to \mathbb{P}_1$ induced by the quotient map $f : C \to \mathbb{P}_1$. It is clear that the only places of $\overline{\mathbb{Q}}(\mathbb{P}_1)$ (potentially) ramified in $\overline{\mathbb{Q}}(\tilde{C})$ are those corresponding to points in the support of $D_1 + D_2 + D_3$. We claim that each of these six places has ramification index 2; this implies that there are two places of $\overline{\mathbb{Q}}(\tilde{C})$ lying over it, and by Riemann-Hurwitz we get

$$2g(\tilde{C}) - 2 = 4(2g(\mathbb{P}_1) - 2) + 6 \cdot 2 \cdot (2-1) \Longrightarrow g(\tilde{C}) = 3.$$

To prove the statement about ramification, observe that

$$\overline{\mathbb{Q}}(\tilde{C}) = \overline{\mathbb{Q}}(\mathbb{P}_1)(\sqrt{q_1(\lambda/\mu,1)q_2(\lambda/\mu,1)}, \sqrt{q_1(\lambda/\mu,1)q_3(\lambda/\mu,1)}, \sqrt{q_2(\lambda/\mu,1)q_3(\lambda/\mu,1)})$$

is a biquadratic extension, and therefore the role of the three divisors D_1, D_2, D_3 is symmetrical. Thus it suffices to prove the claim for a place P of $\overline{\mathbb{Q}}(\mathbb{P}_1)$ corresponding to a point in the support of D_3. The place P is unramified in the intermediate extension $\overline{\mathbb{Q}}(\sqrt{q_1(\lambda/\mu,1)q_2(\lambda/\mu,1)})/\overline{\mathbb{Q}}(\mathbb{P}_1)$, while it is ramified in $\overline{\mathbb{Q}}(\sqrt{q_1(\lambda/\mu,1)q_3(\lambda/\mu,1)})/\overline{\mathbb{Q}}(\mathbb{P}_1)$, hence also in $\overline{\mathbb{Q}}(\tilde{C})/\overline{\mathbb{Q}}(\mathbb{P}_1)$. This immediately implies our claim. □

Remark 7.10. Proposition 7.9 will be applied to the following situation. By Theorem 7.8 (1), the existence of an angle of squared amplitude ζ_n^e in the space $V_{[\lambda:\mu]}$ is equivalent to the existence of a rational point on a curve of the form $u^2 = f_{n,k,e}(\lambda,\mu)$. Thus, the simultaneous existence of angles of squared amplitudes $\zeta_n^{e_1}, \zeta_n^{e_2}$ is equivalent to the existence of a rational point on the fibre product

$$C_{n,k,e_1,e_2} : \begin{cases} u^2 = f_{n,k,e_1}(\lambda,\mu) \\ v^2 = f_{n,k,e_2}(\lambda,\mu), \end{cases}$$

which is precisely the kind of curve studied in Proposition 7.9.

The curves C_{n,k,e_1,e_2} were originally found in a completely different way, namely, as (desingularisations of) irreducible components of the intersection of the two conjugates of a surface S defined over $\mathbb{Q}(\zeta_n)^+$, see Remark 7.6. The techniques of [Lom19, CMSV19, CLV21] revealed that their Jacobians are all split, which led us to look for the more conceptual interpretation of Proposition 7.9.

7.3. $n = 8$.

Theorem 7.11. For $(\lambda,\mu) \in \mathbb{Q}^2 \setminus \{(0,0)\}$, let

$$\tau_{\lambda,\mu} := \lambda(\zeta_8^2 - \zeta_8^3) + \mu(1+\zeta_8) = \lambda \cdot \sqrt{2-\sqrt{2}}\zeta_{16} + \mu \cdot \sqrt{2+\sqrt{2}}\zeta_{16}.$$

The following hold:

(1) The space $\langle 1, \tau_{\lambda,\mu}\rangle_{\mathbb{Q}}$ only depends on $[\lambda:\mu] \in \mathbb{P}_1(\mathbb{Q})$. We denote it by $V_{[\lambda:\mu]}$.
(2) Let $\tau \in \mathbb{Q}(\zeta_8)$ generate a space V such that the squared angle between 1 and τ is ζ_8. There exists $(\lambda,\mu) \in \mathbb{Q}^2 \setminus \{(0,0)\}$ such that $\tau = \tau_{\lambda,\mu}$.
(3) Conversely, for every $(\lambda,\mu) \in \mathbb{Q}^2 \setminus \{(0,0)\}$, the squared angle between 1 and τ is ζ_8.
(4) For every $[\lambda:\mu] \in \mathbb{P}_1(\mathbb{Q})$, the complex conjugate $\overline{V_{[\lambda:\mu]}}$ is homothetic to $V_{[\lambda:\mu]}$.

(5) The map

$$\begin{array}{rccc} f: & \mathbb{P}_1(\mathbb{Q}) & \to & \{\text{homothety classes of spaces}\} \\ & [\lambda:\mu] & \mapsto & \text{class of } V_{[\lambda:\mu]} \end{array}$$

is 2-to-1. The involution $\iota : [\lambda : \mu] \mapsto [\lambda + \mu : \lambda - \mu]$ preserves the fibres of f.

(6) Let $S = \{[0:1], [1:0], [1:1], [-1:1]\}$. Notice that $\iota([0:1]) = [1:-1]$ and $\iota([1:0]) = [1:1]$. The spaces corresponding to $[\lambda : \mu] \in S$ have the following structure:

- $[\lambda : \mu] = [0:1]$ or $[-1:1]$. The space is of type (4). A rational quadruple of $V_{[0:1]}$ is given by $\{1, \tau_{0,1}, \tau_{0,1} - 1, \tau_{0,1} - 2\}$. The various angles in this quadruple realise all the squared amplitudes $\zeta_8, \zeta_8^2, \zeta_8^3$ and ζ_8^4; this is the configuration formed by the sides and diagonals of a rhombus with angles $\pi/4$ and $3\pi/4$.
- $[\lambda : \mu] = [1:0]$ or $[1:1]$. The space is of type $2 \cdot$ (3). The two rational triples of $V_{[1:0]}$ are given by $\{\tau_{1,0}+1, \tau_{1,0}-2, \tau_{1,0}-\frac{1}{2}\}$ and $\{1, \tau_{1,0}, \tau_{1,0}-1\}$. This is the space of Theorem 5.6 (2).

(7) If $[\lambda : \mu] \notin S$, the space $V_{[\lambda:\mu]}$ contains precisely two angles with squared amplitude ζ_8 and precisely two angles with squared amplitude ζ_8^3. These are given respectively by

$$\zeta_8 : \quad (1, \tau) \text{ and } (\tau + a, \tau + b),$$

where

$$a = \frac{\lambda^2 - 2\lambda\mu - \mu^2}{\lambda + \mu}, \quad b = -\frac{2\left(\lambda^3 + 2\lambda^2\mu - \lambda\mu^2\right)}{\lambda^2 - 2\lambda\mu - \mu^2}$$

and by

$$\zeta_8^3 : \quad (\tau + c, \tau + d) \text{ and } (\tau + c', \tau + d')$$

where

$$c = \frac{-\lambda^2 - 2\lambda\mu + \mu^2}{2\lambda}, \quad d = -2(\lambda + \mu)$$
$$c' = -2\mu, \quad d' = \frac{\lambda^2 + 2\lambda\mu - \mu^2}{\mu - \lambda}.$$

(8) Let $[\lambda : \mu] \notin S$. The space $V_{[\lambda:\mu]}$ contains an angle with squared amplitude ζ_8^2 if and only if $\lambda(\lambda^3 + \lambda^2\mu + \lambda\mu^2 + \mu^3)$ is the square of a rational number u. In this case, setting

$$e_\pm = \frac{\lambda^2 - \mu^2 \mp u}{\mu}, \quad f_\pm = \frac{-2\lambda\mu \pm u}{\lambda},$$

the space $V_{[\lambda:\mu]}$ contains precisely two angles with squared amplitude ζ_8^2, given by $(\tau + e_+, \tau + f_+)$ and $(\tau + e_-, \tau + f_-)$. The curve in weighted projective space

$$u^2 = \lambda(\lambda^3 + \lambda^2\mu + \lambda\mu^2 + \mu^3)$$

has genus 1 and a rational point, so it defines an elliptic curve. It is in particular isomorphic over $\mathbb{Q}$ to the elliptic curve with Weierstrass equation $y^2 = x^3 + x^2 + x + 1$, whose Mordell-Weil group over $\mathbb{Q}$ is $\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. A generator for the free part of the Mordell-Weil group is given by $(0, 1)$. The set of $[\lambda : \mu]$ such that $V_{[\lambda:\mu]}$ contains an angle of squared amplitude ζ_8^2 is stable under ι, and the map $[\lambda : \mu : u] \mapsto [\lambda + \mu : \lambda - \mu : 2u]$ is a lift of ι to an involution of the elliptic curve. It corresponds to translation by the non-trivial 2-torsion point $[1 : -1 : 0]$.

(9) Let $[\lambda : \mu] \notin S$. The space $V_{[\lambda:\mu]}$ contains an angle with squared amplitude ζ_8^4 if and only if $3\lambda^4 + 2\lambda^3\mu + 4\lambda^2\mu^2 + 2\lambda\mu^3 + \mu^4$ is the square of a rational number v. In this case, setting

$$g_\pm = \frac{\lambda^2 - 2\lambda\mu - \mu^2 \mp v}{\lambda + \mu}, \quad h_\pm = \frac{\lambda^2 - 2\lambda\mu - \mu^2 \pm v}{\lambda + \mu},$$

the space $V_{[\lambda:\mu]}$ contains precisely two angles with squared amplitude ζ_8^4 are precisely $(\tau + g_+, \tau + h_+)$ and $(\tau + g_-, \tau + h_-)$. The curve in weighted projective space

$$v^2 = 3\lambda^4 + 2\lambda^3\mu + 4\lambda^2\mu^2 + 2\lambda\mu^3 + \mu^4$$

has genus 1 and a rational point, so it defines an elliptic curve. It is in particular isomorphic over $\mathbb{Q}$ to the elliptic curve with Weierstrass equation $y^2 + 2xy - 2y = x^3 - 6x^2 - 3x$, whose Mordell-Weil group over $\mathbb{Q}$ is $\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. A generator for the free part of the Mordell-Weil group is given by $(0, 2)$. Notice that $g_+ g_- = ab$, so that this case falls within the scope of Theorem 6.4.

(10) Let $[\lambda : \mu] \notin S$. The space $V_{[\lambda:\mu]}$ does not have both an angle with squared amplitude ζ_8^2 and an angle with squared amplitude ζ_8^4.

Remark 7.12. The reader may wonder why the curve appearing in part (9) of the above theorem has genus 1, while all the curves appearing in the classification of spaces with $ab = cd$ (see Theorem 6.3) have genus 0. The reason is that in Theorem 6.4 we do not impose any restrictions on the field generated by τ, which is (usually) a quadratic extension of the field $\mathbb{Q}(x, y, z)$, while in Theorem 7.11 we require that τ belongs to the field $\mathbb{Q}(x, y, z) = \mathbb{Q}(\zeta_8)$. Thus we find, with a different geometric structure, a subfamily of the rectangular spaces having additional angles.

Remark 7.13. Part (7) of Theorem 7.11 shows that the angles with amplitude ζ_8^3 can be rationally parametrised. An avatar of the same fact appears in the computations that verify Theorem 7.7, see the output file `222.out`. Indeed, one finds that (for $x = y = \zeta_8$ and $z = \zeta_8^3$) the intersection of the conjugates of the surface S described in Remark 7.6 contains a rational curve. Similarly, a (highly singular) model of the curve of genus 1 that appears in part (8) can also be obtained as the intersection of two conjugates of the surface S for $x = y = \zeta_8$, $z = \zeta_8^2$. Similar comments apply to the theorems below: the various curves of genera 1, 3 and 5 that we describe could also be realised (though with a very singular model) as intersections of conjugates of the surface S, for suitable values of the parameters x, y, z.

7.4. $n = 5$ **and** $n = 10$**.** Consider a solution (a, b, c, d, x, y, z) to Equation (2) with $ab \neq cd$ and x, y, z having common order $n = 10$. In this case, Equation (4) shows that τ belongs to $\mathbb{Q}(\zeta_{10})$. One of the following necessarily holds: one among x, y, z is a primitive 10th root of unity (in which case, up to the Galois action and permutations, we may assume that $x = \zeta_{10}$), or two of the three roots of unity have order 5 and the remaining one has order 2 (other combinations are ruled out by Lemma 3.1), and as above we may assume $x = \zeta_{10}^2$. We study the two cases in Theorem 7.14 and 7.15 respectively. Notice that a solution with common order $n = 5$ falls within the scope of Theorem 7.15, by simply requiring that the only rational angles that appear in the lattice are given by $\zeta_{10}^{\pm 2}$ and $\zeta_{10}^{\pm 4}$, see Remark 7.16.

Theorem 7.14. For $(\lambda, \mu) \in \mathbb{Q}^2 \setminus \{(0, 0)\}$ let

$$\tau_{\lambda,\mu} = \lambda(-2 - 2\zeta_{10}^2 + \zeta_{10}^3) + \mu(1 + \zeta_{10}).$$

The following hold:

(1) The space $V = \langle 1, \tau \rangle_{\mathbb{Q}}$ only depends on $[\lambda : \mu] \in \mathbb{P}_1(\mathbb{Q})$. We denote it by $V_{[\lambda,\mu]}$.
(2) Let $\tau \in \mathbb{Q}(\zeta_{10})$ generate a space V such that the squared angle between 1 and τ is ζ_{10}. Then there exist $(\lambda, \mu) \in \mathbb{Q}^2 \setminus \{(0, 0\}$ such that $\tau = \tau_{\lambda,\mu}$.
(3) Conversely, for every $(\lambda, \mu) \in \mathbb{Q}^2 \setminus \{(0, 0\}$, the squared angle between 1 and τ is ζ_{10}.
(4) For every $[\lambda : \mu] \in \mathbb{P}_1(\mathbb{Q})$, the complex conjugate $\overline{V_{[\lambda,\mu]}}$ is homothetic to $V_{[\lambda,\mu]}$.
(5) The map

$$\begin{array}{rccc} f: & \mathbb{P}_1(\mathbb{Q}) & \to & \{\text{homothety classes of spaces}\} \\ & [\lambda : \mu] & \mapsto & \text{class of } V_{[\lambda:\mu]} \end{array}$$

is 2-to-1. The involution $\iota : [\lambda : \mu] \mapsto [3\lambda - \mu : 4\lambda - 3\mu]$ preserves the fibres of f.
(6) Let $S = \{[0 : 1], [1 : 3]\}$. Notice that $\iota([0 : 1]) = [1 : 3]$ and $\iota([1 : 3]) = [0 : 1]$. The space corresponding to $[\lambda : \mu] \in S$ is of type (4). A rational quadruple of $V_{[0:1]}$ is given by $\{1, \tau_{0,1}, \tau_{0,1} - 1, \tau_{0,1} - 2\}$. The various angles in this quadruple realise all the squared amplitudes $\zeta_{10}, \zeta_{10}^2, \zeta_{10}^4$ and ζ_{10}^5.
(7) If $[\lambda : \mu] \notin S$, the space $V_{[\lambda:\mu]}$ contains precisely two angles with squared amplitude ζ_{10}. These are given by

$$\zeta_{10}: \quad (1, \tau) \text{ and } (\tau + a, \tau + b),$$

where

$$a = \frac{4\lambda^2 - 6\lambda\mu + \mu^2}{3\lambda - \mu}, \quad b = -\frac{5\lambda(\lambda^2 + \lambda\mu - \mu^2)}{4\lambda^2 - 6\lambda\mu + \mu^2}.$$

(8) Let $[\lambda : \mu] \notin S$. The space $V_{[\lambda:\mu]}$ contains an angle with squared amplitude ζ_{10}^2 if and only if $\lambda(3\lambda - \mu)(11\lambda^2 - 9\lambda\mu + 4\mu^2)$ is the square of a rational number u. In this case, setting

$$c_\pm = \frac{-2\mu^2 + 7\lambda\mu - 3\lambda^2 \mp u}{2\mu - 4\lambda}, \quad d_\pm = \frac{-5\lambda\mu + 5\lambda^2 \mp u}{2\lambda}$$

the rational angles in V with squared amplitude ζ_{10}^2 are precisely $(\tau + c_+, \tau + d_+)$ and $(\tau + c_-, \tau + d_-)$. The curve in weighted projective space

$$u^2 = \lambda(3\lambda - \mu)\left(11\lambda^2 - 9\lambda\mu + 4\mu^2\right)$$

has genus 1 and a rational point $[0:1:0]$, so it defines an elliptic curve. It is in particular isomorphic over $\mathbb{Q}$ to the elliptic curve with Weierstrass equation $y^2+xy+y=x^3-x^2+2$, whose Mordell-Weil group over $\mathbb{Q}$ is $\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$. A generator for the free part of the Mordell-Weil group is given by $(4,5)$.

(9) Let $[\lambda:\mu]\notin S$. The space $V_{[\lambda:\mu]}$ contains an angle with squared amplitude ζ_{10}^3 if and only if $41\lambda^4-38\lambda^3\mu+9\lambda^2\mu^2+8\lambda\mu^3-4\mu^4$ is the square of a rational number u. In this case, setting

$$e_\pm=-\frac{\lambda^2+11\lambda\mu-6\mu^2+u}{2\lambda-4\mu},\quad f_\pm=\frac{9\lambda^2-\lambda\mu-4\mu^2+u}{4\lambda+2\mu}$$

the rational angles in $V_{[\lambda:\mu]}$ with squared amplitude ζ_{10}^3 are precisely $(\tau+e_+,\tau+f_+)$ and $(\tau+e_-,\tau+f_-)$. The curve in weighted projective space

$$u^2=41\lambda^4-38\lambda^3\mu+9\lambda^2\mu^2+8\lambda\mu^3-4\mu^4$$

has genus 1 and a rational point $[1:1:4]$, so it defines an elliptic curve. It is in particular isomorphic over $\mathbb{Q}$ to the elliptic curve with Weierstrass equation $y^2+xy+y=x^3-x^2+20x+22$, whose Mordell-Weil group over $\mathbb{Q}$ is $\mathbb{Z}\oplus\mathbb{Z}/4\mathbb{Z}$. A generator for the free part of the Mordell-Weil group is given by $(79,660)$. A torsion point of order 4 is $(4,10)$.

(10) Let $[\lambda:\mu]\notin S$. The space $V_{[\lambda:\mu]}$ contains an angle with squared amplitude ζ_{10}^4 if and only if $(14\lambda^2-6\lambda\mu+\mu^2)(2\lambda^2-2\lambda\mu+\mu^2)$ is the square of a rational number u. In this case, setting

$$g_\pm=\frac{6\lambda^2-4\lambda\mu-\mu^2\pm u}{4\lambda},\quad h_\pm=\frac{-2\lambda^2+8\lambda\mu-3\mu^2\pm u}{2(\mu-\lambda)}$$

the rational angles in $V_{[\lambda:\mu]}$ with squared amplitude ζ_{10}^4 are precisely $(\tau+g_+,\tau+h_+)$ and $(\tau+g_-,\tau+h_-)$. The curve in weighted projective space

$$u^2=(14\lambda^2-6\lambda\mu+\mu^2)(2\lambda^2-2\lambda\mu+\mu^2)$$

has genus 1 and a rational point $[0:1:1]$, so it defines an elliptic curve. It is in particular isomorphic over $\mathbb{Q}$ to the elliptic curve with Weierstrass equation $y^2=x^3+x^2-3x-2$, whose Mordell-Weil group over $\mathbb{Q}$ is $\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$. A generator for the free part of the Mordell-Weil group is given by $(3,-5)$.

(11) Let $[\lambda:\mu]\notin S$. The space $V_{[\lambda:\mu]}$ contains an angle with squared amplitude ζ_{10}^5 if and only if $31\lambda^4-38\lambda^3\mu+24\lambda^2\mu^2-7\lambda\mu^3+\mu^4$ is the square of a rational number u. In this case, setting

$$i_\pm=\frac{4\lambda^2-6\lambda\mu+\mu^2\mp u}{3\lambda-\mu},\quad j_\pm=\frac{4\lambda^2-6\lambda\mu+\mu^2\pm u}{3\lambda-\mu}$$

the rational angles in $V_{[\lambda:\mu]}$ with squared amplitude ζ_{10}^5 are precisely $(\tau+i_+,\tau+j_+)$ and $(\tau+i_-,\tau+j_-)$. We have $i_+j_+=i_-j_-=ab$ (where a,b are as in (4)), so that this case falls into the classification of Theorem 6.4.

(12) If such a space has, in addition to the angles of squared amplitude ζ_{10}, rational angles of two different amplitudes, then (up to a nonzero rational number) τ is one of the following:

(a) $\tau_1=\frac{1}{2}(\zeta_{10}^3-2\zeta_{10}^2+2\zeta_{10})$. The corresponding space is of type (3) + (3), with the two rational triples given by $(1,\tau,\tau-\frac{1}{2})$ (angles of squared amplitudes $\zeta_{10}^{\pm1},\zeta_{10}^{\pm2},\zeta_{10}^{\pm3}$) and $(\tau-\frac{5}{8},\tau-2,\tau-\frac{1}{6})$ (angles of squared amplitudes $\zeta_{10}^{\pm1},\zeta_{10}^{\pm2},\zeta_{10}^{\pm3}$).

(b) $\tau_2=\frac{1}{2}(-\zeta_{10}^3+2\zeta_{10}^2+2\zeta_{10}+4)$. The corresponding space is of type (3) + (3), with the two rational triples given by $(1,\tau,\tau-\frac{5}{2})$ (angles of squared amplitudes $\zeta_{10}^{\pm1},\zeta_{10}^{\pm2},\zeta_{10}^{\pm3}$) and $(\tau-\frac{5}{8},\tau-\frac{15}{2},\tau-2)$ (angles of squared amplitudes $\zeta_{10}^{\pm1},\zeta_{10}^{\pm2},\zeta_{10}^{\pm3}$).

(c) $\tau_3=\frac{1}{3}(\zeta_{10}^3-2\zeta_{10}^2+3\zeta_{10}+1)$. The corresponding space is of type (4), with a rational 4-tuple given by $(1,\tau,\tau-\frac{5}{3},\tau-\frac{5}{6})$ (angles of squared amplitudes $\zeta_{10}^{\pm1},\zeta_{10}^{\pm2},\zeta_{10}^{\pm4},\zeta_{10}^{5}$).

(d) $\tau_4=\zeta_{10}+1$. The corresponding space is of type (4), with a rational 4-tuple given by $(1,\tau,\tau-1,\tau-2)$ (angles of squared amplitudes $\zeta_{10}^{\pm1},\zeta_{10}^{\pm2},\zeta_{10}^{\pm4},\zeta_{10}^{5}$).

(e) $\tau_5=\zeta_{10}^3-2\zeta_{10}^2+\zeta_{10}-1$. The corresponding space is of type (3) + (3), with the two rational triples given by $(1,\tau,\tau+1)$ (angles of squared amplitudes $\zeta_{10}^{\pm1},\zeta_{10}^{\pm3},\zeta_{10}^{\pm4}$) and $(\tau-\frac{1}{2},\tau+\frac{4}{3},\tau+5)$ (angles of squared amplitudes $\zeta_{10}^{\pm1},\zeta_{10}^{\pm3},\zeta_{10}^{\pm4}$).

(f) $\tau_6 = 2\zeta_{10}^3 - 4\zeta_{10}^2 + \zeta_{10} - 3$. The corresponding space is of type (3) + (3), with the two rational triples given by $(1, \tau, \tau+5)$ (angles of squared amplitudes $\zeta_{10}^{\pm 1}, \zeta_{10}^{\pm 3}, \zeta_{10}^{\pm 4}$) and $(\tau - 10, \tau+1, \tau+\frac{15}{4})$ (angles of squared amplitudes $\zeta_{10}^{\pm 1}, \zeta_{10}^{\pm 3}, \zeta_{10}^{\pm 4}$).

In addition, we have $\langle 1, \tau_1\rangle_{\mathbb{Q}} \sim \langle 1, \tau_2\rangle_{\mathbb{Q}}$, $\langle 1, \tau_3\rangle_{\mathbb{Q}} \sim \langle 1, \tau_4\rangle_{\mathbb{Q}}$, $\langle 1, \tau_5\rangle_{\mathbb{Q}} \sim \langle 1, \tau_6\rangle_{\mathbb{Q}}$, and these are all the pairs of homothetic spaces in this list. Finally, concerning the comparison with Theorem 5.6, we have

$$\langle 1, \tau_1\rangle_{\mathbb{Q}} \sim \langle 1, \tau_2\rangle_{\mathbb{Q}} \sim \langle 1, 2\zeta_5^3 + \zeta_5 + 2\rangle_{\mathbb{Q}} \text{ and } \langle 1, \tau_5\rangle_{\mathbb{Q}} \sim \langle 1, \tau_6\rangle_{\mathbb{Q}} \sim \langle 1, \zeta_5^2 + 2\zeta_5 + 2\rangle_{\mathbb{Q}}$$

Theorem 7.15. For $(\lambda, \mu) \in \mathbb{Q}^2 \setminus \{(0,0)\}$ let

$$\tau_{\lambda,\mu} = \lambda(1 + \zeta_{10}^2) + \mu\zeta_{10}.$$

The following hold:

(1) The space $V = \langle 1, \tau\rangle_{\mathbb{Q}}$ only depends on $[\lambda : \mu] \in \mathbb{P}_1(\mathbb{Q})$. We denote it by $V_{[\lambda,\mu]}$.
(2) Let $\tau \in \mathbb{Q}(\zeta_{10})$ generate a space V such that the squared angle between 1 and τ is ζ_{10}^2. Then there exist $(\lambda, \mu) \in \mathbb{Q}^2 \setminus \{(0,0\}$ such that $\tau = \tau_{\lambda,\mu}$.
(3) Conversely, for every $(\lambda, \mu) \in \mathbb{Q}^2 \setminus \{(0,0\}$, the squared angle between 1 and τ is ζ_{10}^2.
(4) For every $[\lambda : \mu] \in \mathbb{P}_1(\mathbb{Q})$, the complex conjugate $\overline{V_{[\lambda,\mu]}}$ is homothetic to $V_{[\lambda,\mu]}$.
(5) The map

$$\begin{array}{rccc} f: & \mathbb{P}_1(\mathbb{Q}) & \to & \{\text{homothety classes of spaces}\} \\ & [\lambda : \mu] & \mapsto & \text{class of } V_{[\lambda:\mu]} \end{array}$$

is 2-to-1. The involution $\iota : [\lambda : \mu] \mapsto [-\lambda : \lambda + \mu]$ preserves the fibres of f.
(6) Let $S = \{[0:1], [1:0], [1:-1], [2:-1]\}$. Notice that $\iota([0:1]) = [0:1]$, $\iota([2:-1]) = [2:-1]$ and $\iota([1:0]) = [1:-1]$. The spaces corresponding to $[\lambda : \mu] \in S$ have the following structure:
- $[\lambda : \mu] = [0 : 1]$. The space is of type (4). A rational quadruple of $V_{[0:1]}$ is given by $\{1, \tau_{0,1}, \tau_{0,1} - 1, \tau_{0,1} + 1\}$. The various angles in this quadruple realise all the squared amplitudes $\zeta_{10}, \zeta_{10}^2, \zeta_{10}^4$ and ζ_{10}^5.
- $[\lambda : \mu] = [1 : 0]$ or $[1 : -1]$. The space is of type (4). A rational quadruple of $V_{[1:0]}$ is given by $\{1, \tau_{1,0}, \tau_{1,0} - 1, \tau_{1,0} - 2\}$. The various angles in this quadruple realise all the squared amplitudes $\zeta_{10}^2, \zeta_{10}^3, \zeta_{10}^4$ and ζ_{10}^5.
- $[\lambda : \mu] = [2 : -1]$. The space is of type (2), with only one rational angle $(1, \tau_{2,-1})$ of squared amplitude ζ_{10}^2.

(7) If $[\lambda : \mu] \notin S$, the space $V_{[\lambda:\mu]}$ contains precisely two angles with squared amplitude ζ_{10}^2. These are given by

$$(1, \tau) \text{ and } (\tau + a, \tau + b),$$

where

$$a = \frac{-2\lambda\mu - \mu^2}{\lambda + \mu}, \quad b = \frac{\lambda(\lambda^2 + \lambda\mu - \mu^2)}{\mu(2\lambda + \mu)}.$$

(8) Let $[\lambda : \mu] \notin S$. The space $V_{[\lambda:\mu]}$ contains an angle with squared amplitude ζ_{10}^4 if and only if $\lambda(\lambda^3 + 2\lambda^2\mu + 5\lambda\mu^2 + 4\mu^3)$ is the square of a rational number u. The curve in weighted projective space

$$u^2 = \lambda(\lambda^3 + 2\lambda^2\mu + 5\lambda\mu^2 + 4\mu^3)$$

has genus 1 and a rational point $(0 : 1 : 0)$, so it defines an elliptic curve. It is in particular isomorphic over $\mathbb{Q}$ to the elliptic curve with Weierstrass equation $y^2 + xy + y = x^3 + x^2$, whose Mordell-Weil group over $\mathbb{Q}$ is cyclic of order 4, generated by $(0, 0)$. The only four rational points are given by $[\lambda : \mu : u] = [0 : 1 : 0], [1 : 0 : \pm 1], [1 : -1 : 0]$. The spaces corresponding to these rational points are all of type (4).
(9) Let $[\lambda : \mu] \notin S$. The space $V_{[\lambda:\mu]}$ contains an angle with squared amplitude ζ_{10}^5 if and only if $\lambda^4 + 2\lambda^3\mu + 4\lambda^2\mu^2 + 3\lambda\mu^3 + \mu^4$ is the square of a rational number u. In this case, setting

$$e_\pm = \frac{-2\lambda\mu - \mu^2 \mp u}{\lambda + \mu}, \quad f_\pm = \frac{-2\lambda\mu - \mu^2 \pm u}{\lambda + \mu}$$

the rational angles in $V_{[\lambda:\mu]}$ with squared amplitude ζ_{10}^5 are precisely $(\tau + e_+, \tau + f_+)$ and $(\tau + e_-, \tau + f_-)$. We have $e_+ f_+ = e_- f_- = ab$ (where a, b are as in (4)), so that this case falls into the classification of Theorem 6.4.

(10) No such space has both an angle with squared amplitude ζ_{10}^4 and an angle with squared amplitude ζ_{10}^5.

Remark 7.16. If (a,b,c,d,x,y,z) is a solution to Equation (2) with $ab \neq cd$ and x,y,z having common order 5, then (up to symmetry equivalence) we may assume $x = \zeta_5 = \zeta_{10}^2$, and therefore the corresponding value of $\tau \in \mathbb{Q}(\zeta_5) = \mathbb{Q}(\zeta_{10})$ generates a space of the type considered in Theorem 7.15. In order for this space to have other rational angles besides those of squared amplitude ζ_{10}^2, it needs to contain an angle with squared amplitude ζ_{10}^4. The theorem implies that the corresponding space is of type (4).

7.5. $n = 12$. Consider a solution (a,b,c,d,x,y,z) to Equation (2) with $ab \neq cd$ and x,y,z having common order $n = 12$. In this case, Equation (4) shows that τ belongs to $\mathbb{Q}(\zeta_{12})$. One of the following necessarily holds: one among x,y,z is a primitive 12th root of unity (in which case, up to the Galois action and permutations, we may assume that $x = \zeta_{12}$), or one of the three roots of unity has order 4 (in which case, again up to the Galois action and permutations, we may assume $x = \zeta_{12}^3$). Notice that, if neither of these conditions holds, then all the three roots of unity x,y,z have order dividing 6, contradicting the fact that their common order is 12. Finally, in the case when x is ζ_{12}^3 and none among x,y,z is a primitive 12th root of unity, then at least one between y and z needs to be $\zeta_{12}^{\pm 2}$ or $\zeta_{12}^{\pm 4}$ in order to ensure that the common order of x,y,z is exactly 12.

We study the case $x = \zeta_{12}$ in Theorem 7.17 and the case $x = \zeta_{12}^3$ (with neither y nor z primitive) in Theorem 7.18.

Theorem 7.17. For $(\lambda,\mu) \in \mathbb{Q}^2 \setminus \{(0,0)\}$, let

$$\tau_{\lambda,\mu} := \lambda(\zeta_{12} + \zeta_{12}^2 - \zeta_{12}^3) + \mu(1 + \zeta_{12}).$$

The following hold:

(1) The space $\langle 1, \tau_{\lambda,\mu}\rangle_{\mathbb{Q}}$ only depends on $[\lambda:\mu] \in \mathbb{P}_1(\mathbb{Q})$. We denote it by $V_{[\lambda:\mu]}$.
(2) Let $\tau \in \mathbb{Q}(\zeta_{12})$ generate a space V such that the squared angle between 1 and τ is ζ_{12}. There exists $(\lambda,\mu) \in \mathbb{Q}^2 \setminus \{(0,0)\}$ such that $\tau = \tau_{\lambda,\mu}$.
(3) Conversely, for every $(\lambda,\mu) \in \mathbb{Q}^2 \setminus \{(0,0)\}$, the squared angle between 1 and τ is ζ_{12}.
(4) For every $[\lambda:\mu] \in \mathbb{P}_1(\mathbb{Q})$, the complex conjugate $\overline{V_{[\lambda:\mu]}}$ is homothetic to $V_{[\lambda:\mu]}$.
(5) The map

$$\begin{array}{rccc} f: & \mathbb{P}_1(\mathbb{Q}) & \to & \{\text{homothety classes of spaces}\} \\ & [\lambda:\mu] & \mapsto & \text{class of } V_{[\lambda:\mu]} \end{array}$$

is 2-to-1. The involution $\iota : [\lambda:\mu] \mapsto [\lambda+\mu : -\mu]$ preserves the fibres of f.
(6) Let $S = \{[0:1],[1:0],[1:1],[-1:1],[-1,2],[-2,1]\}$. Notice that $\iota([0:1]) = [-1:1]$, $\iota([1:0]) = [1:0]$,$\iota([1:1]) = [-2:1]$, and $\iota([-1:2]) = [-1:2]$. The spaces corresponding to $[\lambda:\mu] \in S$ have the following structure:
   - $[\lambda:\mu] = [0:1]$ or $[-1:1]$. The space is of type (4). A rational quadruple of $V_{[0:1]}$ is given by $\{1, \tau_{0,1}, \tau_{0,1}-1, \tau_{0,1}-2\}$. The various angles in this quadruple realise all the squared amplitudes $\zeta_{12}, \zeta_{12}^2, \zeta_{12}^5$ and ζ_{12}^6
   - $[\lambda:\mu] = [1:0]$. The space is of type (4). A rational quadruple of $V_{[1:0]}$ is given by $\{1, \tau_{1,0}, \tau_{1,0}-1, \tau_{1,0}-2\}$. The various angles in this quadruple realise all the squared amplitudes $\zeta_{12}, \zeta_{12}^2, \zeta_{12}^3$ and ζ_{12}^5
   - $[\lambda:\mu] = [1:1]$ or $[2:-1]$. The space is of type $2\cdot$(3). Two rational triples of $V_{[1:1]}$ are given by $\{1, \tau_{1,1}, \tau_{1,1}-3\}$ and $\{\tau_{1,1}+1, \tau_{1,1}-4, \tau_{1,1}-3/2\}$. In both triples the various angles realise all the squared amplitudes $\zeta_{12}, \zeta_{12}^4, \zeta_{12}^5$
   - $[\lambda:\mu] = [-1:2]$. The space is of type $2\cdot$(3). Two rational triples of $V_{[-1:2]}$ are given by $\{1, \tau_{-1,2}, \tau_{-1,2}+1\}$ and $\{\tau_{-1,2}-3, \tau_{-1,2}+3/4, \tau_{-1,2}+2\}$. In both triples the various angles realise all the squared amplitudes $\zeta_{12}, \zeta_{12}^4, \zeta_{12}^5$.
(7) If $[\lambda:\mu] \notin S$, the space $V_{[\lambda:\mu]}$ contains precisely two angles with squared amplitude ζ_{12} and two angles with squared amplitude ζ_{12}^5. These are given by

$$\zeta_{12}: \quad (1,\tau) \text{ and } (\tau+a, \tau+b),$$

where

$$a = -\mu\frac{2\lambda+\mu}{\lambda+\mu}, \quad b = \lambda\frac{2\lambda^2+2\lambda\mu-\mu^2}{\mu(2\lambda+\mu)}$$

and by

$$\zeta_{12}^5: \quad (\tau + c, \tau + d) \text{ and } (\tau + c', \tau + d')$$

where

$$c = -\lambda - 2\mu, \quad d = \frac{-2\lambda^2 - 2\lambda\mu + \mu^2}{\lambda - \mu}$$

$$c' = \frac{-2\lambda^2 - 2\lambda\mu + \mu^2}{2\lambda}, \quad d' = -2(\lambda + \mu).$$

(8) Let $[\lambda : \mu] \notin S$. The space $V_{[\lambda:\mu]}$ contains an angle with squared amplitude ζ_{12}^2 if and only if $\lambda(\lambda+\mu)\left(\lambda^2+\lambda\mu+\mu^2\right)$ is the square of a rational number u. The curve in weighted projective space

$$u^2 = \lambda(\lambda + \mu)\left(\lambda^2 + \lambda\mu + \mu^2\right)$$

has genus 1 and a rational point $[0 : 1 : 0]$, so it defines an elliptic curve. It is in particular isomorphic over $\mathbb{Q}$ to the elliptic curve with Weierstrass equation $y^2 = 1 + 2x + 2x^2 + x^3$, whose Mordell-Weil group over $\mathbb{Q}$ is $\mathbb{Z}/4\mathbb{Z}$. The only four rational points are given by $[\lambda : \mu : u] = [1 : 0 : \pm 1], [0 : 1 : 0], [-1 : 1 : 0]$ and they correspond to the three lattices of type (4) described above in part (6).

(9) Let $[\lambda : \mu] \notin S$. The space $V_{[\lambda:\mu]}$ contains an angle with squared amplitude ζ_{12}^3 if and only if $(2\lambda^2 + 2\lambda\mu + \mu^2)(2\lambda^2 + 2\lambda\mu + 5\mu^2)$ is the square of a rational number u. The curve in weighted projective space

$$u^2 = (2\lambda^2 + 2\lambda\mu + \mu^2)(2\lambda^2 + 2\lambda\mu + 5\mu^2)$$

has genus 1 and a rational point $[1 : 0 : 2]$, so it defines an elliptic curve. It is in particular isomorphic over $\mathbb{Q}$ to the elliptic curve with Weierstrass equation $y^2 + 2xy = x^3 - 6x^2 + 4x$, whose Mordell-Weil group over $\mathbb{Q}$ is $(\mathbb{Z}/2\mathbb{Z})^2$. The only four rational points are given by $[\lambda : \mu : u] = [1 : 0 : \pm 2], [1 : -2 : \pm 6]$.

(10) Let $[\lambda : \mu] \notin S$. The space $V_{[\lambda:\mu]}$ contains an angle with squared amplitude ζ_{12}^4 if and only if $(\lambda^2 + \lambda\mu + \mu^2)(5\lambda^2 + 5\lambda\mu + 2\mu^2)$ is the square of a rational number u. In this case, setting

$$e_\pm = \frac{\lambda^2 - 2\lambda\mu - 2\mu^2 \pm u}{\lambda + 2\mu}, \quad f_\pm = -\frac{\lambda^2 + 4\lambda\mu + \mu^2 \pm u}{2\lambda + \mu}$$

the rational angles in $V_{[\lambda:\mu]}$ with squared amplitude ζ_{12}^6 are precisely $(\tau + e_+, \tau + f_+)$ and $(\tau + e_-, \tau + f_-)$. The curve in weighted projective space

$$u^2 = \left(\lambda^2 + \lambda\mu + \mu^2\right)\left(5\lambda^2 + 5\lambda\mu + 2\mu^2\right)$$

has genus 1 and a rational point $[1 : 1 : 6]$, so it defines an elliptic curve. It is in particular isomorphic over $\mathbb{Q}$ to the elliptic curve with Weierstrass equation $y^2 = x^3 - 18x - 27$, whose Mordell-Weil group over $\mathbb{Q}$ is $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}$. A generator for the free part of the Mordell-Weil group is given by $(-2, 1)$.

(11) Let $[\lambda : \mu] \notin S$. The space $V_{[\lambda:\mu]}$ contains an angle with squared amplitude ζ_{12}^6 if and only if $(\lambda^2 + \lambda\mu + \mu^2)(2\lambda^2 + 2\lambda\mu + \mu^2)$ is the square of a rational number u. In this case, setting

$$g_\pm = -\frac{\mu(2\lambda + \mu) \pm u}{\lambda + \mu}, \quad h_\pm = \frac{-\mu(2\lambda + \mu) \pm u}{\lambda + \mu}$$

the rational angles in V with squared amplitude ζ_{12}^6 are precisely $(\tau + g_+, \tau + h_+)$ and $(\tau + g_-, \tau + h_-)$. The curve in weighted projective space

$$u^2 = (\lambda^2 + \lambda\mu + \mu^2)(2\lambda^2 + 2\lambda\mu + \mu^2)$$

has genus 1 and a rational point $[1 : -1 : 1]$, so it defines an elliptic curve. It is in particular isomorphic over $\mathbb{Q}$ to the elliptic curve with Weierstrass equation $y^2 = x^3 - x^2 - 4x - 2$, whose Mordell-Weil group over $\mathbb{Q}$ is $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}$. A generator for the free part of the Mordell-Weil group is given by $\left(-\frac{3}{4}, \frac{1}{8}\right)$.

(12) If such a space has, in addition to the angles of squared amplitude ζ_{12} and ζ_{12}^5, rational angles of two different amplitudes, then (up to a nonzero rational number) τ is one of the following:

(a) $\tau_1 = \zeta_{12}^3 - \zeta_{12}^2 + 1$. The corresponding space is of type (4), with a rational 4-tuple given by $(1, \tau, \tau - 1, \tau - \frac{1}{2})$ (angles of squared amplitudes $\zeta_{12}^{\pm 1}, \zeta_{12}^{\pm 2}, \zeta_{12}^{\pm 5}, \zeta_{12}^6$).

(b) $\tau_2 = \zeta_{12} + 1$. The corresponding space is of type (4), with a rational 4-tuple given by $(1, \tau, \tau - 1, \tau - 2)$ (angles of squared amplitudes $\zeta_{12}^{\pm 1}, \zeta_{12}^{\pm 2}, \zeta_{12}^{\pm 5}, \zeta_{12}^{6}$).

(c) $\tau_3 = \frac{1}{2}(\zeta_{12}^3 - \zeta_{12}^2 + \zeta_{12} + 2)$. The corresponding space is of type (4), with a rational 4-tuple given by $(1, \tau, \tau - 1, \tau - \frac{3}{2})$ (angles of squared amplitudes $\zeta_{12}^{\pm 1}, \zeta_{12}^{\pm 3}, \zeta_{12}^{\pm 4}, \zeta_{12}^{\pm 5}, \zeta_{12}^{6}$).

In addition, we have $\langle 1, \tau_1 \rangle_{\mathbb{Q}} \sim \langle 1, \tau_2 \rangle_{\mathbb{Q}}$, and this is the only pair of homothetic spaces in this list.

Theorem 7.18. For $(\lambda, \mu) \in \mathbb{Q}^2 \setminus \{(0,0)\}$, let

$$\tau_{\lambda,\mu} := \lambda(\zeta_{12} + \zeta_{12}^2) + \mu(1 + \zeta_{12}^3).$$

The following hold:

(1) The space $\langle 1, \tau_{\lambda,\mu} \rangle_{\mathbb{Q}}$ only depends on $[\lambda : \mu] \in \mathbb{P}_1(\mathbb{Q})$. We denote it by $V_{[\lambda:\mu]}$.

(2) Let $\tau \in \mathbb{Q}(\zeta_{12})$ generate a space V such that the squared angle between 1 and τ is ζ_{12}^3. There exists $(\lambda, \mu) \in \mathbb{Q}^2 \setminus \{(0,0)\}$ such that $\tau = \tau_{\lambda,\mu}$.

(3) Conversely, for every $(\lambda, \mu) \in \mathbb{Q}^2 \setminus \{(0,0)\}$, the squared angle between 1 and τ is ζ_{12}^3.

(4) For every $[\lambda : \mu] \in \mathbb{P}_1(\mathbb{Q})$, the complex conjugate $\overline{V_{[\lambda:\mu]}}$ is homothetic to $V_{[\lambda:\mu]}$.

(5) The map

$$\begin{array}{rccc} f: & \mathbb{P}_1(\mathbb{Q}) & \to & \{\text{homothety classes of spaces}\} \\ & [\lambda : \mu] & \mapsto & \text{class of } V_{[\lambda:\mu]} \end{array}$$

is 2-to-1. The involution $\iota : [\lambda : \mu] \mapsto [-\lambda : \lambda + \mu]$ preserves the fibres of f.

(6) Let $S = \{[-2, 1]\}$. Notice that $\iota([-2 : 1]) = [-2 : 1]$. The spaces corresponding to $[\lambda : \mu] \in S$ has type (2) with a single rational angle given by $(1, \tau_{-2,1})$ with squared amplitude ζ_{12}^3.

(7) If $[\lambda : \mu] \notin S$, the space $V_{[\lambda:\mu]}$ contains precisely two angles with squared amplitude ζ_{12}^3. These are given by

$$(1, \tau) \text{ and } (\tau + a, \tau + b),$$

where

$$a = -\lambda - 2\mu, \quad b = \frac{\lambda^2 - 2\lambda\mu - 2\mu^2}{\lambda + 2\mu}.$$

(8) Let $[\lambda : \mu] \notin S$. The space $V_{[\lambda:\mu]}$ contains an angle with squared amplitude ζ_{12}^2 if and only if $(\lambda^2 - \lambda\mu - \mu^2)(\lambda^2 + \lambda\mu + \mu^2)$ is the square of a rational number u. In this case, setting

$$c_\pm = -\frac{2\lambda\mu + \mu^2 \pm u}{\lambda + \mu}, \quad d_\pm = \frac{\lambda^2 - \mu^2 \mp u}{\mu}$$

the rational angles in V with squared amplitude ζ_{12}^2 are precisely $(\tau + c_+, \tau + d_+)$ and $(\tau + c_-, \tau + d_-)$. The curve in weighted projective space

$$u^2 = (\lambda^2 - \lambda\mu - \mu^2)(\lambda^2 + \lambda\mu + \mu^2)$$

has genus 1 and a rational point $(1 : 0 : 1)$, so it defines an elliptic curve. It is in particular isomorphic over $\mathbb{Q}$ to the elliptic curve with Weierstrass equation $y^2 = x^3 - x^2 + 4x$, whose Mordell-Weil group over $\mathbb{Q}$ is $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}$. A generator for the free part of the Mordell-Weil group is given by $(1, 2)$.

(9) Let $[\lambda : \mu] \notin S$. The space $V_{[\lambda:\mu]}$ contains an angle with squared amplitude ζ_{12}^4 if and only if $(5\lambda^2 - \lambda\mu - \mu^2)(\lambda^2 + \lambda\mu + \mu^2)$ is the square of a rational number u. In this case, setting

$$e_\pm = -\frac{\lambda^2 + 4\lambda\mu + \mu^2 \pm u}{2\lambda + \mu}, \quad f_\pm = \frac{-2\lambda^2 - 2\lambda\mu + \mu^2 \pm u}{\lambda - \mu}$$

the rational angles in $V_{[\lambda:\mu]}$ with squared amplitude ζ_{12}^4 are precisely $(\tau + e_+, \tau + f_+)$ and $(\tau + e_-, \tau + f_-)$. The curve in weighted projective space

$$u^2 = (5\lambda^2 - \lambda\mu - \mu^2)(\lambda^2 + \lambda\mu + \mu^2)$$

has genus 1 and a rational point $(1 : 1 : 3)$, so it defines an elliptic curve. It is in particular isomorphic over $\mathbb{Q}$ to the elliptic curve with Weierstrass equation $y^2 = x^3 + 9x + 54$, whose Mordell-Weil group over $\mathbb{Q}$ is $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}$. A generator for the free part of the Mordell-Weil group is given by $(6, 18)$.

(10) The space $V_{[\lambda:\mu]}$ does not contain any angles with squared amplitude ζ_{12}^6. Indeed, this happens if and only if $2(\lambda^2 + \lambda\mu + \mu^2)$ is the square of a rational number u, but the genus-0 curve $u^2 = 2(\lambda^2 + \lambda\mu + \mu^2)$ in $\mathbb{P}_{2,\mathbb{Q}}$ has no rational points (since it has no $\mathbb{Q}_2$- and no $\mathbb{Q}_3$-points).

(11) No such space has both an angle with squared amplitude ζ_{12}^2 and an angle with squared amplitude ζ_{12}^4.
(12) If such a space has, in addition to the angles of squared amplitude ζ_{12}^3, rational angles of two different amplitudes, then:
  (a) $[\lambda : \mu] = [1, 0]$ or $[1 : -1]$, which correspond to a space of type (4), with a rational 4-tuple given by $(1, \tau_{1,0}, \tau_{1,0} - 1, \tau_{1,0} + 1)$ (angles of squared amplitudes $\zeta_{12}^{\pm 1}, \zeta_{12}^{\pm 2}, \zeta_{12}^{\pm 3}, \zeta_{12}^{\pm 5}$).
  (b) $[\lambda : \mu] = [1, 1]$ or $[-1 : 2]$, which correspond to a space of type (4), with a rational 4-tuple given by $(1, \tau_{1,1}, \tau_{1,1} - 1, \tau_{1,1} - 3)$ (angles of squared amplitudes $\zeta_{12}^{\pm 1}, \zeta_{12}^{\pm 3}, \zeta_{12}^{\pm 4}, \zeta_{12}^{\pm 5}$).

Remark 7.19. If $E \to \mathbb{P}^1$ is one of the elliptic families, and if $\iota : \mathbb{P}^1 \to \mathbb{P}^1$ is the involution (coming from complex conjugation) which exchanges homothetic spaces, a geometric interpretation can be given to the action of ι on E:

(1) ι lifts to an involution of E;
(2) every involution of an elliptic curve is either of the form $x \mapsto Q - x$ or of the form $x \mapsto x + T_2$ with T_2 a 2-torsion point;
(3) the lift must preserve the branch locus of $E \to \mathbb{P}^1$, hence it sends the 0 of E to a ramification point of $E \to \mathbb{P}^1$, that is, a 2-torsion point;
(4) hence, if it is of the form $x \mapsto Q - x$, then Q is a 2-torsion point;
(5) given a lift φ of ι, also $-\varphi$ is a lift, so there is always a lift of the form $x \mapsto x + T_2$;
(6) in particular, if E admits a unique non-trivial 2-torsion point, then translation by that point induces ι on $\mathbb{P}^1$.

8. Computational details

8.1. **Enumerating triples of roots of unity.** In carrying out several of the computational arguments outlined above we are faced with the problem of enumerating triples (x, y, z) of roots of unity with a given common order, up to the equivalence relation given by Definition 1.1. To make this process more efficient, we rely on the following observations:

Lemma 8.1. Let (x, y, z) be a triple with common order n. Write $x = \zeta_n^{e_1}$, $y = \zeta_n^{e_2}$ and $z = \zeta_n^{e_3}$ with $e_1, e_2, e_3 \in [1, n]$, and identify (x, y, z) with (e_1, e_2, e_3).

(1) Up to the Galois action, (e_1, e_2, e_3) is equivalent to a triple with $e_1 \mid n$;
(2) Up to the action of the group $G \times \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ of Definition 1.1, (e_1, e_2, e_3) is equivalent to a triple with

$$e_1 \mid n, \qquad (e_2, n) \geq e_1, \qquad (e_3, n) \geq e_1, \qquad e_2 \leq n/2, \qquad e_3 \leq \frac{n}{2}, \qquad e_3 \geq e_2.$$

Proof. Recall that the Galois group of $\mathbb{Q}(\zeta_n)/\mathbb{Q}$ can be canonically identified with $(\mathbb{Z}/n\mathbb{Z})^\times$: we denote by σ_i the Galois automorphism sending ζ_n to ζ_n^i. Under the identification $(x, y, z) \leftrightarrow (e_1, e_2, e_3)$, the Galois action translates to $\sigma_i(e_1, e_2, e_3) = (ie_1, ie_2, ie_3)$. It is an elementary arithmetic fact that any two elements a, b in $\mathbb{Z}/n\mathbb{Z}$ such that $(n, a) = (n, b)$ differ by multiplication by an element in $(\mathbb{Z}/n\mathbb{Z})^\times$. This shows (1). As for (2), observe that up to the action $(e_1, e_2, e_3) \mapsto (\pm e_1 \bmod n, \pm e_2 \bmod n, \pm e_3 \bmod n)$ we can assume that each e_i lies in the interval $[0, \frac{n}{2}]$. Using the permutation action we can reorder (e_1, e_2, e_3) so that $(n, e_1) \geq (n, e_2), (n, e_3)$. The Galois action can then be used as in (1) to get an equivalent triple in which $e_1 \mid n$. Finally, reordering again if necessary we can assume $e_3 \geq e_2$. □

Remark 8.2. A further reduction in the number of triples to be considered comes from the following observation. Fix a root of unity among x, y, z, say x for simplicity. We can consider the polynomial P of equation (2) as a polynomial of degree 2 in x. Writing $P = x^2 P_2(a, b, c, d, y, z) + xP_1(a, b, c, d, y, z) + P_0(a, b, c, d, y, z)$, we see that x lies in an at most quadratic extension of $\mathbb{Q}(y, z)$, unless we have $P_2(a, b, c, d, y, z) = P_1(a, b, c, d, y, z) = P_0(a, b, c, d, y, z) = 0$. One checks that these equations can only be satisfied by a space of type (4), hence we can further assume that $\mathbb{Q}(x, y, z)$ has degree at most 2 over $\mathbb{Q}(y, z)$. The same holds for the degree of $\mathbb{Q}(x, y, z)$ over its subfields $\mathbb{Q}(x, z)$ and $\mathbb{Q}(x, y)$.

Remark 8.3. Working with the reduced set of triples suggested by Lemma 8.1 and Remark 8.2 leads to a tremendous speed-up in practice: for $n = 144$ (the largest value we need to consider in the computational proof of Theorem 7.7), the reduced set of triples has size 1869, less than 0.1% of the 2,515,968 triples that satisfy $(e_1, e_2, e_3, 144) = 1$.

8.2. **Finding defining equations over $\mathbb{Q}$.** Let K be a finite extension of $\mathbb{Q}$ and let $p_1, \dots, p_r \in K[x_1, \dots, x_n]$ be polynomials with coefficients in K. Suppose we are interested in finding all solutions to the equations

$$p_1(x_1, \dots, x_n) = \cdots = p_r(x_1, \dots, x_n) = 0 \tag{16}$$

where $x_1, \dots, x_n$ are *rational* numbers. Fix a basis $\beta_1, \dots, \beta_d$ of K as $\mathbb{Q}$-vector space, where $d = [K : \mathbb{Q}]$. We can write each p_i as $p_i(x_1, \dots, x_n) = \sum_{j=1}^{d} \beta_j p_{ij}(x_1, \dots, x_n)$, where each p_{ij} now has rational coefficients. Any solution to (16) with $x_1, \dots, x_n \in \mathbb{Q}$ then corresponds to a rational solution of the system of equations

$$p_{ij}(x_1, \dots, x_n) = 0 \quad \forall i = 1, \dots, r,\ j = 1, \dots, d, \tag{17}$$

which is now given by equations with rational coefficients. In order to efficiently convert equations as in (16) to equations as in (17), we use the following simple observation:

Lemma 8.4. With the previous notation, the ideal of $\mathbb{Q}[x_1, \dots, x_n]$ generated by the polynomials p_{ij} is also generated by the polynomials $\mathrm{tr}_{K/\mathbb{Q}}(p_i\beta_j)$ for $i = 1, \dots, r$ and $j = 1, \dots, d$. Here the *trace* of a polynomial is taken coefficient-wise.

Proof. As $K/\mathbb{Q}$ is a separable extension, it is well-known that the trace form $K \times K \to \mathbb{Q}$ is nondegenerate. Let $\beta_i^\vee$ be the dual basis of β_i with respect to the trace form. By definition, p_{ij} is then $\mathrm{tr}_{K/\mathbb{Q}}(p_i\beta_j^\vee)$. On the other hand, as $\{\beta_i\}$ and $\{\beta_i^\vee\}$ are two bases for the $\mathbb{Q}$-vector space K, there is an invertible matrix M_{jk} with rational coefficients such that $\sum_k M_{jk}\beta_k^\vee = \beta_j$. This easily implies that the $\mathbb{Q}$-span of the polynomials $\mathrm{tr}_{K/\mathbb{Q}}(p_i\beta_j^\vee)$ coincides with the $\mathbb{Q}$-span of the polynomials $\mathrm{tr}_{K/\mathbb{Q}}(p_i\beta_j)$, which in turn implies the lemma. □

8.3. **Rational points on curves.** The higher-genus curves we encounter in Section 7 all have genus 3 or 5, as proved in Proposition 7.9. In order to determine their rational points, however, we often reduce to studying certain auxiliary curves of genus 2, so we begin with this case. Given a (smooth) projective curve $C/\mathbb{Q}$ of genus 2 with Jacobian $J/\mathbb{Q}$, in order to determine $C(\mathbb{Q})$ we proceed as follows:

(1) Since every such curve is hyperelliptic over $\mathbb{Q}$, we fix an equation for C of the form $y^2 = f(x)$ with $f(x) \in \mathbb{Q}[x]$ of degree 5 or 6 and having no repeated factors (by a slight abuse of notation we identify this affine curve with the unique smooth projective curve having the same function field). Up to rescaling, we can in fact assume that $f(x)$ has integral coefficients.
(2) We preliminary run a quick test for local solubility: we loop over primes $3 \le p \le 13$ that do not divide the discriminant of $f(x)$ and check whether the (projective) curve C_p obtained from C by reduction modulo p has any $\mathbb{F}_p$-rational points. Since we have a well-defined reduction map $C(\mathbb{Q}) \to C_p(\mathbb{F}_p)$, the fact that $C_p(\mathbb{F}_p) = \emptyset$ implies $C(\mathbb{Q}) = \emptyset$. To explain why we stop at $p = 13$, notice that for $p > 13$ (and not dividing the discriminant of $f(x)$) the Weil bounds imply $\#C_p(\mathbb{F}_p) \ge p + 1 - 4\sqrt{p} > 0$, that is, the reduced curve automatically has $\mathbb{F}_p$-points.
(3) If C passes this preliminary test, we use standard descent techniques to prove that the Mordell-Weil group $J(\mathbb{Q})$ has rank at most 1. If this is the case, then Chabauty's method (which is fully implemented in MAGMA for genus 2 curves with $\mathrm{rk}\, J(\mathbb{Q}) \le 1$) can be used to provably determine $C(\mathbb{Q})$.

All the genus-2 curves we meet can be handled by this combination of techniques.

Remark 8.5 ('Modular' interpretation of the automorphisms of C)**.** The curves C we studied in Section 7 all possess three independent involutions. Two are given by $u \mapsto -u$ and $v \mapsto -v$; the third one is different in each case, but always of the form $[\lambda : \mu : u : v] \mapsto [\alpha\lambda + \beta\mu, \gamma\lambda + \delta\mu, hu, ku]$ for a certain $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ in $\mathrm{GL}_2(\mathbb{Q})$ and $h, k \in \mathbb{Q}^\times$. This fact is not an accident, and is implicitly used in the determination of the rational points of these curves (indeed, this extra involution induces an involution on the quotients of C, which is then exploited to obtain $C(\mathbb{Q})$), so we take some time to explain its geometric origin.

To simplify the notation, we consider the special case $n = 10$, where we look at angles of squared amplitude ζ_{10}^2 and ζ_{10}^3 (but the argument applies unchanged to all other cases). Consider, therefore, the curve

$$\begin{cases} u^2 = f_2(\lambda, \mu) = \lambda(3\lambda - \mu)(11\lambda^2 - 9\lambda\mu + 4\mu^2) \\ v^2 = f_3(\lambda, \mu) = 41\lambda^4 - 38\lambda^3\mu + 9\lambda^2\mu^2 + 8\lambda\mu^3 - 4\mu^4. \end{cases}$$

A rational point $[\lambda : \mu : u : v]$ of C has the following 'modular' interpretation: it gives rise to $\tau = \lambda(-2 - 2\zeta_{10}^2 + \zeta_{10}^3) + \mu(1 + \zeta_{10})$, and to rational numbers

$$a = \frac{4\lambda^2 - 6\lambda\mu + \mu^2}{3\lambda - \mu}, \quad b = -\frac{5\lambda(\lambda^2 + \lambda\mu - \mu^2)}{4\lambda^2 - 6\lambda\mu + \mu^2},$$

$$a'_\pm = \frac{-3\lambda^2 + 7\lambda\mu - 2\mu^2 \mp u}{-4\lambda + 2\mu}, \quad b'_\pm = \frac{5\lambda^2 - 5\lambda\mu \mp u}{2\lambda}$$

and

$$a''_\pm = \frac{\lambda^2 + 11\lambda\mu - 6\mu^2 \mp u}{-2\lambda + 4\mu}, \quad b''_\pm = \frac{9\lambda^2 - \lambda\mu - 4\mu^2 \mp u}{4\lambda + 2\mu},$$

such that in the space $\langle 1, \tau\rangle_{\mathbb{Q}}$ there are six distinguished pairs of rational angles, namely those given by the rays

$$(1, \tau), (\tau + a, \tau + b); \quad (\tau + a'_\pm, \tau + b'_\pm); \quad (\tau + a''_\pm, \tau + b''_\pm).$$

The automorphism $u \mapsto -u$ exchanges a'_+ with a'_- and b'_+ with b'_-, thus exchanging the two angles in the above list with squared amplitude ζ_{10}^2: the 'modular' interpretation of this automorphism is the fact that there is no preferred way to order these two angles, so the labels $\pm$ are arbitrary, and can be exchanged freely. Similarly, $v \mapsto -v$ simply exchanges $a''_+ \leftrightarrow a''_-$ and $b''_+ \leftrightarrow b''_-$, and therefore swaps the two angles with amplitude ζ_{10}^3.

Finally, there is a *third* geometric construction giving rise to an automorphism of C. Indeed, if V is a space having angles of squared amplitudes $\zeta_{10}, \zeta_{10}^2, \zeta_{10}^3$, the same is true for the complex conjugate $\overline{V}$. If we let σ_i (for $(i, 10) = 1$) be the automorphism of $\mathbb{Q}(\zeta_{10})$ sending $\zeta_{10} \mapsto \zeta_{10}^i$, then we have

$$\overline{V} = \langle 1, \overline{\tau}\rangle_{\mathbb{Q}} \sim \langle \frac{1}{\overline{\tau}}, 1\rangle_{\mathbb{Q}} = \left\langle \frac{\sigma_1(\tau)\sigma_5(\tau)\sigma_7(\tau)}{\sigma_1(\tau)\sigma_5(\tau)\sigma_7(\tau)\sigma_{11}(\tau)}, 1 \right\rangle_{\mathbb{Q}} = \langle \sigma_1(\tau)\sigma_5(\tau)\sigma_7(\tau), 1\rangle_{\mathbb{Q}},$$

where the last equality follows from the fact that $\sigma_1(\tau)\sigma_5(\tau)\sigma_7(\tau)\sigma_{11}(\tau) = N_{\mathbb{Q}(\zeta_{10})/\mathbb{Q}}(\tau)$ is a nonzero rational number. Notice that $1/\overline{\tau}$ is in $\mathbb{Q}(\zeta_{10})$ and forms with the positive real axis the same angle that τ forms with 1, so $1/\overline{\tau}$ is again of the form considered in Theorem 7.14. In particular, it is a linear combination with rational coefficients of $-2 - 2\zeta_{10}^2 + \zeta_{10}^3$ and $1 + \zeta_{10}$. A short calculation shows that

$$\sigma_1(\tau)\sigma_5(\tau)\sigma_7(\tau) = (\lambda^2 + \lambda\mu - \mu^2) \cdot \left((4\lambda - 3\mu)(1 + \zeta_{10}) + (-\mu + 3\lambda)(-2 - 2\zeta_{10}^2 + \zeta_{10}^3)\right),$$

so that $\overline{V} \sim \langle 1, (-\mu + 3\lambda)(-2 - 2\zeta_{10}^2 + \zeta_{10}^3) + (4\lambda - 3\mu)(1 + \zeta_{10})\rangle_{\mathbb{Q}}$. Of course one can also apply the geometric transformations performed above (the symmetry corresponding to complex conjugation and the various homotheties) to the lines in V forming rational angles; in this way, from the original rational point $[\lambda : \mu : u : v]$ we obtain a new point, which can be seen to be $[3\lambda - \mu : 4\lambda - 3\mu : 5u : 5v]$. We note that the signs of $5u, 5v$ in this formula are somewhat arbitrary, since they depend on the precise labelling of the angles in the transformed lattice, but any choice of signs would work, since $u \mapsto -u$ and $v \mapsto -v$ are also automorphisms of C.

Although this argument formally applies only to the *rational points* of C, all we have done is algebraic manipulation, so it is not hard to see that this is indeed an automorphism of C (of course, this can also be checked directly on the equations). Finally, the geometric interpretation given above (or the explicit formula) makes it clear that the automorphism constructed in this way commutes with $u \mapsto -u$ and with $v \mapsto -v$. This proves that all the curves C we have to consider admit an action of $(\mathbb{Z}/2\mathbb{Z})^3$.

All the higher-genus curves we encounter in Section 7 are of one of the two types considered in Proposition 7.9, hence we only explain how we find all the rational points on such curves. Let C be such a curve, and – in case it is singular of genus 3 – let $f : \tilde{C} \to C$ be its desingularisation. It is clear that we have $C(\mathbb{Q}) = f(\tilde{C}(\mathbb{Q})) \cup C_{\mathrm{sing}}(\mathbb{Q})$, where C_{sing} is the 0-dimensional singular locus of C. The determination of $C_{\mathrm{sing}}(\mathbb{Q})$ is straightforward, so we can assume to be working with the smooth curve $\tilde{C}$. Furthermore, if $g(C) = 5$, Proposition 7.9 yields the existence of an explicit map $C \to C'$ with $g(C') = 3$, and – if we can determine $C'(\mathbb{Q})$ – it is a simple matter to obtain from this also the set $C(\mathbb{Q})$. Thus we just need to explain how we determine $C(\mathbb{Q})$ when C is a smooth curve of genus 3 over $\mathbb{Q}$. It should be stressed that there is no known algorithm capable of achieving this for *all* smooth curves of genus 3 over $\mathbb{Q}$: we simply describe a procedure that *happens* to apply to all the curves that we consider in this work.

(1) We determine whether C is hyperelliptic over $\overline{\mathbb{Q}}$ (in other words, we check whether the canonical map f of C is *not* an embedding).

(2) Suppose that f is not an embedding. In this case, f realises C as a double cover of a smooth genus-0 curve $\mathcal{L}$. Using standard techniques, we determine whether $\mathcal{L}$ has $\mathbb{Q}$-rational points: if $\mathcal{L}(\mathbb{Q}) = \emptyset$, then clearly also $C(\mathbb{Q}) = \emptyset$. Otherwise, using a rational point on $\mathcal{L}$ we can identify $\mathcal{L}$ with $\mathbb{P}_{1,\mathbb{Q}}$, and from this obtain a hyperelliptic model $C_h : y^2 = f(x)$ for C over $\mathbb{Q}$ (where, again, with slight abuse of notation we identify this affine curve with the corresponding smooth projective curve).
(3) If, on the other hand, the canonical map of C is an embedding, then its image is a plane quartic $Q \subseteq \mathbb{P}_{2,\mathbb{Q}}$.
(4) We compute the automorphism group G over $\mathbb{Q}$ of C_h or of Q, respectively:
   (a) in the hyperelliptic case this is not hard, since every such automorphism is of the form $(x, y) \mapsto (\frac{ax+b}{cx+d}, \frac{ey}{(cx+d)^3})$ for some $a, b, c, d, e \in \mathbb{Q}$;
   (b) in the plane quartic case, we exploit the fact that – since we are working with the canonical embedding – every automorphism of Q lifts to an automorphism of the ambient $\mathbb{P}_{2,\mathbb{Q}}$ (this follows from the fact that every automorphism of C acts on the three-dimensional vector space of regular differentials by linear automorphisms). Thus we simply need to consider linear transformations of $\mathbb{P}_{2,\mathbb{Q}}$ that preserve – up to nonzero scalars – the single quartic equation defining Q.
(5) For every α in the automorphism group G, we consider the quotient map $g_\alpha : C \to C/\langle\alpha\rangle =: C_\alpha$ under the action of the finite group $\langle\alpha\rangle$. This has strictly lower genus than C.
(6) For every C_α of genus 1, using MAGMA's machinery for elliptic curves we check whether $C_\alpha(\mathbb{Q})$ is a finite set. If this is the case, pulling back the rational points via g_α we determine $C(\mathbb{Q})$.
(7) For every C_α of genus 2, we use the techniques explained above to try and determine $C_\alpha(\mathbb{Q})$. If they succeed, we again obtain the set $C(\mathbb{Q})$ by pulling back $C_\alpha(\mathbb{Q})$ along g_α.

It is a (lucky) accident of nature that, for all the curves C we consider in Section 7, the above procedure always finds a curve C_α for which the set $C_\alpha(\mathbb{Q})$ is finite and can be determined, and therefore $C(\mathbb{Q})$ can also be determined.

Angle	$u^2 = f(\lambda, \mu)$	$a_\pm, b_\pm$	P	E	$E(\mathbb{Q})$	Generators
ζ_8^2	$\lambda(\lambda+\mu)(\lambda^2+\mu^2)$	$\dfrac{\lambda^2-\mu^2 \pm u}{\mu}, \dfrac{-2\lambda\mu \mp u}{\lambda}$	$[0:1:0]$	$y^2 = (x+1)(x^2+1)$	$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}$	$(-1,0), (1,2)$
ζ_8^4	$(\lambda^2+\mu^2)(3\lambda^2+2\lambda\mu+\mu^2)$	$\dfrac{\lambda^2-2\lambda\mu-\mu^2 \pm u}{\lambda+\mu}, \dfrac{\lambda^2-2\lambda\mu-\mu^2 \mp u}{\lambda+\mu}$	$[0:1:1]$	$y^2 = (x+3)(x^2-2x-7)$	$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}$	$(-3,0), (5,8)$

TABLE 1. Rational angles in lattices with squared angle $(1, \tau) = \zeta_8$

Angle	$u^2 = f(\lambda, \mu)$	$a_\pm, b_\pm$	P	E	$E(\mathbb{Q})$	Generators
ζ_{10}^4	$\mu(\lambda+\mu)(4\lambda^2+\lambda\mu+\mu^2)$	$\dfrac{-2\lambda^2-\lambda\mu+\mu^2 \pm u}{2\lambda}, \dfrac{-3\lambda\mu-\mu^2 \mp u}{2\mu}$	$[0:1:1]$	$y^2+xy+y = x^3+x^2$	$\mathbb{Z}/4\mathbb{Z}$	$(0,0)$
ζ_{10}^5	$\lambda^4+3\lambda^3\mu+4\lambda^2\mu^2+2\lambda\mu^3+\mu^4$	$\dfrac{-\lambda^2-2\lambda\mu \pm u}{\lambda+\mu}, \dfrac{-\lambda^2-2\lambda\mu \mp u}{\lambda+\mu}$	$[0:1:1]$	$y^2 = (x+2)(x^2-x-1)$	$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}$	$(-2,0), (3,-5)$

TABLE 2. Rational angles in lattices with squared angle $(1, \tau) = \zeta_{10}^2$

Angle	$u^2 = f(\lambda, \mu)$	$a_\pm, b_\pm$	P	E	$E(\mathbb{Q})$	Generators
ζ_{12}^2	$(\lambda^2-\lambda\mu-\mu^2)(\lambda^2+\lambda\mu+\mu^2)$	$\dfrac{-2\lambda\mu-\mu^2 \mp u}{\lambda+\mu}, \dfrac{\lambda^2-\mu^2 \mp u}{\mu}$	$[1:-1:1]$	$y^2 = x(x^2-x+4)$	$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}$	$(0,0), (4,8)$
ζ_{12}^4	$(\lambda^2+\lambda\mu+\mu^2)(5\lambda^2-\lambda\mu-\mu^2)$	$\dfrac{-\lambda^2-4\lambda\mu-\mu^2 \mp u}{2\lambda+\mu}, \dfrac{2\lambda^2+2\lambda\mu-\mu^2 \mp u}{-\lambda+\mu}$	$[1:1:3]$	$y^2 = (x+3)(x^2-3x+18)$	$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}$	$(-3,0), (6,18)$

TABLE 3. Rational angles in lattices with squared angle $(1, \tau) = \zeta_{12}^3$

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Angle	$u^2 = f(\lambda, \mu)$	$a_\pm, b_\pm$	P	E	$E(\mathbb{Q})$	Generators
ζ_{10}^2	$\lambda(3\lambda-\mu)(11\lambda^2-9\lambda\mu+4\mu^2)$	$\frac{-3\lambda^2+7\lambda\mu-2\mu^2 \mp u}{-4\lambda+2\mu}$ $\frac{5\lambda^2-5\lambda\mu \mp u}{2\lambda}$	$[0:1:0]$	$y^2+xy+y=x^3-x^2+2$	$\mathbb{Z}/2\mathbb{Z}\oplus\mathbb{Z}$	$(-1,0),(4,5)$
ζ_{10}^3	$41\lambda^4-38\lambda^3\mu+9\lambda^2\mu^2+8\lambda\mu^3-4\mu^4$	$\frac{\lambda^2+11\lambda\mu-6\mu^2 \mp u}{-2\lambda+4\mu}$ $\frac{9\lambda^2-\lambda\mu-4\mu^2 \mp u}{4\lambda+2\mu}$	$[1:1:4]$	$y^2+xy+y=x^3-x^2+20x+22$	$\mathbb{Z}/4\mathbb{Z}\oplus\mathbb{Z}$	$(4,10),(79,-740)$
ζ_{10}^4	$(2\lambda^2-2\lambda\mu+\mu^2)(14\lambda^2-6\lambda\mu+\mu^2)$	$\frac{6\lambda^2-4\lambda\mu-\mu^2 \mp u}{4\lambda}$ $\frac{-2\lambda^2+8\lambda\mu-3\mu^2 \mp u}{-2\lambda+2\mu}$	$[0:1:1]$	$y^2=(x+2)(x^2-x-1)$	$\mathbb{Z}/2\mathbb{Z}\oplus\mathbb{Z}$	$(-2,0),(3,-5)$
ζ_{10}^5	$31\lambda^4-38\lambda^3\mu+24\lambda^2\mu^2-7\lambda\mu^3+\mu^4$	$\frac{-4\lambda^2+6\lambda\mu-\mu^2 \pm u}{-3\lambda+\mu}$ $\frac{-4\lambda^2+6\lambda\mu-\mu^2 \mp u}{-3\lambda+\mu}$	$[0:1:1]$	$y^2=(x+5)(x^2-5x-25)$	$\mathbb{Z}/2\mathbb{Z}\oplus\mathbb{Z}$	$(-5,0),(-\frac{15}{4},\frac{25}{8})$

TABLE 4. Rational angles in lattices with squared angle $(1,\tau)=\zeta_{10}$

Angle	$u^2 = f(\lambda, \mu)$	$a_\pm, b_\pm$	P	E	$E(\mathbb{Q})$	Generators
ζ_{12}^2	$\lambda(\lambda+\mu)(\lambda^2+\lambda\mu+\mu^2)$	$\frac{\lambda^2-\mu^2 \pm u}{\mu}$ $\frac{-\lambda^2-2\lambda\mu \mp u}{\lambda}$	$[0:1:0]$	$y^2 = x(x^2-x+1)$	$\mathbb{Z}/4\mathbb{Z}$	$(1,1)$
ζ_{12}^3	$(2\lambda^2+2\lambda\mu+\mu^2)(2\lambda^2+2\lambda\mu+5\mu^2)$	$\frac{-2\lambda^2-6\lambda\mu-\mu^2 \pm u}{4\lambda+2\mu}$ $\frac{2\lambda^2-2\lambda\mu-3\mu^2 \mp u}{2\mu}$	$[1:0:2]$	$y^2 = (x-2)(x+1)(x+2)$	$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	$(2,0), (-1,0)$
ζ_{12}^4	$(\lambda^2+\lambda\mu+\mu^2)(5\lambda^2+5\lambda\mu+2\mu^2)$	$\frac{\lambda^2-2\lambda\mu-2\mu^2 \pm u}{\lambda+2\mu}$ $\frac{-\lambda^2-4\lambda\mu-\mu^2 \mp u}{2\lambda+\mu}$	$[1:1:6]$	$y^2 = (x+3)(x^2-3x-9)$	$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}$	$(-3,0), (6,9)$
ζ_{12}^6	$(\lambda^2+\lambda\mu+\mu^2)(2\lambda^2+2\lambda\mu+\mu^2)$	$\frac{-2\lambda\mu-\mu^2 \pm u}{\lambda+\mu}$ $\frac{-2\lambda\mu-\mu^2 \mp u}{\lambda+\mu}$	$[0:1:1]$	$y^2 = (x+1)(x^2-2x-2)$	$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}$	$(-1,0), (-\frac{3}{4}, \frac{1}{8})$

TABLE 5. Rational angles in lattices with squared angle $(1, \tau) = \zeta_{12}$

10. Pictures of the four spaces of type 3+3

In this section we show that the four spaces of type (3)+(3) can be constructed geometrically from the diagonals of regular polygons. This resembles the constructions given in [DVZ22] for the two dodecagonal spaces, which arise from similar configurations of incident diagonals in a dodecagon. The relationship between intersection of diagonals and diophantine-trigonometric equations was explored for example in [PR98].

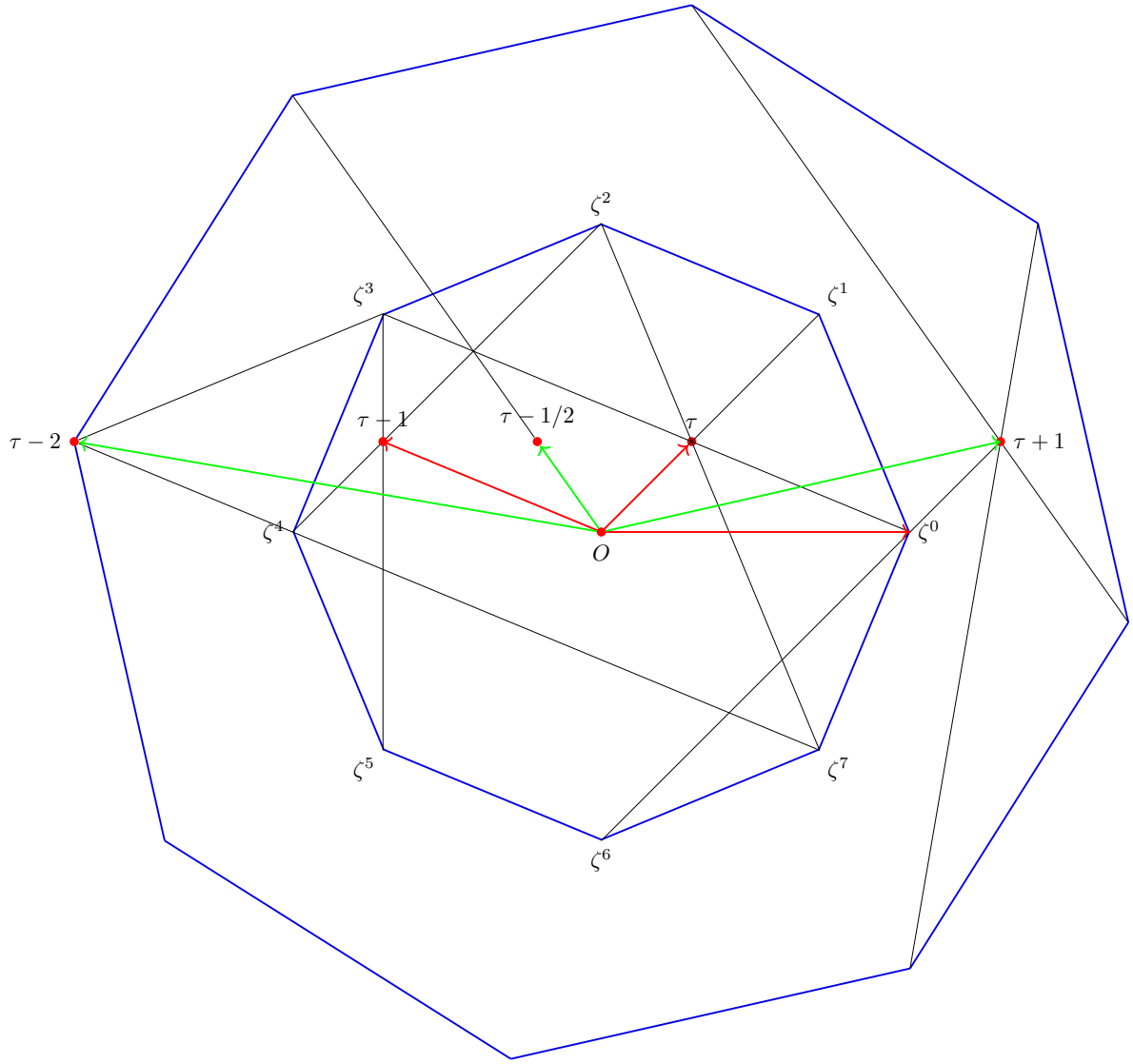


Figure 1. $\tau = \zeta_8^2 - \zeta_8 + 1$

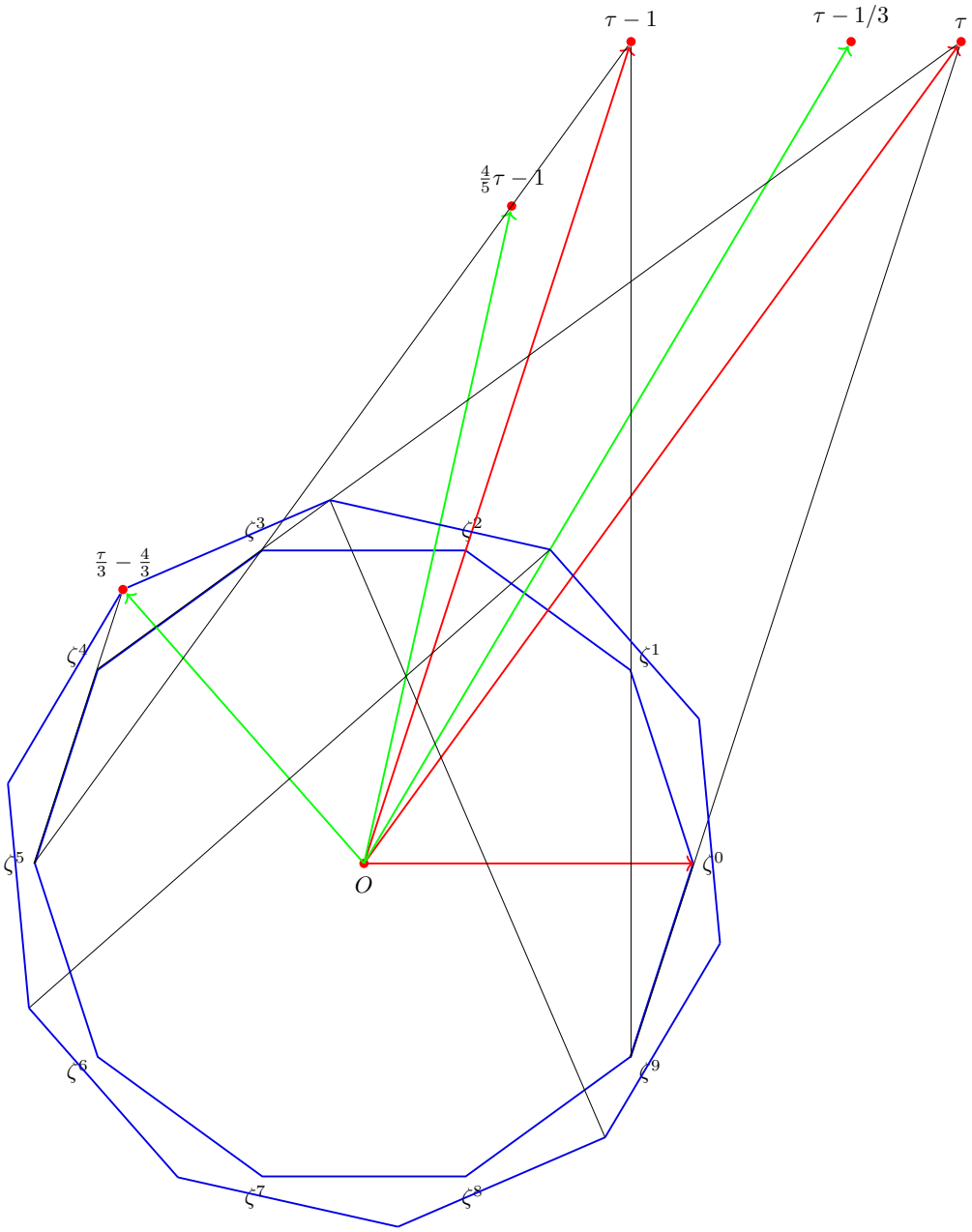


FIGURE 2. $\tau = \zeta_{10}^4 + 2\zeta_{10} + 2$

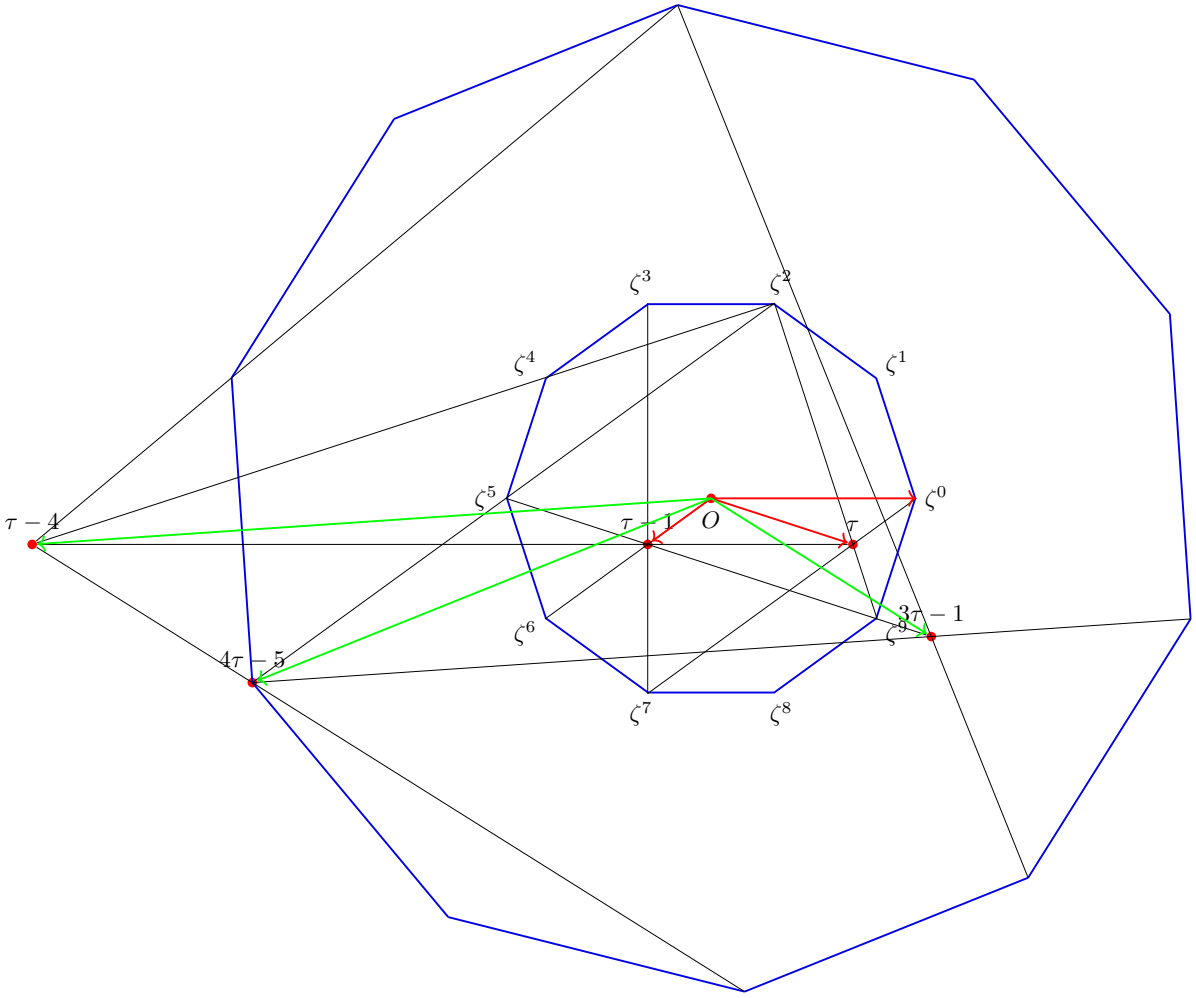


FIGURE 3. $\tau = \zeta_{10}^2 - 2\zeta_{10} + 2$

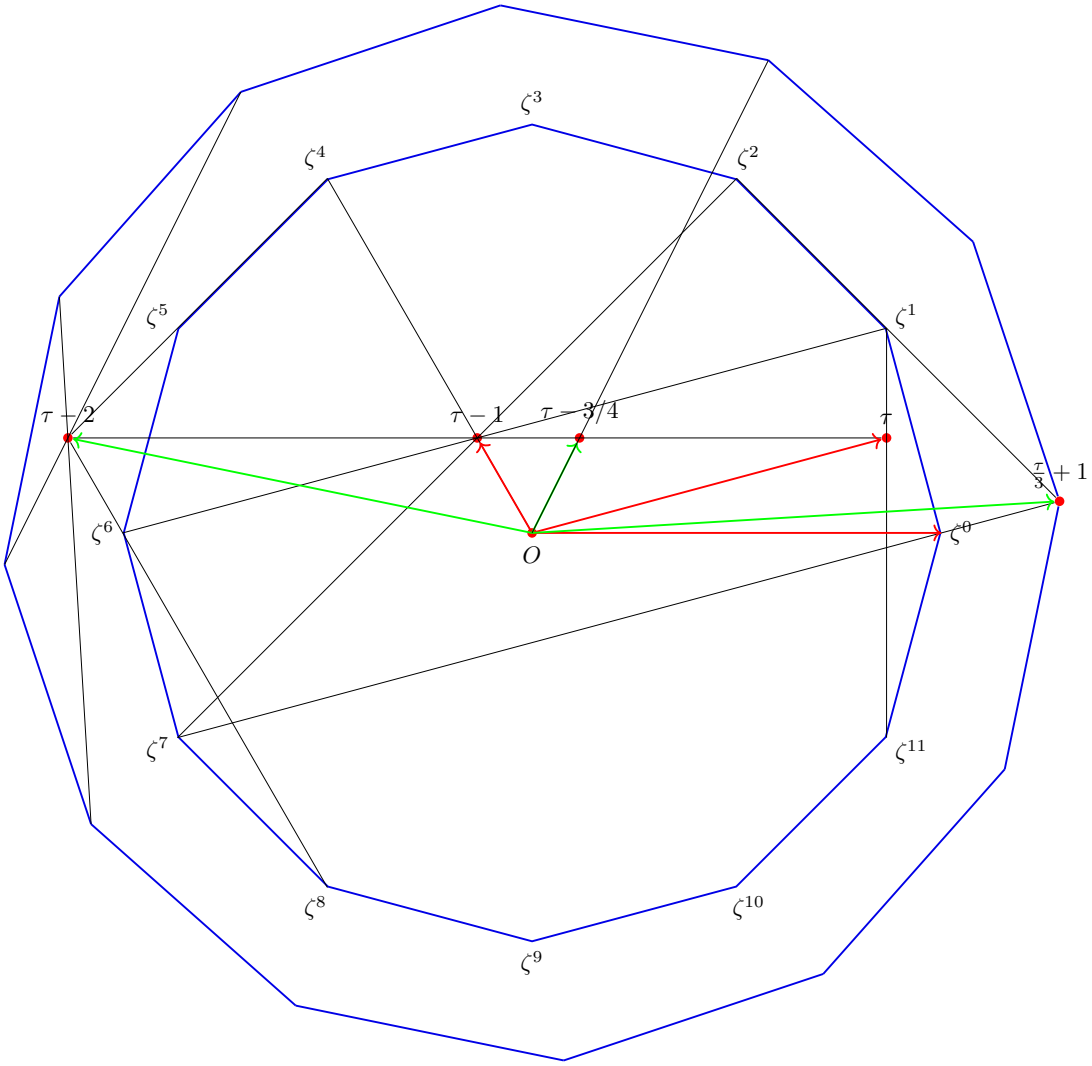


FIGURE 4. $\tau = -2\zeta_{12}^3 + 2\zeta_{12}^2 + \zeta_{12} - 1$

Appendix A. A euclidean proof

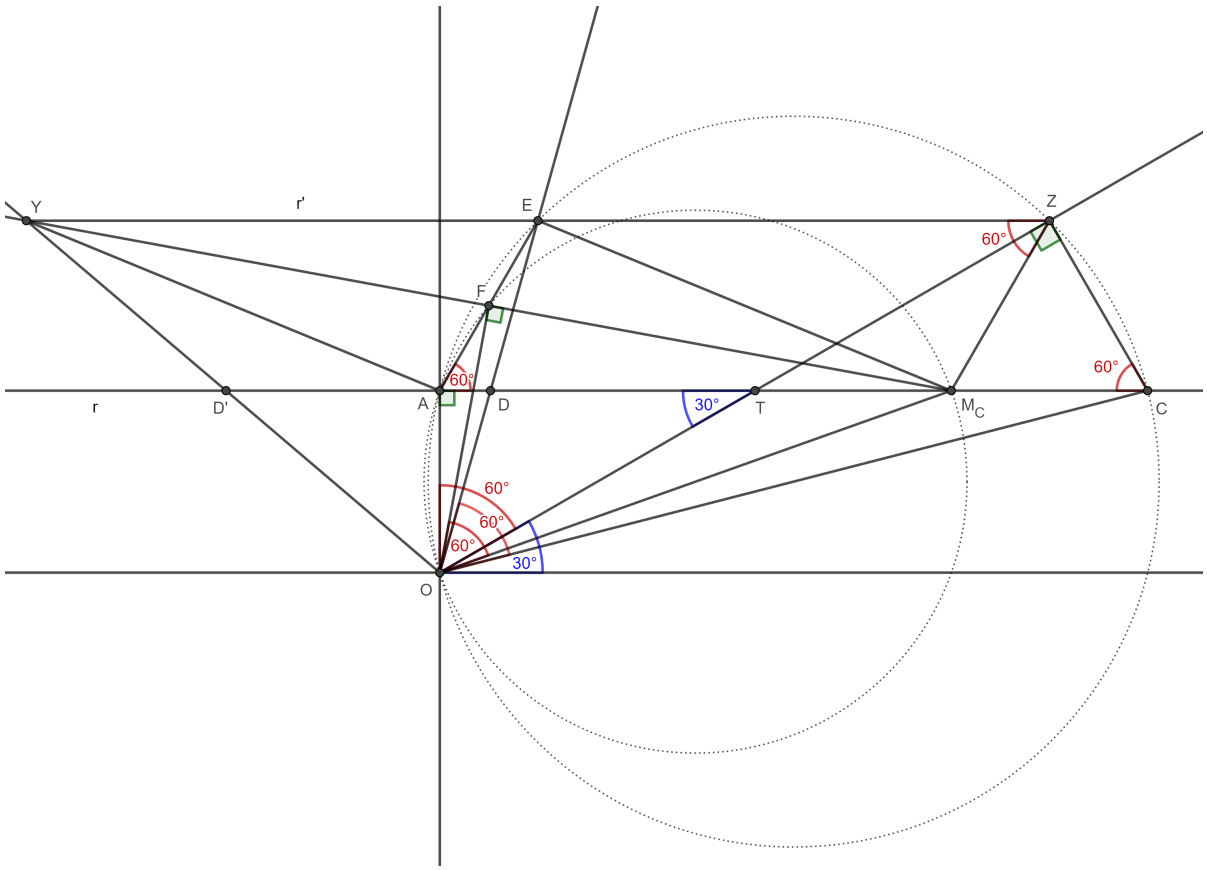
In keeping with the elementary nature of the question and of the objects involved, we present here a Euclidean explanation for the family $(\zeta_6, \zeta_6^2, \zeta_6^2, 2d, \frac{1}{2}c, c, d)$ from Theorem 6.4 (6), at least in the case $cd < 0$. In the following geometric construction the points D', D, T, M_C, C, O correspond respectively to $\tau + 2d, \tau + d, \tau, \tau + \frac{1}{2}c, \tau + c, 0$.

Proposition A.1. Consider 5 points on a line r, labelled D', D, T, M_C, C in this order, and let O be a point that does not lie on r. Assume that:

(1) the angle $\angle OTD$ is 30°;
(2) the angle $\angle COD$ is 60°;
(3) M_C is the midpoint of TC;
(4) the angle $\angle M_C OD'$ is 120°.

Then D is the midpoint of the segment $D'T$.

Proof.



Letting A be the projection of O on r, define a point E by requiring that the triangles EAO and CTO are similar. In other words, EAO is the image of CTO under the unique spiral similarity of centre O that carries T to A. Under this spiral similarity, M_C is sent to the midpoint F of AE. The similarity implies $\angle OAE = \angle OTC = \angle OAT + \angle TOA = \angle OAT + (90° - 30°)$, and on the other hand $\angle OAE = \angle OAT + \angle TAE$, whence $\angle TAE = 60°$. We also have $\angle M_C OF = (\angle COD - \angle COM_C) + \angle DOF$, and the spiral similarity gives $\angle DOF = \angle COM_C$, so $\angle M_C OF = \angle COD = 60°$. Thus, in the quadrilateral $FAOM_C$ we have $\angle M_C OF = 60° = \angle M_C AF$, so the points F, A, O, M_C all lie on the same circle. In particular, $\angle OFM_C = \angle OAM_C$ is a right angle. We further have $\angle COE = \angle COD = 60° = \angle CAE$, so the quadrilateral $COAE$ is also cyclic.

Let Y be the intersection of the ray OD' with the ray $M_C F$. The angle $\angle M_C OY = \angle M_C OD'$ is equal to 120° by assumption. Since $FAOM_C$ is cyclic, the angle $\angle YM_C O = \angle FM_C O$ is supplementary to $\angle FAO = \angle FAM_C + \angle M_C AO = 150°$, so $\angle YM_C O = 30°$. Thus, looking at the triangle $M_C OY$ we obtain $\angle OYM_C = 180° - \angle M_C OY - \angle YM_C O = 30°$, hence the triangle $M_C OY$ is isosceles. We

already know that OF is the altitude relative to its base, hence we obtain that OF is also its median: $M_CF = FY$. As already noticed, we also have $EF = FA$. Thus, the diagonals of the quadrilateral AM_CEY bisect each other, and therefore AM_CEY is a parallelogram. In particular, the line r' through E and Y is parallel to r. Let Z be the point of intersection between r' and the ray OT. We now show that $ZEAM_C$ is a parallelogram.

Note first that since r, r' are parallel we have $\angle TZE = \angle OTD = 30° = 180° - \angle OAE$, so Z lies on the same circle as E, A, O (which we know to also pass through C). Hence, $\angle CZO = \angle CAO = 90°$. In particular, the triangle TZC has $\angle TZC = 90°$ and $\angle CTZ = \angle OTA = 30°$, so $\angle ZCT = 60°$. In this triangle, the segment ZM_C is the median relative to the hypotenuse, so $ZM_C = \frac{1}{2}TC = M_CC$ and $CZ = \frac{1}{2}CT = M_CC$: the triangle ZCM_C is therefore equilateral, and $M_CZC = 60°$. Looking at the circle through Z, E, A, O, C we obtain $\angle EZC = 180° - \angle COE = 180° - \angle COD = 120°$, and by difference with $\angle M_CZC = 60°$ we get $\angle M_CZE = 60°$. Thus, M_CZ is parallel to EA, and since ZE is parallel to AM_C we obtain as desired that $ZEAM_C$ is a parallelogram.

Finally, the triangles TOD' and ZOY are similar (their sides being parallel in pairs), and the homotethy of centre O that takes T to Z and D' to Y also takes D to E. Hence, D is the midpoint of $D'T$ if and only if E is the midpoint of YZ. We have already shown that $EYAM_C$ and $ZEAM_C$ are parallelograms, so we have $YE = AM_C = EZ$, and the claim follows. □

Conflict of interest statement

On behalf of all authors, the corresponding author states that there is no conflict of interest.

Data availability statement

We do not analyse or generate any datasets, because our work proceeds within a theoretical and mathematical approach. The source code and output files relative to the computations required in the paper are available at

https://github.com/DavideLombardoMath/rational-angles-lattices.

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